Soluzioni II Esonero/I Appello - 30/01/2017
Analisi Matematica 1 (canale Dam-K)

$\boxed{\textbf{1)}\text{ Studiare il grafico della funzione } f(x) = \dfrac{x^2-16}{x+5}.}$

Dominio=$\{x \neq -5\}$ Segno di $f(x)$: $f > 0$ in $(-5,-4) \cup (4,+\infty)$

Asintoti verticali: $\lim_{x\to(-5)^{\pm}} f(x) = \pm\infty$

Asintoti orizz.i/obl.: $\lim_{x\to\pm\infty} \dfrac{f(x)}{x} = 1$, $\lim_{x\to\pm\infty}[f(x)-x] = -\lim_{x\to\pm\infty} \dfrac{5x+16}{x+5} = -5$

$f'(x) = \frac{(x+2)(x+8)}{(x+5)^2}$ Segno di $f'(x)$: $f' > 0$ in $(-\infty,-8) \cup (-2,+\infty)$

$\boxed{\textbf{1)}\text{ Studiare il grafico della funzione } f(x) = \dfrac{x^2-6x}{x+2}.}$

Dominio=$\{x \neq -2\}$ Segno di $f(x)$: $f > 0$ in $(-2,0) \cup (6,+\infty)$

Asintoti verticali: $\lim_{x\to(-2)^{\pm}} f(x) = \pm\infty$

Asintoti orizz./obl.: $\lim_{x\to\pm\infty} \dfrac{f(x)}{x} = 1$, $\lim_{x\to\pm\infty}[f(x)-x] = -8 \lim_{x\to\pm\infty} \dfrac{x}{x+2} = -8$

$f'(x) = \frac{(x-2)(x+6)}{(x+2)^2}$ Segno di $f'(x)$: $f' > 0$ in $(-\infty,-6) \cup (2,+\infty)$

$\boxed{\textbf{2)}\text{ Calcolare il limite } \lim_{x\to 0} \dfrac{1-\cos(\sqrt{2}x)-\sin^2 x}{(e^{x^4}-1)\sqrt[3]{8-x}}.}$

$$\lim_{x\to 0} \frac{1-\cos(\sqrt{2}x)-\sin^2 x}{(e^{x^4}-1)\sqrt[3]{8-x}} = \lim_{x\to 0} \frac{x^2 - \frac{x^4}{6} + O(x^6) - (x - \frac{x^3}{6} + O(x^5))^2}{2x^4} \frac{x^4}{e^{x^4}-1}$$
$$= \lim_{x\to 0} \frac{-\frac{x^4}{6} + \frac{x^4}{3} + O(x^6)}{2x^4} = \frac{1}{12}$$

$\boxed{\textbf{2)}\text{ Calcolare il limite } \lim_{x\to 0} \dfrac{e^{-x^2}-\cos^2 x}{\tan(x^4)\sin(\frac{\pi}{2}+x)}.}$

$$\lim_{x\to 0} \frac{e^{-x^2}-\cos^2 x}{\tan(x^4)\sin(\frac{\pi}{2}+x)} = \lim_{x\to 0} \frac{1 - x^2 + \frac{x^4}{2} + O(x^6) - (1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6))^2}{x^4} \frac{x^4}{\sin(x^4)} \cos(x^4)$$
$$= \lim_{x\to 0} \frac{\frac{x^4}{2} - \frac{x^4}{4} - \frac{x^4}{12} + O(x^6)}{x^4} = \frac{1}{6}$$

$\boxed{\textbf{3)} \text{ Calcolare il limite } \lim_{x\to 0} \frac{(1+x)\ln(1+x)-\sin x}{1-\cos x}.}$

$$\lim_{x\to 0} \frac{(1+x)\ln(1+x)-\sin x}{1-\cos x} = \lim_{x\to 0} \frac{\ln(1+x)+1-\cos x}{\sin x} = \lim_{x\to 0}[\frac{\ln(1+x)}{x}\frac{x}{\sin x} + x\frac{1-\cos x}{x^2}\frac{x}{\sin x}] = 1$$

$\boxed{\textbf{3)} \text{ Calcolare il limite } \lim_{x\to 0} \frac{e^x\cos x - 1 - x\cos x}{\ln(1+x^2)}.}$

$$\lim_{x\to 0} \frac{e^x\cos x - 1 - x\cos x}{\ln(1+x^2)} = \lim_{x\to 0} \frac{-\sin x(e^x - x) + \cos x(e^x-1)}{\frac{2x}{1+x^2}} = \lim_{x\to 0}[-(e^x-x)\frac{\sin x}{2x} + \cos x\frac{e^x-1}{2x}] = 0$$

$\boxed{\textbf{4)} \text{ Calcolare il limite } \lim_{n\to+\infty} \ln(n!)\left[\sqrt{n^6+n} - \sqrt{n^6+1}\right].}$

$$\lim_{n\to+\infty} \ln(n!)\left[\sqrt{n^6+n} - \sqrt{n^6+1}\right] = \lim_{n\to+\infty} \frac{(n-1)\ln(n!)}{\sqrt{n^6+n}+\sqrt{n^6+1}} = 0$$

poiché

$$|\frac{(n-1)\ln(n!)}{\sqrt{n^6+n}+\sqrt{n^6+1}}| \le \frac{(n-1)n\ln n}{\sqrt{n^6+n}+\sqrt{n^6+1}} \le \frac{\ln n}{2n} \to 0$$

$\boxed{\textbf{4)} \text{ Calcolare il limite } \lim_{n\to+\infty} (1+\frac{1}{n^2})^{\ln(n!)}.}$

$$\lim_{n\to+\infty} (1+\frac{1}{n^2})^{\ln(n!)} = \lim_{n\to+\infty} [(1+\frac{1}{n^2})^{n^2}]^{\frac{\ln(n!)}{n^2}} = 1$$

poiché

$$|\frac{\ln(n!)}{n^2}| \le \frac{n\ln n}{n^2} = \frac{\ln n}{n} \to 0$$

$\boxed{\textbf{5)} \text{ Calcolare } \int \frac{1+2\cos x}{\sin x(3-\cos x)}dx.}$

Poniamo $t = \tan\frac{x}{2}$ ottenendo

$$\int \frac{1+2\cos x}{\sin x(3-\cos x)}dx = \frac{1}{2}\int \frac{3-t^2}{t(2t^2+1)}dt = \frac{1}{2}\int [\frac{3}{t} - \frac{7t}{2t^2+1}]dt = \frac{3}{4}\ln\frac{1-\cos x}{1+\cos x} - \frac{7}{8}\ln\frac{3-\cos x}{1+\cos x} + c$$

$\boxed{\textbf{5)} \text{ Calcolare } \int \frac{\cos^2 x + \sin x\cos x}{-\cos^2 x + \sin x \cos x}dx.}$ Poniamo $t = \tan x$ ottenendo

$$\int \frac{\cos^2 x + \sin x\cos x}{-\cos^2 x + \sin x \cos x}dx = \int \frac{t+1}{(t^2+1)(t-1)}dt = \int [\frac{1}{t-1} - \frac{t}{t^2+1}]dt = \ln|\sin x - \cos x| + c$$

$\boxed{\textbf{6)}\text{ Calcolare } \int \frac{dx}{4(\sqrt{x}-2\sqrt[4]{x}-3)}.}$ Poniamo $t=\sqrt[4]{x}$ ottenendo

$$\begin{aligned}\int \frac{dx}{4(\sqrt{x}-2\sqrt[4]{x}-3)} &= \int \frac{t^3}{t^2-2t-3}dt = \int [t+2+\frac{7t+6}{(t-3)(t+1)}]dt = \int [t+2+\frac{27}{4}\frac{1}{t-3}+\frac{1}{4}\frac{1}{t+1}]dt \\ &= \frac{\sqrt{x}}{2}+2\sqrt[4]{x}+\frac{27}{4}\ln|\sqrt[4]{x}-3|+\frac{1}{4}\ln|\sqrt[4]{x}+1|+c\end{aligned}$$

$\boxed{\textbf{6)}\text{ Calcolare } \int \frac{x+2}{x\sqrt[3]{x+1}}dx.}$ Poniamo $t=\sqrt[3]{x+1}$ ottenendo

$$\begin{aligned}&\int \frac{x+2}{x\sqrt[3]{x+1}}dx = 3\int \frac{t(t^3+1)}{t^3-1}dt = \int [3t+\frac{6t}{(t-1)(t^2+t+1)}]dt = \int [3t+\frac{2}{t-1}-2\frac{t-1}{t^2+t+1}]dt \\ &= \int [3t+\frac{2}{t-1}-\frac{2t+1}{t^2+t+1}+\frac{4}{(\frac{2t+1}{\sqrt{3}})^2+1}]dt \\ &= \frac{3}{2}(x+1)^{\frac{2}{3}}+2\ln|\sqrt[3]{x+1}-1|-\ln[(x+1)^{\frac{2}{3}}+(x+1)^{\frac{1}{3}}+1]+2\sqrt{3}\arctan(\frac{2\sqrt[3]{x+1}+1}{\sqrt{3}})+c\end{aligned}$$

$\boxed{\textbf{7)}\text{ Discutere la convergenza di } \sum_{n=2}^{\infty}(\frac{n+1}{n-1})^{-n^2}.}$ Converge dal criterio della radice $n-$esima:

$$\sqrt[n]{a_n} = (\frac{n-1}{n+1})^n = [(1-\frac{2}{n+1})^{-\frac{n+1}{2}}]^{-\frac{2n}{n+1}} \to e^{-2} < 1$$

$\boxed{\textbf{7)}\text{ Discutere la convergenza di } \sum_{n=1}^{\infty}(1-\sin\frac{1}{n})^{n^2}.}$ Converge dal criterio della radice $n-$esima:

$$\sqrt[n]{a_n} = (1-\sin\frac{1}{n})^n = [(1-\sin\frac{1}{n})^{-\frac{1}{\sin\frac{1}{n}}}]^{-\frac{\sin\frac{1}{n}}{\frac{1}{n}}} \to e^{-1} < 1$$

$\boxed{\textbf{8)}\text{ Calcolare } \int \frac{dx}{2x+\sqrt{4x^2+x}}.}$ Poniamo $4x^2+x=(2x+t)^2$, ossia $x=\frac{t^2}{1-4t}$, ottenendo

$$\int \frac{dx}{2x+\sqrt{4x^2+x}} = 2\int \frac{1-2t}{1-4t}dt = \int [1+\frac{1}{1-4t}]dt = \sqrt{4x^2+x}-2x-\frac{1}{4}\ln|4\sqrt{4x^2+x}-8x-1|+c$$

$\boxed{\textbf{9)}\text{ Discutere la convergenza di } \sum_{n=1}^{\infty}\frac{1}{2^{2\ln n}}.}$ Converge poiché $2\ln 2>1$:

$$\sum_{n=1}^{\infty}\frac{1}{2^{2\ln n}} = \sum_{n=1}^{\infty}\frac{1}{n^{2\ln 2}} < +\infty$$