Generalized gluing for Einstein constraint equations

We construct a family of new solutions to the Einstein constraint equations by performing the generalized connected sum of two known compact \( m \)-dimensional constant mean curvature solutions \((M_1, g_1, \Pi_1)\) and \((M_2, g_2, \Pi_2)\) along a common isometrically embedded \( k \)-dimensional sub-manifold \((K, g_K)\). Away from the gluing locus the metric and the second fundamental form of the new solutions can be chosen as close as desired to the ones of the original solutions. The proof is essentially based on the conformal method and the geometric construction involves conformal transformations along the slices of the normal fiber bundle of \( K \), for this reason the codimension \( n := m - k \) of \( K \) in \( M_1 \) and \( M_2 \) is required to be \( \geq 3 \). In this sense our result is a generalization of the Isenberg-Mazzeo-Pollack gluing, which works for connected sum at points and in dimension 3. The solution we obtain for the Einstein constraint equations can be used to produce new short time vacuum solutions of the Einstein system on a Lorentzian \((m + 1)\)-dimensional manifold, as guaranteed by a well known result of Choquet-Bruhat.