BLOW-UP SOLUTIONS FOR LINEAR PERTURBATIONS OF THE YAMABE EQUATION

PIERPAOLO ESPOSITO, ANGELA PISTOIA, AND JÉRÔME VÉTOIS

Abstract. For a smooth, compact Riemannian manifold (M,g) of dimension $N \geq 3$, we are interested in the critical equation

$$\Delta_g u + \left(\frac{N-2}{4(N-1)} S_g + \varepsilon h\right) u = u^{\frac{N+2}{N-2}} \quad \text{in } M\,, \quad u > 0 \quad \text{in } M\,,$$

where Δ_g is the Laplace–Beltrami operator, S_g is the Scalar curvature of (M,g), $h \in C^{0,\alpha}(M)$, and ε is a small parameter.

1. Introduction

Letting (M,g) be a smooth, compact Riemannian N–manifold, $N \geq 3$, we consider the solutions $u \in C^{2,\alpha}$ of the problem

$$\Delta_g u + \kappa u = c u^p, \quad u > 0 \quad \text{in } M\,, \tag{1.1}$$

where $\Delta_g := -\operatorname{div}_g \nabla$ is the Laplace–Beltrami operator, $\kappa \in C^{0,\alpha}(M)$, $\alpha \in (0,1)$, $c \in \mathbb{R}$, and $p > 1$.

When $\kappa = \alpha_N S_g$ and $p = 2^* - 1$, where $\alpha_N := \frac{N-2}{4(N-1)}$, S_g is the Scalar curvature of (M,g) and $2^* := \frac{2N}{N-2}$ is the critical Sobolev exponent, equation (1.1) reads as

$$\Delta_g u + \frac{N-2}{4(N-1)} S_g u = c u^{\frac{N+2}{N-2}}, \quad u > 0 \quad \text{in } M\,, \tag{1.2}$$

and is referred to in the literature as the Yamabe problem. The constant c can be restricted to the values $-1/1$ or 0 depending on whether the *Yamabe invariant* of (M,g), namely

$$\mu_g(M) = \inf_{\widetilde{g} \in [g]} \left(\operatorname{Vol}_{\widetilde{g}}(M)^{\frac{2-N}{N}} \int_M S_{\widetilde{g}} dv_{\widetilde{g}} \right)$$

has negative/positive sign or vanishes, respectively, where $[g] = \{\phi g : \phi \in C^\infty(M),\, \phi > 0\}$ is the conformal class of g and $\operatorname{Vol}_{\widetilde{g}}(M)$ is the volume of the manifold $(M,\widetilde{g})$. If u is a solution of (1.2), then the metric $\widetilde{g} = u^{4/(N-2)} g$ has constant Scalar curvature and belongs to $[g]$.

The Yamabe problem, raised by H. Yamabe [42] in '60, was firstly solved by Trudinger [41] when $\mu_g(M) \leq 0$. In this case, the solution is unique (up to a normalization when $\mu_g(M) = 0$). In general, a solution of (1.2) can be found by a direct constrained minimization method. As shown by Aubin [1], the inequality

$$\mu_g(M) < \mu_{g_0}(\mathbb{S}^N), \tag{1.3}$$

where $(\mathbb{S}^N, g_0)$ is the round sphere, is the key ingredient to show compactness of minimizing sequences, a non-trivial fact in view of the non-compactness of the Sobolev embedding $H_1^2(M) \hookrightarrow L^{2^*}(M)$. $(\mathbb{S}^N, g_0)$ has already constant Scalar curvature. For manifolds (M,g)

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which are not conformally equivalent to $(\mathbb{S}^N, g_0)$ ($(M,g) \neq (\mathbb{S}^N, g_0)$ for short) with $\mu_g(M) > 0$, the Yamabe equation (1.2) has been solved via (1.3) by:

- Aubin [1] in the non-locally conformally flat case with $N \geq 6$, by exploiting the non-vanishing of the Weyl curvature tensor Weyl_g of (M,g) in the construction of local test functions;
- Schoen [37] when either $N = 3, 4, 5$ or $(M,g) \neq (\mathbb{S}^N, g_0)$ is locally conformally flat, by exploiting the Positive Mass Theorem by Schoen–Yau [39, 40] in the construction of global test functions

(see also Lee–Parker [22] for a unified approach).

From now on, we restrict our attention to the case where (M,g) has *positive Yamabe invariant* $\mu_g(M) > 0$. When $(M,g) \neq (\mathbb{S}^N, g_0)$, Schoen [38] addressed the question of the compactness of Yamabe metrics, and he proved the compactness to be true in the locally conformally flat case [38]. Recently, compactness of Yamabe metrics has been proved to be true for a general manifold $(M,g) \neq (\mathbb{S}^N, g_0)$ of dimension $N \leq 24$ by Khuri–Marques–Schoen [21]. Unexpectedly, compactness of Yamabe metrics has revealed to be false in general in dimensions $N \geq 25$ by Brendle [5] and Brendle–Marques [6]. Previous contributions where the compactness of Yamabe metrics is proved in lower dimensions are by Li–Zhu [27] ($N = 3$), Druet [10] ($N \leq 5$), Marques [28] ($N \leq 7$), and Li–Zhang [24–26] ($N \leq 11$). In all these results, it is shown that sequences of solutions $(u_k)_{k\in\mathbb{N}}$ of (1.1) with $\kappa \equiv \alpha_N S_g$, $c = 1$, and exponents $(p_k)_{k\in\mathbb{N}}$ in $[1+\varepsilon_0, 2^*-1]$, $\varepsilon_0 > 0$ fixed, are pre-compact in $C^{2,\alpha}(M)$, $\alpha \in (0,1)$.

When $\kappa \not\equiv \alpha_N S_g$, the situation is different. When $\kappa < \alpha_N S_g$, Druet [9, 10] (see also Druet–Hebey [13] and Druet–Hebey–Vétois [16]) proved that compactness does hold for equation (1.1) with $c = 1$ and exponents p in the range $[1+\varepsilon_0, 2^*-1]$, for all dimensions $N \geq 3$ (in case $N = 3$, it is possible to write a more refined condition on the mass, see Li–Zhu [27]). As shown in Micheletti–Pistoia–Vétois [29] and Pistoia–Vétois [32], in dimensions $N \geq 4$, such a compactness result does not hold when $\kappa(\xi_0) > \alpha_n S_g(\xi_0)$ at some point $\xi_0 \in M$ with a nondegeneracy assumption at ξ_0, and, see [29], compactness does not hold either in the supercritical range $p > 2^*-1$ when $\kappa(\xi_0) < \alpha_N S_g(\xi_0)$ at some point $\xi_0 \in M$. We also refer to Robert–Vétois [36, Theorem 2.3] where a special non-compactness result is obtained in dimension $N = 6$ for potentials $\kappa > \alpha_N S_g$ (see also Druet [9] and Druet–Hebey [11, 12] in case of $(M,g) = (\mathbb{S}^N, g_0)$ with $N = 6$). In the locally conformally flat case with $N \geq 4$, Hebey–Vaugon [19] proved that there always exists $\widetilde{g} \in [g]$ such that the equation $\Delta_{\widetilde{g}} u + \alpha_N \max_M(S_{\widetilde{g}})u = u^{2^*-1}$ in M is not compact. In case $(M,g) = (\mathbb{S}^N, g_0)$ with $N \geq 5$ and when $(\kappa - \alpha_N S_g)$ is a positive constant, Chen–Wei–Yan [8] proved that equation (1.1) with $c = 1$ and $p = 2^*-1$ is not compact (see also the constructions by Hebey–Wei [20] in case $N = 3$).

When the potential κ varies, for manifolds $(M,g) \neq (\mathbb{S}^N, g_0)$ with $\mu_g(M) > 0$, Druet [10] (see also Druet–Hebey [14]) proved that sequences of solutions $(u_k)_{k\in\mathbb{N}}$ of (1.1) with $c = 1$, exponents $(p_k)_{k\in\mathbb{N}}$ in $[1+\varepsilon_0, 2^*-1]$, and potentials $(\kappa_k)_{k\in\mathbb{N}}$, are pre-compact in $C^{2,\alpha}(M)$, $\alpha \in (0,1)$, when $n = 3, 4, 5$ provided that $\kappa_k \leq \alpha_n S_g$. The same result is strongly expected to be true in the locally conformally flat case and generally for $N \leq 24$.

The aim of the paper is to investigate the effect of positive perturbations of the geometric potential by exhibiting the failure of compactness properties for the equation

$$\Delta_g u + (\alpha_N S_g + \varepsilon h)u = u^{2^*-1}, \quad u > 0 \quad \text{in } M\,, \tag{1.4}$$

where $h \in C^{0,\alpha}(M)$, $\alpha \in (0,1)$, with $\max_M h > 0$ and $\varepsilon > 0$ is a small parameter.

A family $(u_\varepsilon)_\varepsilon$ of solutions to equation (1.4) is said to *blow up* at some point $\xi_0 \in M$ if there holds $\sup_U u_\varepsilon \to +\infty$ as $\varepsilon \to 0$, for all neighborhoods U of ξ_0 in M. Letting

$$E(\xi) := \frac{h(\xi)}{\left|\mathrm{Weyl}_g(\xi)\right|_g},$$

our main result is:

Theorem 1.1. *Let $(M,g) \neq (\mathbb{S}^N, g_0)$ be a smooth, compact, non-locally conformally flat Riemannian manifold with $N \geq 6$ and $\mu_g(M) > 0$. Let $h \in C^{0,\alpha}(M)$, $\alpha \in (0,1)$, so that $\max_M h > 0$ and $\inf\{|\mathrm{Weyl}_g(x)|_g : h(x) > 0\} > 0$. Then for $\varepsilon > 0$ small, equation (1.4) has a solution u_ε such that the family $(u_\varepsilon)_\varepsilon$ blows up, up to a sub-sequence, as $\varepsilon \to 0$ at some point ξ_0 so that $E(\xi_0) = \max_M E$.*

Introducing the "reduced energy" $\widetilde{E} : (0,\infty) \times M \to \mathbb{R}$ defined as

$$\widetilde{E}(d,\xi) = c_2 d^2 h(\xi) - c_3 d^4 \left|\mathrm{Weyl}_g(\xi)\right|_g^2$$

with $c_2, c_3 > 0$, Theorem 1.1 is an easy consequence of the following more general result:

Theorem 1.2. *Let $(M,g) \neq (\mathbb{S}^N, g_0)$ be a smooth, compact, non-locally conformally flat Riemannian manifold with $N \geq 6$ and $\mu_g(M) > 0$, and $h \in C^{0,\alpha}(M)$, $\alpha \in (0,1)$. Assume that there exists a C^0–stable critical set $\mathcal{D} \subset (0,\infty) \times M$ of $\widetilde{E}$. Then for $\varepsilon > 0$ small, equation (1.4) has a solution u_ε such that the family $(u_\varepsilon)_\varepsilon$ blows up, up to a sub-sequence, at some $\xi_0 \in \pi(\mathcal{D})$, where $\pi : (0,\infty) \times M \to M$ is the projection operator onto the second component.*

According to Li [23], we say that a compact set $\mathcal{D} \subset (0,\infty) \times M$ of critical points of $\widetilde{E}$ is a C^0*–stable critical set* of $\widetilde{E}$ if for any compact neighborhood U of $\mathcal{D}$ in $(0,\infty) \times M$, there exists $\delta > 0$ such that, if $\mathcal{J} \in C^1(U)$ and $\|\mathcal{J} - \widetilde{E}\|_{C^0(U)} \leq \delta$, then $\mathcal{J}$ has at least one critical point in U.

Given $\xi \in M$ so that $h(\xi) > 0$, define $d(\xi)$ as

$$d(\xi) = \left(\frac{c_2 h(\xi)}{2c_3 \left|\mathrm{Weyl}_g(\xi)\right|_g^2} \right)^{1/2}$$

with the convention that $d(\xi) = +\infty$ if $\mathrm{Weyl}_g(\xi) = 0$. Given $\xi \in M$ with $h(\xi) > 0$, the function $\widetilde{E}$ is increasing for $d \in (0, d(\xi))$ and, if $d(\xi) < +\infty$, achieves its global maximum in d at $d(\xi)$. Since

$$\widetilde{E}(d(\xi),\xi) = \frac{c_2^2 h^2(\xi)}{4c_3 \left|\mathrm{Weyl}_g(\xi)\right|_g^2} = \frac{c_2^2}{4c_3} E(\xi)^2,$$

in order to derive Theorem 1.1, the set $\mathcal{D}$ in Theorem 1.2 is constructed as

$$\mathcal{D} = \{(d(\xi),\xi) : \xi \in M \text{ s.t. } E(\xi) = \max_M E\},$$

which is clearly a C^0*–stable critical set* of $\widetilde{E}$. Since $d(\xi)$ is a maximum point of $\widetilde{E}$ in d, neither minimum points of E, nor saddle points of E can provide any C^0*–stable critical set* of $\widetilde{E}$.

Let us finally compare problem (1.4) with its Euclidean counter-part on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 4$, with homogeneous Dirichlet boundary condition:

$$\Delta_{\mathrm{Eucl}} u + \lambda u = u^{2^*-1} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = \text{ on } \partial\Omega. \tag{1.5}$$

For $\lambda \geq 0$, a direct minimization method (for the corresponding Rayleigh quotient) never gives rise to any solution of (1.5), and no solutions exist at all if Ω is star-shaped as shown by Pohožaev [33]. Moreover, following the arguments developed by Ben Ayed–El Mehdi–Grossi–Rey [3], problem (1.5) has never any solution with a single blow-up point as $\lambda \to 0^+$. The effect of the geometry, which is crucial to provide a solution for the Yamabe problem (corresponding to $\lambda = 0$ in (1.5)) by minimization, is also relevant to producing solutions of (1.4) (corresponding to $\lambda \to 0^+$ in (1.5)) with a single blow-up point as stated in Theorem 1.1. When $\lambda < 0$, solutions of (1.5) can be found by direct minimization as shown by Brezis–Nirenberg [7], and exhibit a single blow-up point as $\lambda \to 0^-$ as shown by Han [18], in contrast with the compactness property proved by Druet [9,10]. Solutions of (1.5) with a single blow-up point, see Rey [34,35], and with multiple blow-up points, see Bahri–Li–Rey [2] and Musso–Pistoia [30], as $\lambda \to 0^-$ have been constructed in a very general way.

We attack the existence issue of blowing-up solutions by a perturbative method, referred to in the literature as the non-linear Lyapunov–Schmidt reduction. Such a method is well known and the main point is to produce a suitable ansatz for the solutions. In the non-locally conformally flat case with $N \geq 6$ the basic ansatz is like in Aubin [1], but, see Section 2, needs to be slightly corrected via linearization so to account for the local geometry. A similar idea has been used for the prescribed $Q-$curvature problem by Pistoia–Vaira [31], the fourth-order analogue of the Yamabe problem. An alternative and more geometrical approach can be devised based on the conformal covariance of $\Delta_g + \alpha_N S_g$. The main point is to allow the metric g to vary in the conformal class so to gain flatness at each point $\xi \in M$, and this approach allows us, see Esposito–Pistoia–Vétois [17], to cover in an unified way also the remaining cases $N = 4, 5$ or (M, g) locally conformally flat with $N \geq 6$ (the case $N = 3$ is always excluded by the compactness result of Li–Zhu [27]). The aim of this paper is at the same time to advertise the general result contained in [17], and to provide a simpler and more intuitive proof in a special case. Thanks to the solvability theory of the linearized operator, we are led to study critical points of a finite-dimensional functional $\mathcal{J}_\varepsilon$, and a key step is to obtain in Section 3 an asymptotic expansion of $\mathcal{J}_\varepsilon$ by identifying the "reduced energy" $\widetilde{E}$ as the main order term. In Section 4, we describe the main steps of the non-linear Lyapunov-Schmidt reduction, and we deduce our general result Theorem 1.2.

2. The correcting term towards an improved ansatz

Letting

$$U(r) = \left(\frac{\sqrt{N(N-2)}}{1+r^2} \right)^{\frac{N-2}{2}}, \tag{2.1}$$

we aim to solve

$$\Delta V + pU^{p-1}V = \frac{1}{3} \sum_{i,j=1}^{N} R_{ij}(\xi) \frac{y^i y^j}{|y|} \partial_r U + \alpha_N S_g(\xi) U\,, \tag{2.2}$$

where $p = \frac{N+2}{N-2}$ and R_{ij} are the components of the Ricci tensor Ric_g of (M, g) in geodesic coordinates. Here, $\Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial y_i^2}$ is the Euclidean laplacian with the standard sign convention, and $U(|y|)$ is the unique positive radial solution of $-\Delta U = U^p$ with $U(0) = \max_{\mathbb{R}^N} U = [N(N-2)]^{\frac{N-2}{4}}$.

Since $S_g(\xi)=\sum_{i=1}^{N} R_{ii}(\xi)$, a straightforward computation shows that

$$V(y)=[N(N-2)]^{\frac{N-2}{4}}\left(\frac{|y|^2+3}{12(1+|y|^2)^{\frac{N}{2}}}\sum_{i,j=1}^{N} R_{ij}(\xi)y^iy^j-\frac{S_g(\xi)}{24(N-1)}\frac{|y|^4+3}{(1+|y|^2)^{\frac{N}{2}}}\right) \tag{2.3}$$

is a solution of (2.2) as we were searching for.

Let $0<r_0<i_g(M)$, where $i_g(M)$ is the injectivity radius of (M,g). Take χ a smooth cutoff function such that $0\le\chi\le 1$ in $\mathbb{R}$, $\chi\equiv 1$ in $[-r_0/2,r_0/2]$, and $\chi\equiv 0$ out of $[-r_0,r_0]$. For any point ξ in M and for any positive real number μ, we define the functions $\mathcal{U}_{\mu,\xi}$ and $\mathcal{V}_{\mu,\xi}$ on M by

$$\mathcal{U}_{\mu,\xi}(z)=\chi\left(d_g(z,\xi)\right)U_\mu\left(d_g(z,\xi)\right)\,,\ \ \mathcal{V}_{\mu,\xi}(z)=\chi\left(d_g(z,\xi)\right)V_\mu\left(\exp_\xi^{-1}(z)\right),$$

where d_g is the geodesic distance in (M,g) and $\exp_\xi^{-1}$ is the geodesic coordinate system. Here, U_μ and V_μ are defined as

$$U_\mu(x)=\mu^{-\frac{N-2}{2}}U\left(\frac{x}{\mu}\right)\,,\quad V_\mu(x)=\mu^{-\frac{N-2}{2}}V\left(\frac{x}{\mu}\right),$$

obtained by scaling U and V in (2.1) and (2.3), respectively. Since $\mu_g(M)>0$ implies the coercivity of the conformal laplacian $\Delta_g+\alpha_N S_g$, let $i^*:L^{\frac{2N}{N+2}}(M)\to H^1_g(M)$ be the bounded operator defined as follows: the function $u=i^*(w)$ is the unique solution in $H^1_g(M)$ of the equation $\Delta_g u+\alpha_N S_g u=w$ in M. Problem (1.4) re-writes as

$$u=i^*\left[u_+^p-\varepsilon hu\right]\,, \tag{2.4}$$

and we look for solutions of (2.4) in the form

$$u_\varepsilon(z)=\mathcal{W}_{\mu,\xi}(z)+\phi_\varepsilon(z),\quad \mathcal{W}_{\mu,\xi}=\mathcal{U}_{\mu,\xi}+\mu^2\mathcal{V}_{\mu,\xi}\,, \tag{2.5}$$

where $\xi\in M$, $\mu>0$ is small and ϕ_ε is a small remainder term.

First of all, we introduce the error term

$$\mathcal{R}_{\mu,\xi}=\mathcal{W}_{\mu,\xi}-i^*\left[(\mathcal{W}_{\mu,\xi})_+^p-\varepsilon h\mathcal{W}_{\mu,\xi}\right]\,. \tag{2.6}$$

We want to point out that the choice of the ansatz in (2.5) with the extra term $\mathcal{V}_{\mu,\xi}$ is motivated by the need that the error term has to be small enough. Indeed, the error term is estimated as follows.

Lemma 2.1. *Let $N\ge 6$. There exists a positive constant $C_0>0$ such that for any μ small and ξ in M there holds*

$$\|\mathcal{R}_{\mu,\xi}\|\le C_0\begin{cases}\mu^{\frac{N-2}{2}}+\varepsilon\mu^2|\ln\mu|^{\frac{2}{3}} & \text{if } N=6\\ \mu^{\frac{N-2}{2}}+\varepsilon\mu^2 & \text{if } N=7\\ \mu^3|\ln\mu|^{\frac{5}{8}}+\varepsilon\mu^2 & \text{if } N=8\\ \mu^3+\varepsilon\mu^2 & \text{if } N=9\\ \mu^{2\frac{N+2}{N-2}}+\varepsilon\mu^2 & \text{if } N\ge 10.\end{cases} \tag{2.7}$$

Proof. It is enough to estimate the $L^{\frac{2N}{N+2}}$–norm of

$$\Delta_g\mathcal{W}_{\mu,\xi}+(\alpha_N S_g+\varepsilon h)\mathcal{W}_{\mu,\xi}-(\mathcal{W}_{\mu,\xi})_+^p\,.$$

Since $\mathcal{U}_{\mu,\xi}\circ\exp_\xi$ is radially symmetric in $B_0(r_0)$, we have that

$$\Delta_g\mathcal{U}_{\mu,\xi}\left(\exp_\xi x\right)=-\Delta\left(\mathcal{U}_{\mu,\xi}\circ\exp_\xi\right)(x)-\frac{1}{2}\partial_r(\ln|g|)\partial_r\left(\mathcal{U}_{\mu,\xi}\circ\exp_\xi\right)(x),$$

where $|g| := \det g$. In geodesic coordinates, we have the Taylor expansion

$$|g| = 1 - \frac{1}{3}\sum_{i,j=1}^{N} R_{ij}(\xi)x^i x^j + \mathrm{O}(|x|^3) \tag{2.8}$$

(see for example Lee–Parker [22]), yielding to

$$\begin{aligned}\Delta_g \mathcal{U}_{\mu,\xi}\left(\exp_\xi x\right) &= -\chi(|x|)\Delta U_\mu(x) + \frac{\chi(|x|)}{3}\sum_{i,j=1}^{N}\frac{R_{ij}(\xi)x^i x^j}{|x|}\partial_r U_\mu(x)\\ &\quad + \mathrm{O}\left(\mu^{\frac{N-2}{2}} + |x|^2|\nabla U_\mu|\right)\\ &= \mathcal{U}^p_{\mu,\xi}\left(\exp_\xi x\right) + \frac{\chi(|x|)}{3}\sum_{i,j=1}^{N}\frac{R_{ij}(\xi)x^i x^j}{|x|}\partial_r U_\mu(x) + \mathrm{O}\left(\mu^{\frac{N-2}{2}} + |x|^2|\nabla U_\mu|\right) \end{aligned}\tag{2.9}$$

in view of $-\Delta U_\mu = U^p_\mu$. Similarly, we have that

$$\Delta_g \mathcal{V}_{\mu,\xi}\left(\exp_\xi x\right) = -\chi(|x|)\Delta V_\mu(x) + \mathrm{O}\left(\mu^{\frac{N-6}{2}} + |x||\nabla V_\mu|\right).$$

Since by (2.2) we have that

$$\Delta(\mu^2 V_\mu) + pU_\mu^{p-1}(\mu^2 V_\mu) = \frac{1}{3}\sum_{i,j=1}^{N} R_{ij}(\xi)\frac{x^i x^j}{|x|}\partial_r U_\mu + \alpha_N S_g(\xi)U_\mu, \tag{2.10}$$

by (2.9)–(2.10) we get that

$$\|\Delta_g \mathcal{W}_{\mu,\xi} + \alpha_N S_g \mathcal{W}_{\mu,\xi} - \mathcal{U}^p_{\mu,\xi} - p\mu^2\mathcal{U}^{p-1}_{\mu,\xi}\mathcal{V}_{\mu,\xi}\|_{L^{\frac{2N}{N+2}}(M)} = \begin{cases} \mathrm{O}\left(\mu^{\frac{N-2}{2}}\right) & \text{if } N = 6,7\\ \mathrm{O}\left(\mu^3|\ln\mu|^{\frac{5}{8}}\right) & \text{if } N = 8\\ \mathrm{O}\left(\mu^3\right) & \text{if } N \ge 9.\end{cases} \tag{2.11}$$

Since

$$\|h\mathcal{W}_{\mu,\xi}\|_{L^{\frac{2N}{N+2}}(M)} = \begin{cases} \mathrm{O}\left(\mu^2|\ln\mu|^{\frac{2}{3}}\right) & \text{if } N = 6\\ \mathrm{O}\left(\mu^2\right) & \text{if } N \ge 7\end{cases}$$

and

$$\|\left(\mathcal{W}_{\mu,\xi}\right)^p_+ - \mathcal{U}^p_{\mu,\xi} - p\,\mathcal{U}^{p-1}_{\mu,\xi}\left(\mu^2\mathcal{V}_{\mu,\xi}\right)\|_{L^{\frac{2N}{N+2}}(M)} = \begin{cases} \mathrm{O}\left(\mu^4|\ln\mu|^{\frac{2}{3}}\right) & \text{if } N = 6\\ \mathrm{O}\left(\mu^{2\frac{N+2}{N-2}}\right) & \text{if } N \ge 7\end{cases}$$

in view of $|(a+b)^p_+ - a^p - pa^{p-1}b| = \mathrm{O}(|b|^p)$ for all $a > 0$ and $b \in \mathbb{R}$, by (2.11) we deduce the validity of (2.7). □

3. The reduced energy

Introduce the Euler-Lagrange functional $J_\varepsilon : \mathrm{H}^1_g(M) \to \mathbb{R}$ corresponding to equation (1.4):

$$J_\varepsilon(u) := \frac{1}{2}\int_M |\nabla u|^2_g dv_g + \frac{1}{2}\int_M \left(\alpha_N S_g + \varepsilon h\right)u^2 dv_g - \frac{1}{p+1}\int_M u_+^{p+1} dv_g\,.$$

The aim is to find an asymptotic expansion of $J_\varepsilon\left(\mathcal{W}_{\mu,\xi}\right)$. We have that:

Proposition 3.1. *The following expansions do hold as* ϵ, $\mu \to 0$*:*

$$J_\varepsilon\left(\mathcal{W}_{\mu,\xi}\right)=\frac{K_6^{-6}}{6}+\frac{4}{5}\omega_5\left|\mathrm{Weyl}_g(\xi)\right|_g^2\mu^4\ln\mu+\frac{5}{24}K_6^{-6}h(\xi)\varepsilon\mu^2+\mathrm{o}\left(\mu^4\ln\mu+\varepsilon\mu^2\right) \tag{3.1}$$

when $N=6$*, and*

$$J_\varepsilon\left(\mathcal{W}_{\mu,\xi}\right)=\frac{K_N^{-N}}{N}-\frac{K_N^{-N}}{24N(N-4)(N-6)}\left|\mathrm{Weyl}_g(\xi)\right|_g^2\mu^4+\frac{2(N-1)K_N^{-N}h(\xi)}{N(N-2)(N-4)}\varepsilon\mu^2+\mathrm{o}\left(\mu^4+\varepsilon\mu^2\right) \tag{3.2}$$

when $N\geq 7$*, uniformly with respect to* $\xi\in M$*, where* K_N *is the best constant for the embedding of* $D^{1,2}\left(\mathbb{R}^N\right)$ *into* $L^{2^*}\left(\mathbb{R}^N\right)$*.*

Proof. First, we have that

$$\begin{aligned}
J_\varepsilon&\left(\mathcal{U}_{\mu,\xi}+\mu^2\mathcal{V}_{\mu,\xi}\right)-J_\varepsilon\left(\mathcal{U}_{\mu,\xi}\right)=\mu^2\int_M\left[\langle\nabla\mathcal{U}_{\mu,\xi},\nabla\mathcal{V}_{\mu,\xi}\rangle_g+(\alpha_N S_g+\varepsilon h)\,\mathcal{U}_{\mu,\xi}\mathcal{V}_{\mu,\xi}-\mathcal{U}_{\mu,\xi}^p\mathcal{V}_{\mu,\xi}\right]\\
&+\frac{1}{2}\mu^4\int_M\left[\left|\nabla\mathcal{V}_{\mu,\xi}\right|_g^2-p\mathcal{U}_{\mu,\xi}^{p-1}\mathcal{V}_{\mu,\xi}^2\right]dv_g+\frac{1}{2}\mu^4\int_M(\alpha_N S_g+\varepsilon h)\ \mathcal{V}_{\mu,\xi}^2dv_g\\
&-\frac{1}{p+1}\int_M\left[\left(\mathcal{U}_{\mu,\xi}+\mu^2\mathcal{V}_{\mu,\xi}\right)_+^{p+1}-\mathcal{U}_{\mu,\xi}^{p+1}-(p+1)\mathcal{U}_{\mu,\xi}^p\mu^2\mathcal{V}_{\mu,\xi}-\frac{1}{2}p(p+1)\mathcal{U}_{\mu,\xi}^{p-1}\mu^4\mathcal{V}_{\mu,\xi}^2\right]dv_g\\
&=\mu^2\int_M\left[\Delta_g\mathcal{U}_{\mu,\xi}+\alpha_N S_g\mathcal{U}_{\mu,\xi}-\mathcal{U}_{\mu,\xi}^p\right]\mathcal{V}_{\mu,\xi}dv_g+\frac{1}{2}\mu^4\int_M\left[\Delta_g\mathcal{V}_{\mu,\xi}-p\mathcal{U}_{\mu,\xi}^{p-1}\mathcal{V}_{\mu,\xi}\right]\mathcal{V}_{\mu,\xi}dv_g\\
&+\begin{cases}\mathrm{o}\left(\mu^4\ln\mu\right) & \text{if } N=6\\ \mathrm{o}\left(\mu^4\right) & \text{if } N\geq 7\end{cases}
\end{aligned} \tag{3.3}$$

as $\mu\to 0$, in view of

$$\begin{aligned}
&\int_M\left|\left(\mathcal{U}_{\mu,\xi}+\mu^2\mathcal{V}_{\mu,\xi}\right)_+^{p+1}-\mathcal{U}_{\mu,\xi}^{p+1}-(p+1)\mathcal{U}_{\mu,\xi}^p\mu^2\mathcal{V}_{\mu,\xi}-\frac{1}{2}p(p+1)\mathcal{U}_{\mu,\xi}^{p-1}\mu^4\mathcal{V}_{\mu,\xi}^2\right|dv_g\\
&\qquad=\mathrm{O}\left(\mu^{\frac{4N}{N-2}}\int_M\left|\mathcal{V}_{\mu,\xi}\right|^{\frac{2N}{N-2}}dv_g\right)=\mathrm{o}\left(\mu^4\right)
\end{aligned}$$

and $\int_M\mathcal{V}_{\mu,\xi}^2dv_g=\begin{cases}\mathrm{O}(1) & \text{if } N=6\\ \mathrm{o}(1) & \text{if } N\geq 7\end{cases}$ as $\mu\to 0$. Now, observe that there holds

$$\begin{aligned}
\mu^2&\int_M\left[\Delta_g\mathcal{U}_{\mu,\xi}+\alpha_N S_g\mathcal{U}_{\mu,\xi}-\mathcal{U}_{\mu,\xi}^p\right]\mathcal{V}_{\mu,\xi}dv_g+\mu^4\int_M\left[\Delta_g\mathcal{V}_{\mu,\xi}-p\mathcal{U}_{\mu,\xi}^{p-1}\mathcal{V}_{\mu,\xi}\right]\mathcal{V}_{\mu,\xi}dv_g\\
&=\mu^2\int_M\left[\Delta_g\mathcal{W}_{\mu,\xi}+\alpha_N S_g\mathcal{W}_{\mu,\xi}-\mathcal{U}_{\mu,\xi}^p-p\mu^2\mathcal{U}_{\mu,\xi}^{p-1}\mathcal{V}_{\mu,\xi}\right]\mathcal{V}_{\mu,\xi}dv_g+\mathrm{O}\left(\mu^4\int_M\mathcal{V}_{\mu,\xi}^2dv_g\right)\\
&=\begin{cases}\mathrm{o}\left(\mu^4\ln\mu\right) & \text{if } N=6\\ \mathrm{o}\left(\mu^4\right) & \text{if } N\geq 7\end{cases}
\end{aligned}$$

as $\mu\to 0$, in view of (2.11). By (2.8) and

$$\begin{aligned}
\Delta_g\mathcal{V}_{\mu,\xi}(\exp_\xi x)&=-\Delta\left(\mathcal{V}_{\mu,\xi}\circ\exp_\xi\right)(x)\\
&\quad+\mathrm{O}\left(|x|\left|\nabla\left(\mathcal{V}_{\mu,\xi}\circ\exp_\xi\right)(x)\right|+|x|^2\left|\nabla^2\left(\mathcal{V}_{\mu,\xi}\circ\exp_\xi\right)(x)\right|\right)
\end{aligned}$$

we deduce that

$$\int_M\left[\Delta_g\mathcal{V}_{\mu,\xi}-p\mathcal{U}_{\mu,\xi}^{p-1}\mathcal{V}_{\mu,\xi}\right]\mathcal{V}_{\mu,\xi}dv_g=-\int_{B_0\left(\frac{r_0}{2\mu}\right)}\left(\Delta V+pU^{p-1}V\right)V\,dy+\begin{cases}\mathrm{O}(1) & \text{if } N=6\\ \mathrm{o}(1) & \text{if } N\geq 7\end{cases} \tag{3.4}$$

as $\mu \to 0$. By (3.3) and (3.4), we get that

$$J_\varepsilon\left(\mathcal{U}_{\mu,\xi}+\mu^2\mathcal{V}_{\mu,\xi}\right)=J_\varepsilon\left(\mathcal{U}_{\mu,\xi}\right)+\frac{1}{2}\mu^4\int_{B_0(\frac{r_0}{2\mu})}\left(\Delta V+pU^{p-1}V\right)V\,dy+\begin{cases}\mathrm{o}\left(\mu^4\ln\mu\right) & \text{if } N=6\\ \mathrm{o}\left(\mu^4\right) & \text{if } N\geq 7\end{cases} \tag{3.5}$$

as $\mu \to 0$. By (2.2)–(2.3) and easy symmetry properties we deduce that

$$\begin{aligned}&\int_{B_0(\frac{r_0}{2\mu})}\left(\Delta V+pU^{p-1}V\right)V\,dy\\ &=-\frac{[N(N-2)]^{\frac{N-2}{2}}(N-2)}{36}\int_{B_0(\frac{r_0}{2\mu})}\left(\sum_{i,j=1}^N R_{ij}(\xi)y^iy^j\right)^2\frac{|y|^2+3}{(1+|y|^2)^N}dy\\ &+\frac{[N(N-2)]^{\frac{N-2}{2}}\alpha_N}{72N(N-1)}S_g^2(\xi)\int_{B_0(\frac{r_0}{2\mu})}\frac{(7N-10)|y|^6+3(7N-8)|y|^4+3(7N-10)|y|^2-9N}{(1+|y|^2)^N}dy\\ &=-\frac{[N(N-2)]^{\frac{N-2}{2}}(N-2)}{36}\int_{B_0(\frac{r_0}{2\mu})}\sum_{i,j,k,s=1}^N E_{ij}(\xi)E_{ks}(\xi)y^iy^jy^ky^s\frac{|y|^2+3}{(1+|y|^2)^N}dy\\ &-\omega_{N-1}\frac{[N(N-2)]^{\frac{N-2}{2}}(N-2)}{576N^2(N-1)^2}S_g^2(\xi)\Big[(N-2)(N-4)I_N^{\frac{N+4}{2}}+3(N^2-8N+8)I_N^{\frac{N+2}{2}}\\ &-3N(7N-10)I_N^{\frac{N}{2}}+9N^2I_N^{\frac{N-2}{2}}\Big]+\mathrm{o}(1)\end{aligned} \tag{3.6}$$

as $\mu \to 0$, where the E_{ij}'s are the components of the traceless part $\mathrm{E}_g=\mathrm{Ric}_g-\frac{S_g}{N}g$ of the Ricci curvature Ric_g of (M,g) in geodesic coordinates and

$$I_p^q=\begin{cases}\displaystyle\int_0^{+\infty}\frac{r^q}{(1+r)^p}dr & \text{if } p-q>1\\ \displaystyle\int_0^{\frac{r_0^2}{4\mu^2}}\frac{r^q}{(1+r)^p}dr & \text{if } p-q\leq 1.\end{cases}$$

Since integration by parts yields to

$$I_{p+1}^q=\frac{p-q-1}{p}I_p^q \quad \text{and} \quad I_{p+1}^{q+1}=\frac{q+1}{p-q-1}I_{p+1}^q \tag{3.7}$$

as soon as $p-q>1$, we have that

$$I_N^{\frac{N}{2}}=\frac{N}{N-2}I_N^{\frac{N-2}{2}}=\frac{N-4}{N+2}I_N^{\frac{N+2}{2}} \quad \text{and} \quad I_N^{\frac{N+4}{2}}=\begin{cases}-2\ln\mu+\mathrm{O}(1) & \text{if } N=6\\ \frac{(N+2)(N+4)}{(N-4)(N-6)}I_N^{\frac{N}{2}} & \text{if } N\geq 7\end{cases} \tag{3.8}$$

as $\mu \to 0$, and it can be easily checked that

$$I_N^{\frac{N}{2}}=\frac{N\omega_N}{2^{N-1}(N-2)\,\omega_{N-1}}=\frac{2K_N^{-N}}{[N(N-2)]^{\frac{N-2}{2}}(N-2)^2\omega_{N-1}} \tag{3.9}$$

(see Aubin [1]). Since for all $i\neq j$ there holds

$$\int_{S^{N-1}}(y^i)^4dv_{g_0}=3\int_{S^{N-1}}(y^i)^2(y^j)^2dv_{g_0}=\frac{3}{N(N+2)}\int_{S^{N-1}}|y|^4dv_{g_0}\,,$$

by (3.6) and (3.8)–(3.9) we deduce that

$$\int_{B_0(\frac{r_0}{2\mu})} \left(\Delta V + pU^{p-1}V\right) V dy = \frac{8}{3}\omega_5 |E_g(\xi)|_g^2 \ln\mu + \frac{16}{225}\omega_5 S_g^2(\xi)\ln\mu + \mathrm{O}(1) \tag{3.10}$$

if $N = 6$, and

$$\int_{B_0(\frac{r_0}{2\mu})} \left(\Delta V + pU^{p-1}V\right) V dy = -\frac{2N-7}{9N(N-2)(N-4)(N-6)} K_N^{-N} |E_g(\xi)|_g^2 + \frac{(N-2)(N-7)}{36N^2(N-1)(N-4)(N-6)} K_N^{-N} S_g^2(\xi) + \mathrm{o}(1) \tag{3.11}$$

if $N \geq 7$. Inserting (3.10)–(3.11) into (3.5), by Lemma 3.2 below we deduce the validity of (3.1)–(3.2). □

We are left with proving the following:

Lemma 3.2. *The following expansions do hold as ϵ, $\mu \to 0$:*

$$J_\varepsilon(U_{\mu,\xi}) = \frac{K_6^{-6}}{6} + \left[\frac{4}{5}\left|\mathrm{Weyl}_g(\xi)\right|_g^2 - \frac{4}{3}|E_g(\xi)|_g^2 - \frac{8}{225}S_g^2(\xi)\right]\omega_5\mu^4\ln\mu + \frac{5}{24}K_6^{-6}h(\xi)\varepsilon\mu^2 + \mathrm{o}\left(\mu^4\ln\mu + \varepsilon\mu^2\right)$$

when $N = 6$, and

$$J_\varepsilon(U_{\mu,\xi}) = \frac{K_N^{-N}}{N} + \left[-\frac{K_N^{-N}}{24N(N-4)(N-6)}\left|\mathrm{Weyl}_g(\xi)\right|_g^2 + \frac{(2N-7)K_N^{-N}}{18N(N-2)(N-4)(N-6)}\left|\mathrm{E}_g(\xi)\right|_g^2 - \frac{(N-2)(N-7)K_N^{-N}}{72N^2(N-1)(N-4)(N-6)} S_g(\xi)^2\right]\mu^4 + \frac{2(N-1)K_N^{-N}}{N(N-2)(N-4)} h(\xi)\varepsilon\mu^2 + \mathrm{o}\left(\mu^4 + \varepsilon\mu^2\right)$$

when $N \geq 7$, uniformly with respect to $\xi \in M$.

Proof. There hold

$$\frac{1}{\omega_{N-1}r^{N-1}}\int_{\partial B_\xi(r)} h d\sigma_g = h(\xi) + \mathrm{O}(r), \tag{3.12}$$

$$\frac{1}{\omega_{N-1}r^{N-1}}\int_{\partial B_\xi(r)} S_g d\sigma_g = S_g(\xi) - \frac{1}{2N}\Lambda_g(\xi) r^2 + \mathrm{O}\left(r^4\right), \tag{3.13}$$

$$\frac{1}{\omega_{N-1}r^{N-1}}\int_{\partial B_\xi(r)} d\sigma_g = 1 - \frac{1}{6N}S_g(\xi) r^2 + A_g(\xi) r^4 + \mathrm{O}\left(r^5\right), \tag{3.14}$$

as $r \to 0$, uniformly with respect to ξ, where $d\sigma_g$ is the volume element of $\partial B_\xi(r)$, ω_{N-1} is the volume of the unit $(N-1)$–sphere, and where (see (3.17)–(3.18))

$$\Lambda_g(\xi) = \Delta_g S_g(\xi) + \frac{1}{3}S_g(\xi)^2 \tag{3.15}$$

and

$$A_g(\xi) = \frac{18\Delta_g S_g(\xi) + 8\left|\mathrm{Ric}_g(\xi)\right|_g^2 - 3\left|\mathrm{Rm}_g(\xi)\right|_g^2 + 5S_g(\xi)^2}{360N(N+2)}. \tag{3.16}$$

The orthogonal decomposition of Riemann curvature is given by

$$|\mathrm{Rm}_g(\xi)|_g^2 = \left|\mathrm{Weyl}_g(\xi)\right|_g^2 + \frac{4}{N-2}\left|\mathrm{E}_g(\xi)\right|_g^2 + \frac{2}{N(N-1)} S_g(\xi)^2, \tag{3.17}$$

where Weyl_g is the Weyl curvature of g and $\mathrm{E}_g = \mathrm{Ric}_g - \frac{S_g}{N} g$ is the traceless part of the Ricci curvature of g. Moreover, we get

$$|\mathrm{Ric}_g(\xi)|_g^2 = |\mathrm{E}_g(\xi)|_g^2 + \frac{1}{N} S_g(\xi)^2. \tag{3.18}$$

By (3.8) and (3.14), we compute

$$\begin{aligned}
\int_M |\nabla U_{\mu,\xi}|_g^2 dv_g &= [N(N-2)]^{\frac{N-2}{2}}(N-2)^2 \int_0^{\frac{r_0}{2}} \frac{\mu^{N-2} r^2}{(\mu^2+r^2)^N} \int_{\partial B_\xi(r)} d\sigma_g dr + \mathrm{O}\left(\mu^{N-2}\right) \qquad (3.19)\\
&= [N(N-2)]^{\frac{N-2}{2}}(N-2)^2 \omega_{N-1}\\
&\quad \times \int_0^{\frac{r_0}{2\mu}} \frac{r^{N+1}}{(1+r^2)^N}\left(1 - \frac{1}{6N} S_g(\xi)\mu^2 r^2 + A_g(\xi)\mu^4 r^4 + \mathrm{O}\left(\mu^5 r^5\right)\right) dr + \mathrm{O}\left(\mu^{N-2}\right)\\
&= \frac{[N(N-2)]^{\frac{N-2}{2}}(N-2)^2}{2}\omega_{N-1}\\
&\quad \times \left(I_N^{\frac{N}{2}} - \frac{1}{6N} I_N^{\frac{N+2}{2}} S_g(\xi)\mu^2 + I_N^{\frac{N+4}{2}} A_g(\xi)\mu^4 + \mathrm{O}\left(I_N^{\frac{N+5}{2}}\mu^5 + \mu^{N-2}\right)\right)\\
&= \begin{cases} K_N^{-N}\left(1 - \dfrac{N+2}{6N(N-4)} S_g(\xi)\mu^2\right) - 9216\,\omega_5 A_g(\xi)\mu^4 \ln\mu + \mathrm{O}\left(\mu^4\right) & \text{if } N=6\\ K_N^{-N}\left(1 - \dfrac{N+2}{6N(N-4)} S_g(\xi)\mu^2 + \dfrac{(N+2)(N+4)}{(N-4)(N-6)} A_g(\xi)\mu^4\right) + \mathrm{O}\left(\mu^5\right) & \text{if } N \geq 7 \end{cases}
\end{aligned}$$

in view of (3.9). Since by (3.7) there hold

$$I_{N-2}^{\frac{N-2}{2}} = \frac{4(N-1)(N-2)}{N(N-4)} I_N^{\frac{N}{2}} \quad \text{and} \quad I_{N-2}^{\frac{N}{2}} = \begin{cases} -2\ln\mu + \mathrm{O}(1) & \text{if } N=6\\ \frac{4(N-1)(N-2)}{(N-4)(N-6)} I_N^{\frac{N}{2}} & \text{if } N \geq 7 \end{cases}$$

as $\mu \to 0$, by (3.13) we compute

$$\begin{aligned}
\int_M S_g U_{\mu,\xi}^2 dv_g &= [N(N-2)]^{\frac{N-2}{2}} \int_0^{\frac{r_0}{2}} \frac{\mu^{N-2}}{(\mu^2+r^2)^{N-2}} \int_{\partial B_\xi(r)} S_g d\sigma_g dr + \mathrm{O}\left(\mu^{N-2}\right)\\
&= [N(N-2)]^{\frac{N-2}{2}} \omega_{N-1}\mu^2 \int_0^{\frac{r_0}{2\mu}} \frac{r^{N-1}}{(1+r^2)^{N-2}}\left(S_g(\xi) - \frac{1}{2N}\Lambda_g(\xi)\mu^2 r^2 + \mathrm{O}\left(\mu^4 r^4\right)\right) dr\\
&\quad + \mathrm{O}\left(\mu^{N-2}\right)\\
&= \frac{[N(N-2)]^{\frac{N-2}{2}}}{2}\omega_{N-1}\mu^2\left(I_{N-2}^{\frac{N-2}{2}} S_g(\xi) - \frac{1}{2N} I_{N-2}^{\frac{N}{2}}\Lambda_g(\xi)\mu^2 + \mathrm{O}\left(\mu^4 I_{N-2}^{\frac{N+2}{2}} + \mu^{N-2}\right)\right)\\
&= \begin{cases} \dfrac{5K_6^{-6}}{12}\mu^2 S_g(\xi) + 48\omega_5 \Lambda_g(\xi)\mu^4 \ln\mu + \mathrm{O}\left(\mu^4\right) & \text{if } N=6\\ \dfrac{4(N-1)K_N^{-N}}{N(N-2)(N-4)}\mu^2\left(S_g(\xi) - \dfrac{1}{2(N-6)}\Lambda_g(\xi)\mu^2\right) + \mathrm{O}\left(\mu^5\right) & \text{if } N \geq 7 \end{cases}
\end{aligned} \tag{3.20}$$

in view of (3.9). Similarly, by (3.12), we have that

$$\varepsilon \int_M hU_{\mu,\xi}^2 dv_g = \frac{4(N-1)K_N^{-N}}{N(N-2)(N-4)} h(\xi)\,\varepsilon\mu^2 + \mathrm{o}\left(\varepsilon\mu^2\right) \tag{3.21}$$

By (3.8) and (3.14), we compute

$$\begin{aligned}
\int_M U_{\mu,\xi}^{2^*} dv_g &= [N(N-2)]^{\frac{N}{2}} \int_0^{\frac{r_0}{2}} \frac{\mu^N}{\left(\mu^2+r^2\right)^N} \int_{\partial B_\xi(r)} d\sigma_g dr + \mathrm{O}\left(\mu^N\right) \\
&= [N(N-2)]^{\frac{N}{2}} \omega_{N-1} \int_0^{\frac{r_0}{2\mu}} \frac{r^{N-1}}{\left(1+r^2\right)^N} \left(1 - \frac{1}{6N} S_g(\xi)\,\mu^2 r^2 + A_g(\xi)\,\mu^4 r^4\right) dr + \mathrm{O}\left(\mu^5\right) \\
&= \frac{[N(N-2)]^{\frac{N}{2}}}{2} \omega_{N-1} \left(I_N^{\frac{N-2}{2}} - \frac{1}{6N} I_N^{\frac{N}{2}} S_g(\xi)\,\mu^2 + I_N^{\frac{N+2}{2}} A_g(\xi)\,\mu^4\right) + \mathrm{O}\left(\mu^5\right) \\
&= K_N^{-N} \left(1 - \frac{1}{6(N-2)} S_g(\xi)\,\mu^2 + \frac{N(N+2)}{(N-2)(N-4)} A_g(\xi)\,\mu^4\right) + \mathrm{O}\left(\mu^5\right)
\end{aligned} \tag{3.22}$$

in view of (3.9). Finally, the claimed expansions follow by (3.19), (3.20), (3.21) and (3.22) in view of (3.15)–(3.18). □

4. The Lyapunov-Schmidt reduction argument

Since equation (1.4) can be re-written as (2.4), the function $u = \mathcal{W}_{\mu,\xi} + \phi$ does solve (1.4) as soon as

$$\hat{L}_{\mu,\xi}(\phi) = -\mathcal{R}_{\mu,\xi} - N_{\mu,\xi}(\phi), \tag{4.1}$$

where $\mathcal{R}_{\mu,\xi}$ is given in (2.6),

$$N_{\mu,\xi}(\phi) = -i^* \left[(\mathcal{W}_{\mu,\xi}+\phi)_+^p - (\mathcal{W}_{\mu,\xi})_+^p - p\,(\mathcal{W}_{\mu,\xi})_+^{p-1}\phi\right]$$

is the nonlinear term (quadratic in ϕ) and

$$\begin{array}{rccl}
\hat{L}_{\mu,\xi}: & H_g^1(M) & \to & H_g^1(M) \\
& \phi & \mapsto & \phi - i^*\left[p\,(\mathcal{W}_{\mu,\xi})_+^{p-1}\phi - \varepsilon h\phi\right]
\end{array}$$

is the linearized operator of (2.4) at $\mathcal{W}_{\mu,\xi}$.

Since $\mathcal{W}_{\mu,\xi}$ is a small perturbation of $\mathcal{U}_{\mu,\xi}$, as $\varepsilon, \mu \to 0$ the operator $\hat{L}_{\mu,\xi}$ in balls with radii of order μ looks pretty much as a scaling of the limiting operator $L_\infty:\ \Phi \to \Phi + (\Delta)^{-1}\left[pU^{p-1}\Phi\right]$, where U is given in (2.1). It is well known (see Bianchi–Egnell [4]) that

$$\ker L_\infty = \mathrm{Span}\ \left\{\Phi^0, \Phi^1, \dots, \Phi^N\right\},$$

where

$$\Phi^0(y) = \frac{1-|y|^2}{\left(1+|y|^2\right)^{\frac{N}{2}}}, \qquad \Phi^i(y) = \frac{y^i}{\left(1+|y|^2\right)^{\frac{N}{2}}} \quad \forall\ i = 1, \dots, N. \tag{4.2}$$

Since there is no hope for the full invertibility of $\hat{L}_{\mu,\xi}$ in $H_g^1(M)$, let us introduce the "asymptotic kernel" $K_{\mu,\xi}$ and its "orthogonal space" $K_{\mu,\xi}^\perp$ as

$$K_{\mu,\xi} = \mathrm{Span}\ \left\{Z_{\mu,\xi}^0, \dots, Z_{\mu,\xi}^N\right\}$$

and

$$K_{\mu,\xi}^\perp = \left\{\phi \in H_g^1(M):\ \int_M \mathcal{U}_{\mu,\xi}^{p-1} Z_{\mu,\xi}^i \phi\, d\mu_g = 0 \quad \forall\ i = 0, \dots, N\right\},$$

where

$$Z^i_{\mu,\xi}(z) = \chi\left(d_g(z,\xi)\right)\mu^{\frac{2-N}{2}}\Phi^i\left(\frac{\exp_\xi^{-1}(z)}{\mu}\right)$$

for $i = 0, \ldots, N$, with Φ^i given by (4.2). Letting $\Pi_{\mu,\xi}$ and $\Pi^\perp_{\mu,\xi}$ be the projectors of $H^1_g(M)$ onto the respective subspaces, equation (4.1) is equivalent to solving

$$L_{\mu,\xi}(\phi) = -\Pi^\perp_{\mu_\xi}\left(\mathcal{R}_{\mu,\xi} + N_{\mu,\xi}(\phi)\right), \tag{4.3}$$

$$\Pi_{\mu_\xi}\left(\hat{L}_{\mu,\xi}(\phi)\right) = -\Pi_{\mu_\xi}\left(\mathcal{R}_{\mu,\xi} + N_{\mu,\xi}(\phi)\right) \tag{4.4}$$

for some $\phi \in K^\perp_{\mu,\xi}$, where $L_{\mu,\xi} = \Pi^\perp_{\mu,\xi} \circ \hat{L}_{\mu,\xi} : K^\perp_{\mu,\xi} \to K^\perp_{\mu,\xi}$. First we can solve equation (4.3), a rather standard result in this context (see for example Musso–Pistoia [30]):

Lemma 4.1. *There exists a positive constant C_0 such that, for any ε, μ small and any $\xi \in M$, there holds*

$$\|L_{\mu,\xi}(\phi)\| \geq C_0 \|\phi\|$$

for all $\phi \in K^\perp_{\mu,\xi}$. As a consequence, (4.3) *admits a unique solution $\phi_{\mu,\xi} \in K^\perp_{\mu,\xi}$, which is continuously differentiable in μ and ξ, so that*

$$\|\phi_{\mu,\xi}\| = \begin{cases} \mathrm{o}\left(\mu^2\sqrt{|\ln\mu|} + \sqrt{\varepsilon}\mu\right) & \text{if } N = 6 \\ \mathrm{o}\left(\mu^2 + \sqrt{\varepsilon}\mu\right) & \text{if } N \geq 7. \end{cases} \tag{4.5}$$

Let us just stress out that the estimate (4.5) heavily depends on (2.7). The need of improving the ansatz in Section 2 comes out from getting the correct smallness rate of ϕ as expressed by (4.5). Finally, we have all the ingredients to prove our main result.

Proof of Theorem 1.2. A first well known fact (see for example Musso–Pistoia [30]) is the equivalence between equation (4.4) and the search of critical points for

$$\widetilde{\mathcal{J}}_\varepsilon(\mu,\xi) = J_\varepsilon\left(\mathcal{W}_{\mu,\xi} + \phi_{\mu,\xi}\right),$$

where $\phi_{\mu,\xi}$ is given by Lemma 4.1. We just need to prove that

$$J_\varepsilon\left(\mathcal{W}_{\mu,\xi} + \phi_{\mu,\xi}\right) - J_\varepsilon\left(\mathcal{W}_{\mu,\xi}\right) = \begin{cases} \mathrm{o}\left(\mu^4|\ln\mu| + \varepsilon\mu^2\right) & \text{if } N = 6 \\ \mathrm{o}\left(\mu^4 + \varepsilon\mu^2\right) & \text{if } N \geq 7 \end{cases} \tag{4.6}$$

as $\varepsilon, \mu \to 0$. Indeed, we have that

$$\begin{aligned} J_\varepsilon\left(\mathcal{W}_{\mu,\xi} + \phi_{\mu,\xi}\right) - J_\varepsilon\left(\mathcal{W}_{\mu,\xi}\right) &= \int_M \Big(\left\langle \nabla\mathcal{R}_{\mu,\xi}, \nabla\phi_{\mu,\xi}\right\rangle_g + \alpha_N S_g \mathcal{R}_{\mu,\xi}\phi_{\mu,\xi} - \left(W_{\mu,\xi}\right)^p_+ \phi_{\mu,\xi}\Big) dv_g \\ &+ \frac{1}{2}\int_M |\nabla\phi_{\mu,\xi}|^2_g dv_g + \frac{1}{2}\int_M \left(\alpha_N S_g + \varepsilon h\right)\phi^2_{\mu,\xi} dv_g \\ &- \frac{1}{p+1}\int_M \left[\left(\mathcal{W}_{\mu,\xi} + \phi_{\mu,\xi}\right)^{p+1}_+ - \left(\mathcal{W}_{\mu,\xi}\right)^{p+1}_+ - (p+1)\left(\mathcal{W}_{\mu,\xi}\right)^p_+ \phi_{\mu,\xi}\right] dv_g \\ &= \mathrm{O}\left(\|\mathcal{R}_{\mu,\xi}\|\,\|\phi_{\mu,\xi}\| + \|\phi_{\mu,\xi}\|^2 + \int_M \left(\mathcal{W}_{\mu,\xi}\right)^{p-1}_+ \phi^2_{\mu,\xi} dv_g + \int_M \phi^{p+1}_{\mu,\xi} dv_g\right) \\ &= \mathrm{O}\left(\|\mathcal{R}_{\mu,\xi}\|\,\|\phi_{\mu,\xi}\| + \|\phi_{\mu,\xi}\|^2\right) \end{aligned}$$

by the Sobolev embedding $H^1_g(M) \hookrightarrow L^{p+1}(M)$ and the Hölder's inequality. By (2.7) and (4.5) we then deduce the validity of (4.6). Setting

$$\mu(d) = d\begin{cases} l^{-1}(\varepsilon) & \text{if } N = 6 \\ \sqrt{\varepsilon} & \text{if } N \geq 7, \end{cases}$$

where $l:(0,e^{-\frac{1}{2}})\to(0,\frac{e^{-1}}{2})$ is defined as $l(\mu)=-\mu^2\ln\mu$, by Proposition 3.1 and (4.6) we deduce the following asymptotic estimates:

$$\mathcal{J}(d,\xi):=\frac{\widetilde{\mathcal{J}}_\varepsilon(\mu(d),\xi)-K_N^{-N}}{\varepsilon^2}\left(\ln l^{-1}(\varepsilon)\right)^\gamma=c_2d^2h(\xi)-c_3d^4\left|\mathrm{Weyl}_g(\xi)\right|_g^2+\mathrm{o}(1)$$

as $\varepsilon\to0$, uniformly with respect to $\xi\in M$ and to d in compact subsets of $(0,\infty)$, where $c_2,c_3>0$ are suitable constants, $\gamma=1$ when $N=6$ and $\gamma=0$ when $N\geq7$. Letting $\mathcal{D}\subset(0,\infty)\times M$ be a C^0*–stable critical set* of $\widetilde{E}$ and U be a compact neighborhood of $\mathcal{D}$ in $(0,\infty)\times M$, by the definition of stability it follows that $\mathcal{J}$ has a critical point $(d_\varepsilon,\xi_\varepsilon)\in U\subset(0,\infty)\times M$, for ε small. Up to a subsequence and taking U smaller and smaller, we can assume that $(d_\varepsilon,\xi_\varepsilon)\to(t_0,\xi_0)$ as $\varepsilon\to0$ with $\xi_0\in\pi(\mathcal{D})$. By elliptic regularity theory $u_\varepsilon=\mathcal{W}_{\mu(d_\varepsilon),\xi_\varepsilon}+\phi_{\mu(d_\varepsilon),\xi_\varepsilon}$ is a solution of (1.4). Since $\xi_\varepsilon\to\xi_0$ and $\left\|\phi_{\mu(d_\varepsilon),\xi_\varepsilon}\right\|\to0$ as $\varepsilon\to0$, it is easily seen that $u_\varepsilon>0$ and $u_\varepsilon^{2^*}\rightharpoonup K_N^{-N}\delta_{\xi_0}$ in the measures sense as $\varepsilon\to0$ (see for example Rey [35]), where δ_ξ denotes the Dirac mass measure at ξ. From very basic facts concerning the asymptotic analysis of solutions of Yamabe-type equations (see for example Druet–Hebey [12] and Druet–Hebey–Robert[15]), we get that the family $(u_\varepsilon)_\varepsilon$ blows up at the point ξ_0 as $\varepsilon\to0$. □

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Pierpaolo Esposito, Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S. Leonardo Murialdo 1, 00146 Roma, Italy

E-mail address: esposito@mat.uniroma3.it

Angela Pistoia, Dipartimento SBAI, Università di Roma "La Sapienza", via Antonio Scarpa 16, 00161 Roma, Italy

E-mail address: pistoia@dmmm.uniroma1.it

Jérôme Vétois, Université de Nice - Sophia Antipolis, Laboratoire J.-A. Dieudonné, UMR CNRS-UNS 7351, Parc Valrose, 06108 Nice Cedex 2, France

E-mail address: vetois@unice.fr