Non-topological condensates for the self-dual Chern-Simons-Higgs model

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Abstract

For the abelian self-dual Chern-Simons-Higgs model we address existence issues of periodic vortex configurations – the so-called condensates– of non-topological type as $k \to 0$, where $k > 0$ is the Chern-Simons parameter. We provide a positive answer to the long-standing problem on the existence of non-topological condensates with magnetic field concentrated at some of the vortex points (as a sum of Dirac measures) as $k \to 0$, a question which is of definite physical interest.

Keywords: **AMS subject classification**:

1 Introduction and statement of main results

The Chern-Simons vortex theory is a planar theory which is physically relevant in connection with high critical temperature superconductivity, the quantum Hall effect and anyonic particle physics, as widely discussed by Dunne [19]. Hong-Kim-Pac [24] and Jackiw-Weinberg [25] have proposed an abelian self-dual model where the electrodynamics is governed only by the Chern-Simons term. Over the Minkowski space $(\mathbb{R}^{1+2}, g)$, with metric tensor $g = \text{diag}\,(1, -1, -1)$, the model is described by the following Lagrangean density:

$$\mathcal{L}_(\mathcal{A}, \phi) = \frac{k}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + D_\alpha\phi\overline{D^\alpha\phi} - \frac{1}{k^2}|\phi|^2\left(|\phi|^2 - 1\right)^2,$$

where the Chern-Simons coupling parameter $k > 0$ measures the strenght of the Chern-Simons term and the antisymmetric Levi-Civita tensor $\epsilon^{\alpha\beta\gamma}$ is fixed with $\epsilon^{012} = 1$. The metric tensor g is used to lower and raise indices in the usual way, and the standard summation convention over repeated indices is adopted. The gauge potential $\mathcal{A} = -iA_\alpha dx^\alpha$ is a 1-form (a connection over the principal bundle $\mathbb{R}^{1+2} \times U(1)$), $A_\alpha : \mathbb{R}^{1+2} \to \mathbb{R}$

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for $\alpha = 0, 1, 2$, and the Higgs field $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the matter field. The gauge field $F_{\mathcal{A}} = -\frac{i}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$ is a $2-$form (the curvature of $\mathcal{A}$), where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, and the Higgs field ϕ is weakly coupled with the gauge potential $\mathcal{A}$ through the covariant derivative D_A as follows: $D_A \phi = D_\alpha \phi \, dx^\alpha$, $D_\alpha \phi = \partial_\alpha \phi - i A_\alpha \phi$ for $\alpha = 0, 1, 2$.

The self-dual regime has been identified by Hong-Kim-Pac [24] and Jackiw-Weinberger [25] through the choice of the "triple-well" potential $\frac{1}{k^2}|\phi|^2(|\phi|^2 - 1)^2$ which yields to a Bogomol'nyi reduction [5] for the Chern-Simons-Higgs model, as we discuss below. Vortices are time-independent (x^0 is the time-variable) configurations $(\mathcal{A}, \phi)$ which solve the Euler-Lagrange equations

$$\begin{cases} D_\mu D^\mu \phi = -\dfrac{1}{k^2}(|\phi|^2 - 1)(3|\phi|^2 - 1)\phi \\ \dfrac{k}{2}\epsilon^{\mu\alpha\beta} F_{\alpha\beta} = J^\mu := i\left(\phi \overline{D^\mu \phi} - \overline{\phi} D^\mu \phi\right) \end{cases} \tag{1.1}$$

and have finite energy. In the self-dual regime, for energy-minimizing vortices (at given magnetic flux) the second-order Euler-Lagrange equations are equivalent to the first-order self-dual equations

$$\begin{cases} D_\pm \phi = 0 \\ F_{12} \pm \frac{2}{k^2}|\phi|^2(|\phi|^2 - 1) = 0 \\ kF_{12} + 2A_0|\phi|^2 = 0, \end{cases} \tag{1.2}$$

where $D_\pm = D_1 \pm i D_2$ and the last equation is usually referred to as the Gauss law. In the sequel, we restrict our attention to energy-minimizing vortices (at given magnetic flux), and we will simply refer to them as vortices.

In the physical interpretation, the electric field $\vec{E} = (\partial_1 A_0, \partial_2 A_0, 0)$ is planar, the magnetic field $\vec{B} = (0, 0, F_{1,2})$ is in the orthogonal direction, and J^0, $\vec{J} = (J^1, J^2)$ can be identified with the charge density, current density, respectively, as in the classical Maxwell theory. Thanks to the Gauss law, vortices are both electrically and magnetically charged, a physical relevant property which was absent in the abelian Maxwell-Higgs model [26, 36]. Notice that $\mathcal{A}$ and ϕ are not observable quantities, as they are defined only up to a gauge transformation, whereas the electric and magnetic fields as well as the magnitude $|\phi|$ of the Higgs field define gauge-independent quantities. The second and third equations in (1.2) only involve observable quantities, whereas the first one $D_+\phi = 0$ (or $D_-\phi = 0$) – a gauge invariant version of the Cauchy-Riemann equations– implies holomorphic-type properties for the Higgs field ϕ (or $\bar{\phi}$) in a suitable gauge. Following an approach first developed by Taubes [36] for the abelian Maxwell-Higgs model, vortices $(\phi, \mathcal{A})$ can be found in the form:

$$\phi = e^{\frac{u}{2} \pm i \sum_{j=1}^N Arg(z - p_j)}, \quad A_0 = \pm\frac{1}{k}(|\phi|^2 - 1), \quad A_1 \pm iA_2 = -i(\partial_1 \pm i\partial_2)\log\phi \tag{1.3}$$

as soon as $u = \log|\phi|^2$ does solve the elliptic problem

$$-\Delta u = \frac{1}{\epsilon^2} e^u (1 - e^u) - 4\pi \sum_{j=1}^N \delta_{p_j}, \tag{1.4}$$

where $\epsilon = \frac{k}{2}$ and $p_1, \ldots, p_N$ are the zeroes of ϕ (repeated according to their multiplicities)– usually referred to as the vortex points (with the convention $N = 0$ if $\phi \neq 0$). We refer the interested reader to [35, 39] and the references therein for more details and for an extensive discussion of several gauge field theories.

For planar vortices, the finite energy condition $\int_{\mathbb{R}^2} e^u(1 - e^u) < +\infty$ imposes two possible asymptotic behaviors at infinity. The topological behavior $|\phi|^2 = e^u \to 1$ as $|z| \to \infty$ gives the vortex number N the topological meaning of winding number for ϕ at infinity (up to a $\pm$ sign, depending on whether $D_+\phi = 0$ or $D_-\phi = 0$), yielding to quantization effects for the energy E, the magnetic flux Φ and the electric charge Q in the class of topological $N-$vortices: $E = 2\pi N$, $\Phi = \pm 2\pi N$ and $Q = \pm 2\pi k N$. The existence of planar topological vortices has been addressed in [23, 33, 38]. The non-topological behavior $|\phi|^2 = e^u \to 0$ as $|z| \to \infty$ has no counterpart in the abelian Maxwell-Higgs model, and the possible coexistence of topological and non-topological $N-$vortices is the main new feature in Chern-Simons theories. After the seminal work [32] in a radial setting with a single vortex point (see also [10] for related results), it has been a challenging problem to find planar non-topological $N-$vortices [7, 8] for an arbitrary configuration of $p_1, \dots, p_N$. Surprisingly, two different classes have been found by using different limiting problems: the singular Liouville equation in [7] or the Chern-Simons equation $-\Delta U = e^U(1 - e^U) - 4\pi\delta_0$ in [8]. Since the latter problem has no scale-invariance, in [8] the points $p_1, \dots, p_N$ are taken along the vertices of a regular $N-$polygon in order to glue together $U(\frac{x-p_j}{\epsilon})$, $j = 1, \dots, N$, for there is no freedom to adjust the height at each p_j to account for the interaction, but the approximating function has invertible linearized operator.

Since the theoretical prediction by Abrikosov [2], the appearance of lattice structure, in the form of spatially periodic vortices, has been experimentally observed. To account for it, the model is formulated on

$$\Omega = \{z = t\omega_1 + s\omega_2 : \ (t, s) \in (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})\},$$

where ω_1, $\omega_2 \in \mathbb{C} \setminus \{0\}$ satisfy Im $(\frac{\omega_2}{\omega_1}) > 0$. Condensates are time-independent configurations $(\mathcal{A}, \phi)$ which solve the Euler-Lagrange equations (1.1), have finite energy and satisfy the 't Hooft boundary conditions [37]:

$$e^{i\xi_k(z+\omega_k)}\phi(z + \omega_k) = e^{i\xi_k(z)}\phi(z), \quad A_0(z + \omega_k) = A_0(z), \quad (A_j + \partial_j\xi_k)(z + \omega_k) = (A_j + \partial_j\xi_k)(z) \qquad (1.5)$$

for all $z \in \Gamma^1 \cup \Gamma^2 \setminus \Gamma^k$ and $k = 1, 2$, where $\Gamma^1 = \{z = t\omega_1 - \frac{1}{2}\omega_2 : \ |t| < \frac{1}{2}\}$, $\Gamma^2 = \{z = -\frac{1}{2}\omega_1 + t\omega_2 : \ |t| < \frac{1}{2}\}$ and ξ_1, ξ_2 are real-valued smooth functions defined in a neighborhood of $\Gamma^2 \cup \{\omega_1 + \Gamma^2\}$, $\Gamma^1 \cup \{\omega_2 + \Gamma^1\}$, respectively. For energy-minimizing vortices (at given magnetic flux) the Euler-Lagrange equations (1.1) are still equivalent to the self-dual ones (1.2). Since (1.5) just reduces to a double periodicity for the observable quantities F_{12} and $|\phi|$ in Ω, a configuration $(\mathcal{A}, \phi)$ in the form (1.3) does solve (1.2) as soon as $u = \log|\phi|^2$ is a doubly-periodic solution of (1.4) in Ω, see [6, 34] for an exact derivation.

Hereafter, up to a translation, let us assume that $\phi \neq 0$ on $\partial\Omega$ (i.e. $p_1, \dots, p_N \in \Omega$) in such a way the winding number deg $(\phi, \partial\Omega, 0)$ is well-defined, and the vortex number N is simply given by $|\text{deg}\ (\phi, \partial\Omega, 0)|$. By (1.5) we still have quantization effects as in the case of planar topological vortices: $E = 2\pi N$, $\Phi = \pm 2\pi N$ and $Q = \pm 2\pi k N$, where the $\pm$ sign depends on whether $D_+\phi = 0$ or $D_-\phi = 0$. Hereafter, up to change ϕ with $\bar{\phi}$, let us assume that $D_+\phi = 0$ and restrict our attention to energy-minimizing condensates (at given magnetic flux), simply referred to as condensates.

Letting $G(z, p)$ be the Green function of $-\Delta$ in Ω with pole at p:

$$\begin{cases} -\Delta G(z, p) = \delta_p - \frac{1}{|\Omega|} & \text{in } \Omega \\ \int_\Omega G(z, p) dz = 0, \end{cases}$$

one is led to consider the following equivalent regular version of (1.4):

$$-\Delta v = \frac{1}{\epsilon^2} e^{u_0+v}(1 - e^{u_0+v}) - \frac{4\pi N}{|\Omega|} \qquad \text{in } \Omega \qquad (1.6)$$

in terms of $v = u - u_0$, where $u_0 = -4\pi \sum_{j=1}^N G(z, p_j)$ and the potential e^{u_0} is a smooth non-negative function which vanishes exactly at $p_1, \dots, p_N$. By translation invariance, notice that $G(z, p) = G(z - p, 0)$, and

$G(z,0)$ can be decomposed as $G(z,0) = -\frac{1}{2\pi}\log|z| + H(z)$, where H is a (not doubly-periodic) function with $\Delta H = \frac{1}{|\Omega|}$ in Ω. If v is a solution of (1.6), by integration over Ω notice that

$$\int_\Omega e^{u_0+v}(1-e^{u_0+v}) = \int_\Omega |\phi|^2(1-|\phi|^2) = 2\epsilon^2 \int_\Omega F_{12} = 4\pi N\epsilon^2 \tag{1.7}$$

in view of (1.2), yielding to the necessary condition

$$16\pi N\epsilon^2 = |\Omega| - 4\int_\Omega \left(e^{u_0+v} - \frac{1}{2}\right)^2 < |\Omega|$$

for the solvability. According to [6], Caffarelli and Yang show the existence of $0 < \epsilon_c < \sqrt{\frac{|\Omega|}{16\pi N}}$ so that (1.4) has a maximal doubly-periodic solution u_ϵ for $0 < \epsilon < \epsilon_c$, while no solution exists for $\epsilon > \epsilon_c$. Notice that (1.6) admits a variational structure with energy functional

$$J_\epsilon(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2\epsilon^2}\int_\Omega \left(e^{u_0+v}-1\right)^2 + \frac{4\pi N}{|\Omega|}\int_\Omega v$$

where $v \in H^1(\Omega) = \{v \in H^1_{\text{loc}}(\mathbb{R}^2) : v \text{ doubly periodic in } \Omega\}$. Later, Tarantello [34] shows that the maximal solution u_ϵ is a local minimum for J_ϵ in $H^1(\Omega)$, and a second solution u^ϵ is found as a mountain-pass critical point for J_ϵ.

To each solution u of (1.4) we can associate the $N-$condensate $(\mathcal{A},\phi)$ in the form (1.3) (with the $+$ sign as we agreed), and let $(\mathcal{A}_\epsilon,\phi_\epsilon)$, $(\mathcal{A}^\epsilon,\phi^\epsilon)$ be the ones corresponding to u_ϵ, u^ϵ. Concerning the asymptotic behavior as $\epsilon \to 0$, by (1.7) we can expect two classes of $N-$condensates:

- $|\phi| \to 1$ as $\epsilon \to 0$ ("topological" behavior),
- $|\phi| \to 0$ as $\epsilon \to 0$ ("non-topological" behavior),

to be understood in suitable norms. For example, $(\mathcal{A}_\epsilon,\phi_\epsilon)$ exhibits "topological" behavior:

$$|\phi_\epsilon| \to 1 \text{ in } C_{\text{loc}}(\bar\Omega \setminus \{p_1,\dots,p_N\}),$$

with

$$(F_{12})_\epsilon \rightharpoonup 2\pi\sum_{j=1}^N \delta_{p_j} \quad \text{in the sense of measures} \tag{1.8}$$

as $\epsilon \to 0$ according to (1.7), see [34]. The concentration property (1.8) for the magnetic field has a definite physical interest, and supports the use of the terminology "vortex points" for the zeroes $p_1,\dots,p_N$ of the Higgs field ϕ. The $N-$condensate $(\mathcal{A}^\epsilon,\phi^\epsilon)$ has in general a different asymptotic behavior as $\epsilon \to 0$:

(i) when $N=1$, $|\phi^\epsilon| \to 0$ in $C^m(\bar\Omega)$, for all $m \geq 0$, and $(F_{12})^\epsilon$ is a compact sequence in $L^1(\Omega)$ (see [34]);

(ii) when $N=2$, $|\phi^\epsilon| \to 0$ in $C(\bar\Omega)$ and either $(F_{12})^\epsilon$ is a compact sequence in $L^1(\Omega)$ or $(F_{12})^\epsilon \rightharpoonup 4\pi\delta_q$ in the sense of measures, for some $q \neq p_1,\ p_2$ with $u_0(q) = \max_\Omega u_0$, depending on whether

$$I(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - 8\pi\log\left(\int_\Omega e^{u_0+v}\right) + \frac{8\pi}{|\Omega|}\int_\Omega v$$

attains its infimum or not in $H^1(\Omega)$ (see [31], and also [18]);

(iii) when $N \geq 3$, $|\phi^\epsilon| \to 0$ in $C(\bar\Omega)$ and $(F_{12})^\epsilon \rightharpoonup 2\pi N \delta_q$ in the sense of measures, for some $q \neq p_1, \ldots, p_N$ with $u_0(q) = \max_\Omega u_0$ (see [12]).

In [17] it is shown the existence of $N-$condensates $(\mathcal{A}, \phi)$ so that $|\phi| \to 0$ a.e. in Ω as $\epsilon \to 0$. Concerning the case $N = 2$, it is a very difficult question, which has been discussed in [9, 27] for $p_1 = p_2$, to know whether or not I attains the infimum in $H^1(\Omega)$. An alternative approach of perturbative type has revelead to be successful for $N = 2$ [29] (see also [20] among other things) by constructing a sequence of $2-$condensates for which the second alternative in (ii) does hold, for a critical point q of u_0. The same approach works as well for $N \geq 3$, provided the concentration points of the magnetic field are not vortex points.

The existence of non-topological $N-$condensates with magnetic field concentrated at vortex points as $\epsilon \to 0$ (like in (1.8)) is the main issue from a physical viewpoint and has not received an answer so far. A first partial answer has been provided by Lin and Yan [28] who construct $N-$condensates $(\mathcal{A}_\epsilon, \phi_\epsilon)$ so that $(F_{12})_\epsilon \rightharpoonup 2\pi N \delta_{p_j}$ in the sense of measures as $\epsilon \to 0$, as soon as $N > 4$ and p_j is a simple vortex point in $\{p_1, \ldots, p_N\}$. As in [8], they make use of the Chern-Simons equation $-\Delta U = e^U(1 - e^U) - 4\pi\delta_0$ as limiting problem, which is not suitable to manage multiple concentration points. Moreover, such a condensate does satisfy $\max_\Omega |\phi_\epsilon| \geq c > 0$ for ϵ small and $|\phi_\epsilon| \to 0$ in $C_{\text{loc}}(\bar\Omega \setminus \{p_j\})$, which fits the notion of "non-topological" behavior in a weak sense. Our aim is to extend to $N-$condensates the perturbative approach developed by Chae and Imanuvilov [7] for planar $N-$vortices, based on the use of the singular Liouville equation as limiting problem. As far as non-topological behavior, let us stress that the problem on the torus is much more rigid than the planar case, as well illustrated by the quantization property $\Phi = 2\pi N$ (valid just in the doubly-periodic situation). For example, when F_{12} is concentrated like a Dirac measure at a vortex point p_l, by the use of Liouville profiles it is natural, as we will see, to have $4\pi(n_l + 1)$ as concentration mass of F_{12} at p_l, where n_l is the multiplicity of p_l in the set $\{p_1, \ldots, p_N\}$, and then the relation $2\pi N = 4\pi \sum_{l=1}^m (n_l + 1)$ does hold as soon as $F_{12} \rightharpoonup 4\pi \sum_{l=1}^m (n_l + 1)\delta_{p_l}$ in the sense of measures. In particular, the concentration of the magnetic field can not take place at all the vortex points $p_1, \ldots, p_N$ as in the planar case [7]. Let us stress that the $N-$condensates constructed in [30] have exactly such a concentration property and then violate the balancing condition (1.9).

Our aim is to provide a general answer to the long-standing question on the existence of non-topological $N-$condensates with magnetic field concentrated at some vortex points. Compared with [7], our main result is rather surprising and reads as follows.

Theorem 1.1. *Let $\{p_1, \ldots, p_m\}$ be a subset of the vortex set $\{p_1, \ldots, p_N\} \subset \Omega$, $\{p_j\}_j$ be the remaining points and n_l, n_j be the corresponding multiplicities so that*

$$2\pi N = 4\pi \sum_{l=1}^m (n_l + 1). \tag{1.9}$$

Letting $\mathcal{H}_0$ be a meromorphic function in Ω so that $|\mathcal{H}_0(z)|^2 = e^{u_0 + 8\pi \sum_{l=1}^m (n_l+1) G(z,p_l)}$ (which exists and is unique up to rotations), assume that $\mathcal{H}_0$ has zero residue at each $p_1, \ldots, p_m$. Letting $\sigma_0(z) = -\left(\int^z \mathcal{H}_0(w)dw\right)^{-1}$ (a well-defined meromorphic function), assume that

$$D_0 = \frac{1}{\pi}\left[\int_{\Omega \setminus \sigma_0^{-1}(B_\rho(0))} e^{u_0 + 8\pi \sum_{l=1}^m (n_l+1) G(z,p_l)} - \sum_{l=1}^m (n_l + 1) \int_{\mathbb{R}^2 \setminus B_\rho(0)} \frac{dy}{|y|^4}\right] < 0 \tag{1.10}$$

for small $\rho > 0$ and the "non-degeneracy condition" $\det A \neq 0$, where A is given by (6.11). Then, for $\epsilon > 0$ small there exists $N-$condensate $(\mathcal{A}_\epsilon, \phi_\epsilon)$ so that $|\phi_\epsilon| \to 0$ in $C(\bar\Omega)$ and

$$(F_{12})_\epsilon \rightharpoonup 4\pi \sum_{l=1}^m (n_l+1)\delta_{p_l} \tag{1.11}$$

weakly in the sense of measures, as $\epsilon \to 0$.

Notice that we can also allow some concentration point not to be a vortex point, by simply adding it to the vortex set with null multiplicity. In section 5 we will see that in the double-vortex case $N = 2$ Theorem 1.1 essentially recovers the result in [20, 29] concerning single-point concentration, for the assumptions just reduce to have the concentration point $q \neq p_1, p_2$ as a non-degenerate critical point of u_0 with $D_0 < 0$ (for similar results concerning the Liouville equation, see [4, 16, 21] in case of bounded domains with Dirichlet b.c. and [22] in case of a flat two-torus). Despite of the complex statement, for a rectangle Ω with $p_1 = 0$, $p_2 = \frac{\omega_1}{2}$, $p_3 = \frac{\omega_2}{2}$ and $p_4 = \frac{\omega_1+\omega_2}{2}$, and n_1, n_2, n_3, n_4 even multiplicities with $\frac{n_4}{2}$ odd, we will check in section 5 that the assumptions of Theorem 1.1 do hold for $m = 1$ and concentration point p_1, up to perform a small translation so to have $p_j \in \Omega$. For computational simplicity, the "non-degeneracy condition" will be checked just for a square with $n = n_3 = 2$ and $(n_1, n_2) = (2, 0)$ or viceversa. Even more important, examples with $m \geq 2$ will be discussed in section 6.

Following an approach developed by Tarantello [34] and exploited in [31], (1.6) can be seen as a perturbed mean-field equation (2.2) with potential e^{u_0} and unperturbed part

$$-\Delta w = 4\pi N \left(\frac{e^{u_0+w}}{\int_\Omega e^{u_0+w}} - \frac{1}{|\Omega|} \right). \tag{1.12}$$

Since e^{u_0} vanishes like $|z-p_l|^{2n_l}$ near each p_l, $l = 1, \dots, m$, the Liouville equation $-\Delta U = |z|^{2n} e^U$ will play a central role in the construction of an approximating function in the perturbative approach. Since $U_{\delta,\sigma_0} = \log \frac{8\delta^2}{(\delta^2+|\sigma_0|^2)^2}$ does solve $-\Delta U = |\sigma_0'|^2 e^U$ in $\Omega \setminus \{\text{poles of } \sigma_0\}$, a natural choice is $\sigma_0 = z^{n+1}$ when $m = 1$ and $p_1 = 0$. Letting P be a projection operator on the space of doubly-periodic functions, the approximation rate of $PU_{\delta,z^{n+1}}$ is unfortunately not sufficiently small to carry out the argument, a problem which often arises in perturbation arguments and is usually overcome by refining the ansatz via linear theory around the approximating function. However, such a procedure would require several subsequent refinements, yielding in general to a high level of complexity. Inspired by [14], in section 2 we will take advantage of the Liouville formula to use the inner parameter σ_0, present in the Liouville formula, to get improved profiles. Since $PU_{\delta,\sigma_0} \sim U_{\delta,\sigma_0} - \log(8\delta^2) + \log|\sigma_0|^4 + 8\pi(n+1)G(z,0)$ as $\delta \to 0$, PU_{δ,σ_0} is a good approximate solution of (1.12) if $\frac{|\sigma_0'|^2}{|\sigma_0|^4} = |(\frac{1}{\sigma_0})'|^2 = e^{u_0+8\pi(n+1)G(z,0)}$. By definition of $\mathcal{H}_0$, it is enough to find a meromorphic σ_0 with $(\frac{1}{\sigma_0})' = \mathcal{H}_0$, a solvable equation if and only if $\mathcal{H}_0$ has zero residue at its unique pole 0. As we will discuss precisely in Remark 4.4, the assumption on the residues of $\mathcal{H}_0$ is then necessary in our context. Moreover, since $\mathcal{H}_0$ has a pole at 0 of multiplicity $n+2$ and zeroes p_j's of multiplicities n_j, by the property $\mathcal{H}_0(z+\omega_j) = e^{i\theta_j}\mathcal{H}_0(z)$, $j = 1, 2$, near $\partial\Omega$ for some $\theta_1, \theta_2 \in \mathbb{R}$ we deduce that

$$0 = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\mathcal{H}_0'}{\mathcal{H}_0} dz = n + 2 - \sum_j n_j = 2(n+1) - N,$$

providing (1.9) as a necessary and sufficient condition for the existence of such $\mathcal{H}_0$ (the sufficient part in shown in next section). $D_0 < 0$ and the "non-degeneracy condition" will be necessary to determine δ and a, a sort of small translation parameter accounting for the perturbation term in (2.2), according to the asymptotic expansion for the corresponding "reduced equations" as derived in section 3. Theorem 1.1 is proved in section 4 when $m = 1$ and in section 6 when $m \geq 2$.

2 Improved Liouville profiles

Let us decompose any solution v of (1.6) as $v = w + c$, where $c = \frac{1}{|\Omega|}\int_\Omega v$. In this way, w has zero average: $\int_\Omega w dz = 0$, and by (1.7) one has

$$e^{2c}\int_\Omega e^{2u_0+2w} - e^c\int_\Omega e^{u_0+w} + 4\pi N\epsilon^2 = 0.$$

This last identity then provides a relation between c and w in the form $c = c_\pm(w)$, where

$$e^{c_\pm(w)} = \frac{8\pi N\epsilon^2}{\int_\Omega e^{u_0+w} \mp \sqrt{(\int_\Omega e^{u_0+w})^2 - 16\pi N\epsilon^2\int_\Omega e^{2u_0+2w}}}, \tag{2.1}$$

whenever $\left(\int_\Omega e^{u_0+w}\right)^2 - 16\pi N\epsilon^2\int_\Omega e^{2u_0+2w} \geq 0$. The two possible choice of "plus" or "minus" sign in (2.1) is another indication of multiple solutions for (1.6). In [34], topological solutions are characterized to satisfy (2.1) with the "plus" sign. Since we are interested to non-topological solutions, it is natural to restrict the attention to the case $c = c_-(w)$, reducing problem (1.6) to the following equation in Ω:

$$\begin{cases} -\Delta w = & 4\pi N\left(\frac{e^{u_0+w}}{\int_\Omega e^{u_0+w}} - \frac{1}{|\Omega|}\right) \\ & +\frac{64\pi^2N^2\epsilon^2\int_\Omega e^{2u_0+2w}}{(\int_\Omega e^{u_0+w} + \sqrt{(\int_\Omega e^{u_0+w})^2 - 16\pi N\epsilon^2\int_\Omega e^{2u_0+2w}})^2}\left(\frac{e^{u_0+w}}{\int_\Omega e^{u_0+w}} - \frac{e^{2u_0+2w}}{\int_\Omega e^{2u_0+2w}}\right) \\ \int_\Omega w = 0. & \end{cases} \tag{2.2}$$

Here and in the next sections, we first discuss the case $m = 1$ in Theorem 1.1. Assume that p is present $n-$times in $\{p_1, \dots, p_N\}$, and denote by p_j's the remaining points in the set $\{p_1, \dots, p_N\}$ with corresponding multiplicities n_j's. Up to a translation, we are assuming that $p_j \in \Omega$ for $j = 1, \dots, N$, a crucial property which will simplify the arguments below. Since the assumptions in Theorem 1.1 for the concentration at p are just local properties, for simplicity in the notations let us simply consider the case $p = 0$.

Since e^{u_0} behaves like $|z|^{2n}$ as $z \to 0$, the local profile of w near 0 will be given in terms of solutions of the "singular" Liouville equation:

$$-\Delta U = |z|^{2n}e^U. \tag{2.3}$$

Recall that by Liouville formula the function

$$\log\frac{8|F'|^2}{(1+|F|^2)^2}$$

does solve $-\Delta U = e^U$ in the set $\{F' \neq 0\}$, for any holomorphic map F. For entire solutions of (2.3) with finite-energy: $\int_{\mathbb{R}^2}|z|^{2n}e^U < +\infty$, it is well known that necessarily $F(z) = \frac{z^{n+1}-a}{\delta}$, and then all the entire finite-energy solutions of (2.3) are classified as

$$U_{\delta,a}(z) = \log\frac{8(n+1)^2\delta^2}{(\delta^2 + |z^{n+1}-a|^2)^2}, \quad \delta > 0,\ a \in \mathbb{C}.$$

Moreover, we have that $\int_{\mathbb{R}^2}|z|^{2n}e^{U_{\delta,a}} = 8\pi(n+1)$. Since by construction the corresponding $v = w + c_-(w)$ will satisfy

$$\frac{1}{\epsilon^2}e^{u_0+v}\left(1 - e^{u_0+v}\right) \rightharpoonup 8\pi(n+1)\delta_0$$

in the sense of measures, the balance condition

$$2\pi N = 4\pi(n+1) \tag{2.4}$$

is necessary in view of (1.7).

Assume for simplicity $e^{u_0} = |z|^{2n}$. Since $\int_\Omega |z|^{2n} e^{U_{\delta,a}} \to 8\pi(n+1)$ as $\delta \to 0$, by (2.4) we have the asymptotic matching of $-\Delta U_{\delta,a} = |z|^{2n} e^{U_{\delta,a}}$ and $4\pi N \frac{|z|^{2n} e^{U_{\delta,a}}}{\int_\Omega |z|^{2n} e^{U_{\delta,a}}}$ as $\delta \to 0$. To correct $U_{\delta,a}$ into a doubly-periodic function, we consider the projection $PU_{\delta,a}$ of $U_{\delta,a}$ as the solution of

$$\begin{cases} -\Delta PU_{\delta,a} = -\Delta U_{\delta,a} + \frac{1}{|\Omega|}\int_\Omega \Delta U_{\delta,a} & \text{in } \Omega \\ \int_\Omega PU_{\delta,a} = 0. \end{cases}$$

In this way, we gain the constant term

$$\frac{1}{|\Omega|}\int_\Omega \Delta U_{\delta,a} = -\frac{1}{|\Omega|}\int_\Omega |z|^{2n} e^{U_{\delta,a}} \to -\frac{4\pi N}{|\Omega|} \qquad \text{as } \delta \to 0$$

in view of (2.4), and we still need to check that the difference between $-\Delta U_{\delta,a} = |z|^{2n} e^{U_{\delta,a}}$ and $4\pi N \frac{|z|^{2n} e^{PU_{\delta,a}}}{\int_\Omega |z|^{2n} e^{PU_{\delta,a}}}$ is asymptotically small. Thanks to an asymptotic expansion of $PU_{\delta,a}$ in terms of $U_{\delta,a}$, we will see that the difference is small (i.e. $PU_{\delta,a}$ is an approximating function of (2.2)) but behaves at most like $|z|^{2n} e^{U_{\delta,a}} O(|z|+\delta^2)$ which is not sufficiently small. A first refinement of the ansatz via the linear theory around $PU_{\delta,a}$ could improve the pointwise error estimate into $|z|^{2n} e^{U_{\delta,a}} O(|z|^2+\delta^2)$, which unfortunately is in general still not enough. Since there is a strong mismatch between the dependence of $U_{\delta,a}$ on z^{n+1} and that of the error on z (or even on z^2), we should push such a procedure through several subsequent refinements. Instead, we play directly with the inner parameters present in the Liouville formula, for we have more flexibility in the choice of $F(z)$ on bounded domains. Hereafter, let us fix an open simply-connected domain $\tilde\Omega$ so that $\overline{\Omega} \subset \tilde\Omega$ and $\tilde\Omega \cap (\omega_1\mathbb{Z} + \omega_2\mathbb{Z}) = \{0\}$, and set $\mathcal{M}(\overline{\Omega}) = \{\sigma\big|_{\overline{\Omega}} : \sigma \text{ meromorphic in } \tilde\Omega\}$. Let $\delta \in (0,+\infty)$, $a \in \mathbb{C}$ and $\sigma \in \mathcal{M}(\overline{\Omega})$ be a function which vanishes only at 0 with multiplicity $n+1$. Since $\log|\sigma'(z)|^2$ is harmonic in $\{\sigma' \neq 0\}$, the choice $F(z) = \frac{\sigma(z)-a}{\delta}$ yields to solutions

$$U_{\delta,a,\sigma}(z) = \log \frac{8\delta^2}{(\delta^2 + |\sigma(z)-a|^2)^2}$$

of $-\Delta U = |\sigma'(z)|^2 e^U$ in $\Omega \setminus \{\text{poles of } \sigma\}$, for $U_{\delta,a,\sigma}$ is a smooth function up to $\{\sigma' = 0\}$.

The guess is so to find a better local approximating function $PU_{\delta,a,\sigma}$ for a suitable choice of σ, where $PU_{\delta,a,\sigma}$ does solve

$$\begin{cases} -\Delta PU_{\delta,a,\sigma} = |\sigma'(z)|^2 e^{U_{\delta,a,\sigma}} - \frac{1}{|\Omega|}\int_\Omega |\sigma'(z)|^2 e^{U_{\delta,a,\sigma}} & \text{in } \Omega \\ \int_\Omega PU_{\delta,a,\sigma} = 0. \end{cases} \tag{2.5}$$

Notice that $PU_{\delta,a,\sigma}$ is well-defined and smooth as long as $\sigma \in \mathcal{M}(\overline{\Omega})$, no matter σ has poles or not.

Recall that $G(z,0)$ can be thought as a doubly-periodic function in $\mathbb{C}$ with singularities on the lattice vertices $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, and $H(z) = G(z,0) + \frac{1}{2\pi}\log|z|$ is then a smooth function in 2Ω with $\Delta H = \frac{1}{|\Omega|}$. Since 2Ω is simply-connected, we can find an holomorphic function H^* in 2Ω having the harmonic function $H - \frac{|z|^2}{4|\Omega|}$ as real part. Since $p_j \in \Omega$, take $\tilde\Omega$ close to Ω so that $\tilde\Omega - p_j \subset 2\Omega$ for all $j = 1, \dots, N$. The function

$$\mathcal{H}(z) = \prod_j (z-p_j)^{n_j} \exp\left(4\pi(n+1)H^*(z) - 2\pi\sum_{j=1}^N H^*(z-p_j) - \frac{\pi}{2|\Omega|}\sum_{j=1}^N |p_j|^2 + \frac{\pi}{|\Omega|} z \overline{\sum_{j=1}^N p_j}\right) \tag{2.6}$$

is holomorphic in $\tilde{\Omega}$ with

$$|\mathcal{H}(z)|^2 = \frac{1}{|z|^{2n}} e^{u_0+8\pi(n+1)H(z)} = e^{4\pi(n+2)H(z)-4\pi\sum_j n_j G(z,p_j)} \qquad \text{in } \tilde{\Omega} \tag{2.7}$$

in view of (2.4). The meromorphic function $\mathcal{H}_0(z) = \frac{\mathcal{H}(z)}{z^{n+2}}$ does satisfy $|\mathcal{H}_0(z)|^2 = e^{u_0+8\pi(n+1)G(z,0)}$ in $\tilde{\Omega}$.

Remark 2.1. *For simplicity in the notations, we are considering the case $p = 0$. When $p \neq 0$, by assuming $\tilde{\Omega} - p \subset 2\Omega$ the function*

$$\begin{aligned}\mathcal{H}^p(z) &= \prod_j (z-p_j)^{n_j} \, exp\left(4\pi(n+1)H^*(z-p) + \frac{\pi(n+1)}{|\Omega|}|p|^2 - \frac{2\pi(n+1)}{|\Omega|} z\bar{p}\right) \times \\ &\quad \times exp\left(-2\pi \sum_{j=1}^N H^*(z-p_j) - \frac{\pi}{2|\Omega|}\sum_{j=1}^N |p_j|^2 + \frac{\pi}{|\Omega|} z \overline{\sum_{j=1}^N p_j}\right)\end{aligned}$$

is holomorphic in $\tilde{\Omega}$ with

$$|\mathcal{H}^p(z)|^2 = \frac{1}{|z-p|^{2n}} e^{u_0+8\pi(n+1)H(z-p)} = e^{4\pi(n+2)H(z-p)-4\pi\sum_j n_j G(z,p_j)} \qquad \text{in } \tilde{\Omega}$$

in view of (2.4). The meromorphic function $\mathcal{H}_0^p(z) = \frac{\mathcal{H}^p(z)}{(z-p)^{n+2}}$ does satisfy $|\mathcal{H}_0^p(z)|^2 = e^{u_0+8\pi(n+1)G(z,p)}$ in $\tilde{\Omega}$.

Hereafter, for a meromorphic function g in $\tilde{\Omega}$ the notation $\int^z g(w)dw$ stands for the anti-derivative of $g(z)$, which is a well-defined meromorphic function in the simply-connected domain $\tilde{\Omega}$ as soon as g has zero residues at each of its poles. Since $\mathcal{H}(0) \neq 0$ by (2.7), we define

$$\sigma_0(z) = -\left(\int^z \mathcal{H}_0(w) e^{-c_0 w^{n+1}} dw\right)^{-1} = -\left(\int^z \frac{\mathcal{H}(w) e^{-c_0 w^{n+1}}}{w^{n+2}} dw\right)^{-1}, \tag{2.8}$$

where

$$c_0 = \frac{1}{\mathcal{H}(0)(n+1)!} \frac{d^{n+1}\mathcal{H}}{dz^{n+1}}(0) \tag{2.9}$$

guarantees that the residue of $\mathcal{H}_0(z)e^{-c_0 z^{n+1}}$ at 0 vanishes. By construction $\sigma_0 \in \mathcal{M}(\overline{\Omega})$ vanishes only at zero with multiplicity $n+1$, as needed, with

$$\lim_{z\to 0} \frac{z^{n+1}}{\sigma_0(z)} = \frac{\mathcal{H}(0)}{n+1}, \tag{2.10}$$

and does solve

$$|\sigma_0'(z)|^2 = |\sigma_0(z)|^4 e^{u_0+8\pi(n+1)G(z,0)} e^{-2\,\mathrm{Re}[c_0 z^{n+1}]} \tag{2.11}$$

in view of (2.7).

Let $\sigma \in \mathcal{M}(\overline{\Omega})$ be a function which vanishes only at zero with multiplicity $n+1$. For $a \in \mathbb{C}$ small there exist $a_0, \ldots, a_n$ so that $\{z \in \tilde{\Omega} : \sigma(z) = a\} = \{a_0, \ldots, a_n\}$ (distinct points when $a \neq 0$). For a small the function

$$\begin{aligned}\mathcal{H}_{a,\sigma}(z) &= \prod_j (z-p_j)^{n_j} \exp\left(4\pi \sum_{k=0}^n H^*(z-a_k) - \frac{2\pi}{|\Omega|} z \overline{\sum_{k=0}^n a_k} - 2\pi \sum_{j=1}^N H^*(z-p_j)\right. \\ &\quad \left. - \frac{\pi}{2|\Omega|}\sum_{j=1}^N |p_j|^2 + \frac{\pi}{|\Omega|} z \overline{\sum_{j=1}^N p_j}\right)\end{aligned} \tag{2.12}$$

is holomorphic in $\tilde\Omega$ with

$$|\mathcal{H}_{a,\sigma}(z)|^2 = \frac{1}{|z|^{2n}} e^{u_0 + 8\pi \sum_{k=0}^n H(z-a_k) - \frac{2\pi}{|\Omega|}\sum_{k=0}^n |a_k|^2} \qquad \text{in } \tilde\Omega \tag{2.13}$$

in view of (2.4). The advantage in our construction of $\mathcal{H}_{a,\sigma}$, which might be carried over in a simpler and more direct way, is the holomorphic/anti-holomorphic dependence in the a_k's as well as in z, a crucial property as we will see in Appendix A. When $a = 0$, then $a_0 = \cdots = a_n = 0$ and $\mathcal{H} = \mathcal{H}_{0,\sigma}$.

Endowed with the norm $\|\sigma\| := \|\frac{\sigma}{\sigma_0}\|_{\infty,\tilde\Omega}$, the set $\mathcal{M}'(\overline{\Omega}) = \{\sigma \in \mathcal{M}(\overline{\Omega}) : \|\sigma\| < \infty\}$ is a Banach space, and let $\mathcal{B}_r$ be the closed ball centered at σ_0 and radius $r > 0$, i.e.

$$\mathcal{B}_r = \left\{\sigma \in \mathcal{M}(\overline{\Omega}) : \left\|\frac{\sigma}{\sigma_0} - 1\right\|_{\infty,\tilde\Omega} \le r\right\}. \tag{2.14}$$

For $a \neq 0$ and r small, the aim is to find a solution $\sigma_a \in \mathcal{B}_r$ of

$$\sigma(z) = -\left[\int^z \left(\frac{\sigma(w) - a}{\prod_{k=0}^n (w - a_k)} \frac{w^{n+1}}{\sigma(w)}\right)^2 \frac{\mathcal{H}_{a,\sigma}(w)}{w^{n+2}} e^{-c_{a,\sigma} w^{n+1}} dw\right]^{-1}$$

for a suitable coefficient $c_{a,\sigma}$. To be more precise, letting

$$g_{a,\sigma}(z) = \frac{\sigma(z) - a}{\prod_{k=0}^n (z - a_k)}$$

for $|a| < \rho$ and $\sigma \in \mathcal{B}_r$, by Lemma A.1 we have that $g_{a,\sigma} \in \mathcal{M}(\overline{\Omega})$ never vanishes, and the problem above gets re-written as

$$\sigma(z) = -\left[\int^z \frac{g_{a,\sigma}^2(w)}{g_{0,\sigma}^2(w)} \frac{\mathcal{H}_{a,\sigma}(w)}{w^{n+2}} e^{-c_{a,\sigma} w^{n+1}} dw\right]^{-1}. \tag{2.15}$$

The choice

$$c_{a,\sigma} = \frac{1}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \left[\frac{g_{a,\sigma}^2(z) g_{0,\sigma}^2(0)}{g_{a,\sigma}^2(0) g_{0,\sigma}^2(z)} \frac{\mathcal{H}_{a,\sigma}(z)}{\mathcal{H}_{a,\sigma}(0)}\right](0) \tag{2.16}$$

lets vanish the residue of the integrand function in (2.15) making the R.H.S. well-defined. Since $\sigma_a \in \mathcal{B}_r$, the function σ_a vanishes only at zero with multiplicity $n+1$, and satisfies

$$|\sigma_a'(z)|^2 = |\sigma_a(z) - a|^4 \exp\left(u_0 + 8\pi \sum_{k=0}^n G(z, a_k) - \frac{2\pi}{|\Omega|}\sum_{k=0}^n |a_k|^2 - 2\,\mathrm{Re}[c_{a,\sigma_a} z^{n+1}]\right) \tag{2.17}$$

in view of (2.13). The resolution of problem (2.15)-(2.16) will be addressed in Appendix A.

We have the following expansion for $PU_{\delta,a,\sigma}$ as $\delta \to 0$:

Lemma 2.2. *There holds*

$$PU_{\delta,a,\sigma} = U_{\delta,a,\sigma} - \log(8\delta^2) + 4\log|g_{a,\sigma}| + 8\pi \sum_{k=0}^n H(z - a_k) + \Theta_{\delta,a,\sigma} + 2\delta^2 f_{a,\sigma} + O(\delta^4) \tag{2.18}$$

in $C(\overline{\Omega})$, uniformly for $|a| < \rho$ and $\sigma \in \mathcal{B}_r$, where

$$\Theta_{\delta,a,\sigma} = -\frac{1}{|\Omega|}\int_\Omega \log \frac{|\sigma(z) - a|^4}{(\delta^2 + |\sigma(z) - a|^2)^2}$$

and $f_{a,\sigma}$ is defined in (2.22)*. In particular, there holds*

$$PU_{\delta,a,\sigma} = 8\pi \sum_{k=0}^{n} G(z,a_k) + \Theta_{\delta,a,\sigma} + 2\delta^2 \left(f_{a,\sigma} - \frac{1}{|\sigma(z)-a|^2} \right) + O(\delta^4)$$

in $C_{loc}(\overline{\Omega} \setminus \{0\})$, uniformly for $|a| < \rho$ and $\sigma \in \mathcal{B}_r$.

Proof: Define

$$r_{\delta,a,\sigma} = PU_{\delta,a,\sigma} - U_{\delta,a,\sigma} + \log(8\delta^2) - 4\log|g_{a,\sigma}| - 8\pi \sum_{k=0}^{n} H(z-a_k).$$

The function $U_{\delta,a,\sigma}$ does satisfy $-\Delta U_{\delta,a,\sigma} = |\sigma'(z)|^2 e^{U_{\delta,a,\sigma}}$ just in $\Omega \setminus \{\text{poles of } \sigma\}$. At the same time, the function $-4\log|g_{a,\sigma}|$ is harmonic in $\Omega \setminus \{\text{poles of } \sigma\}$, and has exactly the same singular behavior of $U_{\delta,a,\sigma}$ near each pole of σ. It means that

$$-\Delta\left[U_{\delta,a,\sigma} + 4\log|g_{a,\sigma}|\right] = |\sigma'(z)|^2 e^{U_{\delta,a,\sigma}} \tag{2.19}$$

does hold in the whole Ω. Since $\Delta H = \frac{1}{|\Omega|}$, by (2.5) and (2.19) we get that

$$-\Delta r_{\delta,a,\sigma} = \frac{1}{|\Omega|}\left[8\pi(n+1) - \int_\Omega |\sigma'(z)|^2 e^{U_{\delta,a,\sigma}}\right].$$

By the Green's representation formula we have that

$$r_{\delta,a,\sigma}(z) = \frac{1}{|\Omega|}\int_\Omega r_{\delta,a,\sigma} + \int_{\partial\Omega}[\partial_\nu r_{\delta,a,\sigma}(w)G(w,z) - r_{\delta,a,\sigma}(w)\partial_\nu G(w,z)]ds(w), \tag{2.20}$$

where ν is the unit outward normal of $\partial\Omega$ and $ds(w)$ is the line integral element. Since as $\delta \to 0$ there holds

$$r_{\delta,a,\sigma}(w) = PU_{\delta,a,\sigma}(w) - 8\pi\sum_{k=0}^{n} G(w,a_k) + 2\frac{\delta^2}{|\sigma(w)-a|^2} + O(\delta^4)$$

in $C^1(\partial\Omega)$ uniformly in $|a| < \rho$ and $\sigma \in \mathcal{B}_r$, by double-periodicity of $PU_{\delta,a,\sigma} - 8\pi\sum_{k=0}^{n} G(\cdot,a_k)$ we get that

$$\int_{\partial\Omega}[\partial_\nu r_{\delta,a,\sigma}(w)G(w,z) - r_{\delta,a,\sigma}(w)\partial_\nu G(w,z)]ds(w) = 2\delta^2 f_{a,\sigma}(z) + O(\delta^4) \tag{2.21}$$

in $C(\bar{\Omega})$, where

$$f_{a,\sigma}(z) = \int_{\partial\Omega}\left[\partial_\nu \frac{1}{|\sigma(w)-a|^2} G(w,z) - \frac{1}{|\sigma(w)-a|^2}\partial_\nu G(w,z)\right]ds(w). \tag{2.22}$$

Inserting (2.21) into (2.20) we get that

$$r_{\delta,a,\sigma}(z) = \Theta_{\delta,a,\sigma} + 2\delta^2 f_{a,\sigma}(z) + O(\delta^4) \tag{2.23}$$

in $C(\overline{\Omega})$ uniformly in $|a| < \rho$ and $\sigma \in \mathcal{B}_r$, where

$$\Theta_{\delta,a,\sigma} := \frac{1}{|\Omega|}\int_\Omega r_{\delta,a,\sigma} = -\frac{1}{|\Omega|}\int_\Omega \log\frac{|\sigma(z)-a|^4}{(\delta^2+|\sigma(z)-a|^2)^2}.$$

The estimate (2.23) yields to the desired expansion for $PU_{\delta,a,\sigma}$ as $\delta \to 0$. ∎

Letting $\sigma_a \in \mathcal{B}_r$ be the solution of (2.15)-(2.16), we build up the correct approximating function as $W = PU_{\delta,a,\sigma_a}$. We need to control the approximation rate of W for δ and ϵ small enough, by estimating the error term

$$\begin{aligned} R &= \Delta W + 4\pi N\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{1}{|\Omega|}\right) \\ &\quad + \frac{64\pi^2 N^2\epsilon^2 \int_\Omega e^{2u_0+2W}}{\left(\int_\Omega e^{u_0+W} + \sqrt{(\int_\Omega e^{u_0+W})^2 - 16\pi N\epsilon^2 \int_\Omega e^{2u_0+2W}}\right)^2}\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right). \end{aligned} \tag{2.24}$$

In order to simplify the notations, we set $U_{\delta,a} = U_{\delta,a,\sigma_a}$, $c_a = c_{a,\sigma_a}$, $\Theta_{\delta,a} = \Theta_{\delta,a,\sigma_a}$, $f_a = f_{a,\sigma_a}$, and omit the subscript a in σ_a. We have the following crucial result.

Theorem 2.3. *Let $|a| < \frac{\rho}{2}$ and set*

$$\eta = \epsilon^2 \delta^{-\frac{2}{n+1}} \max\left\{1, \frac{|a|}{\delta}\right\}^{\frac{2n}{n+1}}. \tag{2.25}$$

The following expansions do hold

$$\begin{aligned} &\Delta W + 4\pi N\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{1}{|\Omega|}\right) \\ &= |\sigma'(z)|^2 e^{U_{\delta,a}}\left[\frac{e^{2\operatorname{Re}[c_a z^{n+1}]}}{1 + 2\operatorname{Re}[c_a F_a(a)] + |c_a|^2 \operatorname{Re} G_a(a) + \frac{1}{2}|c_a|^2 \Delta \operatorname{Re} G_a(a)\delta^2 \log\frac{1}{\delta} + \frac{\delta^2}{n+1}D_a} - 1\right] \\ &\quad + |\sigma'(z)|^2 e^{U_{\delta,a}} O(\delta^2|z| + \delta^2|a|^{\frac{1}{n+1}} + \delta^2|c_a| + \delta^{\frac{2n+3}{n+1}}) + O(\delta^2) \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} &\frac{64\pi^2 N^2\epsilon^2 \int_\Omega e^{2u_0+2W}}{(\int_\Omega e^{u_0+W} + \sqrt{(\int_\Omega e^{u_0+W})^2 - 16\pi N\epsilon^2 \int_\Omega e^{2u_0+2W}})^2}\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right) \\ &= |\sigma'(z)|^2 e^{U_{\delta,a}}\left[\frac{8(n+1)^2\epsilon^2}{\pi|\alpha_a|^{\frac{2}{n+1}}\delta^{\frac{2}{n+1}}}E_{a,\delta} - \epsilon^2|\sigma'(z)|^2 e^{U_{\delta,a}}\right]\left[1 + O(|c_a||z|^{n+1} + \eta) + o(1)\right] \end{aligned} \tag{2.27}$$

as $\epsilon, \delta \to 0$, where α_a, F_a, G_a, D_a, $E_{a,\delta}$ are given in (2.31), (2.35), (2.43), (2.47), *respectively.*

Proof: Recall that (2.15) implies the validity of (2.17), which, combined with Lemma 2.2, yields to the following crucial estimate:

$$W = U_{\delta,a} - \log(8\delta^2) + \log|\sigma'(z)|^2 - u_0 + \frac{2\pi}{|\Omega|}\sum_{k=0}^{n}|a_k|^2 + 2\operatorname{Re}[c_a z^{n+1}] + \Theta_{\delta,a} + 2\delta^2 f_a + O(\delta^4) \tag{2.28}$$

in $C(\overline{\Omega})$ as $\delta \to 0$, uniformly for $|a| < \rho$. Since by Lemma A.1 $\sigma = q^{n+1}$ in $\sigma^{-1}(B_\rho(0))$, through the change of variables $y = q(z)$ in $\sigma^{-1}(B_\rho(0)) = q^{-1}(B_{\rho^{\frac{1}{n+1}}}(0))$, by (2.28) we have that

$$\begin{aligned} &\frac{8\delta^2}{e^{\frac{2\pi}{|\Omega|}\sum_{k=0}^{n}|a_k|^2 + \Theta_{\delta,a} + 2\delta^2 f_a(0)}}\int_{\sigma^{-1}(B_\rho(0))} e^{u_0+W} = \int_{q^{-1}(B_{\rho^{\frac{1}{n+1}}}(0))} |\sigma'(z)|^2 e^{U_{\delta,a} + 2\operatorname{Re}[c_a z^{n+1}] + O(\delta^2|z| + \delta^4)} \\ &= \int_{B_{\rho^{\frac{1}{n+1}}}(0)} \frac{8(n+1)^2\delta^2|y|^{2n}}{(\delta^2 + |y^{n+1} - a|^2)^2} e^{2\operatorname{Re}[c_a(q^{-1}(y))^{n+1}] + O(\delta^2|y| + \delta^4)}. \end{aligned} \tag{2.29}$$

Since $q^{-1}(y) \sim y$ at $y=0$, the following Taylor expansion does hold

$$e^{c_a (q^{-1}(y))^{n+1}} = 1 + c_a y^{n+1} \sum_{k=0}^{+\infty} \alpha_a^k y^k \tag{2.30}$$

in $B_{\rho^{\frac{1}{n+1}}}(0)$, where the coefficients α_a^k depend on a through $\sigma = \sigma_a$. In particular, we have that $\alpha_a := \alpha_a^0$ takes the form

$$\alpha_a = \lim_{z \to 0} \frac{z^{n+1}}{\sigma(z)} \neq 0. \tag{2.31}$$

By (2.30) we then deduce that

$$e^{2\,\mathrm{Re}[c_a (q^{-1}(y))^{n+1}]} = \left| e^{c_a (q^{-1}(y))^{n+1}} \right|^2 = 1 + 2\,\mathrm{Re}\left[c_a y^{n+1} \sum_{k=0}^{+\infty} \alpha_a^k y^k \right] + |c_a|^2 |y|^{2n+2} \sum_{k,s=0}^{+\infty} \alpha_a^k \overline{\alpha}_a^s y^k \overline{y}^s. \tag{2.32}$$

Since

$$\sum_{j=0}^{n} [e^{i \frac{2\pi}{n+1} j}]^k = \sum_{j=0}^{n} e^{i \frac{2\pi}{n+1} j} = 0$$

for all integer $k \notin (n+1)\mathbb{N}$, by the change of variables $y \to e^{i \frac{2\pi}{n+1} j} y$ we have that

$$\begin{aligned} \int_{B_{\rho^{\frac{1}{n+1}}}(0)} \frac{|y|^m y^k}{(\delta^2 + |y^{n+1} - a|^2)^2} &= \sum_{j=0}^{n} \int_{B_{\rho^{\frac{1}{n+1}}}(0) \cap C_j} \frac{|y|^m y^k}{(\delta^2 + |y^{n+1} - a|^2)^2} \\ &= \int_{B_{\rho^{\frac{1}{n+1}}}(0) \cap C_0} \frac{|y|^m y^k}{(\delta^2 + |y^{n+1} - a|^2)^2} \sum_{j=0}^{n} [e^{i \frac{2\pi}{n+1} j}]^k = 0 \end{aligned} \tag{2.33}$$

for all $m \geq 0$ and integer $k \notin (n+1)\mathbb{N}$, where C_j is the sector of the plane between the angles $e^{i \frac{2\pi}{n+1} j}$ and $e^{i \frac{2\pi}{n+1} (j+1)}$. Formula (2.33) tells us that many terms of the expansion (2.32) will give no contribution when inserted in an integral formula like (2.29). Using the notation $\ldots$ to denote such terms, we can rewrite (2.32) as

$$\begin{aligned} e^{2\,\mathrm{Re}[c_a (q^{-1}(y))^{n+1}]} &= 1 + 2\,\mathrm{Re}\left[c_a \sum_{k=0}^{+\infty} \alpha_a^{k(n+1)} y^{(k+1)(n+1)} \right] + |c_a|^2 |y|^{2n+2} \sum_{k=0}^{+\infty} |\alpha_a^k|^2 |y|^{2k} \\ &\quad + 2|c_a|^2 |y|^{2n+2}\, \mathrm{Re}\left[\sum_{k=0}^{+\infty} \sum_{m=1}^{+\infty} \overline{\alpha}_a^k \alpha_a^{k+m(n+1)} |y|^{2k} y^{m(n+1)} \right] + \ldots \end{aligned} \tag{2.34}$$

Setting

$$F_a(y) = \sum_{k=0}^{+\infty} \alpha_a^{k(n+1)} y^{k+1}, \quad G_a(y) = |y|^2 \left[2 \sum_{k=0}^{+\infty} \sum_{m=1}^{+\infty} \overline{\alpha}_a^k \alpha_a^{k+m(n+1)} |y|^{\frac{2k}{n+1}} y^m + \sum_{k=0}^{+\infty} |\alpha_a^k|^2 |y|^{\frac{2k}{n+1}} \right], \tag{2.35}$$

through the change of variables $y \to y^{n+1}$ we can re-write (2.29) as

$$\frac{8\delta^2}{(n+1)e^{\frac{2\pi}{|\Omega|}\sum_{k=0}^n |a_k|^2+\Theta_{\delta,a}+2\delta^2 f_a(0)}}\int_{\sigma^{-1}(B_\rho(0))} e^{u_0+W}$$
$$= \int_{B_\rho(0)} \frac{8\delta^2}{(\delta^2+|y-a|^2)^2}\Big(1+\mathrm{Re}[2c_aF_a(y)+|c_a|^2G_a(y)]+O(\delta^2|y|^{\frac{1}{n+1}}+\delta^4)\Big)$$
$$= 8\pi-\int_{\mathbb{R}^2\setminus B_\rho(0)}\frac{8\delta^2}{|y|^4}+\int_{B_\rho(0)}\frac{8\delta^2}{(\delta^2+|y-a|^2)^2}\,\mathrm{Re}[2c_aF_a(y)+|c_a|^2G_a(y)]+O(\delta^2|a|^{\frac{1}{n+1}}+\delta^{\frac{2n+3}{n+1}}). \quad (2.36)$$

Since $|a|<\frac{\rho}{2}$ and F is an holomorphic function in $B_{\frac{\rho}{2}}(a)\subset B_\rho(0)$, we can expand F_a in a power series around $y=a$:

$$F_a(y)=\sum_{k=0}^{\infty}\frac{F_a^{(k)}(a)}{k!}(y-a)^k, \quad (2.37)$$

and then get

$$\begin{aligned} 2\int_{B_\rho(0)}\frac{8\delta^2}{(\delta^2+|y-a|^2)^2}\,\mathrm{Re}[c_aF_a(y)] &= 2\int_{B_{\frac{\rho}{2}}(a)}\frac{8\delta^2}{(\delta^2+|y-a|^2)^2}\,\mathrm{Re}[c_aF_a(y)]+O(\delta^2|c_a|)\\ &= 16\pi\,\mathrm{Re}[c_aF_a(a)]+O(\delta^2|c_a|) \end{aligned} \quad (2.38)$$

in view of

$$\int_{B_{\frac{\rho}{2}}(a)}\frac{(y-a)^k}{(\delta^2+|y-a|^2)^2}=0$$

for all integer $k\geq 1$. The map $\mathrm{Re}\,G_a$ is just $C^{2+\frac{2}{n+1}}(B_\rho(0))$ and can be expanded up to second order in $y=a$:

$$\mathrm{Re}\,G_a(y)=\mathrm{Re}\,G_a(a)+\langle\nabla\,\mathrm{Re}\,G_a(a),y-a\rangle+\frac{1}{2}\langle D^2\,\mathrm{Re}\,G_a(a)(y-a),y-a\rangle+O(|y-a|^{\frac{2(n+2)}{n+1}}) \quad (2.39)$$

for $y\in B_{\frac{\rho}{2}}(a)$, yielding to

$$|c_a|^2\int_{B_\rho(0)}\frac{8\delta^2}{(\delta^2+|y-a|^2)^2}\,\mathrm{Re}\,G_a(y)=|c_a|^2\int_{B_{\frac{\rho}{2}}(a)}\frac{8\delta^2}{(\delta^2+|y-a|^2)^2}\,\mathrm{Re}\,G_a(y)+O(\delta^2|c_a|^2)$$
$$=8\pi|c_a|^2\,\mathrm{Re}\,G_a(a)+\frac{|c_a|^2}{4}\Delta\,\mathrm{Re}\,G_a(a)\int_{B_{\frac{\rho}{2}}(a)}\frac{8\delta^2}{(\delta^2+|y-a|^2)^2}|y-a|^2+O(\delta^2|c_a|^2)$$
$$=8\pi|c_a|^2\,\mathrm{Re}\,G_a(a)+4\pi|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2\log\frac{1}{\delta}+O(\delta^2|c_a|^2) \quad (2.40)$$

in view of

$$\int_{B_{\frac{\rho}{2}}(a)}\frac{(y-a)_1}{(\delta^2+|y-a|^2)^2}=\int_{B_{\frac{\rho}{2}}(a)}\frac{(y-a)_2}{(\delta^2+|y-a|^2)^2}=\int_{B_{\frac{\rho}{2}}(a)}\frac{(y-a)_1(y-a_2)}{(\delta^2+|y-a|^2)^2}=0$$
$$\int_{B_{\frac{\rho}{2}}(a)}\frac{(y-a)_1^2}{(\delta^2+|y-a|^2)^2}=\int_{B_{\frac{\rho}{2}}(a)}\frac{(y-a)_2^2}{(\delta^2+|y-a|^2)^2}=\frac{1}{2}\int_{B_{\frac{\rho}{2}}(a)}\frac{|y-a|^2}{(\delta^2+|y-a|^2)^2}.$$

By inserting (2.38), (2.40) into (2.36) we get that

$$\frac{8\delta^2}{(n+1)e^{\frac{2\pi}{|\Omega|}\sum_{k=0}^n |a_k|^2+\Theta_{\delta,a}+2\delta^2 f_a(0)}}\int_{\sigma^{-1}(B_\rho(0))} e^{u_0+W}$$
$$= 8\pi - \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{8\delta^2}{|y|^4} + 16\pi\,\mathrm{Re}[c_a F_a(a)] + 8\pi|c_a|^2\,\mathrm{Re}\,G_a(a) + 4\pi|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2\log\frac{1}{\delta}$$
$$+O(\delta^2|a|^{\frac{1}{n+1}} + \delta^2|c_a| + \delta^{\frac{2n+3}{n+1}}). \tag{2.41}$$

By Lemma 2.2, (2.41) and Lemma A.1 we get that

$$\frac{\delta^2}{\pi(n+1)e^{\frac{2\pi}{|\Omega|}\sum_{k=0}^n |a_k|^2+\Theta_{\delta,a}+2\delta^2 f_a(0)}}\int_\Omega e^{u_0+W} = 1 + 2\,\mathrm{Re}[c_a F_a(a)] + |c_a|^2\,\mathrm{Re}\,G_a(a)$$
$$+\frac{1}{2}|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2\log\frac{1}{\delta} + \frac{\delta^2}{n+1}D_a + O(\delta^2|a|^{\frac{1}{n+1}} + \delta^2|c_a| + \delta^{\frac{2n+3}{n+1}}), \tag{2.42}$$

where

$$\pi D_a = \int_{\Omega\setminus\sigma^{-1}(B_\rho(0))} e^{u_0+8\pi\sum_{k=0}^n G(z,a_k)-\frac{2\pi}{|\Omega|}\sum_{k=0}^n|a_k|^2} - \int_{\mathbb{R}^2\setminus B_\rho(0)}\frac{n+1}{|y|^4}. \tag{2.43}$$

In view of (2.4) and $\int_\Omega |\sigma'(z)|^2 e^{U_{\delta,a}} = 8\pi(n+1) + O(\delta^2)$, by (2.28) and (2.42) we have that

$$\Delta W + 4\pi N\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{1}{|\Omega|}\right)$$
$$= |\sigma'(z)|^2 e^{U_{\delta,a}}\left[4\pi N\frac{e^{2\,\mathrm{Re}[c_a z^{n+1}]+O(\delta^2|z|+\delta^4)}}{8\delta^2 e^{-\frac{2\pi}{|\Omega|}\sum_{k=0}^n|a_k|^2-\Theta_{\delta,a}-2\delta^2 f_a(0)}\int_\Omega e^{u_0+W}} - 1\right] + \frac{1}{|\Omega|}\left(\int_\Omega |\sigma'(z)|^2 e^{U_{\delta,a}} - 4\pi N\right)$$
$$= |\sigma'(z)|^2 e^{U_{\delta,a}}\left[\frac{e^{2\,\mathrm{Re}[c_a z^{n+1}]}}{1 + 2\,\mathrm{Re}[c_a F_a(a)] + |c_a|^2\,\mathrm{Re}\,G_a(a) + \frac{1}{2}|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2\log\frac{1}{\delta} + \frac{\delta^2}{n+1}D_a} - 1\right]$$
$$+|\sigma'(z)|^2 e^{U_{\delta,a}} O(\delta^2|z| + \delta^2|a|^{\frac{1}{n+1}} + \delta^2|c_a| + \delta^{\frac{2n+3}{n+1}}) + O(\delta^2)$$

as $\delta\to 0$, yielding to the validity of (2.26).

Introducing the notation $B(w) = 16\pi N(\int_\Omega e^{2u_0+2w})(\int_\Omega e^{u_0+w})^{-2}$, we can write the following expansion

$$\frac{16\pi N\int_\Omega e^{2u_0+2W}}{(\int_\Omega e^{u_0+W} + \sqrt{(\int_\Omega e^{u_0+W})^2 - 16\pi N\epsilon^2\int_\Omega e^{2u_0+2W}})^2} = \frac{B(W)}{4} + O(\epsilon^2 B^2(W)). \tag{2.44}$$

Arguing as for (2.42), the change of variables $y = \sigma(z)$ yields to

$$\frac{64\delta^{4+\frac{2}{n+1}}}{e^{\frac{4\pi}{|\Omega|}\sum_{k=0}^n|a_k|^2+2\Theta_{\delta,a}}}\int_\Omega e^{2u_0+2W} = \delta^{\frac{2}{n+1}}\int_{\sigma^{-1}(B_\rho(0))}|\sigma'(z)|^4 e^{2U_{\delta,a}+O(|c_a||z|^{n+1}+\delta^2)} + O(\delta^{4+\frac{2}{n+1}})$$
$$= 64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\int_{B_\rho(0)}\frac{\delta^{4+\frac{2}{n+1}}|y|^{\frac{2n}{n+1}}}{(\delta^2+|y-a|^2)^4}\left(1 + O(|c_a||y| + \delta^2 + |y|^{\frac{1}{n+1}})\right) + O(\delta^{4+\frac{2}{n+1}})$$
$$= 64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\int_{B_\rho(0)}\frac{\delta^{4+\frac{2}{n+1}}|y+a|^{\frac{2n}{n+1}}}{(\delta^2+|y|^2)^4}\left(1 + O(\delta^2 + |y|^{\frac{1}{n+1}} + |a|^{\frac{1}{n+1}})\right) + O(\delta^{4+\frac{2}{n+1}}) \tag{2.45}$$

in view of

$$|\sigma'(z)|^2=(n+1)^2|\alpha_a|^{-2}|z|^{2n}(1+O(|z|))=(n+1)^2|\alpha_a|^{-\frac{2}{n+1}}|\sigma(z)|^{\frac{2n}{n+1}}(1+O(|\sigma(z)|^{\frac{1}{n+1}})), \tag{2.46}$$

where α_a is given by (2.31). We have that

$$\int_{B_\rho(0)}\frac{\delta^{4+\frac{2}{n+1}}|y+a|^{\frac{2n}{n+1}}}{(\delta^2+|y|^2)^4}=\int_{\mathbb{R}^2}\frac{|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^4}+O(\delta^{4+\frac{2}{n+1}})$$

if $|a|=O(\delta)$ and

$$\int_{B_\rho(0)}\frac{\delta^{4+\frac{2}{n+1}}|y+a|^{\frac{2n}{n+1}}}{(\delta^2+|y|^2)^4}=\Big(\frac{|a|}{\delta}\Big)^{\frac{2n}{n+1}}\int_{\mathbb{R}^2}\frac{1}{(1+|y|^2)^4}\Big[1+O\Big(\frac{\delta}{|a|}+\delta^6\Big)\Big]$$

if $|a|>>\delta$, where in the latter we have used the inequality:

$$|y+a|^{\frac{2n}{n+1}}=|a|^{\frac{2n}{n+1}}+O(|a|^{\frac{n-1}{n+1}}|y|+|y|^{\frac{2n}{n+1}}).$$

Setting

$$E_{a,\delta}:=\begin{cases}\displaystyle\int_{\mathbb{R}^2}\frac{|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^4} & \text{if } |a|=O(\delta)\\[2ex] \displaystyle\frac{\pi}{3}\Big(\frac{|a|}{\delta}\Big)^{\frac{2n}{n+1}} & \text{if } |a|>>\delta,\end{cases} \tag{2.47}$$

by (2.45) we get that

$$\frac{64\delta^{4+\frac{2}{n+1}}}{e^{\frac{4\pi}{|\Omega|}\sum_{k=0}^n|a_k|^2+2\Theta_{\delta,a}}}\int_\Omega e^{2u_0+2W}=64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}(1+o(1))E_{a,\delta}. \tag{2.48}$$

Since by a combination of (2.42) and (2.48) for $B(W)$ we have that

$$B(W)=32\frac{(n+1)^2}{\pi\delta^{\frac{2}{n+1}}}|\alpha_a|^{-\frac{2}{n+1}}(1+o(1))E_{a,\delta} \tag{2.49}$$

in view of (2.4), by (2.44) and (2.49) we get that

$$\frac{16\pi N\int_\Omega e^{2u_0+2W}}{(\int_\Omega e^{u_0+W}+\sqrt{(\int_\Omega e^{u_0+W})^2-16\pi N\epsilon^2\int_\Omega e^{2u_0+2W}})^2}=8\frac{(n+1)^2}{\pi\delta^{\frac{2}{n+1}}}|\alpha_a|^{-\frac{2}{n+1}}(1+o(1)+O(\eta))E_{a,\delta}, \tag{2.50}$$

where η is given by (2.25). As we have already seen in deriving (2.26), by (2.28) we have that

$$4\pi N\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}=|\sigma'(z)|^2e^{U_{\delta,a}}\left[1+O(|c_a||z|^{n+1})+O(|c_a||a|+\delta^2|\log\delta|)\right], \tag{2.51}$$

and in a similar way one can show that

$$\frac{64(n+1)^3}{\delta^{\frac{2}{n+1}}}|\alpha_a|^{-\frac{2}{n+1}}\frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}E_{a,\delta}=|\sigma'(z)|^4e^{2U_{\delta,a}}\left[1+O(|c_a||z|^{n+1})+o(1)\right] \tag{2.52}$$

in view of (2.48). In conclusion, by (2.50)-(2.52) we have for the ϵ^2-term in R that

$$\frac{64\pi^2 N^2\epsilon^2\int_\Omega e^{2u_0+2W}}{(\int_\Omega e^{u_0+W}+\sqrt{(\int_\Omega e^{u_0+W})^2-16\pi N\epsilon^2\int_\Omega e^{2u_0+2W}})^2}\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}-\frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right)$$
$$=|\sigma'(z)|^2 e^{U_{\delta,a}}\left[\frac{8(n+1)^2\epsilon^2}{\pi|\alpha_a|^{\frac{2}{n+1}}\delta^{\frac{2}{n+1}}}E_{a,\delta}-\epsilon^2|\sigma'(z)|^2e^{U_{\delta,a}}\right]\left[1+O(|c_a||z|^{n+1}+\eta)+o(1)\right]$$

in view of (2.4), yielding to the validity of (2.27). This completes the proof. ■

Let us introduce the following weighted norm

$$\|h\|_*=\sup_{z\in\Omega}\frac{(\delta^2+|\sigma(z)-a|^2)^{1+\frac{\gamma}{2}}}{\delta^\gamma(|\sigma'(z)|^2+\delta^{\frac{2n}{n+1}})}\ |h(z)| \tag{2.53}$$

for any $h\in L^\infty(\Omega)$, where $0<\gamma<1$ is a small fixed constant. We have that

Corollary 2.4. *There exist positive constants δ_0, ϵ_0 and C_0 such that*

$$\|R\|_*\le C_0\left(\delta|c_a|+\delta^{2-\gamma}+\delta^{\frac{2}{n+1}-\gamma}|a|^{2+\gamma}+|c_a||a|^{\frac{n+2}{n+1}}+\eta+\eta^2\right) \tag{2.54}$$

for any $\delta\in(0,\delta_0)$ and $\epsilon\in(0,\epsilon_0)$, where η is given by (2.25).

Proof: Since

$$\frac{e^{2\operatorname{Re}[c_az^{n+1}]}}{1+2\operatorname{Re}[c_aF_a(a)]+|c_a|^2\operatorname{Re}G_a(a)+\frac12|c_a|^2\Delta\operatorname{Re}G_a(a)\delta^2\log\frac1\delta+\frac{\delta^2}{n+1}D_a}-1$$
$$=\frac{e^{2\operatorname{Re}[c_az^{n+1}]}-1}{1+2\operatorname{Re}[c_aF_a(a)]+|c_a|^2\operatorname{Re}G_a(a)+\frac12|c_a|^2\Delta\operatorname{Re}G_a(a)\delta^2\log\frac1\delta+\frac{\delta^2}{n+1}D_a}-2\operatorname{Re}[c_aF_a(a)]$$
$$+O(|c_a|^2|a|^2+\delta^2|\log\delta|)=2\operatorname{Re}[c_a(z^{n+1}-\alpha_aa)]+O(|c_a|^2|z|^{2n+2}+|c_a||a|^2+\delta^2|\log\delta|)$$
$$=2\operatorname{Re}[\alpha_ac_a(\sigma(z)-a)]+O(|c_a||z|^{n+2}+|c_a||a|^2+\delta^2|\log\delta|),$$

by Theorem 2.3 we deduce that

$$R=|\sigma'(z)|^2e^{U_{\delta,a}}O\left(|c_a||\sigma(z)-a|+|c_a||z|^{n+2}+|c_a||a|^2+\delta^2|\log\delta|+\eta+\eta^2\right)+\epsilon^2|\sigma'(z)|^4e^{2U_{\delta,a}}(1+O(\eta))+O(\delta^2)$$

as $\delta\to0$, where η is given in (2.25). In view of the estimates $|z|=O(|\sigma(z)|^{\frac{1}{n+1}})$ and $|\sigma'(z)|^2=O(|\sigma(z)|^{\frac{2n}{n+1}})$

near 0, by setting $y=\sigma(z)$ in $\sigma^{-1}(B_\rho(0))$ we get that

$$\begin{aligned}
\|R\|_* &= O\left(\sup_{y\in B_\rho(0)} \frac{\delta^{2-\gamma}}{(\delta^2+|y-a|^2)^{1-\frac{\gamma}{2}}}\left[|c_a||y-a|+|c_a||y|^{\frac{n+2}{n+1}}+|c_a||a|^2+\delta^2|\log\delta|+\eta+\eta^2\right]\right)\\
&\quad +O\left(\sup_{y\in B_\rho(0)} \frac{\epsilon^2\delta^{4-\gamma}|y|^{\frac{2n}{n+1}}}{(\delta^2+|y-a|^2)^{3-\frac{\gamma}{2}}}[1+O(\eta)]\right)+O\left(\sup_{y\in B_\rho(0)} \frac{\delta^{2-\gamma}(\delta^2+|y-a|^2)^{1+\gamma/2}}{(|y|^{\frac{2n}{n+1}}+\delta^{\frac{2n}{n+1}})}\right)+O(\delta^{2-\gamma})\\
&= O\left(\sup_{y\in B_{2\rho/\delta}(0)} \frac{1}{(1+|y|^2)^{1-\frac{\gamma}{2}}}\left[\delta|c_a||y|+\delta^{\frac{n+2}{n+1}}|c_a||y|^{\frac{n+2}{n+1}}+|c_a||a|^{\frac{n+2}{n+1}}+\delta^2|\log\delta|+\eta+\eta^2\right]\right)\\
&\quad +O\left(\sup_{y\in B_{2\rho/\delta}(0)} \frac{\epsilon^2\delta^{-2}(\delta^{\frac{2n}{n+1}}|y|^{\frac{2n}{n+1}}+|a|^{\frac{2n}{n+1}})}{(1+|y|^2)^{3-\frac{\gamma}{2}}}[1+O(\eta)]\right)\\
&\quad +O\left(\sup_{y\in B_{\rho/\delta}(0)} \frac{\delta^{\frac{2}{n+1}-\gamma}(\delta^{2+\gamma}+|a|^{2+\gamma}+\delta^{2+\gamma}|y|^{2+\gamma})}{(|y|^{\frac{2n}{n+1}}+1)}\right)+O(\delta^{2-\gamma})\\
&= O\left(\delta|c_a|+\delta^{2-\gamma}+\delta^{\frac{2}{n+1}-\gamma}|a|^{2+\gamma}+|c_a||a|^{\frac{n+2}{n+1}}+\eta+\eta^2\right)
\end{aligned}$$

as claimed. ∎

3 The reduced equations

As we will discuss precisely in the next section, it will be crucial to study the system $\int_\Omega RPZ_0=0$ and $\int_\Omega RPZ=0$, where PZ_0 and PZ are the unique solutions with zero average of $\Delta PZ_0=\Delta Z_0-\frac{1}{|\Omega|}\int_\Omega \Delta Z_0$ and $\Delta PZ=\Delta Z-\frac{1}{|\Omega|}\int_\Omega\Delta Z$ in Ω. Here, the functions Z_0 and Z are defined as follows:

$$Z_0(z)=\frac{\delta^2-|\sigma(z)-a|^2}{\delta^2+|\sigma(z)-a|^2}\quad\text{and}\quad Z(z)=\frac{\delta(\sigma(z)-a)}{\delta^2+|\sigma(z)-a|^2},$$

and are (not doubly-periodic) solutions of $-\Delta\phi=|\sigma'(z)|^2e^{U_{\delta,a,\sigma}}\phi$ in Ω. Through the changes of variable $y=\sigma(z)$ and $y\to\frac{y-a}{\delta}$ notice that

$$\begin{aligned}
\int_\Omega \Delta Z_0 &= -\int_{\sigma^{-1}(B_\rho(0))}|\sigma'(z)|^2e^{U_{\delta,a,\sigma}}Z_0+O(\delta^2)=-8(n+1)\delta^2\int_{B_\rho(0)}\frac{\delta^2-|y-a|^2}{(\delta^2+|y-a|^2)^3}+O(\delta^2)\\
&= -8(n+1)\int_{B_{\rho/\delta}(0)}\frac{1-|y|^2}{(1+|y|^2)^3}+O(\delta^2)=O(\delta^2) \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega \Delta Z &= -\int_{\sigma^{-1}(B_\rho(0))}|\sigma'(z)|^2e^{U_{\delta,a,\sigma}}Z+O(\delta^3)=-8(n+1)\delta^3\int_{B_\rho(0)}\frac{y-a}{(\delta^2+|y-a|^2)^3}+O(\delta^3)\\
&= -8(n+1)\int_{B_{\rho/\delta}(0)}\frac{y}{(1+|y|^2)^3}+O(\delta^3)=O(\delta^3) \qquad (3.2)
\end{aligned}$$

in view of

$$\int_{\mathbb{R}^2}\frac{1-|y|^2}{(1+|y|^2)^3}=0\,,\quad \int_{\mathbb{R}^2}\frac{y}{(1+|y|^2)^3}=0.$$

By (3.1)-(3.2) the following expansions, useful in the sequel, are easily deduced:

$$PZ_0 = Z_0 - \frac{1}{|\Omega|}\int_\Omega Z_0 + O(\delta^2)\,, \qquad PZ = Z - \frac{1}{|\Omega|}\int_\Omega Z + O(\delta) \tag{3.3}$$

in $C(\overline{\Omega})$, uniformly in $|a| < \rho$ and $\sigma \in \mathcal{B}_r$.

Notice that up to now there is no relation between a and δ. However, as we will show in Remarks 3.2 and 3.3, the range $|a| >> \delta$ is not compatible with solving simultaneously $\int_\Omega RPZ_0 = 0$ and $\int_\Omega RPZ = 0$. Hence, we shall restrict our attention to the case $a = O(\delta)$ in next sections, so that, we can assume that $\eta = \epsilon^2\delta^{-\frac{2}{n+1}}$ in (2.25) and $E_{a,\delta} = \int_{\mathbb{R}^2} \frac{|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^4}$ in (2.47). We have that

Proposition 3.1. *Assume $|a| \le C_0\delta$ for some $C_0 > 0$. The following expansions do hold as $\delta, \eta \to 0$*

$$\begin{aligned}\int_\Omega RPZ_0 \;=\; & -16\pi(n+1)|\alpha_a|^2|c_a|^2\delta^2\log\frac{1}{\delta} - 8\pi\delta^2 D_a + 64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\eta\int_{\mathbb{R}^2}\frac{(|y|^2-1)|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}\\ & + o(\delta^2+\eta) + O(\delta^2|c_a| + |a|^{\frac{1}{n+1}}\delta^2|\log\delta| + \eta^2),\end{aligned} \tag{3.4}$$

and

$$\int_\Omega RPZ = 4\pi(n+1)\delta\overline{\alpha_a c_a} - 64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\eta\int_{\mathbb{R}^2}\frac{|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}y}{(1+|y|^2)^5} + o(\delta|c_a| + \delta|a| + \eta + \delta^2) + O(\eta^2), \tag{3.5}$$

where $\eta = \epsilon^2\delta^{-\frac{2}{n+1}}$ and $c_a = c_{a,\sigma_a}$, α_a, D_a are given by (2.16), (2.31), (2.43), respectively.

Proof: Through the changes of variable $y = q(z)$ in $\sigma^{-1}(B_\rho(0))$, $y \to y^{n+1}$ and $y \to \frac{y-a}{\delta}$ we get that

$$\begin{aligned}&\int_\Omega \frac{\delta^\gamma(|\sigma'(z)|^2 + \delta^{\frac{2n}{n+1}})}{(\delta^2 + |\sigma(z)-a|^2)^{1+\frac{\gamma}{2}}} = \int_{\sigma^{-1}(B_\rho(0))}\frac{\delta^\gamma(|\sigma'(z)|^2 + \delta^{\frac{2n}{n+1}})}{(\delta^2 + |\sigma(z)-a|^2)^{1+\frac{\gamma}{2}}} + O(\delta^\gamma) \\ &= O\left(\int_{B_{\rho^{\frac{1}{n+1}}}(0)}\frac{\delta^\gamma(|y|^{2n} + \delta^{\frac{2n}{n+1}})}{(\delta^2 + |y^{n+1}-a|^2)^{1+\frac{\gamma}{2}}}\right) + O(\delta^\gamma) = O\left(\int_{B_\rho(0)}\frac{\delta^\gamma(1 + \delta^{\frac{2n}{n+1}}|y|^{-\frac{2n}{n+1}})}{(\delta^2+|y-a|^2)^{1+\frac{\gamma}{2}}}\right) + O(\delta^\gamma) \\ &= O\left(\int_{B_{\rho/\delta}(0)}\frac{1+|y+\frac{a}{\delta}|^{-\frac{2n}{n+1}}}{(1+|y|^2)^{1+\frac{\gamma}{2}}}\right) + O(\delta^\gamma) = O(1)\end{aligned} \tag{3.6}$$

in view of

$$\int_{B_{\rho/\delta}(0)}\frac{|y+\frac{a}{\delta}|^{-\frac{2n}{n+1}}}{(1+|y|^2)^{1+\frac{\gamma}{2}}} \le \int_{B_1(0)}|y|^{-\frac{2n}{n+1}} + \int_{\mathbb{R}^2}\frac{1}{(1+|y|^2)^{1+\frac{\gamma}{2}}} < +\infty.$$

Hence, by Corollary 2.4 we get that

$$\int_\Omega |R| = O\left(\delta|c_a| + \delta^{2-\gamma} + \delta^{\frac{2}{n+1}-\gamma}|a|^{2+\gamma} + |c_a||a|^{\frac{n+2}{n+1}} + \eta + \eta^2\right). \tag{3.7}$$

By (3.3) and (3.7) we deduce that

$$\int_\Omega RPZ_0 = \int_\Omega R(Z_0+1) + o(\delta^2) + O(\eta\delta^2 + \eta^2\delta^2) \tag{3.8}$$

in view of $\int_\Omega R=0$. Since by Hölder inequality

$$\begin{aligned}\int_\Omega |Z_0+1| &= \int_{\sigma^{-1}(B_\rho(0))} \frac{2\delta^2}{\delta^2+|\sigma(z)-a|^2}+O(\delta^2)=O\bigg(\int_{B_\rho(0)} |y|^{-\frac{2n}{n+1}}\frac{\delta^2}{\delta^2+|y-a|^2}\bigg)+O(\delta^2)\\ &= O\bigg(\delta^{\frac{1}{n+1}}\int_{B_\rho(0)}\frac{1}{|y|^{\frac{2n}{n+1}}|y-a|^{\frac{1}{n+1}}}\bigg)+O(\delta^2)\\ &= O\bigg(\delta^{\frac{1}{n+1}}\Big[\int_{B_\rho(0)}\frac{1}{|y|^{\frac{2n+1}{n+1}}}\Big]^{\frac{2n}{2n+1}}\Big[\int_{B_\rho(0)}\frac{1}{|y-a|^{\frac{2n+1}{n+1}}}\Big]^{\frac{1}{2n+1}}\bigg)+O(\delta^2)=O(\delta^{\frac{1}{n+1}}),\end{aligned}$$

by (2.26) we have that

$$\begin{aligned}&\int_\Omega (Z_0+1)\left[\Delta W+4\pi N\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}-\frac{1}{|\Omega|}\right)\right] \qquad (3.9)\\ &=\int_{\sigma^{-1}(B_\rho(0))}|\sigma'(z)|^2 e^{U_{\delta,a}}(Z_0+1)\left[\frac{e^{2\operatorname{Re}[c_a z^{n+1}]}}{1+2\operatorname{Re}[c_aF_a(a)]+|c_a|^2\operatorname{Re}G_a(a)+\frac12|c_a|^2\Delta\operatorname{Re}G_a(a)\delta^2\log\frac1\delta+\frac{\delta^2}{n+1}D_a}-1\right]\\ &+O(\delta^2|c_a|)+o(\delta^2)\\ &=\int_{B_{\rho^{\frac{1}{n+1}}}(0)}\frac{16(n+1)^2\delta^4|y|^{2n}}{(\delta^2+|y^{n+1}-a|^2)^3}\left[\frac{e^{2\operatorname{Re}[c_a(q^{-1}(y))^{n+1}]}}{1+2\operatorname{Re}[c_aF_a(a)]+|c_a|^2\operatorname{Re}G_a(a)+\frac12|c_a|^2\Delta\operatorname{Re}G_a(a)\delta^2\log\frac1\delta+\frac{\delta^2}{n+1}D_a}-1\right]\\ &+O(\delta^2|c_a|)+o(\delta^2).\end{aligned}$$

We have that the expansion (2.34) still holds in this context, where the notation ... stands for terms that give no contribution in the integral term of (3.9) in view of the analogous of formula (2.33):

$$\int_{B_{\rho^{\frac{1}{n+1}}}(0)}\frac{|y|^m y^k}{(\delta^2+|y^{n+1}-a|^2)^3}=0 \qquad (3.10)$$

for all $m\ge 0$ and integer $k\notin (n+1)\mathbb{N}$. Hence, through the changes of variables $y\to y^{n+1}$ and $y\to\frac{y-a}{\delta}$, by the symmetries we have that

$$\begin{aligned}&\int_{B_{\rho^{\frac{1}{n+1}}}(0)}\frac{16(n+1)^2\delta^4|y|^{2n}}{(\delta^2+|y^{n+1}-a|^2)^3}e^{2\operatorname{Re}[c_a(q^{-1}(y))^{n+1}]}=\int_{B_\rho(0)}\frac{16(n+1)\delta^4}{(\delta^2+|y-a|^2)^3}\operatorname{Re}[1+2c_aF_a(y)+|c_a|^2G_a(y)]\\ &=\int_{B_\rho(a)}\frac{16(n+1)\delta^4}{(\delta^2+|y-a|^2)^3}\left[1+2\operatorname{Re}[c_aF_a(a)]+|c_a|^2\operatorname{Re}G_a(a)+\frac14|c_a|^2\Delta\operatorname{Re}G_a(a)|y-a|^2+O(|y-a|^{\frac{2(n+2)}{n+1}})\right]\\ &+O(\delta^4)=8\pi(n+1)\left[1+2\operatorname{Re}[c_aF_a(a)]+|c_a|^2\operatorname{Re}G_a(a)+\frac14|c_a|^2\Delta\operatorname{Re}G_a(a)\delta^2\right]+O(\delta^{\frac{2(n+2)}{n+1}}) \qquad (3.11)\end{aligned}$$

in view of (2.37), (2.39) and

$$\int_{\mathbb{R}^2}\frac{dy}{(1+|y|^2)^3}=\int_{\mathbb{R}^2}\frac{|y|^2}{(1+|y|^2)^3}dy=\frac\pi2,$$

where F_a and G_a are given by (2.35). By (3.11) we can re-write (3.9) as

$$\int_\Omega (Z_0+1)\left[\Delta W + 4\pi N\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{1}{|\Omega|}\right)\right]$$

$$= 8\pi(n+1)\left[\frac{1+2\,\mathrm{Re}[c_aF_a(a)]+|c_a|^2\,\mathrm{Re}\,G_a(a)+\frac14|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2}{1+2\,\mathrm{Re}[c_aF_a(a)]+|c_a|^2\,\mathrm{Re}\,G_a(a)+\frac12|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2\log\frac1\delta+\frac{\delta^2}{n+1}D_a}-1\right]+O(\delta^2|c_a|)$$

$$+o(\delta^2) = -16\pi(n+1)|\alpha_a|^2|c_a|^2\delta^2\log\frac1\delta - 8\pi\delta^2 D_a + O(\delta^2|c_a|+|a|^{\frac{1}{n+1}}\delta^2|\log\delta|)+o(\delta^2) \tag{3.12}$$

in view of $\Delta\,\mathrm{Re}\,G_a(a) = 4|\alpha_a|^2 + O(|a|^{\frac{1}{n+1}})$. By (2.27) we also deduce that

$$\int_\Omega \frac{64\pi^2N^2\epsilon^2\int_\Omega e^{2u_0+2W}}{(\int_\Omega e^{u_0+W}+\sqrt{(\int_\Omega e^{u_0+W})^2-16\pi N\epsilon^2\int_\Omega e^{2u_0+2W}})^2}(Z_0+1)\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}-\frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right)$$

$$= \int_{\sigma^{-1}(B_\rho(0))}|\sigma'(z)|^2e^{U_{\delta,a}}(Z_0+1)\left[\frac{8(n+1)^2\epsilon^2}{\pi|\alpha_a|^{\frac{2}{n+1}}\delta^{\frac{2}{n+1}}}E_{a,\delta}-\epsilon^2|\sigma'(z)|^2e^{U_{\delta,a}}\right]\left[1+O(|c_a||z|^{n+1}+\eta)+o(1)\right]$$

$$+O(\delta^4\eta) = \frac{128(n+1)^3\epsilon^2}{\pi|\alpha_a|^{\frac{2}{n+1}}\delta^{\frac{2}{n+1}}}E_{a,\delta}\int_{B_\rho(0)}\frac{\delta^4}{(\delta^2+|y-a|^2)^3}\left[1+O(|c_a||y|+\eta)+o(1)\right]$$

$$-128(n+1)^3\epsilon^2|\alpha_a|^{-\frac{2}{n+1}}\int_{B_\rho(0)}\frac{\delta^6|y|^{\frac{2n}{n+1}}}{(\delta^2+|y-a|^2)^5}\left[1+O(|y|^{\frac{1}{n+1}}+\eta)+o(1)\right]+O(\delta^4\eta)$$

$$= 64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\epsilon^2\delta^{-\frac{2}{n+1}}E_{a,\delta} - 128(n+1)^3\epsilon^2|\alpha_a|^{-\frac{2}{n+1}}\int_{B_\rho(0)}\frac{\delta^6|y+a|^{\frac{2n}{n+1}}}{(\delta^2+|y|^2)^5}\left[1+O(|y|^{\frac{1}{n+1}}+\eta)+o(1)\right]$$

$$+o(\eta+\delta^2)+O(\eta^2)$$

in view of (2.46). Since

$$\delta^{\frac{2}{n+1}}\int_{B_\rho(0)}\frac{\delta^6|y+a|^{\frac{2n}{n+1}}}{(\delta^2+|y|^2)^5}\left[1+O(|y|^{\frac{1}{n+1}}+\eta)+o(1)\right] = \int_{\mathbb{R}^2}\frac{|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}+o(1)+O(\eta)$$

when $|a| = O(\delta)$, we then have that

$$\int_\Omega \frac{64\pi^2N^2\epsilon^2\int_\Omega e^{2u_0+2W}}{(\int_\Omega e^{u_0+W}+\sqrt{(\int_\Omega e^{u_0+W})^2-16\pi N\epsilon^2\int_\Omega e^{2u_0+2W}})^2}(Z_0+1)\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}-\frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right)$$

$$= 64(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\eta\int_{\mathbb{R}^2}\frac{(|y|^2-1)|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}+o(\eta+\delta^2)+O(\eta^2) \tag{3.13}$$

in view of (2.47). Inserting (3.12) and (3.13) into (3.8), we get the validity of (3.4).

Remark 3.2. *Notice that in the range $|a| >> \delta$ we find that*

$$\delta^{\frac{2}{n+1}}\int_{B_\rho(0)}\frac{\delta^6|y+a|^{\frac{2n}{n+1}}}{(\delta^2+|y|^2)^5}\left[1+O\Big(|y|^{\frac{1}{n+1}}+\eta\Big(\frac{|a|}{\delta}\Big)^{\frac{2n}{n+1}}\Big)+o(1)\right] = \frac{\pi}{4}\Big(\frac{|a|}{\delta}\Big)^{\frac{2n}{n+1}}\left[1+o(1)+O\Big(\eta\Big(\frac{|a|}{\delta}\Big)^{\frac{2n}{n+1}}\Big)\right]$$

in view of the inequality $|y+a|^{\frac{2n}{n+1}} = |a|^{\frac{2n}{n+1}} + O(|a|^{\frac{n-1}{n+1}}|y|+|y|^{\frac{2n}{n+1}})$, so that the main order of $\int_\Omega RPZ_0$ in this range is essentially given by

$$-16\pi(n+1)|\alpha_a|^2|c_a|^2\delta^2\log\frac1\delta - 8\pi\delta^2D_a - \frac{32\pi}{3}(n+1)^3|\alpha_a|^{-\frac{2}{n+1}}\eta\Big(\frac{|a|}{\delta}\Big)^{\frac{2n}{n+1}}.$$

By (3.3) and (3.7) we deduce that

$$\int_\Omega R\,PZ = \int_\Omega RZ + o(\delta|c_a| + \delta|a| + \eta + \delta^2) + O(\eta^2\delta) \tag{3.14}$$

in view of $\int_\Omega R = 0$. Since as before

$$\begin{aligned}\int_\Omega |Z| &= \int_{\sigma^{-1}(B_\rho(0))} \frac{\delta|\sigma(z)-a|}{\delta^2+|\sigma(z)-a|^2} + O(\delta) = O\Big(\int_{B_\rho(0)} |y|^{-\frac{2n}{n+1}} \frac{\delta|y-a|}{\delta^2+|y-a|^2}\Big) + O(\delta)\\ &= O\Big(\delta^{\frac{1}{n+1}} \int_{B_\rho(0)} \frac{1}{|y|^{\frac{2n}{n+1}}|y-a|^{\frac{1}{n+1}}}\Big) + O(\delta) = O(\delta^{\frac{1}{n+1}}),\end{aligned}$$

by (2.26) we have that

$$\begin{aligned}&\int_\Omega Z\left[\Delta W + 4\pi N\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{1}{|\Omega|}\right)\right] \qquad (3.15)\\
&= \int_{\sigma^{-1}(B_\rho(0))} |\sigma'(z)|^2 e^{U_{\delta,a}} Z \left[\frac{e^{2\,\mathrm{Re}[c_a z^{n+1}]}}{1+2\,\mathrm{Re}[c_aF_a(a)] + |c_a|^2\,\mathrm{Re}\,G_a(a) + \frac12|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2\log\frac1\delta + \frac{\delta^2}{n+1}D_a} - 1\right]\\
&+O(\delta^2|c_a|) + o(\delta^2)\\
&= \int_{B_{\rho^{\frac{1}{n+1}}}(0)} \frac{8(n+1)^2\delta^3|y|^{2n}(y^{n+1}-a)}{(\delta^2+|y^{n+1}-a|^2)^3} \frac{e^{2\,\mathrm{Re}[c_a(q^{-1}(y))^{n+1}]}}{1+2\,\mathrm{Re}[c_aF_a(a)] + |c_a|^2\,\mathrm{Re}\,G_a(a) + \frac12|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2\log\frac1\delta + \frac{\delta^2}{n+1}D_a}\\
&- \int_{B_\rho(0)} \frac{8(n+1)\delta^3(y-a)}{(\delta^2+|y-a|^2)^3} + O(\delta^2|c_a|) + o(\delta^2)\\
&= \frac{\int_{B_{\rho^{\frac{1}{n+1}}}(0)} \frac{8(n+1)^2\delta^3|y|^{2n}(y^{n+1}-a)}{(\delta^2+|y^{n+1}-a|^2)^3} e^{2\,\mathrm{Re}[c_a(q^{-1}(y))^{n+1}]}}{1+2\,\mathrm{Re}[c_aF_a(a)] + |c_a|^2\,\mathrm{Re}\,G_a(a) + \frac12|c_a|^2\Delta\,\mathrm{Re}\,G_a(a)\delta^2\log\frac1\delta + \frac{\delta^2}{n+1}D_a} + O(\delta^2|c_a|) + o(\delta^2)\end{aligned}$$

in view of

$$\int_{B_\rho(a)} \frac{8(n+1)\delta^3(y-a)}{(\delta^2+|y-a|^2)^3} = 0.$$

Since expansion (2.34) is still valid in view of (3.10), through the changes of variables $y \to y^{n+1}$ and $y \to \frac{y-a}{\delta}$, by the symmetries we have that

$$\begin{aligned}&\int_{B_{\rho^{\frac{1}{n+1}}}(0)} \frac{8(n+1)^2\delta^3|y|^{2n}(y^{n+1}-a)}{(\delta^2+|y^{n+1}-a|^2)^3} e^{2\,\mathrm{Re}[c_a(q^{-1}(y))^{n+1}]}\\
&= \int_{B_\rho(0)} \frac{8(n+1)\delta^3(y-a)}{(\delta^2+|y-a|^2)^3}\,\mathrm{Re}[1+2c_aF_a(y) + |c_a|^2G_a(y)]\\
&= \int_{B_\rho(a)} \frac{8(n+1)\delta^3}{(\delta^2+|y-a|^2)^3}\left[\overline{c_aF_a'(a)}|y-a|^2 + \frac12|c_a|^2(\partial_1+i\partial_2)\,\mathrm{Re}\,G_a(a)|y-a|^2 + O(|c_a|^2|y-a|^3)\right] + O(\delta^3)\\
&= 4\pi(n+1)\delta\left[\overline{c_aF_a'(a)} + \frac12|c_a|^2(\partial_1+i\partial_2)\,\mathrm{Re}\,G_a(a)\right] + O(\delta^2|c_a|^2+\delta^3) \qquad (3.16)\end{aligned}$$

in view of (2.37), (2.39) and $\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^3} dy = \frac{\pi}{2}$, where F_a and G_a are given by (2.35). By (3.16) we can re-write (3.15) as

$$\int_\Omega Z \left[\Delta W + 4\pi N \left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{1}{|\Omega|} \right)\right] = 4\pi(n+1)\delta \left[\overline{c_a F_a'(a)} + \frac{1}{2}|c_a|^2 (\partial_1 + i\partial_2) \operatorname{Re} G_a(a) \right]$$
$$+ o(\delta|c_a| + \delta^2) = 4\pi(n+1)\delta \overline{\alpha_a c_a} + o(\delta|c_a| + \delta^2) \tag{3.17}$$

in view of $F_a'(a) = \alpha_a + O(|a|)$ and $\frac{1}{2}(\partial_1 + i\partial_2) \operatorname{Re} G_a(a) = O(|a|)$. As far as the second term of R, by (2.27) we have that

$$\int_\Omega \frac{64\pi^2 N^2 \epsilon^2 \int_\Omega e^{2u_0+2W}}{(\int_\Omega e^{u_0+W} + \sqrt{(\int_\Omega e^{u_0+W})^2 - 16\pi N \epsilon^2 \int_\Omega e^{2u_0+2W}})^2} Z \left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}} \right)$$
$$= \int_{\sigma^{-1}(B_\rho(0))} |\sigma'(z)|^2 e^{U_{\delta,a}} Z \left[\frac{8(n+1)^2\epsilon^2}{\pi |\alpha_a|^{\frac{2}{n+1}} \delta^{\frac{2}{n+1}}} E_{a,\delta} - \epsilon^2 |\sigma'(z)|^2 e^{U_{\delta,a}} \right] \left[1 + O(|c_a||z|^{n+1} + \eta) + o(1)\right]$$
$$+ O(\delta^3 \eta) = \frac{64(n+1)^3 \epsilon^2}{\pi |\alpha_a|^{\frac{2}{n+1}} \delta^{\frac{2}{n+1}}} E_{a,\delta} \int_{B_\rho(0)} \frac{\delta^3 (y-a)}{(\delta^2 + |y-a|^2)^3} dy \left[1 + O(|c_a||y| + \eta) + o(1)\right]$$
$$-64(n+1)^3 \epsilon^2 |\alpha_a|^{-\frac{2}{n+1}} \int_{B_\rho(0)} \frac{\delta^5 |y|^{\frac{2n}{n+1}} (y-a)}{(\delta^2 + |y-a|^2)^5} \left[1 + O(|y|^{\frac{1}{n+1}} + \eta) + o(1)\right] + O(\delta^3 \eta)$$
$$= -64(n+1)^3 |\alpha_a|^{-\frac{2}{n+1}} \eta \int_{\mathbb{R}^2} \frac{|y + \frac{a}{\delta}|^{\frac{2n}{n+1}} y}{(1+|y|^2)^5} + o(\eta) + O(\eta^2) \tag{3.18}$$

in view of (2.46) and

$$\int_{B_\rho(0)} \frac{\delta^3 (y-a)}{(\delta^2 + |y-a|^2)^3} dy = \int_{B_\rho(a)} \frac{\delta^3 (y-a)}{(\delta^2 + |y-a|^2)^3} dy + O(\delta^3) = O(\delta^3).$$

Inserting (3.17) and (3.18) into (3.14), we get the validity of (3.5). ∎

Remark 3.3. *Since for $|a| >> \delta$ and $n > 1$*

$$\delta^{\frac{2}{n+1}} \int_{B_\rho(0)} \frac{\delta^5 |y|^{\frac{2n}{n+1}} (y-a)}{(\delta^2 + |y-a|^2)^5} = \delta^{\frac{2}{n+1}} \int_{B_\rho(0)} \frac{\delta^5 |y+a|^{\frac{2n}{n+1}} y}{(\delta^2 + |y|^2)^5} + o(1) = \frac{\pi n}{12(n+1)} \left(\frac{|a|}{\delta}\right)^{-\frac{2}{n+1}} \frac{a}{\delta} [1 + o(1)]$$

in view of

$$\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^5} = \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^4} - \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^5} = \frac{\pi}{12}$$

and the inequality

$$|y+a|^{\frac{2n}{n+1}} = |a|^{\frac{2n}{n+1}} + \frac{n}{n+1} |a|^{-\frac{2}{n+1}} (a\overline{y} + \overline{a} y) + O(|a|^{-\frac{2}{n+1}} |y|^2 + |y|^{\frac{2n}{n+1}}),$$

notice that the main order of $\int_\Omega RPZ$, in this range, is essentially given by

$$4\pi(n+1)\delta \overline{\alpha_a c_a} - \frac{16}{3} \pi n (n+1)^2 \epsilon^2 \delta^{-\frac{2}{n+1}} |\alpha_a|^{-\frac{2}{n+1}} \left(\frac{|a|}{\delta}\right)^{-\frac{2}{n+1}} \frac{a}{\delta}.$$

Since α_a is uniformly away from zero, the vanishing of $\int_\Omega RPZ$, which is equivalent to have $\epsilon^2 \delta^{-\frac{2}{n+1}} (\frac{|a|}{\delta})^{\frac{2n}{n+1}} \sim \overline{\alpha_a c_a a}$, is generally not compatible in the range $|a| >> \delta$ with the vanishing of $\int_\Omega RPZ_0$ in view of Remark 3.2, which can take place only if $c_0 = 0$ (in which case $c_a \sim a$). Indeed, the vanishing of $\int_\Omega RPZ$ and $\int_\Omega RPZ_0$ in the range $|a| >> \delta$ implies the contradiction $|a|^2 \sim \delta^2$. This explains why we don't consider the case $|a| >> \delta$.

4 Proof of the main results

In the previous section, we have built up an approximating function $W = PU_{\delta,a,\sigma_a}$. We will now look for solutions w of the form $w = W + \phi$, where ϕ is a small correcting term. In terms of ϕ, problem (2.2) is equivalent to find a doubly-periodic solution ϕ of

$$L(\phi) = -[R + N(\phi)] \qquad \text{in } \Omega \tag{4.1}$$

with $\int_\Omega \phi = 0$. Recalling the notation $B(w) = 16\pi N(\int_\Omega e^{2u_0+2w})(\int_\Omega e^{u_0+w})^{-2}$, the linear operator L is given by

$$L(\phi) = \Delta\phi + \mathcal{K}\phi + \tilde{\gamma}(\phi),$$

where

$$\mathcal{K} = 4\pi N \frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} + \frac{4\pi N\epsilon^2 B(W)}{\left(1+\sqrt{1-\epsilon^2 B(W)}\right)^2}\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - 2\frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right)$$

and

$$\begin{aligned}\tilde{\gamma}(\phi) = &-4\pi N \frac{e^{u_0+W}\int_\Omega e^{u_0+W}\phi}{(\int_\Omega e^{u_0+W})^2} - \frac{4\pi N\epsilon^2 B(W)}{\left(1+\sqrt{1-\epsilon^2 B(W)}\right)^2}\frac{e^{u_0+W}}{(\int_\Omega e^{u_0+W})^2}\int_\Omega e^{u_0+W}\phi \\ &+ \frac{8\pi N\epsilon^2 B(W)}{\left(1+\sqrt{1-\epsilon^2 B(W)}\right)^2}\frac{e^{2u_0+2W}}{(\int_\Omega e^{2u_0+2W})^2}\int_\Omega e^{2u_0+2W}\phi \\ &+ 4\pi N\epsilon^2 \frac{DB(W)[\phi]}{(1+\sqrt{1-\epsilon^2 B(W)})^2\sqrt{1-\epsilon^2 B(W)}}\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right)\end{aligned}$$

with

$$DB(W)[\phi] = 2B(W)\left(\frac{\int_\Omega e^{2u_0+2W}\phi}{\int_\Omega e^{2u_0+2W}} - \frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}}\right).$$

The nonlinear term $N(\phi)$, which is quadratic in ϕ, is given by

$$\begin{aligned}N(\phi) = &\, 4\pi N\left[\frac{e^{u_0+W+\phi}}{\int_\Omega e^{u_0+W+\phi}} - \frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}\left(\phi - \frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}}\right)\right] \\ &+ \left[\frac{4\pi N\epsilon^2 B(W+\phi)}{(1+\sqrt{1-\epsilon^2 B(W+\phi)})^2} - \frac{4\pi N\epsilon^2 B(W)}{(1+\sqrt{1-\epsilon^2 B(W)})^2} - \frac{4\pi N\epsilon^2 DB(W)[\phi]}{(1+\sqrt{1-\epsilon^2 B(W)})^2\sqrt{1-\epsilon^2 B(W)}}\right]\times \\ &\times\left(\frac{e^{u_0+W+\phi}}{\int_\Omega e^{u_0+W+\phi}} - \frac{e^{2(u_0+W+\phi)}}{\int_\Omega e^{2(u_0+W+\phi)}}\right) \\ &+ \frac{4\pi N\epsilon^2 B(W)}{\left(1+\sqrt{1-\epsilon^2 B(W)}\right)^2}\left[\frac{e^{u_0+W+\phi}}{\int_\Omega e^{u_0+W+\phi}} - \frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}\left(\phi - \frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}}\right)\right] \\ &- \frac{4\pi N\epsilon^2 B(W)}{\left(1+\sqrt{1-\epsilon^2 B(W)}\right)^2}\left[\frac{e^{2(u_0+W+\phi)}}{\int_\Omega e^{2(u_0+W+\phi)}} - \frac{e^{2(u_0+W)}}{\int_\Omega e^{2(u_0+W)}} - 2\frac{e^{2(u_0+W)}}{\int_\Omega e^{2(u_0+W)}}\left(\phi - \frac{\int_\Omega e^{2(u_0+W)}\phi}{\int_\Omega e^{2(u_0+W)}}\right)\right] \\ &+ \frac{4\pi N\epsilon^2 DB(W)[\phi]}{(1+\sqrt{1-\epsilon^2 B(W)})^2\sqrt{1-\epsilon^2 B(W)}}\left(\frac{e^{u_0+W+\phi}}{\int_\Omega e^{u_0+W+\phi}} - \frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{2(u_0+W+\phi)}}{\int_\Omega e^{2(u_0+W+\phi)}} + \frac{e^{2(u_0+W)}}{\int_\Omega e^{2(u_0+W)}}\right).\end{aligned} \tag{4.2}$$

Notice that we can re-write $\tilde{\gamma}(\phi)$ as

$$\tilde{\gamma}(\phi) = -\mathcal{K}\frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}} + \frac{8\pi N\epsilon^2 B(W)}{(1+\sqrt{1-\epsilon^2 B(W)})^2\sqrt{1-\epsilon^2 B(W)}}\left(\frac{\int_\Omega e^{2(u_0+W)}\phi}{\int_\Omega e^{2(u_0+W)}} - \frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}}\right)\left[\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}\right.$$
$$\left. +(\sqrt{1-\epsilon^2 B(W)}-1)\frac{e^{2(u_0+W)}}{\int_\Omega e^{2(u_0+W)}}\right]$$
$$= \mathcal{K}\left[-\frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}} + \frac{\epsilon^2 B(W)}{(1+\sqrt{1-\epsilon^2 B(W)})\sqrt{1-\epsilon^2 B(W)}}\left(\frac{\int_\Omega e^{2(u_0+W)}\phi}{\int_\Omega e^{2(u_0+W)}} - \frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}}\right)\right],$$

and L as

$$L(\phi) = \Delta\phi + \mathcal{K}\left[\phi + \gamma(\phi)\right], \tag{4.3}$$

where

$$\gamma(\phi) = -\frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}} + \frac{\epsilon^2 B(W)}{(1+\sqrt{1-\epsilon^2 B(W)})\sqrt{1-\epsilon^2 B(W)}}\left(\frac{\int_\Omega e^{2(u_0+W)}\phi}{\int_\Omega e^{2(u_0+W)}} - \frac{\int_\Omega e^{u_0+W}\phi}{\int_\Omega e^{u_0+W}}\right).$$

Let us observe that

$$\int_\Omega R = \int_\Omega L(\phi) = \int_\Omega N(\phi) = 0.$$

Since the operator L is not invertible, equation $L(\phi) = -R - N(\phi)$ is not generally solvable. The linear theory we will develop in Appendix B states that L has a kernel which is almost generated by PZ_0, PZ and $\overline{PZ}$, yielding to

Proposition 4.1. *Let $M_0 > 0$. There exists $\eta_0 > 0$ small such that for any $0 < \delta \leq \eta_0$, $|\log\delta|\epsilon^2 \leq \eta_0\delta^{\frac{2}{n+1}}$, $|a| \leq M_0\delta$ and $h \in L^\infty(\Omega)$ with $\int_\Omega h = 0$ there is a unique solution ϕ, $d_0 \in \mathbb{R}$ and $d \in \mathbb{C}$ to*

$$\begin{cases} L(\phi) = h + d_0\Delta PZ_0 + \mathrm{Re}[d\Delta PZ] & \text{in } \Omega \\ \int_\Omega \phi = \int_\Omega \phi\Delta PZ_0 = \int_\Omega \phi\Delta PZ = 0. \end{cases} \tag{4.4}$$

Moreover, there is a constant $C > 0$ such that

$$\|\phi\|_\infty \leq C\left(\log\frac{1}{\delta}\right)\|h\|_*, \qquad |d_0| + |d| \leq C\|h\|_*.$$

As a consequence, in Appendix C we will show

Proposition 4.2. *Let $M_0 > 0$. There exists $\eta_0 > 0$ small such that for any $0 < \delta \leq \eta_0$, $|\log\delta|^2\epsilon^2 \leq \eta_0\delta^{\frac{2}{n+1}}$ and $|a| \leq M_0\delta$ there is a unique solution $\phi = \phi(\delta, a)$, $d_0 = d_0(\delta, a) \in \mathbb{R}$ and $d = d(\delta, a) \in \mathbb{C}$ to*

$$\begin{cases} L(\phi) = -[R + N(\phi)] + d_0\Delta PZ_0 + \mathrm{Re}[d\Delta PZ] & \text{in } \Omega \\ \int_\Omega \phi = \int_\Omega \phi\Delta PZ_0 = \int_\Omega \phi\Delta PZ = 0. \end{cases} \tag{4.5}$$

Moreover, the map $(\delta, a) \mapsto \phi(\delta, a)$ is C^1 with

$$\|\phi\|_\infty \leq C|\log\delta|\|R\|_*. \tag{4.6}$$

The function $W + \phi$ will be a true solution of equation (2.2) once we adjust δ and a to have $d_0(\delta, a) = d(\delta, a) = 0$. The crucial point is the following:

Lemma 4.3. *Let $\phi=\phi(\delta,a)$, $d_0=d_0(\delta,a)\in\mathbb{R}$ and $d=d(\delta,a)\in\mathbb{C}$ be the solution of (4.5) given by Proposition 4.2. There exists $\eta_0>0$ such that if $0<\delta\le\eta_0$, $|a|\le\eta_0$ and*

$$\int_\Omega (L(\phi)+N(\phi)+R)PZ_0=0,\qquad \int_\Omega (L(\phi)+N(\phi)+R)PZ=0 \tag{4.7}$$

do hold, then $W+\phi$ is a solution of (2.2), i.e. $d_0(\delta,a)=d(\delta,a)=0$.

Proof: Since by (3.3) and $\|Z_0\|_\infty+\|Z\|_\infty\le 2$ there hold

$$\begin{aligned}\int_\Omega \Delta PZ_0 PZ_0 &= \int_\Omega \Delta Z_0 PZ_0=-\int_{\sigma^{-1}(B_\rho(0))}|\sigma'(z)|^2 e^{U_{\delta,a}}Z_0(Z_0+1)+O(\delta^2)\\ &= -16(n+1)\delta^4\int_{B_\rho(0)}\frac{\delta^2-|y-a|^2}{(\delta^2+|y-a|^2)^4}+O(\delta^2)=-\frac{8\pi}{3}(n+1)+O(\delta^2)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \Delta PZPZ_0 &= \int_\Omega \Delta ZPZ_0=-\int_{\sigma^{-1}(B_\rho(0))}|\sigma'(z)|^2 e^{U_{\delta,a}}Z(Z_0+1)+O(\delta^2)\\ &= -\int_{B_\rho(0)}\frac{16(n+1)\delta^5(y-a)}{(\delta^2+|y-a|^2)^4}+O(\delta^2)=-\int_{B_\rho(0)}\frac{16(n+1)\delta^5 y}{(\delta^2+|y|^2)^4}+O(\delta^2)=O(\delta^2)\end{aligned}$$

in view of (3.1)-(3.2) and

$$\int_{\mathbb{R}^2}\frac{1-|y|^2}{(1+|y|^2)^4}dy=2\int_{\mathbb{R}^2}\frac{dy}{(1+|y|^2)^4}-\int_{\mathbb{R}^2}\frac{dy}{(1+|y|^2)^3}=\frac{\pi}{6},$$

by (4.5) we rewrite the first of (4.7) as

$$0=d_0\int_\Omega \Delta PZ_0PZ_0+\int_\Omega \mathrm{Re}[d\Delta PZPZ_0]=-\frac{8}{3}\pi(n+1)d_0+O(\delta^2|d_0|+\delta^2|d|).$$

Similarly, the second of (4.7) gives that

$$\begin{aligned}0 &= d_0\int_\Omega \Delta PZ_0PZ+\int_\Omega\frac12\left[d\Delta PZ+\bar d\Delta\overline{PZ}\right]PZ=-\int_{\sigma^{-1}(B_\rho(0))}\frac12|\sigma'(z)|^2e^{U_{\delta,a}}\left[dZ+\bar d\,\overline{Z}\right]Z\\ &\quad +O(\delta^2|d_0|+\delta|d|)=-4(n+1)\bar d\int_{\mathbb{R}^2}\frac{|y|^2}{(1+|y|^2)^4}+O(\delta^2|d_0|+\delta|d|)\end{aligned}$$

in view of $\int_\Omega \Delta PZ_0PZ=\int_\Omega\Delta PZPZ_0=O(\delta^2)$, (3.2) and (3.3). Hence, (4.7) can be simply re-written as $d_0+O(\delta^2|d_0|+\delta^2|d|)=0$, $d+O(\delta^2|d_0|+\delta|d|)=0$. Summing up the two relations, we then obtain $|d_0|+|d|=\delta O(|d_0|+|d|)$ which implies $d_0=d=0$. ∎

Remark 4.4. *Since ϕ is sufficiently small, the system (4.7) will be a perturbation of the reduced equations $\int_\Omega RPZ_0=0$, $\int_\Omega RPZ=0$. The integral coefficient in (3.4) is negative for all $\frac{a}{\delta}$, as we will see in Appendix D. Since $\alpha_a\to\alpha_0=\frac{\mathcal{H}(0)}{n+1}\neq 0$ and $c_a\to c_0$ as $a\to 0$, we can always exclude the case $c_0\neq 0$. Indeed, in such a case the equation $\int_\Omega RPZ_0=0$ yields to $\epsilon^2\delta^{-\frac{2}{n+1}}\sim\delta^2|\log\delta|$ as $\delta\to 0$ by means of (3.4) (we are implicitly assuming $\epsilon^2\delta^{-\frac{2}{n+1}}\to 0$, which is a natural range for solving the reduced equations through (3.4)-(3.5)). This is not compatible with $\int_\Omega RPZ=0$, which allows at most $\delta=O(\epsilon^2\delta^{-\frac{2}{n+1}})$ by means of (3.5).*

The last ingredient is an expansion of the system (4.7) with the aid of Proposition 3.1:

Proposition 4.5. *Assume $c_0 = 0$ and $|a| \le M_0\delta$ for some $M_0 > 0$. The following expansions do hold as $\delta \to 0$ and $\epsilon \to 0$*

$$\begin{aligned}\int_\Omega (L(\phi)+N(\phi)+R)PZ_0 &= -8\pi\delta^2 D_0 + 64(n+1)^{\frac{3n+5}{n+1}}|\mathcal{H}(0)|^{-\frac{2}{n+1}}\epsilon^2\delta^{-\frac{2}{n+1}}\int_{\mathbb{R}^2}\frac{(|y|^2-1)|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}\\ &\quad + o(\delta^2+\epsilon^2\delta^{-\frac{1}{n+1}}) + O(\epsilon^4\delta^{-\frac{2}{n+1}}|\log\delta|^2+\epsilon^8\delta^{-\frac{4}{n+1}}|\log\delta|^2) \qquad (4.8)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega (R+L(\phi)+N(\phi))PZ &= 4\pi\delta(\bar\Upsilon a+\bar\Gamma\bar a) - 64(n+1)^{\frac{3n+5}{n+1}}|\mathcal{H}(0)|^{-\frac{2}{n+1}}\epsilon^2\delta^{-\frac{2}{n+1}}\int_{\mathbb{R}^2}\frac{|y+\frac{a}{\delta}|^{\frac{2n}{n+1}}y}{(1+|y|^2)^5}\\ &\quad + o(\delta^2+\epsilon^2\delta^{-\frac{2}{n+1}}) + O(\epsilon^4\delta^{-\frac{2}{n+1}}|\log\delta|^2+\epsilon^8\delta^{-\frac{4}{n+1}}|\log\delta|^2), \qquad (4.9)\end{aligned}$$

where D_0 and Γ, Υ are defined in (1.10) and Lemma A.2, respectively.

Proof: First, note that from the assumptions and (2.54), we find that $\|R\|_* = O(\delta^{2-\gamma}+\eta+\eta^2)$, where $\eta = \epsilon^2\delta^{-\frac{2}{n+1}}$. Hence, since $|\gamma(\phi)| = O((1+\eta)\|\phi\|_\infty)$ in view of (2.49), by (4.6), (B.9), (B.10) and (C.3) we have that

$$\begin{aligned}\int_\Omega (R+L(\phi)+N(\phi))PZ_0 &= \int_\Omega RPZ_0 + O\left((1+\eta)\Big\|\tilde L\Big(PZ_0+\frac{1}{|\Omega|}\int_\Omega Z_0\Big)\Big\|_*\|\phi\|_\infty+\|\phi\|_\infty^2\right) \qquad (4.10)\\ &= \int_\Omega RPZ_0 + o(\delta^2+\eta)+O(\eta^2+\eta^4)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega (R+L(\phi)+N(\phi))PZ &= \int_\Omega RPZ + O\left((1+\eta)\Big\|\tilde L\Big(PZ+\frac{1}{|\Omega|}\int_\Omega Z\Big)\Big\|_*\|\phi\|_\infty+\|\phi\|_\infty^2\right) \qquad (4.11)\\ &= \int_\Omega RPZ + o(\delta^2+\eta)+O(\eta^2+\eta^4)\end{aligned}$$

in view of $PZ_0 = O(1)$ and $PZ = O(1)$, where $\tilde L(\phi) = \Delta\phi + \mathcal{K}\phi$. Since by Lemma A.2 $\mathcal{H}(0)c_a = \Gamma a + \Upsilon\bar a + o(|a|)$ as $a \to 0$ in view of $c_0 = 0$, the desired expansions (4.8)-(4.9) follow by a combination of (3.4)-(3.5) and (4.10)-(4.11). We have used that $\alpha_a \to \alpha_0 = \frac{\mathcal{H}(0)}{n+1}$ as $a \to 0$ in view of (2.10), where α_a is given by (2.31), and $D_a \to D_0$ as $a \to 0$, where D_a is given by (2.43). ■

Thanks to (4.8)-(4.9), the aim is to find $(\delta(\epsilon), a(\epsilon))$ so that (4.7) does hold. To simplify the notations, we denote

$$\varphi_0(\delta,a,\epsilon) = \int_\Omega (L(\phi)+N(\phi)+R)PZ_0 \qquad \varphi(\delta,a,\epsilon) = \overline{\int_\Omega (L(\phi)+N(\phi)+R)PZ},$$

and (4.7) reduces to find a solution of

$$\varphi_0(\delta(\epsilon),a(\epsilon),\epsilon) = \varphi(\delta(\epsilon),a(\epsilon),\epsilon) = 0 \qquad (4.12)$$

for ϵ small. We are now ready to prove our first main result, which clearly implies the validity of Theorem 1.1 with $m = 1$.

Theorem 4.6. *Let $\mathcal{H}_0 = \frac{\mathcal{H}}{z^{n+2}}$, where $\mathcal{H}$ is given in (2.6), be a meromorphic function in Ω with $|\mathcal{H}_0(z)|^2 = e^{u_0+8\pi(n+1)G(z,0)}$ (which exists in view of (2.4) and is unique up to rotations), and $\sigma_0(z) = -(\int^z \mathcal{H}_0(w)dw)^{-1}$. Assume that*

$$\frac{d^{n+1}\mathcal{H}}{dz^{n+1}}(0) = 0 \tag{4.13}$$

and for some small $\rho > 0$

$$D_0 := \frac{1}{\pi}\left[\int_{\Omega\setminus\sigma_0^{-1}(B_\rho(0))} e^{u_0+8\pi(n+1)G(z,0)} - \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{n+1}{|y|^4}\right] < 0. \tag{4.14}$$

If the "non-degeneracy condition"

$$|\Gamma| \neq \left|\Upsilon + \frac{n(2n+3)}{n+1}D_0\right| \tag{4.15}$$

does hold, where Γ and Υ are given in Lemma A.2, for $\epsilon > 0$ small there exist $a(\epsilon)$, $\delta(\epsilon) > 0$ small so that $w_\epsilon = PU_{\delta(\epsilon),a(\epsilon),\sigma_{a(\epsilon)}} + \phi(\delta(\epsilon), a(\epsilon))$ does solve (2.2) with

$$4\pi N\frac{e^{u_0+w_\epsilon}}{\int_\Omega e^{u_0+w_\epsilon}} + \frac{64\pi^2N^2\epsilon^2\int_\Omega e^{2u_0+2w_\epsilon}}{(\int_\Omega e^{u_0+w_\epsilon} + \sqrt{(\int_\Omega e^{u_0+w_\epsilon})^2 - 16\pi N\epsilon^2\int_\Omega e^{2u_0+2w_\epsilon}})^2}\left(\frac{e^{u_0+w_\epsilon}}{\int_\Omega e^{u_0+w_\epsilon}} - \frac{e^{2u_0+2w_\epsilon}}{\int_\Omega e^{2u_0+2w_\epsilon}}\right)$$
$$\rightharpoonup 8\pi(n+1)\delta_0$$

in the sense of measures as $\epsilon \to 0$.

Remark 4.7. *For simplicity, we are considering the case $p = 0$ in Theorem 4.6, which however is still true for $p \neq 0$ by simply replacing in the statement $\mathcal{H}$, $\mathcal{H}_0$ and corresponding quantities with $\mathcal{H}^p$, $\mathcal{H}_0^p$ and corresponding quantities at p, where the latter have been defined in Remark 2.1.*

Proof: Since the equation $\varphi_0(\delta, a, \epsilon) = 0$ naturally requires $\delta^2 \sim \epsilon^2\delta^{-\frac{2}{n+1}}$ in view of (4.8), we make the following change of variables: $\delta = [\frac{(n+1)\epsilon^{n+1}}{|\mathcal{H}(0)|}]^{\frac{1}{n+2}}\mu$ and $\zeta = \frac{a}{\delta}$. The system (4.12) is equivalent to find zeroes of

$$\Gamma_\epsilon(\mu,\zeta) := \left[\frac{(n+1)\epsilon^{n+1}}{|\mathcal{H}(0)|}\right]^{-\frac{2}{n+2}}\left(-\frac{1}{8}\varphi_0, \frac{1}{4\pi\mu^2}\varphi\right)\left(\left[\frac{(n+1)\epsilon^{n+1}}{|\mathcal{H}(0)|}\right]^{\frac{1}{n+2}}\mu, \left[\frac{(n+1)\epsilon^{n+1}}{|\mathcal{H}(0)|}\right]^{\frac{1}{n+2}}\mu\zeta, \epsilon\right),$$

which has the expansion $\Gamma_\epsilon(\mu,\zeta) = \Gamma_0(\mu,\zeta) + o(1)$ as $\epsilon \to 0^+$, uniformly for μ in compact subsets of $(0,+\infty)$, in view of (4.8)-(4.9), where the map $\Gamma_0 : \mathbb{R}\times\mathbb{C} \to \mathbb{R}\times\mathbb{C}$ is defined as

$$\Gamma_0(\mu,\zeta) = \left(\pi D_0\mu^2 - \frac{8(n+1)^3}{\mu^{\frac{2}{n+1}}}\int_{\mathbb{R}^2}\frac{(|y|^2-1)|y+\zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}, \Gamma\zeta + \Upsilon\bar{\zeta} - \frac{16(n+1)^3}{\pi\mu^{\frac{2(n+2)}{n+1}}}\int_{\mathbb{R}^2}\frac{|y+\zeta|^{\frac{2n}{n+1}}\bar{y}}{(1+|y|^2)^5}\right).$$

We need to exhibit "stable" zeroes of Γ_0 in $(0,+\infty)\times\mathbb{C}$, which will persist under $L^\infty-$small perturbations yielding to zeroes of Γ_ϵ as required. The easiest case is given by the point $(\mu_0, 0)$, that solves $\Gamma_0 = 0$ for $\mu_0 = (\frac{8(n+1)^3 I_0}{\pi D_0})^{\frac{n+1}{2(n+2)}} > 0$ in view of the assumption (4.14) and (see (D.7))

$$I_0 := \int_{\mathbb{R}^2}\frac{(|y|^2-1)|y|^{\frac{2n}{n+1}}}{(1+|y|^2)^5} < 0.$$

Regarding Γ_0 as a map from $\mathbb{R}^3$ into $\mathbb{R}^3$ and setting $\Gamma = \Gamma_1 + i\Gamma_2$, $\Upsilon = \Upsilon_1 + i\Upsilon_2$, we have that

$$D\Gamma_0(\mu_0, 0) = \begin{pmatrix} \frac{2(n+2)}{n+1}\pi D_0\mu_0 & 0 & 0 \\ 0 & \Gamma_1 + \Upsilon_1 + \frac{n(2n+3)}{n+1}D_0 & \Upsilon_2 - \Gamma_2 \\ 0 & \Gamma_2 + \Upsilon_2 & \Gamma_1 - \Upsilon_1 - \frac{n(2n+3)}{n+1}D_0 \end{pmatrix}$$

in view of (D.7) and

$$\int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}}}{(1+|y|^2)^5} dy = \pi \int_0^\infty \frac{\rho^{\frac{n}{n+1}}}{(1+\rho)^5} d\rho = \pi I_5^{\frac{n}{n+1}}.$$

Since

$$\det D\Gamma_0(\mu_0, 0) = \frac{2(n+2)}{n+1}\pi D_0\mu_0 \left(|\Gamma|^2 - \left| \Upsilon + \frac{n(2n+3)}{n+1}D_0 \right|^2 \right) \neq 0$$

in view of assumption (4.15), the point $(\mu_0, 0)$ is an isolated zero of Γ_0 with non-trivial local index. Since $D\Gamma_0(\mu_0, 0)$ is an invertible matrix, there exists $\nu > 0$ small so that $|D\Gamma_0(\mu_0, 0)(\mu - \mu_0, \zeta)| \geq \nu|(\mu - \mu_0, \zeta)|$. By a Taylor expansion of Γ_0 we can find $r_0 > 0$ small so that

$$|\Gamma_\epsilon(\mu, \zeta)| = |\Gamma_0(\mu, \zeta)| + o(1) \geq \nu|(\mu - \mu_0, \zeta)| + O\left((\mu - \mu_0)^2 + |\zeta|^2\right) + o(1) \geq \frac{\nu}{2}|(\mu - \mu_0, \zeta)|$$

for all $(\mu, \zeta) \in \partial B_r(\mu_0, 0)$ and all $r \leq r_0$, for ϵ sufficiently small depending on r. Then, the map Γ_ϵ has in $B_{r_0}(\mu_0, 0)$ well-defined degree for all ϵ small, and it then coincides with the local index of Γ_0 at $(\mu_0, 0)$. In this way, the map Γ_ϵ has a zero of the form $(\mu_\epsilon, \zeta_\epsilon)$ with $\mu_\epsilon \to \mu_0$ and $|\zeta_\epsilon| \to 0$ as $\epsilon \to 0$. Therefore, we have solved (4.12) for $\delta(\epsilon) = [\frac{(n+1)\epsilon^{n+1}}{|\mathcal{H}(0)|}]^{\frac{1}{n+2}}\mu_\epsilon$ and $a(\epsilon) = \delta(\epsilon)\zeta_\epsilon$, and the corresponding w_ϵ does solve (2.2) and satisfy the required concentration property as stated in Theorem 4.6. ■

Remark 4.8. *With some extra work, it is rather standard to see that* (4.8) *does hold in a C^1−sense. For ζ in a bounded set, by IFT we can find $\epsilon > 0$ small so that the first equation in $\Gamma_\epsilon(\mu, \zeta) = 0$ can be solved by $\mu(\epsilon, \zeta)$, depending continuously in ζ, so that*

$$\mu(\epsilon, \zeta) \to \mu(\zeta) := \left(\frac{8(n+1)^3}{\pi D_0} \int_{\mathbb{R}^2} \frac{(|y|^2 - 1)|y + \zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5} \right)^{\frac{n+1}{2(n+2)}}$$

as $\epsilon \to 0$. In Appendix D it is proved that $\int_{\mathbb{R}^2} \frac{(|y|^2-1)|y+\zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5} < 0$ for all $\zeta \in \mathbb{C}$, yielding to $\mu(\zeta) > 0$ when $D_0 < 0$. Plugging $\mu(\epsilon, \zeta)$ into the second equation in $\Gamma_\epsilon(\mu, \zeta) = 0$ we are reduced to find a "stable" zero of

$$\int_{\mathbb{R}^2} \frac{(|y|^2 - 1)|y + \zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5} \left(\bar{\Upsilon}\zeta + \bar{\Gamma}\bar{\zeta}\right) - 2D_0 \int_{\mathbb{R}^2} \frac{|y + \zeta|^{\frac{2n}{n+1}} y}{(1+|y|^2)^5} = 0.$$

Notice that $\bar{\Upsilon}\zeta + \bar{\Gamma}\bar{\zeta}$ acts in real notation as the multiplication for the matrix

$$A = \begin{pmatrix} \mathrm{Re}(\Gamma + \Upsilon) & \mathrm{Im}(\Upsilon - \Gamma) \\ -\mathrm{Im}(\Gamma + \Upsilon) & \mathrm{Re}(\Upsilon - \Gamma) \end{pmatrix}.$$

Since by Appendix D we have that

$$\int_{\mathbb{R}^2} \frac{(|y|^2 - 1)|y + \zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5} = f(|\zeta|), \qquad \int_{\mathbb{R}^2} \frac{|y + \zeta|^{\frac{2n}{n+1}} y}{(1+|y|^2)^5} = g(|\zeta|)\zeta,$$

we can re-write the above equation as $A\zeta = \frac{2D_0 g(|\zeta|)}{f(|\zeta|)}\zeta$. *Letting* (λ_1, e_1) *be an eigen-pair of* A *with* $|e_1| = 1$, *we can find a solution* $\zeta_0 = |\zeta_0| e_1$ *as soon as* $|\zeta_0| \neq 0$ *does solve* $\frac{2D_0 g(|\zeta_0|)}{f(|\zeta_0|)} = \lambda_1$. *Since by Appendix D we know that* $f < 0 < g$, *we can find solutions* $(\mu_\epsilon, \zeta_\epsilon)$ *of* $\Gamma_\epsilon(\mu, \zeta) = 0$ *with* ζ_ϵ *bifurcating from* $\zeta_0 \neq 0$ *as soon as one of the eigenvalues of* A *positive and belongs to* $\frac{2D_0 g}{f}(0, +\infty)$. *In particular, by* (D.7)-(D.8) *and* (D.10)-(D.11) *we have that*

$$\frac{g(0)}{f(0)} = -\frac{(2n+3)(3n+1)}{4(n+1)}, \qquad \frac{g(|\zeta|)}{f(|\zeta|)} \to -\frac{51}{356} \text{ as } |\zeta| \to \infty,$$

and the condition above is fullfilled if one of the eigenvalues of A *lies in* $(\frac{51}{178}|D_0|, \frac{(2n+3)(3n+1)}{2(n+1)}|D_0|)$.

5 Examples and comments

In this section, we will discuss the validity of (4.13)-(4.15) by providing some examples. Recall that in Theorem 4.6 we were implicitly assuming that $\{p_1, \dots, p_N\} \subset \Omega$ and denoting for simplicity the concentration point p as 0. The assumption $\{p_1, \dots, p_N\} \subset \Omega$ simplifies the global construction in $\bar{\Omega}$ of $\mathcal{H}$ but (4.13)-(4.15) just require the local existence for such $\mathcal{H}$ at 0 as well as for σ_0 and H^*. In this respect, the only relevant assumption is that the concentration point lies in Ω, and so we will provide examples with $0 \in \{\tilde{p}_1, \dots, \tilde{p}_N\} \subset \bar{\Omega}$. To be more precise, let us explain the general strategy we will adopt below. Since we are in a doubly-periodic setting, the configuration of the vortex points has to be periodic in $\bar{\Omega}$: for all $j = 1, \dots, N$ the points $(\tilde{p}_j + \omega_1\mathbb{Z} + \omega_2\mathbb{Z}) \cap \bar{\Omega}$ belong to $\{\tilde{p}_1, \dots, \tilde{p}_N\}$ and have all the same multiplicity. Then, we can find $J \subset \{1, \dots, N\}$ so that the points $\{\tilde{p}_j : j \in J\}$ are all non-zero, distinct modulo $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ and $(\{\tilde{p}_j : j \in J\} + \omega_1\mathbb{Z} + \omega_2\mathbb{Z}) \cap \bar{\Omega} = \{\tilde{p}_1, \dots, \tilde{p}_N\} \setminus \{0\}$. Take now a translation vector $\tau \in \Omega$ so that $\{\tilde{p}_1 + \tau, \dots, \tilde{p}_N + \tau\} \cap \partial\Omega = \emptyset$, or equivalently $(\{\tilde{p}_1, \dots, \tilde{p}_N\} + \tau + \omega_1\mathbb{Z} + \omega_2\mathbb{Z}) \cap \partial\Omega = \emptyset$. Then, it follows that $(\tilde{p}_j + \tau + \omega_1\mathbb{Z} + \omega_2\mathbb{Z}) \cap \Omega$ is composed by a single point p_j, for all $j = 1, \dots, N$. The idea is to apply Theorem 4.6, as formulated in Remark 4.7, to the translated vortex configuration $\{\tau\} \cup \{p_j : j \in J\} \subset \Omega$ with τ as concentration point. The validity of (4.13)-(4.15) in the translated situation will follow by appropriate assumptions on $\{\tilde{p}_1, \dots, \tilde{p}_N\}$.

Before stating our first result, let us introduce the notion of even vortex configuration: $-\tilde{p}_j \in \{\tilde{p}_1, \dots, \tilde{p}_N\} + \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ with the same multiplicity of $\tilde{p}_j$, for all $j = 1, \dots, N$. In the periodic case, notice that $\{\tilde{p}_j : j \in J\}$ is still an even configuration. The validity of (4.13) is discussed in the following:

Proposition 5.1. *Assume* n *is even and the periodic vortex configuration is even with* $0 \in \{\tilde{p}_1, \dots, \tilde{p}_N\}$. *Let* $\mathcal{H}^\tau$ *be the function corresponding to* $p = \tau$ *and remaining vortex points* $\{p_j : j \in J\} \subset \Omega$, *as given in Remark 2.1. Then, there holds*

$$\frac{d^k \mathcal{H}^\tau}{dz^k}(\tau) = 0$$

for all odd number k.

<u>Proof:</u> Since $-\Omega = \Omega$ and the periodic vortex configuration $\{\tilde{p}_1, \dots, \tilde{p}_N\}$ is even, we have that $G(z)$, $H(z)$ and $e^{-4\pi \sum_{j\in J} n_j G(z, \tilde{p}_j)}$ are even functions in view of $G(z,p) = G(z-p, 0)$. So, it follows that $e^{4\pi(n+2)H(z-\tau) - 4\pi\sum_{j\in J} n_j G(z, \tilde{p}_j + \tau)} = e^{4\pi(n+2)H(z-\tau) - 4\pi\sum_{j\in J} n_j G(z, p_j)}$ takes the same value at $\pm z + \tau$ for all $z \in \Omega$. The function $\mathcal{H}^\tau$ satisfies $|\mathcal{H}^\tau|(z+\tau) = |\mathcal{H}^\tau|(-z+\tau)$ for all $z \in \Omega$, and then $\mathcal{H}^\tau(z+\tau) = \mathcal{H}^\tau(-z+\tau)$ for all z since $\mathcal{H}^\tau$ is an holomorphic function. By differentiating $k-$times at τ, it yields to $\frac{d^k\mathcal{H}^\tau}{dz^k}(\tau) = 0$ when k is odd. ∎

The discussion of (4.14) is more interesting and will make use of the Weierstrass elliptic function $\wp$ to represent D_0 in case of an even periodic vortex configuration. Furthermore, when Ω is a rectangle, the points p_j's are half-periods and all the multiplicities are even numbers, by some ideas in [9] we will show that assumption (4.14) holds if and only if $\frac{n_3}{2}$ is an odd number, where n_3 is the multiplicity of the half-period $\frac{\omega_1+\omega_2}{2}$. Due to the presence of high order derivatives ($2(n+1)$th order) in (4.15), we will verify the validity of the "non-degeneracy" condition in the simplest case $n = n_3 = 2$ and Ω a square torus. As we will see, the validity of (4.15) is just a computational matter which could be carried out in very generality for each case of interest.

We have the following representation formula:

Proposition 5.2. *Assume that the periodic vortex configuration is even with $0 \in \{\tilde{p}_1, \dots, \tilde{p}_N\}$, and n_j is even when $\tilde{p}_j \in \{\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\}$. Let D_0^τ be the coefficient corresponding to $p = \tau$ and remaining vortex points $\{p_j : j \in J\} \subset \Omega$, as given in Theorem 4.6. Then, for τsmall we have that D_0^τ is given by (5.7), and does not depend on τ.*

Proof: The Weierstrass elliptic function

$$\wp(z) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)} \left(\frac{1}{(z + n\omega_1 + m\omega_2)^2} - \frac{1}{(n\omega_1 + m\omega_2)^2} \right)$$

is a doubly-periodic meromorphic function with a single pole in Ω at 0 of multiplicity 2. Moreover, the only branching points of $\wp$ are simple and given by the three half-periods $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$ and $\frac{\omega_3}{2} = \frac{\omega_1+\omega_2}{2}$, i.e. $\wp'(\frac{\omega_j}{2}) = 0$ and $\wp''(\frac{\omega_j}{2}) \neq 0$ for $j = 1, 2, 3$. For $p \in \bar{\Omega} \setminus \{0\}$, note that $2\pi[2G(z,0) - G(z,p) - G(z,-p)]$ is a doubly-periodic harmonic function in Ω with a singular behavior $-2\log|z|$ at $z = 0$. Moreover, it behaves like $\log|z-p|$ at $z = p$ and $\log|z+p|$ at $z = -p$ when $p \neq \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$, and like $2\log|z-p|$ if $p \in \{\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}\}$. Thus, we have that

$$2\pi[2G(z,0) - G(z,p) - G(z,-p)] = \log|\wp(z) - \wp(p)| + \text{const.}$$

no matter p is an half-period or not, in view of $\wp(p) = \wp(-p)$, $\wp'(p) = -\wp'(-p) \neq 0$ if $p \neq \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}$ and $\wp'(p) = 0$, $\wp''(p) \neq 0$ if $p \in \{\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}\}$. Since the periodic vortex configuration is even, take I as the minimal subset of J so that $(\{\tilde{p}_k, -\tilde{p}_k : k \in I\} + \omega_1\mathbb{Z} + \omega_2\mathbb{Z}) \cap \{\tilde{p}_j : j \in J\} = \{\tilde{p}_j : j \in J\}$ and

$$\hat{n}_k = \begin{cases} \frac{n_k}{2} & \text{if } \tilde{p}_k \text{ is an half-period} \\ n_k & \text{otherwise.} \end{cases}$$

Letting $N = n + \sum_{j \in J} n_j$ and $u_0(z) = -4\pi n G(z,0) - 4\pi \sum_{j\in J} n_j G(z, \tilde{p}_j)$, assumption (2.4) implies that

$$u_0 + 8\pi(n+1)G(z,0) = 4\pi \sum_{k\in I} \hat{n}_k [2G(z,0) - G(z,\tilde{p}_k) - G(z,-\tilde{p}_k)],$$

yielding to

$$e^{u_0 + 8\pi(n+1)G(z,0)} = \text{const.} \,\Big| \prod_{k\in I} (\wp(z) - \wp(\tilde{p}_k))^{\hat{n}_k} \Big|^2.$$

The additional assumption that n_j is even when $\tilde{p}_j$ is an half-period is crucial to have $(\wp(z) - \wp(\tilde{p}_j))^{\hat{n}_j}$ as a single-valued function. The function

$$\mathcal{H}_0(z) = \lambda_0 \prod_{k\in I} (\wp(z) - \wp(\tilde{p}_k))^{\hat{n}_k}, \quad \lambda_0 = e^{2\pi(n+2)H(0) - 2\pi \sum_{j\in J} n_j G(0,\tilde{p}_j)} \tag{5.1}$$

is an elliptic function with a single pole at 0 of zero residue, which satisfies

$$|\mathcal{H}_0|^2 = e^{u_0 + 8\pi(n+1)G(z,0)}. \tag{5.2}$$

Then

$$\sigma_0(z) = -\left(\int^z \mathcal{H}_0(w)dw\right)^{-1} = -\lambda_0^{-1}\left(\int^z \prod_{k\in I}(\wp(w)-\wp(\tilde{p}_k))^{\hat{n}_k}dw\right)^{-1} \tag{5.3}$$

is a well-defined meromorphic function in 2Ω which satisfies

$$\left|\left(\frac{1}{\sigma_0}\right)'(z)\right|^2 = |\mathcal{H}_0|^2(z) = e^{u_0+8\pi(n+1)G(z,0)}. \tag{5.4}$$

Switching now to the translated vortex configuration $\{\tau\}\cup\{p_j : j\in J\}$, let us first notice that the total multiplicity is still N, and introduce $u_0^\tau = u_0(z-\tau) = -4\pi n G(z,\tau) - 4\pi\sum_{j\in J} n_j G(z,p_j)$. We have that $\mathcal{H}_0^\tau(z) = \mathcal{H}_0(z-\tau)$ is a meromorphic function in Ω with

$$|\mathcal{H}_0^\tau|^2 = e^{u_0^\tau+8\pi(n+1)G(z,\tau)}$$

in view of (5.2). Since such a function $\mathcal{H}_0^\tau$ is unique up to rotations, we can assume that $\mathcal{H}_0^\tau$ coincides with the function $\mathcal{H}_0$ corresponding to $p=\tau$ and remaining vortex points $\{p_j : j\in J\}\subset\Omega$, as given in Theorem 4.6. Setting $\mathcal{H}(z) = z^{n+2}\mathcal{H}_0(z)$, we also have that

$$\mathcal{H}^\tau(z) = \mathcal{H}(z-\tau) \tag{5.5}$$

for all $z\in\Omega$. Letting

$$\sigma_0^\tau(z) = -\left(\int^z \mathcal{H}_0^\tau(w)dw\right)^{-1}$$

with the correct choice of the constant in the integration $\int^z$, we easily deduce that

$$\sigma_0^\tau(z) = \sigma_0(z-\tau) \tag{5.6}$$

for all $z\in\Omega$ in view of $(\frac{1}{\sigma_0^\tau})'(z) = (\frac{1}{\sigma_0})'(z-\tau)$. Since $(\sigma_0^\tau)^{-1}(B_\rho(0)) - \tau = (\sigma_0)^{-1}(B_\rho(0))$ in view of (5.6), according to (4.14) let us re-write D_0^τ as

$$\begin{aligned}
\pi D_0^\tau &= \int_{\Omega\setminus(\sigma_0^\tau)^{-1}(B_\rho(0))} e^{u_0^\tau+8\pi(n+1)G(z,\tau)} - \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{n+1}{|y|^4}\\
&= \int_{(\Omega-\tau)\setminus(\sigma_0)^{-1}(B_\rho(0))} e^{u_0+8\pi(n+1)G(z,0)} - \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{n+1}{|y|^4}\\
&= \int_{\Omega\setminus(\sigma_0)^{-1}(B_\rho(0))} e^{u_0+8\pi(n+1)G(z,0)} - \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{n+1}{|y|^4}
\end{aligned}$$

by the double-periodicity of $e^{u_0+8\pi(n+1)G(z,0)}$, once we assume for τ small that $(\sigma_0)^{-1}(B_\rho(0))\subset\Omega\cap(\Omega-\tau)$. By (5.4) and the change of variable $z\to\frac{1}{\sigma_0}(z)$ we get that

$$\begin{aligned}
\pi D_0^\tau &= \pi D_0 = \int_{\Omega\setminus(\sigma_0)^{-1}(B_\rho(0))} \left|\left(\frac{1}{\sigma_0}\right)'\right|^2 - \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{n+1}{|y|^4}\\
&= \text{Area}\left[\frac{1}{\sigma_0}\left(\Omega\setminus\sigma_0^{-1}(B_\rho(0))\right)\right] - (n+1)\text{Area}\left(B_{\frac{1}{\rho}}(0)\right).
\end{aligned} \tag{5.7}$$

By the Cauchy argument principle the number of pre-images in $\Omega\setminus\sigma_0^{-1}(B_\rho(0))$ through the map $\frac{1}{\sigma_0}$ is constant for all values in each connected component of $\mathbb{C}\setminus\left(\frac{1}{\sigma_0}(\partial\Omega)\cup\partial B_{\frac{1}{\rho}}(0)\right)$, and the area of each of these components has to be counted in (5.7) according to the multiplicity of pre-images. ∎

Thanks to (5.7), we can now discuss the validity of (4.14).

Proposition 5.3. *Let Ω be a rectangle, and assume that the vortex configuration is the periodic one generated by $\{0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}\}$ with even multiplicities $n, n_1, n_2, n_3 \geq 0$. Suppose that*

$$\frac{n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} = \frac{n}{2} + 1. \tag{5.8}$$

Given D_0^τ as in Propostion 5.2, then $D_0^\tau < 0$ $(> 0)$when $\frac{n_3}{2}$ is odd (even).

<u>Proof:</u> The balance condition (2.4) is satisfied in view of (5.8). Let $\tilde{p}_1 = \frac{\omega_1}{2}$, $\tilde{p}_2 = \frac{\omega_2}{2}$ and $\tilde{p}_3 = \frac{\omega_1+\omega_2}{2}$ be the three half-periods. When Ω is a rectangle, the function $\wp$ takes real values on $\partial\Omega$ and $\wp''(\tilde{p}_j) > 0$ for $j = 1, 2$, $\wp''(\tilde{p}_3) < 0$. As a consequence, we have that

$$\wp(\tilde{p}_1) - \wp(z),\ \wp(z) - \wp(\tilde{p}_2),\ \wp(\pm\tilde{p}_1 + it) - \wp(\tilde{p}_3),\ \wp(\tilde{p}_3) - \wp(\pm\tilde{p}_2 + t) \geq 0 \tag{5.9}$$

for all $z \in \partial\Omega$ and $t \in \mathbb{R}$. Write $\sigma_0(z)$ in (5.3) as

$$\sigma_0(z) = (-1)^{\frac{n+n_2}{2}} \lambda_0^{-1} \left(\int^z (\wp(\tilde{p}_1) - \wp(w))^{\frac{n_1}{2}} (\wp(w) - \wp(\tilde{p}_2))^{\frac{n_2}{2}} (\wp(\tilde{p}_3) - \wp(w))^{\frac{n_3}{2}} dw \right)^{-1}$$

in view of (5.8). Since

$$\frac{d}{dt}\left[\frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0(\pm\tilde{p}_2 + t)}\right] = \lambda_0 (\wp(\tilde{p}_1) - \wp(\pm\tilde{p}_2 + t))^{\frac{n_1}{2}} (\wp(\pm\tilde{p}_2 + t) - \wp(\tilde{p}_2))^{\frac{n_2}{2}} (\wp(\tilde{p}_3) - \wp(\pm\tilde{p}_2 + t))^{\frac{n_3}{2}} \geq 0$$

in view of (5.9), the function $\frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0}$ maps the horizontal sides of $\partial\Omega$ into horizontal segments with same orientation. In the same way, the vertical sides of $\partial\Omega$ are mapped into vertical segments with same/opposite orientation depending on whether $\frac{n_3}{2}$ is an even/odd number. So, $T := \frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0}(\partial\Omega)$ is still a rectangle with same/opposite orientation and $\frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0(\tilde{p}_3)}$ is the right upper/lower corner of T depending on whether $\frac{n_3}{2}$ is an even/odd number. For ρ small, we then have that $\mathbb{C} \setminus \left(\frac{1}{\sigma_0}(\partial\Omega) \cup \partial B_{\frac{1}{\rho}}(0)\right)$ has three connected components: the interior Ω' of $(-1)^{\frac{n+n_2}{2}} T$, $B_{\frac{1}{\rho}}(0) \setminus \overline{\Omega'}$ and $\mathbb{C} \setminus \overline{B_{\frac{1}{\rho}}(0)}$. By Lemma A.1 we have that values in $B_{\frac{1}{\rho}}(0) \setminus \overline{\Omega'}$, $\mathbb{C} \setminus \overline{B_{\frac{1}{\rho}}(0)}$ have exactly $n+1$, 0 pre-images in $\Omega \setminus \sigma_0^{-1}(B_\rho(0))$ through the map $\frac{1}{\sigma_0}$, respectively. By (5.7) we have that $\pi D_0^\tau = [k - (n+1)]\text{Area}(\Omega')$, where k is the number of pre-images corresponding to values in Ω'. Since $\wp(z) - \wp(\tilde{p}_3) = \frac{\wp''(\tilde{p}_3)}{2}(z - \tilde{p}_3)^2 + O(|z - \tilde{p}_3|^3)$ as $z \to \tilde{p}_3$, we obtain that

$$\left[\frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0}\right]'(z) = \mu(z - \tilde{p}_3)^{n_3} + O(|z - \tilde{p}_3|^{n_3+1})$$

and

$$\frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0(z)} - \frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0(\tilde{p}_3)} = \mu \frac{(z - \tilde{p}_3)^{n_3+1}}{n_3 + 1} + O(|z - \tilde{p}_3|^{n_3+2})$$

as $z \to \tilde{p}_3$, where $\mu := \lambda_0 \left(-\frac{\wp''(\tilde{p}_3)}{2}\right)^{\frac{n_3}{2}} [\wp(\tilde{p}_1) - \wp(\tilde{p}_3)]^{\frac{n_1}{2}} [\wp(\tilde{p}_3) - \wp(\tilde{p}_2)]^{\frac{n_2}{2}} > 0$. When $\frac{n_3}{2}$ is an odd number, $\frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0(\tilde{p}_3)}$ is the right lower corner of T and the function $\frac{(-1)^{\frac{n+n_2}{2}}}{\sigma_0}$ maps $\{z = \tilde{p}_3 + \rho e^{i\theta} \mid \pi \leq \theta \leq \frac{3\pi}{2}, 0 \leq \rho < \rho_0\}$ onto a region whose part inside/outside T is covered $\frac{n_3-2}{4}/\frac{n_3-2}{4} + 1$ times, respectively, in view of

$$(n_3 + 1)\pi \leq (n_3 + 1)\theta \leq (n_3 + 1)\frac{3\pi}{2} = (n_3 + 1)\pi + 2\pi\frac{n_3 - 2}{4} + \pi + \frac{\pi}{2}.$$

Hence, near $\tilde p_3$ the map $\frac{1}{\sigma_0}$ covers $\frac{n_3-2}{4}/\frac{n_3-2}{4}+1$ times the interior/exterior part of Ω' near $\frac{1}{\sigma_0(\tilde p_3)}$. Since $\frac{1}{\sigma_0}$ covers $n+1$ times every values in $B_{\frac{1}{\rho}}(0)\setminus\overline{\Omega'}$, there should be $n-\frac{n_3-2}{4}$ distinct points $x\in\Omega\setminus\sigma_0^{-1}(B_\rho(0))$, away from $\tilde p_1,\tilde p_2,\tilde p_3$, so that $\sigma_0(x)=\sigma_0(\tilde p_3)$. Since $\sigma_0'(x)\neq 0$ if $x\neq\tilde p_1,\tilde p_2,\tilde p_3$, it follows that around any such x $\frac{1}{\sigma_0}$ is a local homeomorphism, and then $\frac{1}{\sigma_0}$ covers exactly $n/n+1$ times the interior/exterior part of Ω' near $\frac{1}{\sigma_0(\tilde p_3)}$. Hence, it follows that $k=n$ and $\pi D_0^\tau=-\text{Area}(\Omega')<0$. When $\frac{n_3}{2}$ is even, in a similar way we get that $k=n+2$ and $\pi D_0^\tau=\text{Area}(\Omega')>0$. ■

Now, to discuss (4.15) we further restrict the attention to the case $n=n_3=2$ to get

Proposition 5.4. *Let Ω be a square of side a, $a>0$, and assume that the vortex configuration is the periodic one generated by $\{0,\frac{a}{2},\frac{ia}{2},\frac{a+ia}{2}\}$ with multiplicities $2,n_1,n_2,2$ and $(n_1,n_2)=(2,0)$ (or viceversa). Then, for $\tau\in\Omega$ assumption (4.15) does hold for the vortex configuration $\{\tau\}\cup\{p_j: j\in J\}\subset\Omega$.*

Proof: We are restricting the attention to the cases $(n_1,n_2)=(2,0),(0,2)$ for they are the only possibilities to have even multiplicities satisfying (5.8) for $2,n_1,n_2,2$. Letting $\tilde p_1=\frac{a}{2}$, $\tilde p_2=\frac{ia}{2}$ and $\tilde p_3=\frac{a+ia}{2}$ be the three half-periods, the "non-degeneracy condition" reads as

$$\left|3(\mathcal{H}^\tau)''(\tau)f_3'(\tau)+\mathcal{H}^\tau(\tau)f_3'''(\tau)\right|\neq\left|\frac{6\pi}{a^2}\overline{b_3}(\mathcal{H}^\tau)''(\tau)-\frac{28}{3}D_0^\tau\right| \tag{5.10}$$

in view of $(\mathcal{H}^\tau)'(\tau)=(\mathcal{H}^\tau)'''(\tau)=0$ by Proposition 5.1, where

$$f_l(z)=\frac{1}{l!}\frac{d^l}{dw^l}\left[2\log\frac{w-q_0^\tau(z)}{(q_0^\tau)^{-1}(w)-z}+4\pi H^*(z-(q_0^\tau)^{-1}(w))\right](0)\,,\quad b_l=\frac{1}{l!}\frac{d^l(q_0^\tau)^{-1}}{dw^l}(0).$$

Since $\sigma_0^\tau(z)=\sigma_0(z-\tau)$ by (5.6), we deduce that $q_0^\tau(z)=q_0(z-\tau)$ and $(q_0^\tau)^{-1}=\tau+q_0^{-1}$, where $q_0=z[\frac{\sigma_0(z)}{z^{n+1}}]^{\frac{1}{n+1}}$ is defined out of σ_0 as in Appendix A. Since $\mathcal{H}^\tau(z)=\mathcal{H}(z-\tau)$ in view of (5.5), by (5.7) the "non-degeneracy condition" (5.10) gets re-written in the original variables as:

$$\left|3\mathcal{H}''(0)f_3'(0)+\lambda_0 f_3'''(0)\right|\neq\left|\frac{6\pi}{a^2}\overline{b_3}\mathcal{H}''(0)-\frac{28}{3}D_0\right| \tag{5.11}$$

in view of $\mathcal{H}(0)=\lambda_0$ (see (5.1)), where

$$f_l(z)=\frac{1}{l!}\frac{d^l}{dw^l}\left[2\log\frac{w-q_0(z)}{q_0^{-1}(w)-z}+4\pi H^*(z-q_0^{-1}(w))\right](0)\,,\quad b_l=\frac{1}{l!}\frac{d^l q_0^{-1}}{dw^l}(0).$$

Since $\frac{d^k\mathcal{H}}{dz^k}(0)=0$ for all odd $k\in\mathbb{N}$, we have that

$$\frac{z^3}{\sigma_0(z)}=\frac{\lambda_0}{3}+\frac{\mathcal{H}''(0)}{2}z^2-\frac{\mathcal{H}^{(4)}(0)}{24}z^4-\frac{\mathcal{H}^{(6)}(0)}{2160}z^6+O(z^8),$$

and then

$$\sigma_0(z)=\frac{3}{\lambda_0}z^3-\frac{9\mathcal{H}''(0)}{2\lambda_0^2}z^5+O(z^7),\quad q_0(z)=\frac{3^{\frac13}}{\lambda_0^{\frac13}}z-\frac{3^{\frac13}\mathcal{H}''(0)}{2\lambda_0^{\frac43}}z^3+O(z^5),\quad q_0^{-1}(w)=\frac{\lambda_0^{\frac13}}{3^{\frac13}}w+\frac{\mathcal{H}''(0)}{6}w^3+O(w^5)$$

as $z,w\to 0$. Direct computation shows that $b_3=\frac{\mathcal{H}''(0)}{6}$ and

$$\begin{aligned} f_3(z) &= -\frac{2}{3\sigma_0(z)}+\frac{2\lambda_0}{9z^3}+\frac{2b_3}{z}-\frac{2\pi\lambda_0}{9}(H^*)'''(z)-4\pi b_3(H^*)'(z)\\ &= \frac{\mathcal{H}^{(4)}(0)}{36}z+\frac{\mathcal{H}^{(6)}(0)}{3240}z^3-\frac{2\pi\lambda_0}{9}(H^*)'''(z)-\frac{2\pi}{3}\mathcal{H}''(0)(H^*)'(z)+O(z^5)\end{aligned}$$

as $z \to 0$. Since then

$$f_3'(0) = \frac{\mathcal{H}^{(4)}(0)}{36} - \frac{2\pi\lambda_0}{9}(H^*)^{(4)}(0) - \frac{2\pi}{3}\mathcal{H}''(0)(H^*)''(0), \quad f_3'''(0) = \frac{\mathcal{H}^{(6)}(0)}{540} - \frac{2\pi\lambda_0}{9}(H^*)^{(6)}(0) - \frac{2\pi}{3}\mathcal{H}''(0)(H^*)^{(4)}(0),$$

condition (5.11) is equivalent to

$$\left|\frac{\mathcal{H}''(0)\mathcal{H}^{(4)}(0)}{12} + \frac{\lambda_0\mathcal{H}^{(6)}(0)}{540} - 2\pi(\mathcal{H}''(0))^2(H^*)''(0) - \frac{4\pi\lambda_0}{3}\mathcal{H}''(0)(H^*)^{(4)}(0) - \frac{2\pi\lambda_0^2}{9}(H^*)^{(6)}(0)\right|$$
$$\neq \left|\frac{\pi}{a^2}|\mathcal{H}''(0)|^2 - \frac{28}{3}D_0\right|.$$

By the explicit expression (5.1) of $\mathcal{H}_0$ we have that

$$\mathcal{H}(z) = \lambda_0 z^4(\wp(z) - \wp(\tilde{p}_1))(\wp(z) - \wp(\tilde{p}_3)).$$

Replacing $\mathcal{H}$ with $\frac{\mathcal{H}}{\lambda_0}$, we can assume $\lambda_0 = 1$ and simply study the stronger condition

$$\left|\frac{\mathcal{H}''(0)\mathcal{H}^{(4)}(0)}{4} + \frac{\mathcal{H}^{(6)}(0)}{180} - 6\pi(\mathcal{H}''(0))^2(H^*)''(0) - 4\pi\mathcal{H}''(0)(H^*)^{(4)}(0) - \frac{2\pi}{3}(H^*)^{(6)}(0)\right| < \frac{3\pi}{a^2}|\mathcal{H}''(0)|^2 \quad (5.12)$$

in view of Proposition 5.2 and (5.7). Letting $G_l = \sum_{(n,m)\neq(0,0)} \frac{1}{(n\omega_1 + m\omega_2)^l}$, $l \geq 3$, be the Eisenstein series, the Laurent expansion of $\wp$ near 0 simply re-writes as

$$\wp(z) = \frac{1}{z^2} + \sum_{l=1}^{\infty}(2l+1)G_{2l+2}z^{2l},$$

and then

$$\mathcal{H}(z) = 1 - (\wp(\tilde{p}_1) + \wp(\tilde{p}_3))z^2 + (\wp(\tilde{p}_1)\wp(\tilde{p}_3) + 6G_4)\, z^4 + (10G_6 - 3G_4\wp(\tilde{p}_1) - 3G_4\wp(\tilde{p}_3))\, z^6 + O(z^8)$$

as $z \to 0$. Letting $e_j = \wp(\tilde{p}_j)$ for $j = 1, 2, 3$, recall that

$$e_2 < e_3 \leq 0 < e_1, \quad e_1 + e_2 + e_3 = 0, \quad 15G_4 = -(e_1e_2 + e_1e_3 + e_2e_3), \quad 35G_6 = e_1e_2e_3, \quad (5.13)$$

with $e_3 = 0$ if and only if Ω is a square (see [1]). By the expansion of $\mathcal{H}$ and (5.13), we deduce that

$$\mathcal{H}''(0) = 2e_2,\ \mathcal{H}^{(4)}(0) = 24(e_1e_3 + 6G_4),\ \mathcal{H}^{(6)}(0) = 720(10G_6 + 3G_4e_2),$$

and condition (5.12) gets re-written as

$$\left|460G_6 + 84G_4e_2 - 24\pi e_2^2(H^*)''(0) - 8\pi e_2(H^*)^{(4)}(0) - \frac{2\pi}{3}(H^*)^{(6)}(0)\right| < \frac{12\pi}{a^2}e_2^2 \quad (5.14)$$

in view of (5.13).

From an explicit formula for the Green's function (see [11]) we have that

$$H(z) - \frac{|z|^2}{4|\Omega|} = \mathrm{Re}\left(-\frac{z^2}{4a^2} + \frac{iz}{2a} + \frac{1}{12}\right) - \frac{1}{2\pi}\log\left|\frac{1 - e\left(\frac{z}{a}\right)}{z} \times \prod_{k=1}^{\infty}\left(1 - e\left(\frac{kai + z}{a}\right)\right)\left(1 - e\left(\frac{kai - z}{a}\right)\right)\right|,$$

where $e(z) = e^{2\pi i z}$, yielding to

$$H^*(z) = -\frac{z^2}{4a^2} + \frac{iz}{2a} + \frac{1}{12} - \frac{1}{2\pi}\log\left[\left(\frac{1-e\left(\frac{z}{a}\right)}{z}\right) \times \prod_{k=1}^{\infty}\left(1-e\left(\frac{kai+z}{a}\right)\right)\left(1-e\left(\frac{kai-z}{a}\right)\right)\right].$$

Direct, but tedious, computations show that

$$(H^*)''(0) = -\frac{1}{2a^2} + \frac{\pi}{6a^2} - \frac{4\pi}{a^2}\sum_{k=1}^{\infty}\lambda_k(\lambda_k+1), \quad (H^*)^{(4)}(0) = \frac{\pi^3}{15a^4} + \frac{16\pi^3}{a^4}\sum_{k=1}^{\infty}\lambda_k(\lambda_k+1)(6\lambda_k^2+6\lambda_k+1)$$

$$(H^*)^{(6)}(0) = \frac{8\pi^5}{63a^6} - \frac{64\pi^5}{a^6}\sum_{k=1}^{\infty}\lambda_k(\lambda_k+1)(120\lambda_k^4+240\lambda_k^3+150\lambda_k^2+30\lambda_k+1),$$

where $\lambda_k := \frac{1}{e^{2\pi k}-1}$. On a square torus the Green function $G(z,0)$ has an additional symmetry, the invariance under $\frac{\pi}{2}-$rotations. Therefore, $H^*(iz) = H^*(z)$ for all $z \in \Omega$, and then $(H^*)''(0) = (H^*)^{(6)}(0) = 0$. Since $e_3 = G_6 = 0$, condition (5.14) becomes

$$\left|\frac{28}{5}e_1^2 - 8\pi(H^*)^{(4)}(0)\right| < \frac{12\pi}{a^2}e_1 \tag{5.15}$$

in view of (5.13) and $e_1 = -e_2 > 0$. From the study of the Weierstrass function $\wp$ it is known that (see [3])

$$\sum_{(n,m)\neq(0,0)}\frac{1}{(n+m\tau)^4} = \frac{\pi^4}{45} + \frac{16\pi^4}{3}\sum_{m,k=1}^{\infty}k^3 e^{2\pi i k m\tau}$$

for $\tau \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$. The choice $\tau = i$ yields to

$$15a^4 G_4 = a^4 e_1^2 = \frac{\pi^4}{3} + 80\pi^4\sum_{m,k=1}^{\infty}k^3 e^{-2\pi k m}$$

in view of (5.13), which turns (5.15) into

$$\left|\frac{\pi^4}{3} + 112\pi^4\sum_{m,k=1}^{\infty}k^3e^{-2\pi km} - 32\pi^4\sum_{k=1}^{\infty}\lambda_k(\lambda_k+1)(6\lambda_k^2+6\lambda_k+1)\right| < 3\pi\sqrt{\frac{\pi^4}{3} + 80\pi^4\sum_{m,k=1}^{\infty}k^3e^{-2\pi km}}. \tag{5.16}$$

Since numerically we can approximately compute

$$32\pi^4\sum_{k=1}^{\infty}\lambda_k(\lambda_k+1)(6\lambda_k^2+6\lambda_k+1) \approx 5,9194 \qquad 80\pi^4\sum_{m,k=1}^{\infty}k^3e^{-2\pi km} \approx 14,7985,$$

we get the validity of (5.16), or equivalently (4.15) for the vortex configuration $\{\tau\} \cup \{p_j : j \in J\} \subset \Omega$. ■

As a combination of Propositions 5.1, 5.3 and 5.4 we finally get that

Theorem 5.5. *Let Ω be a square of side a, $a > 0$, and assume that the vortex configuration is the periodic one generated by $\{0, \frac{a}{2}, \frac{ia}{2}, \frac{a+ia}{2}\}$ with multiplicities $2, n_1, n_2, 2$ and $(n_1, n_2) = (2, 0)$ (or viceversa). Then, for τ small the assumption of Theorem 4.6 do hold for the slightly translated vortex configuration $\{-\tau(1+$*

$i), -\tau(1+i)+\frac{a}{2}, -\tau(1+i)+\frac{ia}{2}, -\tau(1+i)+\frac{a+ia}{2}\}$. In particular, for $\epsilon>0$ small we can find $N-$condensate $(\mathcal{A}_\epsilon,\phi_\epsilon)$ so that $|\phi_\epsilon|\to 0$ in $C(\bar\Omega)$ and

$$(F_{12})_\epsilon \rightharpoonup 12\pi\delta_0 \tag{5.17}$$

weakly in the sense of measures, as $\epsilon\to 0$, where $\{0,\frac{a}{2},\frac{ia}{2},\frac{a+ia}{2}\}$ are the zeroes of ϕ_ϵ with multiplicities $2,n_1,n_2,2$ and $(n_1,n_2)=(2,0)$ (or viceversa).

As a final remark, observe that for $n=0$ Theorem 4.6 essentially recovers the result in [29] concerning single-point concentration in any torus Ω (see also [20]). Notice that $n=0$ corresponds to have that the concentration point 0 is not really a singular point and a more simple approach is possible as in the above-mentioned papers. By (2.4) the total multiplicity N is 2 produced by two vortex-points $p_1,p_2\in\Omega\setminus\{0\}$. Assumption (4.13) is equivalent to have $(\log\mathcal{H})'(0)=0$. By the Cauchy-Riemann equations, the last condition can be just re-written as

$$\nabla[2\,\mathrm{Re}\log\mathcal{H}](0)=\nabla\log|\mathcal{H}|^2(0)=\nabla[8\pi H+u_0](0)=0.$$

Since $\nabla H(0)=0$ in view of $H(z)=H(-z)$, we have that (4.13) simply reads as: 0 is a critical point of u_0. As far as (4.14), notice that D_0 does not depend on $\rho>0$ small for

$$\int_{\sigma_0^{-1}(B_\rho(0))\setminus\sigma_0^{-1}(B_r(0))} e^{u_0+8\pi G(z,0)}-\int_{B_\rho(0)\setminus B_r(0)}\frac{dy}{|y|^4}=\text{Area}\Big(B_{\frac{1}{r}}(0)\setminus B_{\frac{1}{\rho}}(0)\Big)-\pi\Big(\frac{1}{r^2}-\frac{1}{\rho^2}\Big)=0$$

for all $0<r\le\rho$, in view of (2.11) with $c_0=0$. Therefore, D_0 can be re-written as

$$D_0=\frac{1}{\pi}\left[\int_{\Omega\setminus\sigma_0^{-1}(B_\rho(0))} e^{u_0+8\pi G(z,0)}-\int_{\mathbb{R}^2\setminus B_\rho(0)}\frac{dy}{|y|^4}\right]=\frac{1}{\pi}\lim_{r\to 0}\left[\int_{\Omega\setminus\sigma_0^{-1}(B_r(0))}\frac{e^{8\pi H(z,0)+u_0}}{|z|^4}-\int_{\mathbb{R}^2\setminus B_r(0)}\frac{1}{|y|^4}\right].$$

Since $\sigma_0(z)=\frac{z}{\lambda_0}+\frac{\mathcal{H}''(0)}{2\lambda_0^2}z^3+O(|z|^5)$ and $\sigma_0^{-1}(z)=\lambda_0 z+O(|z|^3)$ with $\lambda_0=e^{4\pi H(0)-\frac{u_0(0)}{2}}$, notice that $B_{\lambda_0 r-Cr^3}(0)\subset\sigma_0^{-1}(B_r(0))\subset B_{\lambda_0 r+Cr^3}(0)$ for all $r>0$ small, for some constant $C>0$. Thus, there holds

$$\begin{aligned}&\left|\int_{\Omega\setminus\sigma_0^{-1}(B_r(0))}\frac{1}{|z|^4}e^{8\pi[H(z,0)-H(0,0)]+[u_0(z)-u_0(0)]}-\int_{\Omega\setminus B_{\lambda_0 r}(0)}\frac{1}{|z|^4}e^{8\pi[H(z,0)-H(0,0)]+[u_0(z)-u_0(0)]}\right|\\ &=O\left(\int_{B_{\lambda_0 r+Cr^3}(0)\setminus B_{\lambda_0 r-Cr^3}(0)}\frac{1}{|z|^2}\right)=o(1)\end{aligned}$$

as $r\to 0$ in view of $\nabla[8\pi H+u_0](0)=0$, yielding to the same expression for D_0 as in [20, 29]:

$$D_0=\frac{\lambda_0^2}{\pi}\lim_{r\to 0}\left[\int_{\Omega\setminus B_r(0)}\frac{1}{|z|^4}e^{8\pi[H(z,0)-H(0,0)]+[u_0(z)-u_0(0)]}-\int_{\mathbb{R}^2\setminus B_r(0)}\frac{1}{|y|^4}\right].$$

The "non-degeneracy condition" (4.15) reads as

$$\left|\frac{\mathcal{H}''(0)}{\mathcal{H}(0)}-4\pi(H^*)''(0)\right|=|(\log\mathcal{H})''(0)-4\pi(H^*)''(0)|\neq\frac{2\pi}{|\Omega|},$$

in view of $\sigma_0=q_0$, $b_1=\lambda_0$, $f_1(z)=-4\pi\lambda_0(H^*)'(z)+\frac{2\lambda_0}{z}-\frac{2}{\sigma_0(z)}$ and $\mathcal{H}'(0)=0$. Setting $\mathcal{H}_1(z)=e^{-4\pi H^*(z)}\mathcal{H}(z)$, we have that $|\mathcal{H}_1(z)|^2=e^{u_0+\frac{2\pi}{|\Omega|}|z|^2}$ and

$$\begin{aligned}(\log\mathcal{H})''(0)-4\pi(H^*)''(0)&=(\log\mathcal{H}_1)''(0)=2(\mathrm{Re}\log\mathcal{H}_1)''(0)=(\log|\mathcal{H}_1|^2)''(0)=\Big(u_0+\frac{2\pi}{|\Omega|}|z|^2\Big)''(0)\\ &=\frac{1}{4}[(u_0)_{xx}(0)-(u_0)_{yy}(0)-2i(u_0)_{xy}(0)]\end{aligned}$$

in view of (2.6)-(2.7), and the above condition turns into

$$\begin{aligned} 0 \quad &\neq \quad \frac{1}{16}\left|(u_0)_{xx}(0)-(u_0)_{yy}(0)-2i(u_0)_{xy}(0)\right|^2-\frac{4\pi^2}{|\Omega|^2}=\frac{1}{16}\left((u_0)_{xx}(0)-(u_0)_{yy}(0)\right)^2+\frac{1}{4}(u_0)^2_{xy}(0)-\frac{4\pi^2}{|\Omega|^2}\\ &= \quad \frac{1}{16}(\Delta u_0)^2(0)-\frac{1}{4}\det D^2u_0(0)-\frac{4\pi^2}{|\Omega|^2}=-\frac{1}{4}\det D^2u_0(0). \end{aligned}$$

In conclusion, when $n=0$ the assumptions in Theorem 4.6 are equivalent to have 0 as a non-degenerate critical point of $u_0(z)=-4\pi G(z,p_1)-4\pi G(z,p_2)$ with $D_0<0$.

6 A more general result

In this section we deal with the case $m \geq 2$ in Theorem 1.1. For more clearness, let us denote the concentration points as ξ_l, $l=1,\dots,m$, the remaining points in the vortex set as p_j, and by n_l, n_j the corresponding multiplicities.

From section 2 recall that $H(z)=G(z,0)+\frac{1}{2\pi}\log|z|$ is a smooth function in 2Ω with $\Delta H=\frac{1}{|\Omega|}$, and H^* is an holomorphic function in 2Ω with $\operatorname{Re} H^*=H-\frac{|z|^2}{4|\Omega|}$. Up to a translation, we are assuming that $p_j\in\Omega$ for all $j=1,\dots,N$, and taking $\tilde\Omega$ close to Ω so that $\tilde\Omega-p_j\subset 2\Omega$ for all $j=1,\dots,N$. Arguing as for (2.6), the function

$$\begin{aligned} \mathcal{H}(z)=\prod_j (z-p_j)^{n_j}\exp\Bigg(&4\pi\sum_{l=1}^m (n_l+1)H^*(z-\xi_l)-2\pi\sum_{j=1}^N H^*(z-p_j)\\ &+\frac{\pi}{|\Omega|}\sum_{l=1}^m(n_l+1)(\xi_l-2z)\overline{\xi_l}-\frac{\pi}{2|\Omega|}\sum_{j=1}^N|p_j|^2+\frac{\pi}{|\Omega|}z\overline{\sum_{j=1}^N p_j}\Bigg) \end{aligned}$$

is holomorphic in $\tilde\Omega$ and satisfies

$$|\mathcal{H}(z)|^2=\left(\prod_{l=1}^m|z-\xi_l|^{-2n_l}\right)\exp\left(u_0+8\pi\sum_{l=1}^m(n_l+1)H(z-\xi_l)\right)$$

in view of (1.9). For $l=1,\dots,m$ the function

$$\mathcal{H}^l(z)=\mathcal{H}(z)\prod_{l'\neq l}(z-\xi_{l'})^{-(n_{l'}+2)}$$

is holomorphic near ξ_l and satisfies

$$|\mathcal{H}^l(z)|^2=\exp\left(4\pi(n_l+2)H(z-\xi_l)+4\pi\sum_{l'\neq l}(n_{l'}+2)G(z,\xi_{l'})-4\pi\sum_j n_jG(z,p_j)\right). \tag{6.1}$$

To be more clear, let us spend few words to compare the case $m=1$ and $m\geq 2$. When $m=1$ notice that $\mathcal{H}$ satisfies $|\mathcal{H}|^2=e^{u_0+8\pi(n+1)H(z)-2n\log|z|}$ in view of (2.7). The function $e^{u_0+8\pi(n+1)H(z)-2n\log|z|}$ is a sort of effective potential for (2.2) at 0, where $e^{u_0-2n\log|z|}$ is the non-vanishing part of e^{u_0} and $e^{8\pi(n+1)H(z)}$ is the self-interaction of the concentration point 0 driven by $PU_{\delta,0,\sigma_0}$ through (2.18). When $m \geq 2$, (6.1) can be re-written as

$$|\mathcal{H}^l(z)|^2 = \exp\left(u_0 + 8\pi(n_l+1)H(z-\xi_l) + 8\pi\sum_{l'\neq l}(n_{l'}+1)G(z,\xi_{l'}) - 2n_l \log|z-\xi_l|\right)$$

for $l = 1,\dots,m$, yielding to an effective potential for (2.2) at ξ_l exhibiting an additional interaction term $e^{8\pi\sum_{l'\neq l}(n_{l'}+1)G(z,\xi_{l'})}$ generated by the effect of the concentration points $\xi_{l'}$, $l' \neq l$, through (6.12).

Setting $\mathcal{H}_0 = \frac{\mathcal{H}}{(z-\xi_1)^{n_1+2}\dots(z-\xi_m)^{n_m+2}}$, we now define σ_0 as

$$\sigma_0(z) = -\left(\int^z \mathcal{H}_0(w)\exp\left[-\sum_{l=1}^m c_0^l (w-\xi_l)^{n_l+1}\prod_{l'\neq l}(w-\xi_{l'})^{n_{l'}+2}\right]dw\right)^{-1}, \tag{6.2}$$

where

$$c_0^l = \frac{1}{\mathcal{H}_0(\xi_l)(n_l+1)!}\frac{d^{n_l+1}\mathcal{H}^l}{dz^{n_l+1}}(\xi_l), \quad l = 1,\dots,m,$$

guarantee that all the residues of the integrand function in the definition of σ_0 vanish. The presence of the term $\prod_{l'\neq l}(w-\xi_{l'})^{n_{l'}+2}$ is crucial to compute explicitly the c_0^l's for

$$c_0^l(w-\xi_l)^{n_l+1}\prod_{l'\neq l}(w-\xi_{l'})^{n_{l'}+2} = O((w-\xi_{l'})^{n_{l'}+2})$$

has an high-order effect near any other $\xi_{l'}$, $l' \neq l$. By construction $\sigma_0 \in \mathcal{M}(\overline{\Omega})$ vanishes only at the ξ_l's with multiplicity $n_l + 1$ and

$$\lim_{z\to\xi_l}\frac{(z-\xi_l)^{n_l+1}}{\sigma_0(z)} = \frac{\mathcal{H}^l(\xi_l)}{n_l+1},$$

and satisfies

$$|\sigma_0'(z)|^2 = |\sigma_0(z)|^4\exp\left(u_0 + 8\pi\sum_{l=1}^m(n_l+1)G(z,\xi_l) - 2\sum_{l=1}^m \mathrm{Re}\left[c_0^l(z-\xi_l)^{n_l+1}\prod_{l'\neq l}(z-\xi_{l'})^{n_{l'}+2}\right]\right).$$

Under the assumptions of Theorem 1.1, notice that $c_0^l = 0$ for all $l = 1,\dots,m$ and

$$\left|\left(\frac{1}{\sigma_0}\right)'(z)\right|^2 = |\mathcal{H}_0(z)|^2 = e^{u_0+8\pi\sum_{l=1}^m(n_l+1)G(z,\xi_l)}.$$

Since each ξ_l gives a contribution to the dimension of the kernel for the linearized operator (4.3), the parameters δ and a are no longer enough to recover all the degeneracies induced by the ansatz $PU_{\delta,a,\sigma}$, for $\sigma \in \mathcal{M}(\overline{\Omega})$ a function which vanishes only at the points ξ_l, $l = 1,\dots,m$, with multiplicity $n_l + 1$. In our construction, the correct number of parameters to use is $2m + 1$, given by m small complex numbers $a_1,\dots,a_m$ and $\delta > 0$ small, where the latter gives rise to the concentration parameter δ_l at ξ_l, $l = 1,\dots,m$, by means of (6.14). The request that all the δ_l's tend to zero with the same rate is necessary as we will discuss later.

We need to construct an ansatz that looks as $PU_{\delta_l,a_l,\sigma_{a,l}}$ near each ξ_l, for a suitable $\sigma_{a,l}$ which makes the approximation near ξ_l good enough. In order to localize our previous construction, let us define $PU_{\delta_l,a_l,\sigma}$ as the solution of

$$\begin{cases} -\Delta PU_{\delta_l,a_l,\sigma} = \chi(|z-\xi_l|)|\sigma'(z)|^2 e^{U_{\delta_l,a_l,\sigma}} - \frac{1}{|\Omega|}\int_\Omega \chi(|z-\xi_l|)|\sigma'(z)|^2 e^{U_{\delta_l,a_l,\sigma}} & \text{in } \Omega \\ \int_\Omega PU_{\delta_l,a_l,\sigma} = 0, \end{cases}$$

where χ is a smooth radial cut-off function so that $\chi = 1$ in $[-\eta,\eta]$, $\chi = 0$ in $(-\infty,-2\eta]\cup[2\eta,+\infty)$, $0<\eta<\frac{1}{2}\min\{|\xi_l-\xi_{l'}|, \text{dist }(\xi_l,\partial\Omega) : l,l'=1,\dots,m,\ l\neq l'\}$. The approximating function is then built as $W=\sum_{l=1}^m PU_l$, where $U_{\delta_l,a_l,\sigma_{a,l}}$ and $PU_{\delta_l,a_l,\sigma_{a,l}}$ will be simply denoted by U_l and PU_l.

Let us now explain how to find the functions $\sigma_{a,l}$, $l=1,\dots,m$. Setting

$$\mathcal{B}_r^l = \left\{\sigma \text{ holomorphic in } B_{2\eta}(\xi_l) :\ \left\|\frac{\sigma}{\sigma_0}-1\right\|_{\infty,B_{2\eta}(\xi_l)} \le r\right\}$$

for $l=1,\dots,m$, Lemma A.1 still holds in this context for all $\sigma\in\mathcal{B}_r^l$, by simply replacing 0, n with ξ_l, n_l and $\tilde\Omega$ with $B_{2\eta}(\xi_l)$. Then, for all $\sigma=(\sigma_1,\dots,\sigma_m)\in\mathcal{B}_r:=\mathcal{B}_r^1\times\dots\times\mathcal{B}_r^m$ and $a=(a_1,\dots,a_m)\in\mathbb{C}^m$ with $\|a\|_\infty<\rho$ there exist points a_i^l, $l=1,\dots,m$ and $i=0,\dots,n_l$, so that $\{z\in B_{2\eta}(\xi_l):\ \sigma_l(z)=a_l\}=\{\xi_l+a_0^l,\dots,\xi_l+a_{n_l}^l\}$ for all $l=1,\dots,m$. Arguing as for (2.12), for $l=1,\dots,m$ the function

$$\mathcal{H}_{a,\sigma}^l(z)=\prod_j (z-p_j)^{n_j}\prod_{l'\neq l}(z-\xi_{l'})^{n_{l'}}\prod_{l'\neq l}\prod_{i=0}^{n_{l'}}(z-\xi_{l'}-a_i^{l'})^{-2}\exp\Bigg(4\pi\sum_{l'=1}^m\sum_{i=0}^{n_{l'}}H^*(z-\xi_{l'}-a_i^{l'})$$
$$-2\pi\sum_{j=1}^N H^*(z-p_j)+\frac{\pi}{|\Omega|}\sum_{l'=1}^m(n_{l'}+1)(\xi_{l'}-2z)\overline{\xi_{l'}}-\frac{\pi}{2|\Omega|}\sum_{j=1}^N|p_j|^2-\frac{2\pi}{|\Omega|}\sum_{l'=1}^m(z-\xi_{l'})\overline{\sum_{i=0}^{n_{l'}}a_i^{l'}}+\frac{\pi}{|\Omega|}z\overline{\sum_{j=1}^N p_j}\Bigg)$$

is holomorphic near ξ_l and satisfies

$$|\mathcal{H}_{a,\sigma}^l(z)|^2=|z-\xi_l|^{-2n_l}\exp[u_0+8\pi\sum_{i=0}^{n_l}H(z-\xi_l-a_i^l)+8\pi\sum_{l'\neq l}\sum_{i=0}^{n_{l'}}G(z,\xi_{l'}+a_i^{l'})-\frac{2\pi}{|\Omega|}\sum_{l'=1}^m\sum_{i=0}^{n_{l'}}|a_i^{l'}|^2] \quad (6.3)$$

in view of (1.9). Setting

$$g_{a_l,\sigma_l}^l(z)=\frac{\sigma_l(z)-a_l}{\prod_{i=0}^{n_l}(z-\xi_l-a_i^l)},\qquad z\in B_{2\eta}(\xi_l),$$

and

$$c_{a,\sigma}^l=\frac{\prod_{l'\neq l}(\xi_l-\xi_{l'})^{-(n_{l'}+2)}}{(n_l+1)!}\frac{d^{n_l+1}}{dz^{n_l+1}}\left[\Big(\frac{g_{a_l,\sigma_l}^l(z)g_{0,\sigma_l}^l(\xi_l)}{g_{a_l,\sigma_l}^l(\xi_l)g_{0,\sigma_l}^l(z)}\Big)^2\frac{\mathcal{H}_{a,\sigma}^l(z)}{\mathcal{H}_{a,\sigma}^l(\xi_l)}\right](\xi_l), \quad (6.4)$$

the aim is to find a solution $\sigma_a=(\sigma_{a,1},\dots,\sigma_{a,m})\in\mathcal{B}_r$ of the system ($l=1,\dots,m$):

$$\sigma_l(z)=-\left(\int^z\Big(\frac{g_{a_l,\sigma_l}^l(w)}{g_{0,\sigma_l}^l(w)}\Big)^2\frac{\mathcal{H}_{a,\sigma}^l(w)}{(w-\xi_l)^{n_l+2}}\exp\left[-\sum_{l'=1}^m c_{a,\sigma}^{l'}(w-\xi_{l'})^{n_{l'}+1}\prod_{l''\neq l'}(w-\xi_{l''})^{n_{l''}+2}\right]dw\right)^{-1}, \quad (6.5)$$

where the definition of $c^l_{a,\sigma}$ makes null the residue at ξ_l of the integrand function in (6.5). The function $\sigma_{a,l}$ will vanish only at ξ_l with multiplicity n_l+1 and satisfy

$$|\sigma'_{a,l}(z)|^2 = |\sigma_{a,l}(z)-a_l|^4 \exp\Bigg(u_0+8\pi\sum_{l'=1}^m\sum_{i=0}^{n_{l'}} G(z,\xi_{l'}+a_i^{l'}) - \frac{2\pi}{|\Omega|}\sum_{l'=1}^m\sum_{i=0}^{n_{l'}}|a_i^{l'}|^2 \\ -2\sum_{l'=1}^m \mathrm{Re}\Big[c^{l'}_{a,\sigma_a}(z-\xi_{l'})^{n_{l'}+1}\prod_{l''\neq l'}(z-\xi_{l''})^{n_{l''}+2}\Big]\Bigg) \qquad (6.6)$$

in view of (6.3).

Since $\mathcal{H}^l_{0,\sigma}=\mathcal{H}^l$ and $c^l_{0,\sigma}=c^l_0$ for all $l=1,\dots,m$, when $a=0$ the system (6.5) reduces to m-copies of (6.2) in each $B_{2\eta}(\xi_l)$, $l=1,\dots,m$, and it is natural to find σ_a branching off $(\sigma_0,\dots,\sigma_0)$ for a small by IFT. Let us emphasize that each $\sigma_{a,l}$, $l=1,\dots,m$, is close to $\sigma_0\Big|_{B_{2\eta}(\xi_l)}$, a crucial property to have D_0 defined in terms of a unique σ_0 (see (1.10)). Letting $q_{0,l}$ be the function so that $\sigma_0=q_{0,l}^{n_l+1}$ near ξ_l, arguing as in Lemma A.2 we have that

Lemma 6.1. *Up to take ρ smaller, there exists a C^1-map $a\in B_\rho(0)\to\sigma_a\in\mathcal{B}_r$ so that σ_a solves the system* (6.4)-(6.5). *Moreover, the map $a\in B_\rho(0)\to c^l_a:=c^l_{a,\sigma_a}$ is C^1 with*

$$\Gamma^{ll}:=\mathcal{H}(\xi_l)\partial_{a_l}c^l_a\Big|_{a=0}=\frac{1}{n_l!}\frac{d^{n_l+1}}{dz^{n_l+1}}\Big[\mathcal{H}^l(z)f^l_{n_l+1}(z)\Big](\xi_l) \qquad (6.7)$$

$$\Upsilon^{ll}:=\mathcal{H}(\xi_l)\partial_{\bar a_l}c^l_a\Big|_{a=0}=-\frac{2\pi(n_l+1)}{|\Omega|n_l!}\overline{b^l_{n_l+1}}\frac{d^{n_l}\mathcal{H}^l}{dz^{n_l}}(\xi_l) \qquad (6.8)$$

and for $j\neq l$

$$\Gamma^{lj}:=\mathcal{H}(\xi_l)\partial_{a_j}c^l_a\Big|_{a=0}=\frac{n_j+1}{(n_l+1)!}\frac{d^{n_l+1}}{dz^{n_l+1}}\Big[\mathcal{H}^l(z)\tilde f^j_{n_j+1}(z)\Big](\xi_l) \qquad (6.9)$$

$$\Upsilon^{lj}:=\mathcal{H}(\xi_l)\partial_{\bar a_j}c^l_a\Big|_{a=0}=-\frac{2\pi(n_j+1)}{|\Omega|n_l!}\overline{b^j_{n_j+1}}\frac{d^{n_l}\mathcal{H}^l}{dz^{n_l}}(\xi_l), \qquad (6.10)$$

where

$$f^l_{n+1}(z)=\frac{1}{(n+1)!}\frac{d^{n+1}}{dw^{n+1}}\left[2\log\frac{w-q_{0,l}(z)}{q^{-1}_{0,l}(w)-z}+4\pi H^*(z-q^{-1}_{0,l}(w))\right](0), \qquad b^l_{n+1}=\frac{1}{(n+1)!}\frac{d^{n+1}q^{-1}_{0,l}}{dw^{n+1}}(0)$$

and for $j\neq l$

$$\tilde f^j_{n+1}(z)=\frac{1}{(n+1)!}\frac{d^{n+1}}{dw^{n+1}}\Big[-2\log(z-q^{-1}_{0,j}(w))+4\pi H^*(z-q^{-1}_{0,j}(w))\Big](0).$$

Letting $n=\min\{n_l:\ l=1,\dots,m\}$, up to re-ordering, assume that $n=n_1=\dots=n_{m'}<n_l$ for all $l=m'+1,\dots,m$, where $1\le m'\le m$. The matrix A in Theorem 1.1 is the $2m\times 2m-$matrix in the form

$$A=\begin{pmatrix} A^{1,2}_{1,2} & \dots & A^{2m-1,2m}_{1,2}\\ \vdots & \vdots & \vdots\\ A^{1,2}_{2m-1,2m} & \dots & A^{2m-1,2m}_{2m-1,2m}\end{pmatrix}, \qquad (6.11)$$

where the 2×2-blocks are given by

$$A^{2l'-1,2l'}_{2l-1,2l} = \begin{pmatrix} \mathrm{Re}[\Gamma^{ll'}+\Upsilon^{ll'}+\frac{n(2n+3)}{n+1}D_0\frac{|\mathcal{H}^l(\xi_l)|^{-\frac{2}{n+1}}}{\sum_{j=1}^{m'}|\mathcal{H}^j(\xi_j)|^{-\frac{2}{n+1}}}\delta_{ll'}] & \mathrm{Im}[\Upsilon^{ll'}-\Gamma^{ll'}] \\ \mathrm{Im}[\Gamma^{ll'}+\Upsilon^{ll'}] & \mathrm{Im}[\Gamma^{ll'}-\Upsilon^{ll'}-\frac{n(2n+3)}{n+1}D_0\frac{|\mathcal{H}^l(\xi_l)|^{-\frac{2}{n+1}}}{\sum_{j=1}^{m'}|\mathcal{H}^j(\xi_j)|^{-\frac{2}{n+1}}}\delta_{ll'}] \end{pmatrix}$$

when $l=1,\dots,m'$ and by

$$A^{2l'-1,2l'}_{2l-1,2l} = \begin{pmatrix} \mathrm{Re}[\Gamma^{ll'}+\Upsilon^{ll'}] & \mathrm{Im}[\Upsilon^{ll'}-\Gamma^{ll'}] \\ \mathrm{Im}[\Gamma^{ll'}+\Upsilon^{ll'}] & \mathrm{Im}[\Gamma^{ll'}-\Upsilon^{ll'}] \end{pmatrix}$$

when $l=m'+1,\dots,m$, with $\Gamma^{ll'}$ and $\Upsilon^{ll'}$ given by (6.7), (6.9) and (6.8), (6.10), respectively, and $\delta_{ll'}$ the Kronecker's symbol.

Arguing as in Lemma 2.2, for $l=1,\dots,m$ we have that

$$\begin{aligned} PU_{\delta_l,a_l,\sigma_l} &= \chi(|z-\xi_l|)\left[U_{\delta_l,a_l,\sigma_l}-\log(8\delta_l^2)+4\log|g^l_{a_l,\sigma_l}|\right] \\ &\quad +8\pi\sum_{i=0}^{n_l}\left[\frac{1}{2\pi}(\chi(|z-\xi_l|)-1)\log|z-\xi_l-a_i^l|+H(z-\xi_l-a_i^l)\right]+\Theta_{\delta_l,a_l,\sigma_l}+2\delta_l^2 f_{a_l,\sigma_l}+O(\delta_l^4) \end{aligned}$$

and

$$PU_{\delta_l,a_l,\sigma_l} = 8\pi\sum_{i=0}^{n_l}G(z,\xi_l+a_i^l)+\Theta_{\delta_l,a_l,\sigma_l}+2\delta_l^2\left(f_{a_l,\sigma_l}-\frac{\chi(|z-\xi_l|)}{|\sigma_l(z)-a_l|^2}\right)+O(\delta_l^4) \tag{6.12}$$

do hold in $C(\overline{\Omega})$ and $C_{\mathrm{loc}}(\overline{\Omega}\setminus\{\xi_l\})$, respectively, uniformly for $|a|<\rho$ and $\sigma_l\in\mathcal{B}^l_r$, where

$$\Theta_{\delta_l,a_l,\sigma_l} = -\frac{1}{|\Omega|}\int_\Omega \chi(|z-\xi_l|)\log\frac{|\sigma_l(z)-a_l|^4}{(\delta_l^2+|\sigma_l(z)-a_l|^2)^2}$$

and f_{a_l,σ_l} is a smooth function in z (with a uniform control in a_l and σ_l of it and its derivatives in z). Choosing $\sigma_l=\sigma_{a,l}$ and summing up over $l=1,\dots,m$, by (6.6) for our approximating function there hold

$$\begin{aligned} W &= U_{\delta_l,a_l,\sigma_l}-\log(8\delta_l^2)+\log|\sigma_l'|^2-u_0+\frac{2\pi}{|\Omega|}\sum_{l'=1}^m\sum_{i=0}^{n_{l'}}|a_i^{l'}|^2+\Theta^l(a,\delta) \\ &\quad +2\,\mathrm{Re}\left[c^l_{a,\sigma_l}(z-\xi_l)^{n_l+1}\prod_{l'\neq l}(z-\xi_{l'})^{n_{l'}+2}\right]+O(|z-\xi_l|^{n_l+2}\sum_{l'\neq l}|c^{l'}_{a,\sigma_{l'}}|)+\sum_{l'=1}^m O(\delta_{l'}^2|z-\xi_l|+\delta_{l'}^4) \end{aligned} \tag{6.13}$$

and

$$W = 8\pi\sum_{l=1}^m\sum_{i=0}^{n_l}G(z,\xi_l+a_i^l)+O\left(\sum_{l'=1}^m\delta_{l'}^2\log|\delta_{l'}|\right)$$

uniformly in $B_\eta(\xi_l)$ and in $\Omega\setminus\cup_{l=1}^m B_\eta(\xi_l)$, respectively, where

$$\Theta^l(a,\delta) := \sum_{l'=1}^m[\Theta_{\delta_{l'},a_{l'},\sigma_{l'}}+\delta_{l'}^2 f_{a_{l'},\sigma_{l'}}(\xi_l)].$$

As a consequence, we have that

$$\int_\Omega e^{u_0+W} = \sum_{l'=1}^m \left[\int_{B_\rho(0)} \frac{n_{l'}+1}{(\delta_{l'}^2+|y-a_{l'}|^2)^2} + o\Big(\frac{1}{\delta_{l'}^2}\Big)\right] = \pi \sum_{l'=1}^m \frac{n_{l'}+1}{\delta_{l'}^2}[1+o(1)],$$

and then near ξ_l there holds

$$4\pi N \frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} = 4\pi N \frac{|\sigma_l'|^2 e^{U_{\delta_l,a_l,\sigma_l}+O(|z-\xi_l|^{n_l+1})+o(1)}}{8\pi \sum_{l'=1}^m (n_{l'}+1)\delta_l^2 \delta_{l'}^{-2}(1+o(1))}.$$

In order to construct a $N-$condensate $(\mathcal{A}_\epsilon, \phi_\epsilon)$ which satisfies (5.17) as $\epsilon \to 0$, we look for a solution w_ϵ of (2.2) in the form $w_\epsilon = \sum_{l=1}^m PU_{\delta_l,a_l,\sigma_l} + \phi$, where ϕ is a small remainder term and $\delta_l = \delta_l(\epsilon)$, $a_l = a_l(\epsilon)$ are suitable small parameters, so that

$$4\pi N \frac{e^{u_0+w_\epsilon}}{\int_\Omega e^{u_0+w_\epsilon}} + \frac{64\pi^2 N^2 \epsilon^2 \int_\Omega e^{2u_0+2w_\epsilon}}{(\int_\Omega e^{u_0+w_\epsilon} + \sqrt{(\int_\Omega e^{u_0+w_\epsilon})^2 - 16\pi N \epsilon^2 \int_\Omega e^{2u_0+2w_\epsilon}})^2} \left(\frac{e^{u_0+w_\epsilon}}{\int_\Omega e^{u_0+w_\epsilon}} - \frac{e^{2u_0+2w_\epsilon}}{\int_\Omega e^{2u_0+2w_\epsilon}} \right)$$

$$\rightharpoonup 8\pi \sum_{l=1}^m (n_l+1)\delta_{\xi_l}$$

in the sense of measures as $\epsilon \to 0$. Since $|\sigma_l'|^2 e^{U_{\delta_l,a_l,\sigma_l}} \rightharpoonup 8\pi(n_l+1)\delta_{\xi_l}$ as $\delta_l, a_l \to 0$, to have the correct concentration property we need that

$$8\pi \sum_{l'=1}^m (n_{l'}+1)\delta_l^2 \delta_{l'}^{-2} \to 4\pi N$$

for all $l = 1, \dots, m$, and then $\frac{\delta_l}{\delta_{l'}} \to 1$ for all $l, l' = 1, \dots, m$ in view of (1.9). It is then natural to introduce just one parameter δ and to chose the δ_l's as

$$\delta_l = \delta \qquad l = 1, \dots, m. \tag{6.14}$$

We restrict our attention to the case $c_0^l = 0$ for all $l = 1, \dots, m$, which is necessary in our context and is simply a re-formulation of the assumption that $\mathcal{H}_0$ has zero residues at $p_1, \dots, p_m$. As in Theorem 4.6, we will work in the parameter's range:

$$a_l = o(\delta), \qquad \delta \sim \epsilon^{\frac{n+1}{n+2}}$$

as $\epsilon \to 0^+$. Since then

$$K^{-1} \le \frac{\delta^2 + |z-\xi_l|^{2n_l+2}}{\delta^2 + |\sigma_l(z) - a_l|^2} \le K, \qquad K^{-1}|z-\xi_l|^{2n_l} \le |\sigma_l'(z)|^2 \le K|z-\xi_l|^{2n_l}$$

in $B_{2\eta}(\xi_l)$ for all $\sigma_l \in \mathcal{B}_r^l$ and $l = 1, \dots, m$, where $K > 1$, the norm (2.53) can be now simply defined as

$$\|h\|_* = \sup_{z\in\Omega} \left[\sum_{l=1}^m \frac{\delta^\gamma \left(|z-\xi_l|^{2n_l} + \delta^{\frac{2n_l}{n_l+1}} \right)}{(\delta^2 + |z-\xi_l|^{2n_l+2})^{1+\frac{\gamma}{2}}} \right]^{-1} |h(z)|$$

for any $h \in L^\infty(\Omega)$, where $0 < \gamma < 1$ is a small fixed constant. In order to simplify notations, we set $U_l = U_{\delta_l,a_l,\sigma_l}$, $c_a^l = c_{a,\sigma_l}^l$, $\Theta_l = \Theta_{\delta_l,a_l,\sigma_l}$ and $f_l = f_{a_l,\sigma_l}$. We have that

Lemma 6.2. *There exists a constant $C > 0$ independent of δ such that*

$$\|R\|_* \le C\delta^{2-\gamma}. \tag{6.15}$$

Proof: We shall sketch the proof of (6.15), by following ideas used in the proof of Theorem 2.3. Through the change of variable $y = \sigma_l(z)$ in $\sigma_l^{-1}(B_\rho(0))$, by Lemma 6.1, (6.13), (6.14) and $c_0^l = 0$ for all $l = 1, \dots, m$ we find that

$$\frac{8\delta^2}{e^{\frac{2\pi}{|\Omega|}\sum_{l'=1}^m \sum_{i=0}^{n_{l'}} |a_i^{l'}|^2 + \Theta^l(a,\delta)}} \int_{\sigma_l^{-1}(B_\rho(0))} e^{u_0+W} = \int_{\sigma_l^{-1}(B_\rho(0))} |\sigma_l'|^2 e^{U_l + O(|z-\xi_l|^{n_l+1}\sum_{l'=1}^m |c_a^{l'}| + \delta^2|z-\xi_l| + \delta^4)}$$
$$= 8\pi(n_l+1) - \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{8(n_l+1)\delta^2}{|y|^4} + O\Big(\|a\|^2 + \delta\|a\| + \delta^{\frac{2n_l+3}{n_l+1}}\Big),$$

where $\|a\|^2 = \sum_{l=1}^m |a_l|^2$. Setting $\Omega_\rho = \cup_{l=1}^m \sigma_l^{-1}(B_\rho(0))$ we get that

$$\frac{8\delta^2}{e^{\frac{2\pi}{|\Omega|}\sum_{l'=1}^m \sum_{i=0}^{n_{l'}} |a_i^{l'}|^2 + \sum_{l'=1}^m \Theta_{l'}}} \int_\Omega e^{u_0+W} = \sum_{l=1}^m e^{\delta^2 \sum_{l'=1}^m f_{l'}(\xi_l)} \Big[8\pi(n_l+1) - \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{8(n_l+1)\delta^2}{|y|^4}$$
$$+ O(\|a\|^2 + \delta\|a\| + \delta^{\frac{2n_l+3}{n_l+1}}\Big)\Big] + 8\delta^2 \int_{\Omega\setminus\Omega_\rho} e^{u_0 + 8\pi\sum_{l=1}^m \sum_{i=0}^{n_l} G(z,\xi_l + a_i^l)} + O(\delta^4|\log\delta| + \delta^2\|a\|^{\frac{2}{\max_l n_l+1}})$$
$$= \sum_{l=1}^m \Big[8\pi(n_l+1) + 8\pi(n_l+1)\delta^2 \sum_{l'=1}^m f_{l'}(\xi_l) - 8(n_l+1)\delta^2 \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{1}{|y|^4} \Big]$$
$$+ 8\delta^2 \int_{\Omega\setminus\Omega_\rho} e^{u_0 + 8\pi\sum_{l=1}^m \sum_{i=0}^{n_l} G(z,\xi_l + a_i^l)} + o(\delta^2) = 4\pi N \left[1 + \frac{2}{N}\delta^2 D_a + \frac{2}{N}\delta^2 \sum_{l,l'=1}^m (n_l+1) f_{l'}(\xi_l) + o(\delta^2) \right]$$

in view of (1.9), where D_a is given by

$$\pi D_a = \int_{\Omega\setminus\Omega_\rho} e^{u_0 + 8\pi\sum_{l=1}^m \sum_{i=1}^{n_l} G(z,\xi_l + a_i^l)} - \sum_{l=1}^m (n_l+1) \int_{\mathbb{R}^2\setminus B_\rho(0)} \frac{1}{|y|^4}.$$

Hence, for $|z - \xi_l| \le \eta$ we have that

$$\Delta W + 4\pi N \left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{1}{|\Omega|} \right) = |\sigma_l'|^2 e^{U_l} \Big[2\mathrm{Re}\, \Big[c_a^l (z-\xi_l)^{n_l+1} \prod_{l'\ne l} (z-\xi_{l'})^{n_{l'}+2} \Big] \tag{6.16}$$
$$+ \delta^2 \sum_{l'=1}^m f_{l'}(\xi_l) - \frac{2D_a}{N}\delta^2 - \frac{2\delta^2}{N} \sum_{j,l'=1}^m (n_j+1) f_{l'}(\xi_j) + O(\|a\||z-\xi_l|^{n_l+2} + \delta^2|z-\xi_l|) + o(\delta^2) \Big] + O(\delta^2)$$

as $\delta \to 0$, in view of (1.9) and $\int_\Omega \chi_l |\sigma_l'|^2 e^{U_l} = 8\pi(n_l+1) + O(\delta^2)$ for all $l = 1, \dots, m$. For $z \in \Omega \setminus \cup_{l=1}^m B_\eta(\xi_l)$, we have that

$$\Delta W + 4\pi N \left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{1}{|\Omega|} \right) = O(\delta^2). \tag{6.17}$$

On the other hand, arguing as in (2.45), we have that

$$\frac{64\delta^4}{e^{\frac{4\pi}{|\Omega|}\sum_{l'=1}^m \sum_{i=1}^{n_{l'}} |a_i^{l'}|^2 + 2\sum_{l'=1}^m \Theta_{l'}}} \int_\Omega e^{2u_0+2W} = 64 \sum_{l=1}^{m'} \frac{(n+1)^3}{|\alpha_{a,l}|^{\frac{2}{n+1}} \delta^{\frac{2}{n+1}}} \int_{\mathbb{R}^2} \frac{|y + a_l\delta^{-1}|^{\frac{2n}{n+1}}}{(1+|y|^2)^4} + O(\delta^{-\frac{1}{n+1}}),$$

where $\alpha_{a,l} = \lim_{z\to\xi_l} \frac{(z-\xi_l)^{n_l+1}}{\sigma_l(z)}$. Recall that $n = \min\{n_l : l = 1,\dots,m\} = n_1 = \dots = n_{m'} < n_l$ for all $l = m'+1,\dots,m$. Setting

$$\tilde D_{a,\delta} = \sum_{l=1}^{m'} \frac{(n+1)^3}{|\alpha_{a,l}|^{\frac{2}{n+1}}\delta^{\frac{2}{n+1}}} \int_{\mathbb{R}^2} \frac{|y+a_l\delta^{-1}|^{\frac{2n}{n+1}}}{(1+|y|^2)^4}\,dy,$$

we have that

$$\frac{4\pi N\epsilon^2 B(W)}{(1+\sqrt{1-\epsilon^2 B(W)})^2} = 64\epsilon^2 \tilde D_{a,\delta} + o(\epsilon^2\delta^{-\frac{2}{n+1}}),$$

and there hold

$$\frac{4\pi N\epsilon^2 B(W)}{(1+\sqrt{1-\epsilon^2 B(W)})^2}\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right) = |\sigma_l'|^2 e^{U_l}\left[\frac{16\epsilon^2}{\pi N}\tilde D_{a,\delta} - \epsilon^2|\sigma_l'|^2 e^{U_l} + o(\epsilon^2\delta^{\frac{-2}{n+1}})\right] \tag{6.18}$$

in $B_\eta(\xi_l)$, $l = 1,\dots,m$, and

$$\frac{4\pi N\epsilon^2 B(W)}{(1+\sqrt{1-\epsilon^2 B(W)})^2}\left(\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}} - \frac{e^{2u_0+2W}}{\int_\Omega e^{2u_0+2W}}\right) = O(\epsilon^2\delta^{\frac{2n}{n+1}}) \tag{6.19}$$

in $\Omega \setminus \cup_{l=1}^m B_\eta(\xi_l)$. Therefore, we conclude that $\|R\|_* = O(\delta^{2-\gamma} + \|a\|^2 + \epsilon^2\delta^{-\frac{2}{n+1}})$ and (6.15) follows.

■

As mentioned in section 4, when we look for a solution of (2.2) in the form $w = W + \phi$, we are led to study (4.1). In order to state the invertibility of the linear operator L in a suitable functional setting, for $l = 1,\dots,m$ let us introduce the functions:

$$Z_{0l}(z) = \frac{\delta^2 - |\sigma_l(z)-a_l|^2}{\delta^2 + |\sigma_l(z)-a_l|^2}, \quad Z_l(z) = \frac{\delta(\sigma_l(z)-a_l)}{\delta^2 + |\sigma_l(z)-a_l|^2} \qquad z \in B_{2\eta}(\xi_l).$$

Also, let PZ_{0l} and PZ_l be the unique solutions with zero average of

$$\Delta PZ_{0l} = \chi_l \Delta Z_{0l} - \frac{1}{|\Omega|}\int_\Omega \chi_l\Delta Z_{0l}, \qquad \Delta PZ_l = \chi_l\Delta Z_l - \frac{1}{|\Omega|}\int_\Omega \chi_l \Delta Z_l$$

where $\chi_l(z) := \chi(|z-\xi_l|)$, and set $PZ_0 = \sum_{l=1}^m PZ_{0l}$. As in Propositions 4.1-4.2, it is possible to prove:

Proposition 6.3. *Let $M_0 > 0$. There exists $\eta_0 > 0$ small such that for any $0 < \delta \le \eta_0$, $|\log\delta|^2\epsilon^2 \le \eta_0\delta^{\frac{2}{n+1}}$ and $\|a\| \le M_0\delta$ there is a unique solution $\phi = \phi(\delta,a)$, $d_0 = d_0(\delta,a) \in \mathbb{R}$ and $d_l = d_l(\delta,a) \in \mathbb{C}$, $l = 1,\dots,m$, to*

$$\begin{cases} L(\phi) = -[R+N(\phi)] + d_0\Delta PZ_0 + \sum_{l=1}^m \mathrm{Re}[d_l\Delta PZ_l] & \text{in } \Omega \\ \int_\Omega \phi = \int_\Omega \phi\Delta PZ_l = 0 & l = 0,\dots,m. \end{cases}$$

Moreover, the map $(\delta,a) \mapsto \phi(\delta,a)$ is C^1 with

$$\|\phi\|_\infty \le C\delta^{2-\sigma}|\log\delta|. \tag{6.20}$$

The function $W+\phi$ is a solution of (2.2) if we adjust δ and a so to have $d_l(\delta,a)=0$ for all $l=0,1,\dots,m$. Similarly to Lemma 4.3, we have that

Lemma 6.4. *There exists $\eta_0>0$ such that if $0<\delta\le\eta_0$, $\|a\|\le\eta_0\delta$ and*

$$\int_\Omega (L(\phi)+N(\phi)+R)PZ_l=0 \tag{6.21}$$

does hold for all $l=0,\dots,m$, then $W+\phi$ is a solution of (2.2), i.e. $d_l(\delta,a)=0$ for all $l=0,\dots,m$.

Since there hold the expansions

$$PZ_0=\sum_{l=1}^m\left[\chi_l(Z_{0l}+1)-\frac{1}{|\Omega|}\int_\Omega\chi_l(Z_{0l}+1)\right]+O(\delta^2)\,,\quad PZ_l=\chi_lZ_l-\frac{1}{|\Omega|}\int_\Omega\chi_lZ_l+O(\delta)\ \ l=1,\dots,m$$

in $C(\bar\Omega)$, arguing as in Proposition 4.5, by (1.9) and (6.16)-(6.20) we can deduce the following expansion for (6.21):

Lemma 6.5. *Assume $c_0^l=0$ for all $l=1,\dots,m$ and $\|a\|\le\eta_0\delta$. The following expansions do hold as $\epsilon\to0$*

$$\begin{aligned}\int_\Omega(L(\phi)+N(\phi)+R)PZ_0 &= -8\pi D_0\delta^2+64(n+1)^{\frac{3n+5}{n+1}}\epsilon^2\delta^{-\frac{2}{n+1}}\sum_{l=1}^{m'}|\mathcal{H}^l(\xi_l)|^{-\frac{2}{n+1}}\int_{\mathbb{R}^2}\frac{(|y|^2-1)|y+\frac{a_l}{\delta}|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}dy\\ &\quad +o(\delta^2+\epsilon^2\delta^{-\frac{1}{n+1}})+O(\epsilon^4\delta^{-\frac{2}{n+1}}|\log\delta|^2+\epsilon^8\delta^{-\frac{4}{n+1}}|\log\delta|^2)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega(R+L(\phi)+N(\phi))PZ_l &= 4\pi\delta\sum_{l'=1}^m(\overline{\Upsilon^{ll'}}a_{l'}+\overline{\Gamma^{ll'}}\bar a_{l'})-64(n+1)^{\frac{3n+5}{n+1}}\epsilon^2\delta^{-\frac{2}{n+1}}|\mathcal{H}^l(\xi_l)|^{-\frac{2}{n+1}}\chi_M(l)\int_{\mathbb{R}^2}\frac{|y+\frac{a_l}{\delta}|^{\frac{2n}{n+1}}y}{(1+|y|^2)^5}dy\\ &\quad +o(\delta^2+\epsilon^2\delta^{-\frac{2}{n+1}})+O(\epsilon^4\delta^{-\frac{2}{n+1}}|\log\delta|^2+\epsilon^8\delta^{-\frac{4}{n+1}}|\log\delta|^2),\end{aligned}$$

where D_0 is defined in (1.10) and χ_M is the characteristic function of the set $M=\{1,\dots,m'\}$.

Finally, arguing as in the proof of Theorem 4.6, we can establish Theorem 1.1 thanks to $D_0<0$ and the invertibility of the matrix A.

Let us now discuss some examples with $m\ge2$. As already explained at the beginning of section 5, we can consider the case $\xi_1,\dots,\xi_m\in\Omega$ and $p_j\in\bar\Omega$ for all j. In general, it is very difficult to establish the sign of D_0 as required in (1.10). The key idea is to start from a configuration of the vortex points $\{p_1,\dots,p_N\}$ which is obtained in a periodic way by a simpler configuration having just one concentration point. In this case, (1.10) easily follows but Theorem 1.1 is not really needed. One can use Theorem 4.6 to obtain a solution with such a simpler configuration and then repeat it periodically. We then slightly move some of the vortex points in order to:

- keep zero residue of the corresponding $\mathcal{H}_0$ at each concentration point;
- break down the periodicity of the configuration.

In this way, assumption (1.10) is still valid but Theorem 4.6 is no longer applicable in the trivial way we explained above. We now really need to resort to Theorem 1.1. To exhibit some concrete examples, let us focus for simplicity on the case $m=2$ but the general situation can be dealt in the same way. Let Ω be a rectangle generated by $\omega_1=a$ and $\omega_2=ib$, $a,b>0$, and let p_1,p_2,p_3 be the three half-periods. Assume that the vortex set is $\{-\frac{p_1}{2}, \frac{p_1}{2}, 0, p_1, p_2, p_3\}$, and the concentration points are $\xi_1 = -\frac{p_1}{2}$, $\xi_2 = \frac{p_1}{2}$ with multiplicity n. Supposing that 0, p_1 have even multiplicity n_1 and p_2, p_3 have even multiplicity n_2 with $n_1 + n_2 = n+2$, we have that such a configuration is not only $\omega_1 = 2p_1$ periodic but also p_1 periodic: it can be tought as a double repetition (in a p_1-periodic way) of the vortex configuration $\{-\frac{p_1}{2}, 0, p_2\}$ in $\Omega_- := [-\frac{a}{2}, 0] \times [-\frac{b}{2}, \frac{b}{2}]$ with corresponding multiplicities n, n_1 and n_2. If n is even, it is easy to see that $\frac{d^{n+1}\mathcal{H}^i}{dz^{n+1}}(\xi_i) = 0$ for $i = 1, 2$ since the given vortex configuration is even with respect to ξ_1 and ξ_2. Notice that this is still true if we replace 0 and p_1 by $-it$ and $p_1 + it$, respectively, for $t \in \mathbb{R}$, provided they keep the same multiplicity n_1. Arguing as in (5.7), notice that D_0 can be written as

$$\pi D_0 = \text{Area } \left[\frac{1}{\sigma_0}\left(\Omega_- \setminus \sigma_0^{-1}(B_\rho(0))\right)\right] + \text{Area } \left[\frac{1}{\sigma_0}\left(\Omega_+ \setminus \sigma_0^{-1}(B_\rho(0))\right)\right] - 2(n+1)\text{Area}\left(B_{\frac{1}{\rho}}(0)\right),$$

where $\Omega_+ := [0, \frac{a}{2}] \times [-\frac{b}{2}, \frac{b}{2}]$. Since

$$u_0 + 8\pi(n+1)G(z,\xi_1) + 8\pi(n+1)G(z,\xi_2) = -4\pi n_1 \tilde{G}(z,0) - 4\pi n_2 \tilde{G}(z,p_2) + 4\pi(n+2)\tilde{G}(z,\xi_1)$$

in Ω_-, where $\tilde{G}(z,p)$ is the Green function in the torus Ω_- with pole at p, the function $\mathcal{H}_0$ can be expressed as in (5.1) in terms of the Weierstrass function of Ω_+ and the points $-\frac{p_1}{2}$, 0 and p_2. Arguing exactly as in section 5, we have that

$$\text{Area } \left[\frac{1}{\sigma_0}\left(\Omega_- \setminus \sigma_0^{-1}(B_\rho(0))\right)\right] - (n+1)\text{Area}\left(B_{\frac{1}{\rho}}(0)\right) < 0$$

provided the multiplicity n_2 for the corner of Ω_- is so that $\frac{n_2}{2}$ is odd. Arguing similarly in Ω_+, we get that $D_0 < 0$ as soon as $\frac{n_2}{2}$ is an odd number. The example then follows by replacing 0, p_1 with $-it$, $p_1 + it$ with t small for the corresponding $D_{0,t} \to D_0$ as $t \to 0$.

Appendix A: The construction of σ_a

Letting σ_0 be the solution of (2.11) of the form (2.8), where c_0 is given by (2.9), we have that $Q_0(z) = \frac{\sigma_0(z)}{z^{n+1}}$ is an holomorphic function near $z = 0$ so that $Q_0(0) = \frac{n+1}{\mathcal{H}(0)}$ (see (2.10)). Since $Q_0(0) \neq 0$, the $(n+1)-$root $Q_0^{\frac{1}{n+1}}$ of Q_0 is a well-defined holomorhpic function locally at $z = 0$, and it makes sense to define $q_0(z) = zQ_0^{\frac{1}{n+1}}(z)$ near $z = 0$.

For $\sigma \in \mathcal{B}_r$, where $\mathcal{B}_r$ is given in (2.14), in a similar way we have that $Q(z) = \frac{\sigma(z)}{z^{n+1}}$ is an holomorphic function near $z = 0$ with $|\frac{Q(z)}{Q_0(z)} - 1| \leq r$ for all z. Since in particular $|Q(z) - \frac{n+1}{\mathcal{H}(0)}| \leq r|Q_0(z)| + |Q_0(z) - \frac{n+1}{\mathcal{H}(0)}|$, we can find r and $\eta > 0$ small so that $q(z) = zQ^{\frac{1}{n+1}}(z)$ is a well-defined holomorphic function in $B_{3\eta}(0)$ for all $\sigma \in \mathcal{B}_r$, with $\sigma(z) = q^{n+1}(z)$ for all $z \in B_{3\eta}(0)$. Since $q'(0) = Q^{\frac{1}{n+1}}(0)$ satisfies $|q'(0)| \geq [\frac{(1-r)(n+1)}{|\mathcal{H}(0)|}]^{\frac{1}{n+1}} > 0$, then q is locally bi-holomorphic at 0. In order to have uniform invertibility of q for all $\sigma \in \mathcal{B}_r$, let us evaluate the following quantity:

$$\begin{aligned} |1 - \frac{q'(z)}{q'(0)}| &\leq \frac{\sup_{B_\eta(0)} |q''|}{|q'(0)|}|z| \leq \frac{2}{\eta^2}[\frac{(1-r)(n+1)}{|\mathcal{H}(0)|}]^{-\frac{1}{n+1}} \left(\sup_{B_{2\eta}(0)} |q|\right)|z| \\ &\leq \frac{2}{\eta^2}\left(\frac{|\mathcal{H}(0)|}{n+1}\right)^{\frac{1}{n+1}} (\frac{1+r}{1-r})^{\frac{1}{n+1}} \left(\sup_{B_{2\eta}(0)} |q_0|\right)|z| \end{aligned}$$

for all $z \in B_\eta(0)$, in view of the Cauchy's inequality and $|\frac{\sigma(z)}{\sigma_0(z)} - 1| = |\frac{q^{n+1}(z)}{q_0^{n+1}(z)} - 1| \le r$ for all $z \in B_{3\eta}(0)$. Therefore, we can find ρ_1 small so that $|1 - \frac{q'(z)}{q'(0)}| \le \frac{1}{2}$ for all $z \in B_{\rho_1^{\frac{1}{n+1}}}(0)$ and $2\rho_1^{\frac{1}{n+1}}|Q(0)|^{-\frac{1}{n+1}} \le 2\rho_1^{\frac{1}{n+1}}[\frac{|\mathcal{H}(0)|}{n+1}]^{\frac{1}{n+1}}(1-r)^{-\frac{1}{n+1}} \le 2\eta$, uniformly for $\sigma \in \mathcal{B}_r$. Hence, the inverse map q^{-1} of q is defined from $B_{\rho_1^{\frac{1}{n+1}}}(0)$ into $B_{2\rho_1^{\frac{1}{n+1}}|Q(0)|^{-\frac{1}{n+1}}}(0)$: for all $y \in B_{\rho_1^{\frac{1}{n+1}}}(0)$ there exists a unique $z \in B_{2\rho_1^{\frac{1}{n+1}}|Q(0)|^{-\frac{1}{n+1}}}(0)$ so that $q(z) = y$, given by $z = q^{-1}(y)$. Since $\sigma = q^{n+1}$ in $B_{3\eta}(0)$, we have that

$$\text{Card}\,\{z \in B_{2\rho_1^{\frac{1}{n+1}}|Q(0)|^{-\frac{1}{n+1}}}(0) : \sigma(z) = y\} = n+1 \qquad \forall\, y \in B_{\rho_1}(0) \setminus \{0\},$$

for all $\sigma \in \mathcal{B}_r$. Since

$$|\sigma(z)| \ge (1-r) \inf_{\tilde\Omega \setminus B_{2\rho_1^{\frac{1}{n+1}}|Q(0)|^{-\frac{1}{n+1}}}(0)} |\sigma_0(z)| \ge (1-r) \inf_{\tilde\Omega \setminus B_{2\rho_1^{\frac{1}{n+1}}[\frac{|\mathcal{H}(0)|}{n+1}]^{\frac{1}{n+1}}(1+r)^{-\frac{1}{n+1}}}(0)} |\sigma_0(z)| > 0$$

for all $z \in \tilde\Omega \setminus B_{2\rho_1^{\frac{1}{n+1}}|Q(0)|^{-\frac{1}{n+1}}}(0)$ we can find ρ $(\le \rho_1)$ small so that

$$\text{Card}\,\{z \in \tilde\Omega : \sigma(z) = y\} = \text{Card}\,\{z \in B_{2\rho_1^{\frac{1}{n+1}}|Q(0)|^{-\frac{1}{n+1}}}(0) : \sigma(z) = y\} = n+1 \qquad \forall\, y \in B_\rho(0) \setminus \{0\},$$

for all $\sigma \in \mathcal{B}_r$. Since

$$\sigma^{-1}(B_\rho(0)) \subset B_{2\rho_1^{\frac{1}{n+1}}|Q(0)|^{-\frac{1}{n+1}}}(0) \subset B_{2\rho_1^{\frac{1}{n+1}}[\frac{|\mathcal{H}(0)|}{n+1}]^{\frac{1}{n+1}}(1-r)^{-\frac{1}{n+1}}}(0) \subset B_{2\eta}(0),$$

for all $z \in \partial\sigma^{-1}(B_\rho(0)) = \sigma^{-1}(\partial B_\rho(0))$ and $\sigma \in \mathcal{B}_r$ we have that

$$\frac{|z|^{n+1}}{\rho} = \frac{|z|^{n+1}}{|\sigma(z)|} = \frac{1}{|Q(z)|} \ge \frac{1}{(1+r)} \inf_{B_{2\eta}(0)} |Q_0(z)|^{-1} > 0$$

for q_0 is well-defined in $B_{3\eta}(0)$. We can summarize the above discussion as follows:

Lemma A.1. *There exist $r, \rho > 0$ such that $q(z) = zQ(z)^{\frac{1}{n+1}}$ is a locally bi-holomorphic map with $\sigma = q^{n+1}$ and inverse q^{-1} defined on $B_{\rho^{\frac{1}{n+1}}}(0)$, for all $\sigma \in \mathcal{B}_r$. In particular, there exists a neighborhhod V of 0 so that, for all $\sigma \in \mathcal{B}_r$, there hold $V \subset \sigma^{-1}(B_\rho(0))$ and $\sigma : \sigma^{-1}(B_\rho(0)) \to B_\rho(0)$ is a $(n+1)-1$ map in the following sense:*

$$\textit{Card}\,\{z \in \tilde\Omega : \sigma(z) = y\} = n+1 \qquad \forall\, y \in B_\rho(0) \setminus \{0\}.$$

For $|a| < \rho$ and $\sigma \in \mathcal{B}_r$, by Lemma A.1 we have that

$$\sigma^{-1}(a) = \{z \in \tilde\Omega :\ \sigma(z) = a\} = \{a_0, \dots, a_n\},$$

where $a_k = q^{-1}(\hat a_k)$ and $\hat a_k$, $k = 0, \dots, n$, are the $(n+1)-$roots of a, and then $g_{a,\sigma}(z) := \dfrac{\sigma(z) - a}{\prod_{k=0}^n (z - a_k)} \in \mathcal{M}(\overline{\Omega})$ is a non-vanishing function. We are now in position to prove the following.

Lemma A.2. *Up to take ρ smaller, there exists a C^1-map $a \in B_\rho(0) \to \sigma_a \in \mathcal{B}_r$ so that σ_a solves (2.15)-(2.16). Moreover, the map $a \in B_\rho(0) \to c_a = c_{a,\sigma_a}$ is C^1 with*

$$\Gamma := \mathcal{H}(0)\partial_a c_a\Big|_{a=0} = \frac{1}{n!}\frac{d^{n+1}}{dz^{n+1}}\Big[\mathcal{H}(z) f_{n+1}(z)\Big](0)$$

$$\Upsilon := \mathcal{H}(0)\partial_{\bar a} c_a\Big|_{a=0} = -\frac{2\pi(n+1)}{|\Omega| n!}\overline{b_{n+1}}\,\frac{d^n\mathcal{H}}{dz^n}(0),$$

where

$$f_{n+1}(z)=\frac{1}{(n+1)!}\frac{d^{n+1}}{dw^{n+1}}\left[2\log\frac{w-q_0(z)}{q_0^{-1}(w)-z}+4\pi H^*(z-q_0^{-1}(w))\right](0)\,,\qquad b_{n+1}=\frac{1}{(n+1)!}\frac{d^{n+1}q_0^{-1}}{dw^{n+1}}(0).$$

<u>Proof:</u> Given $c_{a,\sigma}$ as in (2.16), equation (2.15) is equivalent to find zeroes of the map $\Lambda:(a,\sigma)\in B_\rho(0)\times\mathcal{B}_r\to\mathcal{M}(\overline{\Omega})$ given as

$$\Lambda(a,\sigma)=\sigma(z)+\left[\int^z\frac{g^2_{a,\sigma}(w)}{g^2_{0,\sigma}(w)}\frac{\mathcal{H}_{a,\sigma}(w)}{w^{n+2}}e^{-c_{a,\sigma}w^{n+1}}dw\right]^{-1}.$$

Observe that the zeroes $a_k=a_k(a,\sigma)=q^{-1}(\hat a_k)$ are continuously differentiable in σ. Differentiating the relation $\sigma(a_k)=a$ at σ_0 along a direction $R\in\mathcal{M}'(\overline{\Omega})$, we have that $\sigma_0'(a_k(a,\sigma_0))\partial_\sigma a_k(a,\sigma_0)[R]+R(a_k(a,\sigma_0))=0$. Since $\sigma_0'(a_k)\sim a_k^n$ and $R(a_k)\sim a_k^{n+1}$ in view of $\|R\|<\infty$, we get that $\partial_\sigma a_k(0,\sigma_0)[R]=0$ for all $R\in\mathcal{M}'(\overline{\Omega})$. For $z\neq0$ the function $\frac{g_{a,\sigma}(z)}{g_{0,\sigma}(z)}$ is continuously differentiable in σ with

$$\partial_\sigma\left(\frac{g_{a,\sigma}(z)}{g_{0,\sigma}(z)}\right)[R]=a\frac{z^{n+1}}{\prod_{k=0}^n(z-a_k)}\frac{R(z)}{\sigma^2(z)}+\frac{\sigma(z)-a}{\prod_{k=0}^n(z-a_k)}\frac{z^{n+1}}{\sigma(z)}\sum_{j=0}^n\frac{1}{z-a_j}\partial_\sigma a_j(a,\sigma)[R]$$

for every $R\in\mathcal{M}'(\overline{\Omega})$. In particular, we get that $\partial_\sigma\left(\frac{g_{a,\sigma}(z)}{g_{0,\sigma}(z)}\right)\Big|_{a=0}[R]=0$ for every $z\neq0$ and $R\in\mathcal{M}'(\overline{\Omega})$. Since we can write $\frac{g_{a,\sigma}(z)}{g_{0,\sigma}(z)}$ as

$$\frac{g_{a,\sigma}(z)}{g_{0,\sigma}(z)}=\frac{z^{n+1}}{\sigma(z)}\prod_{k=0}^n\frac{q(z)-q(a_k)}{z-a_k}=\frac{z^{n+1}}{\sigma(z)}\prod_{k=0}^n\int_0^1q'(a_k+t(z-a_k))dt \tag{A.1}$$

for z small in view of $\sigma=q^{n+1}$, we get that $\frac{g_{a,\sigma}(z)}{g_{0,\sigma}(z)}$ is continuously differentiable in σ and the linear operator $\partial_\sigma\left(\frac{g_{a,\sigma}(z)}{g_{0,\sigma}(z)}\right)$ is continuous at $z=0$. In particular, we get that $\partial_\sigma\left(\frac{g_{a,\sigma_0}(z)}{g_{0,\sigma_0}(z)}\right)\Big|_{a=0}[R]=0$ for every z and $R\in\mathcal{M}'(\overline{\Omega})$. By (2.12) we have that $\mathcal{H}_{a,\sigma}$ is continuously differentiable in σ with $\partial_\sigma\mathcal{H}_{0,\sigma}[R]=0$ for every $R\in\mathcal{M}'(\overline{\Omega})$. We have that $c_{a,\sigma}$ is also continuosuly differentiable in σ with $\partial_\sigma c_{0,\sigma_0}[R]=0$ for every $R\in\mathcal{M}'(\overline{\Omega})$, and so $\Lambda(a,\sigma)$ is with $\partial_\sigma\Lambda(0,\sigma_0)=\mathrm{Id}$.

Since $a_k\sim|a|^{\frac{1}{n+1}}$, the smooth dependence in a is much more delicate, and will be true just for symmetric expressions of the a_k's thanks to the symmetries of $\hat a_k=q(a_k)$. To fully exploit the symmetries, it is crucial that the expression (2.12) of $\mathcal{H}_{a,\sigma}$ is in terms of an holomorphic function H^*. Indeed, we have that

$$\begin{aligned}2\sum_{k=0}^nH^*(z-a_k)-\frac{z}{|\Omega|}\overline{\sum_{k=0}^na_k}&=2\sum_{l=0}^\infty g_l(z)\sum_{k=0}^n\hat a_k^l-\frac{z}{|\Omega|}\overline{\sum_{l=1}^\infty b_l\sum_{k=0}^n\hat a_k^l}\\&=2(n+1)\sum_{l=0}^\infty g_{(n+1)l}(z)a^l-\frac{n+1}{|\Omega|}z\overline{\sum_{l=1}^\infty b_{(n+1)l}a^l}\end{aligned}$$

in view of $\displaystyle\sum_{k=0}^n\hat a_k^l=0$ for all $l\notin(n+1)\mathbb{N}$, where $g_l(z)=\frac{1}{l!}\frac{d^l}{dw^l}[H^*(z-q^{-1}(w))](0)$ and $b_l=\frac{1}{l!}\frac{d^lq^{-1}}{dw^l}(0)$ (recall that $b_0=q^{-1}(0)=0$). Since for z small there holds

$$\sum_{k=0}^n\log\frac{q(z)-q(a_k)}{z-a_k}=\sum_{l=0}^\infty h_l(z)\sum_{k=0}^n\hat a_k^l=(n+1)\sum_{l=0}^\infty h_{(n+1)l}(z)a^l$$

in view of $a_k = q^{-1}(\hat a_k)$, where $h_l(z) = \frac{1}{l!}\frac{d^l}{dw^l}\left[\log \frac{w-q(z)}{q^{-1}(w)-z}\right](0)$, we have that $\frac{g_{a,\sigma}(z)}{g_{0,\sigma}(z)}$ is continuously differentiable in a, $\bar a$ for all z in view of (A.1) (for z far from 0 it is obvious). Hence, by (2.12) $\frac{g^2_{a,\sigma}}{g^2_{0,\sigma}}\mathcal{H}_{a,\sigma}$, $c_{a,\sigma}$ and $\Lambda(a,\sigma)$ are continuously differentiable also in a, $\bar a$, and then Λ is a C^1-map with $\Lambda(0,\sigma_0)=0$, $\partial_\sigma \Lambda(0,\sigma_0)=\mathrm{Id}$. Up to take ρ smaller, by the Implicit Function Theorem we find a C^1-map $a \in B_\rho(0) \to \sigma_a$ so that $\Lambda(a,\sigma_a)=0$, and the function $a \to c_a = c_{a,\sigma_a}$ is C^1. By

$$\partial_a[\frac{g^2_{a,\sigma}(z)g^2_{0,\sigma}(0)}{g^2_{a,\sigma}(0)g^2_{0,\sigma}(z)}\frac{\mathcal{H}_{a,\sigma}(z)}{\mathcal{H}_{a,\sigma}(0)}](0) = \frac{g^2_{0,\sigma}(0)}{g^2_{0,\sigma}(z)}\partial_a[e^{2\log g_{a,\sigma}(z)-2\log g_{a,\sigma}(0)}\frac{\mathcal{H}_{a,\sigma}(z)}{\mathcal{H}_{a,\sigma}(0)}](0) = (n+1)\frac{\mathcal{H}(z)}{\mathcal{H}(0)}[f_{n+1}(z)-f_{n+1}(0)]$$

and

$$\partial_{\bar a}[\frac{g^2_{a,\sigma}(z)g^2_{0,\sigma}(0)}{g^2_{a,\sigma}(0)g^2_{0,\sigma}(z)}\frac{\mathcal{H}_{a,\sigma}(z)}{\mathcal{H}_{a,\sigma}(0)}](0) = -\frac{2\pi(n+1)}{|\Omega|}\frac{\mathcal{H}(z)}{\mathcal{H}(0)}\overline{b_{n+1}z}$$

we deduce the desired expression for Γ and Υ in view of $\partial_\sigma c_{0,\sigma_0}=0$ and (4.13). ∎

Appendix B: The linear theory

In this section, we will prove the invertibility of the linear operator L given by (4.3) under suitable orthogonality conditions. The operator L can be described asymptotically by the following linear operator in $\mathbb{R}^2$

$$L_0(\phi) = \Delta\phi + \frac{8(n+1)^2|y|^{2n}}{(1+|y^{n+1}-\zeta_0|^2)^2}\phi,$$

where $\zeta_0 = \lim \frac{a}{\delta}$. When $\zeta_0 = 0$, as in the case $n=0$ [4], by using a Fourier decomposition of ϕ it can be shown in a rather direct way that the bounded solutions of $L_0(\phi)=0$ in $\mathbb{R}^2$ are precisely linear combinations of

$$Y_0(y) = \frac{1-|y|^{2n+2}}{1+|y|^{2n+2}} \qquad \text{and} \qquad Y_l(y) = \frac{(y^{n+1})_l}{1+|y|^{2n+2}},\ l=1,2.$$

Note that L_0 is the linearized operator at the radial solution $U=U_{1,0}$ of $-\Delta U = |z|^{2n}e^U$.

For the linearized operator at U_{1,ζ_0} with $\zeta_0 \neq 0$, the Fourier decomposition is useless since U_{1,ζ_0} is not radial w.r.t. any point if $n \geq 1$. However, the same property is still true as recently proved in [15], and the argument below could be carried out in full generality in the range $a = O(\delta)$. Since in Theorem 4.6 we are concerned with the case $a = o(\delta)$, for simplicity we will discuss the linear theory just in this case.

Recall that

$$Z_0(z) = \frac{\delta^2 - |\sigma(z)-a|^2}{\delta^2+|\sigma(z)-a|^2}, \qquad Z_l(z) = \frac{\delta[\sigma(z)-a]_l}{\delta^2+|\sigma-a|^2}, \qquad l=1,2,$$

and PZ_l, $l=0,1,2$, denotes the projection of Z_l onto the doubly-periodic functions with zero average:

$$\begin{cases} \Delta PZ_l = \Delta Z_l - \frac{1}{|\Omega|}\int_\Omega \Delta Z_l & \text{in } \Omega \\ \int_\Omega PZ_l = 0. \end{cases}$$

Given $h \in L^\infty(\Omega)$ with $\int_\Omega h = 0$, consider the problem of finding a function ϕ in Ω with zero average and numbers d_l, $l=0,1,2$, such that

$$\begin{cases} L(\phi) = h + \sum_{l=0}^{2} d_l \Delta PZ_l & \text{in } \Omega \\ \int_\Omega \Delta PZ_l \phi = 0 & \forall\, l=0,1,2. \end{cases} \tag{B.1}$$

Since $Z = Z_1 + iZ_2$, observe that (B.1) is equivalent to solve (4.4) with $d = d_1 - id_2$. Let us stress that the orthogonality conditions in (B.1) are taken with respect to the elements of the approximate kernel due to translations and to an extra element which involves dilations. A similar situation already appears in [13].

First, we will prove an a-priori estimate for problem (B.1) when $d_l = 0$ for all $l = 0, 1, 2$, w.r.t. the $\|\cdot\|_*$-norm defined as

$$\|h\|_* = \sup_{z\in\Omega} \frac{(\delta^2 + |\sigma(z) - a|^2)^{1+\gamma/2}}{\delta^\gamma(|\sigma'(z)|^2 + \delta^{\frac{2n}{n+1}})}|h(z)|,$$

where $0 < \gamma < 1$ is a small fixed constant.

Proposition B.1. *There exist $\eta_0 > 0$ small and $C > 0$ such that for any $0 < \delta \le \eta_0$, $\epsilon^2 \le \eta_0 \delta^{\frac{2}{n+1}}$, $|a| \le \eta_0 \delta$ and any solution ϕ to*

$$\begin{cases} L(\phi) = h & \text{in } \Omega \\ \int_\Omega \Delta P Z_l \phi = 0 & \forall\, l = 0, 1, 2 \\ \int_\Omega \phi = 0, \end{cases} \tag{B.2}$$

one has

$$\|\phi\|_\infty \le C \log \frac{1}{\delta} \|h\|_*. \tag{B.3}$$

Proof: The proof of estimate (B.3) consists of several steps. Assume by contradiction the existence of sequences $\delta_k \to 0$, ϵ_k with $\epsilon_k^2 = o(\delta_k^{\frac{2}{n+1}})$, a_k with $a_k = o(\delta_k)$, functions h_k with $|\log \delta_k|\, \|h_k\|_* = o(1)$ as $k \to +\infty$, and solutions ϕ_k of (B.2) with $\|\phi_k\|_\infty = 1$. Since by (4.3) the operator L acts as $L(\phi) = \Delta\phi + \mathcal{K}\,[\phi + \gamma(\phi)]$, where $\gamma(\phi) \in \mathbb{R}$, the function $\psi_k = \phi_k + \gamma(\phi_k)$ does solve

$$\begin{cases} \Delta\psi_k + \mathcal{K}_k \psi_k = h_k & \text{in } \Omega \\ \int_\Omega \Delta P Z_{k,l} \psi_k = 0 & \forall\, l = 0, 1, 2, \end{cases}$$

where W_k, $\mathcal{K}_k$, $Z_{k,l}$ denote the functions W, $\mathcal{K}$, Z_l, respectively, along the given sequence.

Claim 1. $\liminf\limits_{k\to+\infty} \|\psi_k\|_\infty > 0$ *and, up to a subsequence, $\psi_k \to \tilde{c} \in \mathbb{R}$ as $k \to +\infty$ in $C^{1,\alpha}_{loc}(\bar{\Omega} \setminus \{0\})$, for all $\alpha \in (0, 1)$.*

Indeed, assume by contradiction that $\liminf\limits_{k\to+\infty} \|\psi_k\|_\infty = 0$. Up to a subsequence, assume that $\|\psi_k\|_\infty = \|\phi_k + \gamma(\phi_k)\|_\infty \to 0$ as $k \to +\infty$. Since $\epsilon_k^2 = o(\delta_k^{\frac{2}{n+1}})$, by (2.49) it follows that

$$\gamma(\phi_k) = -\frac{\int_\Omega e^{u_0 + W_k}\phi_k}{\int_\Omega e^{u_0+W_k}} + o(1) = O(1).$$

Up to a subsequence we have that $\frac{\int_\Omega e^{u_0+W_k}\phi_k}{\int_\Omega e^{u_0+W_k}} \to c$, and then $\phi_k \to c$ uniformly in Ω as $k \to +\infty$. Since $\int_\Omega \phi_k = 0$, we get $c = 0$ and $\phi_k \to 0$ in $L^\infty(\Omega)$, in contradiction with $\|\phi_k\|_\infty = 1$. Moreover, since $\|\psi_k\|_\infty = O(1)$, by (2.51)-(2.52) we have that $\Delta\psi_k = o(1)$ in $C_{\text{loc}}(\bar{\Omega} \setminus \{0\})$. Up to a subsequence, we have that $\psi_k \to \psi$ as $k \to +\infty$ in $C^{1,\alpha}_{\text{loc}}(\bar{\Omega} \setminus \{0\})$. Since $\|\psi_k\|_\infty = O(1)$, ψ is a bounded function which can be extended to a harmonic doubly-periodic function in Ω. Therefore, $\psi = \tilde{c}$ in Ω with $\tilde{c} = \lim\limits_{k\to+\infty} \gamma(\phi_k)$, since $\frac{1}{|\Omega|}\int_\Omega \psi_k = \gamma(\phi_k)$.

Now, consider the function $\Psi_k(y) = \psi_k(\delta_k^{\frac{1}{n+1}} y)$. Then, Ψ_k satisfies

$$\Delta\Psi_k + K_k(y)\Psi_k = \hat{h}_k(y) \qquad \text{in } \delta_k^{-\frac{1}{n+1}}\Omega,$$

where $K_k(y)=\delta_k^{\frac{2}{n+1}}\mathcal{K}_k(\delta_k^{\frac{1}{n+1}}y)$ and $\hat h_k(y)=\delta_k^{\frac{2}{n+1}}h_k(\delta_k^{\frac{1}{n+1}}y)$. Also, we set $\sigma_k(y)=\delta_k^{-1}\sigma_{a_k}(\delta_k^{\frac{1}{n+1}}y)$ for y in compact subsets of $\mathbb{R}^2$.

Claim 2. *$\Psi_k\to\Psi=0$ in $C_{loc}(\mathbb{R}^2)$ as $k\to+\infty$.*

Indeed, observe that by (2.49) and (2.51)-(2.52) we have the following expansions:

$$\mathcal{K}(z)=|\sigma'(z)|^2e^{U_{\delta,a}}\left[1+O(|c_a||z|^{n+1})+O(|c_a||a|+\delta^2|\log\delta|)\right]+O(\epsilon^2|\sigma'(z)|^4e^{2U_{\delta,a}}). \qquad \text{(B.4)}$$

Since $\epsilon_k^2=o(\delta_k^{\frac{2}{n+1}})$, the first estimate above re-writes along our sequence as

$$K_k(y)=(1+o(1)+O(\delta_k|y|^{n+1}))\frac{8|\sigma_k'(y)|^2}{\left(1+\left|\sigma_k(y)-a_k\delta_k^{-1}\right|^2\right)^2}+o(1)\frac{64|\sigma_k'(y)|^4}{\left(1+\left|\sigma_k(y)-a_k\delta_k^{-1}\right|^2\right)^4}$$

uniformly in $\delta_k^{-\frac{1}{n+1}}\Omega$ as $k\to+\infty$. Since $\sigma=z^{n+1}Q$, we have that $\sigma_k(y)=y^{n+1}Q_{a_k}(\delta_k^{\frac{1}{n+1}}y)$ and $\sigma_k'(y)=(n+1)y^nQ_{a_k}(\delta_k^{\frac{1}{n+1}}y)+\delta_k^{\frac{1}{n+1}}y^{n+1}Q'_{a_k}(\delta_k^{\frac{1}{n+1}}y)$. Since $Q_{a_k}(0)\to\frac{n+1}{\mathcal{H}(0)}=:\gamma\neq0$ and $\|Q'_{a_k}\|_{\infty,\Omega}\le C\|Q_{a_k}\|_{\infty,\tilde\Omega}\le C'$, we have that

$$\sigma_k(y)=y^{n+1}[\gamma+o(1)+O(\delta_k^{\frac{1}{n+1}}|y|)]\,,\qquad \sigma_k'(y)=(n+1)y^n[\gamma+o(1)+O(\delta_k^{\frac{1}{n+1}}|y|)]$$

as $k\to+\infty$. Then we get that

$$K_k(y)=\left[\frac{8(n+1)^2\gamma|^2|y|^{2n}}{\left(1+\left|\sigma_k(y)-a_k\delta_k^{-1}\right|^2\right)^2}+o(1)\frac{64(n+1)^4|\gamma|^4|y|^{4n}}{\left(1+\left|\sigma_k(y)-a_k\delta_k^{-1}\right|^2\right)^4}\right][1+o(1)+O(\delta_k^{\frac{1}{n+1}}|y|)] \qquad \text{(B.5)}$$

uniformly in $\delta_k^{-\frac{1}{n+1}}\Omega$. Choose η small so that $|\sigma_k(y)|\ge\frac{|\gamma|}{2}|y|^{n+1}$ in $B_{\delta_k^{-\frac{1}{n+1}}\eta}(0)$ for k large. Since $\|\Psi_k\|_\infty=O(1)$ and $|\hat h_k(y)|\le C\|h_k\|_*\to0$ on compact sets, by elliptic estimates and (B.5) we get that $\Psi_k(\gamma^{-\frac{1}{n+1}}y)\to\hat\Psi$ in $C_{\rm loc}(\mathbb{R}^2)$ as $k\to+\infty$, where $\hat\Psi$ is a bounded solution of $L_0(\hat\Psi)=0$ (with $\zeta_0=0$). Then $\hat\Psi(y)=\sum_{j=0}^{2}b_jY_j(y)$ for some $b_j\in\mathbb{R}$, $j=0,1,2$.

Since $\Delta Z_{k,l}+|\sigma_k'|^2e^{U_{\delta_k,a_k}}Z_{k,l}=0$ for $l=0,1,2$ (where U_{δ_k,a_k} stands for $U_{\delta_k,a_k,\sigma_{a_k}}$), for $l=1,2$ we have that

$$\int_\Omega\psi_k\Delta Z_{k,l}=-\int_\Omega|\sigma_k'(z)|^2\psi_ke^{U_{\delta_k,a_k}}Z_{k,l}=-\int_{B_{\delta_k^{-\frac{1}{n+1}}\eta}(0)}\frac{8|\sigma_k'(z)|^2(\sigma_k-a_k\delta_k^{-1})\Psi_k}{(1+|\sigma_k-a_k\delta_k^{-1}|^2)^3}\,dy+O(\delta_k^3).$$

Since for all $l=0,1,2$

$$0=\int_\Omega\psi_k\Delta PZ_{k,l}=\int_\Omega\psi_k\left[\Delta Z_{k,l}-\frac{1}{|\Omega|}\int_\Omega\Delta Z_{k,l}\right]=\int_\Omega\psi_k\Delta Z_{k,l}+o(1)$$

as $k\to\infty$ in view of (3.1)-(3.2), by dominated convergence we get that

$$\int_{\mathbb{R}^2}\hat\Psi(y)\,\frac{|y|^{2n}(y^{n+1})_l}{(1+|y|^{2n+2})^3}\,dy=0\qquad\text{for } l=1,2,$$

and we conclude that $b_1 = b_2 = 0$. Similarly, for $l = 0$ we deduce that

$$\int_{\mathbb{R}^2} \hat{\Psi}(y) \frac{|y|^{2n}(1-|y|^{2n+2})}{(1+|y|^{2n+2})^3}\, dy = 0,$$

which implies that $b_0 = 0$. Thus, the claim follows.

On the other hand, from the equation of ψ_k we have the following integral representation

$$\psi_k(z) = \frac{1}{|\Omega|}\int_\Omega \psi_k + \int_\Omega G(y,z)\left[\mathcal{K}_k(y)\psi_k(y) - h_k(y)\right]\, dy. \tag{B.6}$$

Claim 3. $\tilde{c} = 0$

Indeed, Claims 1 and 2 imply that $\psi_k(0) = \Psi_k(0) \to 0$ and $\frac{1}{|\Omega|}\int_\Omega \psi_k = \gamma(\phi_k) \to \tilde{c}$ as $k \to +\infty$ by definition. So, by (B.6) we deduce that

$$\int_\Omega G(y,0)\left[\mathcal{K}_k(y)\psi_k(y) - h_k(y)\right]\, dy \to -\tilde{c}$$

as $k \to +\infty$. Now, we first estimate the integral involving h_k. Since $\int_{B_{\delta_k}(0)} |\log|y||\, dy = O(\delta_k^2 \log \delta_k)$, we get that

$$\left|\int_{B_{\delta_k}(0)} G(y,0)h_k(y)dy\right| \le \frac{C}{\delta_k^2}\|h_k\|_* \int_{B_{\delta_k}(0)} G(y,0)dy \le C|\log\delta_k|\|h_k\|_*.$$

By (3.6) we have that

$$\left|\int_{\Omega\setminus B_{\delta_k}(0)} G(y,0)h_k(y)dy\right| \le C|\log\delta_k| \int_\Omega |h_k| \le C|\log\delta_k|\|h_k\|_*,$$

and we conclude that

$$\left|\int_\Omega G(y,0)h_k(y)dy\right| \le C|\log\delta_k|\|h_k\|_* \to 0$$

in view of $|\log\delta_k|\,\|h_k\|_* = o(1)$ as $k \to +\infty$. By (B.4) we have that

$$\begin{aligned}
&\int_\Omega G(y,0)\mathcal{K}_k(y)\psi_k(y)dy = \int_{B_\eta(0)} G(y,0)\mathcal{K}_k(y)\psi_k(y)dy + O(\delta_k^2)\\
&= \int_{B_{\delta_k^{-\frac{1}{n+1}}\eta}(0)} \left[-\frac{1}{2\pi}\log|y| - \frac{1}{2\pi(n+1)}\log\delta_k + H(\delta_k^{\frac{1}{n+1}}y,0)\right] K_k(y)\Psi_k(y)dy + O(\delta_k^2).
\end{aligned}$$

Since by (B.5) $K_k = O(\frac{|y|^{2n}}{(1+|y|^{2n+2})^2})$ does hold uniformly in $B_{\delta_k^{-\frac{1}{n+1}}\eta}(0) \setminus B_1(0)$ and $K_k(y) \to \frac{8(n+1)^2|y|^{2n}}{(1+|y|^{2n+2})^2}$ as $k \to +\infty$, by dominated convergence we get that

$$\begin{aligned}
&\int_{B_{\delta_k^{-\frac{1}{n+1}}\eta}(0)} \left[-\frac{1}{2\pi}\log|y| + H(\delta_k^{\frac{1}{n+1}}y,0)\right] K_k(y)\Psi_k(y)dy\\
&\to \int_{\mathbb{R}^2} \left[-\frac{1}{2\pi}\log|y| + H(0,0)\right] \frac{8(n+1)^2|y|^{2n}}{(1+|y|^{2n+2})^2}\Psi(y)dy = 0
\end{aligned}$$

as $k \to +\infty$. Since $\int_\Omega h_k = 0$, the integration of the equation satisfied by ψ_k gives that $\int_\Omega \mathcal{K}_k \psi_k = 0$. Then, by (B.4) we get that

$$\int_{B_{\delta_k^{-\frac{1}{n+1}}\eta}(0)} K_k \Psi_k dy = \int_{B_\eta(0)} \mathcal{K}_k \psi_k dy = -\int_{\Omega \setminus B_\eta(0)} \mathcal{K}_k \psi_k = O(\delta_k^2),$$

which implies that

$$\log \delta_k \int_{B_{\delta_k^{-\frac{1}{n+1}}\eta}(0)} K_k \Psi_k dy = O(\delta_k^2 \log \delta_k).$$

In conclusion, we have shown that $\int_\Omega G(y,0)\mathcal{K}_k(y)\psi_k(y)dy \to 0$ as $k \to +\infty$, yielding to $\tilde{c} = 0$.

In the following Claims, we will omit the subscript k. Let us denote $\tilde{L}(\psi) = \Delta\psi + \mathcal{K}\psi$.

Claim 4. *The operator $\tilde{L}$ satisfies the maximum principle in $B_\eta(0) \setminus B_{R\delta^{\frac{1}{n+1}}}(0)$ for R large enough.*

Indeed, as already noticed in the proof of the previous Claim in terms of K_k, there is $C_1 > 0$ such that

$$\mathcal{K}(z) \le C_1 \frac{(n+1)^2 \delta^2 |z|^{2n}}{(\delta^2 + |z|^{2n+2})^2} \tag{B.7}$$

in $B_\eta(0) \setminus B_{\delta^{\frac{1}{n+1}}}(0)$. The function

$$\tilde{Z}(z) = -Y_0\left(\frac{\mu z}{\delta^{\frac{1}{n+1}}}\right) = \frac{\mu^{2n+2}|z|^{2n+2} - \delta^2}{\mu^{2n+2}|z|^{2n+2} + \delta^2}$$

satisfies

$$-\Delta \tilde{Z}(z) = 16(n+1)^2 \frac{\delta^2 \mu^{2n+2}|z|^{2n}(\mu^{2n+2}|z|^{2n+2} - \delta^2)}{(\mu^{2n+2}|z|^{2n+2} + \delta^2)^3}.$$

For R large so that $\mu^{2n+2}R^{2n+2} > \frac{5}{3}$ we have that

$$\begin{aligned}
-\Delta \tilde{Z}(z) &\ge 16(n+1)^2 \frac{\delta^2 \mu^{2n+2}|z|^{2n}}{(\mu^{2n+2}|z|^{2n+2} + \delta^2)^2} \, \frac{\mu^{2n+2}R^{2n+2} - 1}{\mu^{2n+2}R^{2n+2} + 1} \\
&\ge 4(n+1)^2 \frac{\delta^2 \mu^{2n+2} R^{4n+4}}{(\mu^{2n+2}R^{2n+2} + 1)^2} \, \frac{1}{|z|^{2n+4}} \ge \frac{(n+1)^2}{\mu^{2n+2}} \frac{\delta^2}{|z|^{2n+4}}
\end{aligned}$$

in $B_\eta(0) \setminus B_{R\delta^{\frac{1}{n+1}}}(0)$. On the other hand, since $\tilde{Z} \le 1$ we have that

$$\mathcal{K}(z)\tilde{Z}(z) \le C_1 \frac{(n+1)^2 \delta^2 |z|^{2n}}{(\delta^2 + |z|^{2n+2})^2} \le C_1 \frac{(n+1)^2 \delta^2}{|z|^{2n+4}}$$

in $B_\eta(0) \setminus B_{\delta^{\frac{1}{n+1}}}(0)$, and for $0 < \mu < \frac{1}{\sqrt{C_1}}$ we then get that

$$\tilde{L}(\tilde{Z}) \le \left(-\frac{1}{\mu^{2n+2}} + C_1\right) \frac{(n+1)^2 \delta^2}{|z|^{2n+4}} < 0$$

in $B_\eta(0) \setminus B_{R\delta^{\frac{1}{n+1}}}(0)$. Since

$$\tilde{Z}(x) \ge \frac{\mu^{2n+2}R^{2n+2} - 1}{\mu^{2n+2}R^{2n+2} + 1} > \frac{1}{4}$$

for $|z| \ge R\delta^{\frac{1}{n+1}}$, we have provided the existence of a positive super-solution for $\tilde{L}$, a sufficient condition to have that $\tilde{L}$ satisfies the maximum principle.

Claim 5. *There exists a constant $C>0$ such that*

$$\|\psi\|_{\infty,B_\eta(0)\setminus B_{R\delta^{\frac{1}{n+1}}}(0)}\le C[\|\psi\|_i+\|h\|_*],$$

where

$$\|\psi\|_i=\|\psi\|_{\infty,\partial B_{R\delta^{\frac{1}{n+1}}}(0)}+\|\psi\|_{\infty,\partial B_\eta(0)}.$$

Indeed, letting Φ be the solution of

$$\begin{cases} -\Delta\Phi=2\displaystyle\sum_{i=1}^{2}\frac{\delta^{\frac{\sigma_i}{n+1}}}{|z|^{2+\sigma_i}} & \text{for } R\delta^{\frac{1}{n+1}}\le|z|\le r\\ \Phi=0 & \text{for } |z|=r,\ R\delta^{\frac{1}{n+1}}\end{cases}$$

with $r\in(\eta,2\eta)$, $\sigma_1=\sigma(n+1)$ and $\sigma_2=2n+\sigma(n+1)$, we construct a barrier function of the form $\tilde\Phi=4\|\psi\|_i\tilde Z+\|h\|_*\Phi$. A direct computation shows that

$$\Phi(z)=2\sum_{i=1}^{2}\delta^{\frac{\sigma_i}{n+1}}\left[-\frac{1}{\sigma_i^2|z|^{\sigma_i}}+\alpha_i\log|z|+\beta_i\right],$$

where

$$\alpha_i=\frac{1}{\sigma_i^2\log\frac{R\delta^{\frac{1}{n+1}}}{r}}\left(\frac{1}{R^{\sigma_i}\delta^{\frac{\sigma_i}{n+1}}}-\frac{1}{r^{\sigma_i}}\right)<0,\qquad \beta_i=\frac{1}{\sigma_i^2r^{\sigma_i}}-\frac{\log r}{\sigma_i^2\log\frac{R\delta^{\frac{1}{n+1}}}{r}}\left(\frac{1}{R^{\sigma_i}\delta^{\frac{\sigma_i}{n+1}}}-\frac{1}{r^{\sigma_i}}\right)$$

for $i=1,2$. Since

$$0\le\Phi(z)\le2\sum_{i=1}^{2}\delta^{\frac{\sigma_i}{n+1}}\left[-\frac{1}{\sigma_i^2r^{\sigma_i}}+\alpha_i\log R\delta^{\frac{1}{n+1}}+\beta_i\right]=2\sum_{i=1}^{2}\delta^{\frac{\sigma_i}{n+1}}\alpha_i\log\frac{R\delta^{\frac{1}{n+1}}}{r}\le\sum_{i=1}^{2}\frac{2}{\sigma_i^2R^{\sigma_i}},$$

we get that

$$\begin{aligned}\tilde L(\tilde\Phi)&\le\|h\|_*\left[-2\frac{\delta^\sigma}{|z|^{2+\sigma(n+1)}}-2\frac{\delta^{\sigma+\frac{2n}{n+1}}}{|z|^{2+2n+\sigma(n+1)}}+C_1(n+1)^2\frac{\delta^2|z|^{2n}}{(\delta^2+|z|^{2n+2})^2}\sum_{i=1}^{2}\frac{2}{\sigma_i^2R^{\sigma_i}}\right]\\&\le\|h\|_*\left[-2\frac{\delta^\sigma}{|z|^{2+\sigma(n+1)}}-\frac{\delta^{\sigma+\frac{2n}{n+1}}}{(\delta^2+|z|^{2n+2})^{1+\sigma/2}}+\frac{\delta^\sigma|z|^{2n}}{(\delta^2+|z|^{2n+2})^{1+\sigma/2}}\right]\\&\le-\|h\|_*\frac{\delta^\sigma(|z|^{2n}+\delta^{\frac{2n}{n+1}})}{(\delta^2+|z|^{2n+2})^{1+\sigma/2}}\end{aligned}$$

in view of (B.7), for R large so that $C_1(n+1)^2\displaystyle\sum_{i=1}^{2}\frac{2}{\sigma_i^2R^{\sigma_i}}\le1$. Since $|\psi|\le\tilde\Phi$ on $\partial B_{R\delta^{\frac{1}{n+1}}}(0)\cup\partial B_r(0)$ in view of $4\tilde Z\ge1$, by the maximum principle we conclude that $|\psi|\le\tilde\Phi$ in $B_\eta(0)\setminus B_{R\delta^{\frac{1}{n+1}}}(0)$ and the claim follows.

Since Claims 2 and 3 provide that $\|\psi_k\|_i\to0$ as $k\to\infty$, by Claim 5 we conclude that $\|\psi_k\|_\infty=o(1)$ as $k\to+\infty$, a contradiction with $\liminf_{k\to+\infty}\|\psi_k\|_\infty>0$ according to Claim 1. This completes the proof. ∎

We are now in position to solve problem (B.1).

Proposition B.2. *There exists $\eta_0 > 0$ small such that for any $0 < \delta \le \eta_0$, $|\log\delta|\epsilon^2 \le \eta_0 \delta^{\frac{2}{n+1}}$, $|a| \le \eta_0 \delta$ and $h \in L^\infty(\Omega)$ with $\int_\Omega h = 0$ there is a unique solution $\phi := T(h)$, with $\int_\Omega \phi = 0$, and $d_0, d_1, d_2 \in \mathbb{R}$ of problem* (B.1)*. Moreover, there is a constant $C > 0$ so that*

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\delta}\right)\|h\|_*, \quad \sum_{l=0}^{2}|d_l| \le C\|h\|_*. \tag{B.8}$$

<u>Proof:</u> Since $-\Delta Z_l = |\sigma'(z)|^2 e^{U_{\delta,a}} Z_l$ in Ω (where $U_{\delta,a}$ stands for U_{δ,a,σ_a}) and $\int_\Omega \Delta Z_l = O(\delta^2)$ in view of (3.1)-(3.2), we have that $\Delta PZ_l = O(|\sigma'(z)|^2 e^{U_{\delta,a}}) + O(\delta^2)$ in view of $Z_l = O(1)$, yielding to $\|\Delta PZ_l\|_* \le C$ for all $l = 0,1,2$. By Proposition B.1 every solution of (B.1) satisfies

$$\|\phi\|_\infty \le C\left(\log\frac{1}{\delta}\right)\left[\|h\|_* + \sum_{l=0}^{2}|d_l|\right].$$

Set $\langle f, g\rangle = \int_\Omega fg$ and notice that

$$\langle L(\phi), PZ_j\rangle = \langle L(\phi), PZ_j + t\rangle = \langle \phi + \gamma(\phi), \tilde{L}(PZ_j + t)\rangle \tag{B.9}$$

for any $t \in \mathbb{R}$, in view of $\int_\Omega L(\phi) = 0$. To estimate the $|d_l|$'s, let us test equation (B.1) against PZ_j, $j = 0,1,2$, to get

$$\big\langle \phi + \gamma(\phi), \tilde{L}(PZ_j + t_j)\big\rangle = \langle h, PZ_j\rangle + \sum_{l=0}^{2} d_l \langle \Delta PZ_l, PZ_j\rangle$$

where $t_j = \frac{1}{|\Omega|}\int_\Omega Z_j$, $j = 0,1,2$. From the proof of Lemma 4.3 we know that for Z_0 and $Z = Z_1 + iZ_2$ there hold the following:

$$\int_\Omega \Delta PZ_0 PZ_0 = -16(n+1)\int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^4} + O(\delta^2)\,, \qquad \int_\Omega \Delta PZ PZ_0 = O(\delta^2)$$
$$\int_\Omega \Delta PZ\overline{PZ} = -8(n+1)\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4} + O(\delta)\,, \qquad \int_\Omega \Delta PZ PZ = O(\delta)$$

where $\int_{\mathbb{R}^2} \frac{dy}{(1+|y|^2)^4} = 2\int_{\mathbb{R}^2} \frac{1-|y|^2}{(1+|y|^2)^4} = \frac{\pi}{3}$. In terms of the Z_l's we then have that

$$\langle \Delta PZ_l, PZ_j\rangle = -(n+1)C_{ij}\delta_{lj} + O(\delta^2),$$

where δ_{lj} denotes the Kronecker's symbol and $c_{00} = \frac{8\pi}{3}$, $c_{11} = c_{22} = \frac{4\pi}{3}$. For $j = 0,1,2$ let us now estimate $\big\|\tilde{L}(PZ_j + t_j)\big\|_*$:

$$\big\|\tilde{L}(PZ_j + t_j)\big\|_* = \big\| - |\sigma'(z)|^2 e^{U_{\delta,a}} Z_j + \mathcal{K}(PZ_j + t_j) + O(\delta^2)\big\|_* = O(\delta + \epsilon^2\delta^{-\frac{2}{n+1}} + \delta|c_a|) \tag{B.10}$$

in view of (3.1)-(3.3) and (B.4). Since $|\gamma(\phi)| = O(\|\phi\|_\infty)$ in view of (2.49) and $\epsilon^2\delta^{-\frac{2}{n+1}} = o(1)$, by (3.6) we get that

$$\langle \phi + \gamma(\phi), \tilde{L}(PZ_j + t_j)\rangle = O(\delta + \epsilon^2\delta^{-\frac{2}{n+1}})\|\phi\|_\infty,$$

which along the previous estimates yields to

$$|d_j| \le C\left[(\delta + \epsilon^2\delta^{-\frac{2}{n+1}})\|\phi\|_\infty + \|h\|_* + \delta\sum_{l=0}^{2}|d_l|\right] \tag{B.11}$$

in view of $PZ_j = O(1)$. Since (B.11) gives that $\sum_{l=0}^{2} |d_l| = O(\delta + \epsilon^2 \delta^{-\frac{2}{n+1}})\|\phi\|_\infty + O(\|h\|_*)$, we have that every solution of (B.1) satisfies

$$\|\phi\|_\infty \leq C \left(\log \frac{1}{\delta}\right) \left[\|h\|_* + \sum_{l=0}^{2} |d_l|\right] \leq C \log \frac{1}{\delta}(\delta + \epsilon^2 \delta^{-\frac{2}{n+1}})\|\phi\|_\infty + C \log \frac{1}{\delta}\|h\|_*.$$

In view of $\log \frac{1}{\delta}(\delta + \epsilon^2 \delta^{-\frac{2}{n+1}}) = o(1)$ as $\eta_0 \to 0$, the a-priori estimates (B.8) immediately follow.

To solve (B.1), consider now the space

$$H = \left\{ \phi \in H^1(\Omega) \text{ doubly-periodic} : \int_\Omega \phi = 0\,,\ \int_\Omega \Delta PZ_l\, \phi = 0 \text{ for } l = 0, 1, 2 \right\}$$

endowed with the usual inner product $[\phi, \psi] = \int_\Omega \nabla\phi \nabla\psi$. Problem (B.1) is equivalent to finding $\phi \in H$ such that

$$[\phi, \psi] = \int_\Omega \left[\mathcal{K}\left(\phi + \gamma(\phi)\right) - h\right] \psi \qquad \text{for all } \psi \in H.$$

With the aid of Riesz's representation theorem, the equation has the form $(\text{Id} - \text{compact operator})\phi = \tilde{h}$. Fredholm's alternative guarantees unique solvability of this problem for any h provided that the homogeneous equation has only the trivial solution. This is equivalent to (B.1) with $h \equiv 0$, which has only the trivial solution by the a-priori estimates (B.8). The proof is now complete. ■

Appendix C: The nonlinear problem

We consider the following non linear problem

$$\begin{cases} L(\phi) = -[R + N(\phi)] + \sum_{l=0}^{2} d_l \Delta PZ_l \quad \text{in } \Omega \\ \int_\Omega \Delta PZ_l \phi = 0 \text{ for all } l = 0, 1, 2 \\ \int_\Omega \phi = 0, \end{cases} \tag{C.1}$$

where R, $N(\phi)$ and L are given by (2.24), (4.2) and (4.3), respectively. Notice that (4.5) and (C.1) are equivalent by setting $d = d_1 - id_2$.

Lemma C.1. *There exists $\delta_0 > 0$ small such that for any $0 < \delta < \eta_0$, $|\log \delta|^2 \epsilon^2 \leq \eta_0 \delta^{\frac{2}{n+1}}$, $|a| \leq \eta_0 \delta$ problem* (C.1) *admits a unique solution ϕ and d_l, $l = 0, 1, 2$. Moreover, there exists $C > 0$ so that*

$$\|\phi\|_\infty \leq C|\log \delta| \|R\|_*. \tag{C.2}$$

Proof: In terms of the operator T defined in Proposition B.2, problem (C.1) reads as

$$\phi = -T\left(R + N(\phi)\right) := \mathcal{A}(\phi).$$

For a given number $M > 0$, let us consider the space

$$\mathcal{F}_M = \{\phi \in L^\infty(\Omega) \text{ doubly-periodic } : \ \|\phi\|_\infty \leq M|\log \delta|\, \|R\|_*\}.$$

It is a straightforward but tedious computation to show that

$$\|N(\phi_1)-N(\phi_2)\|_* \le C_1(\|\phi_1\|_\infty+\|\phi_2\|_\infty)\|\phi_1-\phi_2\|_\infty. \tag{C.3}$$

Just to give an idea on how (C.3) can be proved, observe that $0 \le \frac{e^{u_0+W+\phi}}{\int_\Omega e^{u_0+W+\phi}} \le e^{2\|\phi\|_\infty}\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}$ and $|\int_\Omega e^{u_0+W+\phi}\phi| \le \|\phi\|_\infty \int_\Omega e^{u_0+W+\phi}$. For $\|\phi\|_\infty \le 1$ we can then get that

$$\|\phi\|_\infty \|D[\frac{e^{u_0+W+\phi}}{\int_\Omega e^{u_0+W+\phi}}][\phi]\|_* + \|D^2[\frac{e^{u_0+W+\phi}}{\int_\Omega e^{u_0+W+\phi}}][\phi,\phi]\|_* = O(\|\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}\|_*\|\phi\|_\infty^2) = O(\|\phi\|_\infty^2)$$

in view of $\|\frac{e^{u_0+W}}{\int_\Omega e^{u_0+W}}\|_* = O(1)$ by (2.51). This exactly what we need to estimate in $\|\cdot\|_*$−norm the difference between the first term of $N(\phi_1)$ and $N(\phi_2)$. For the other terms we can argue in a similar way to get

$$\|\phi\|_\infty \|D[\frac{e^{2(u_0+W+\phi)}}{\int_\Omega e^{2(u_0+W+\phi)}}][\phi]\|_* + \|D^2[\frac{e^{2(u_0+W+\phi)}}{\int_\Omega e^{2(u_0+W+\phi)}}][\phi,\phi]\|_* = O(\|\frac{e^{2(u_0+W)}}{\int_\Omega e^{2(u_0+W)}}\|_*\|\phi\|_\infty^2) = O(\|\phi\|_\infty^2)$$

in view of $\|\frac{e^{2(u_0+W)}}{\int_\Omega e^{2(u_0+W)}}\|_* = O(1)$ by (2.52), and

$$\|\phi\|_\infty \|D[B(W+\phi)][\phi]\|_* + \|D^2[B(W+\phi)][\phi,\phi]\|_* = O(B(W)\|\phi\|_\infty^2) = O(\delta^{-\frac{2}{n+1}}\|\phi\|_\infty^2)$$

in view of (2.49). Since $\epsilon^2\delta^{-\frac{2}{n+1}} = o(1)$ we can deduce the validity of (C.3).

Denote by C' the constant present in (B.8). By Proposition B.2 and (C.3) we get that

$$\|\mathcal{A}(\phi_1)-\mathcal{A}(\phi_2)\|_\infty \le C'|\log\delta|\|N(\phi_1)-N(\phi_2)\|_* \le 2C'C_1M\|R\|_*\log^2\delta\|\phi_1-\phi_2\|_\infty$$

for all $\phi_1,\phi_2 \in \mathcal{F}_M$. By Proposition B.2 we also have that

$$\|\mathcal{A}(\phi)\|_\infty \le C'|\log\delta|\,[\|R\|_* + \|N(\phi)\|_*] \le C'|\log\delta|\|R\|_* + C'C_1|\log\delta|\|\phi\|_\infty^2$$

for all $\phi \in \mathcal{F}_M$. Fix now M as $M = 2C'$, and by (2.54) take η_0 small so that $4(C')^2C_1\log^2\delta\|R\|_* < \frac{1}{2}$ in order to have that $\mathcal{A}$ is a contraction mapping of $\mathcal{F}_M$ into itself. Therefore $\mathcal{A}$ has a unique fixed point ϕ in $\mathcal{F}_M$, which satisfies (C.2) with $C = M$. ∎

Appendix D: The integral coefficients in (3.4)-(3.5)

Letting $\zeta = \frac{a}{\delta}$, we aim to investigate the integral coefficients

$$I := \int_{\mathbb{R}^2} \frac{(|y|^2-1)|y+\zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5}\,dy\,, \qquad K := \int_{\mathbb{R}^2} \frac{|y+\zeta|^{\frac{2n}{n+1}}y}{(1+|y|^2)^5}\,dy$$

which appear in (3.4)-(3.5) or (4.8)-(4.9). We will show below that $I = f(|\zeta|)$ and $K = g(|\zeta|)\zeta$ with $f < 0 < g$, and the asymptotic behavior of f and g as $|\zeta| \to +\infty$ will be identified.

By the change of variable $y \to y+\zeta$ and the Taylor expansion

$$(1-x)^{-5} = \sum_{k=0}^{+\infty} c_k x^k \quad \text{for } |x|<1$$

with $c_k = \frac{(4+k)!}{24\,k!}$, we can re-write I as

$$I = \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}}(|y-\zeta|^2-1)}{(1+|y-\zeta|^2)^5} dy = \sum_{k=0}^{+\infty} c_k \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}}(|y|^2+|\zeta|^2-1-y\bar\zeta-\bar y\zeta)(y\bar\zeta+\bar y\zeta)^k}{(1+|y|^2+|\zeta|^2)^{5+k}} dy$$

in view of

$$(1+|y-\zeta|^2)^{-5} = (1+|y|^2+|\zeta|^2)^{-5}(1-\frac{y\bar\zeta+\bar y\zeta}{1+|y|^2+|\zeta|^2})^{-5}$$

with

$$\frac{|y\bar\zeta+\bar y\zeta|}{1+|y|^2+|\zeta|^2} \le \frac{|y|^2+|\zeta|^2}{1+|y|^2+|\zeta|^2} < 1.$$

Since

$$(y\bar\zeta+\bar y\zeta)^k = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} y^j \bar\zeta^j \bar y^{k-j}\zeta^{k-j} = \sum_{1\le j<\frac{k}{2}} \begin{pmatrix} k \\ j \end{pmatrix} \zeta^{k-2j}\bar y^{k-2j}|\zeta|^{2j}|y|^{2j} + \sum_{\frac{k}{2}<j\le k} \begin{pmatrix} k \\ j \end{pmatrix} \bar\zeta^{2j-k}y^{2j-k}|\zeta|^{2k-2j}|y|^{2k-2j}$$

for k odd and

$$(y\bar\zeta+\bar y\zeta)^k = \sum_{1\le j<\frac{k}{2}} \begin{pmatrix} k \\ j \end{pmatrix} \zeta^{k-2j}\bar y^{k-2j}|\zeta|^{2j}|y|^{2j} + \sum_{\frac{k}{2}<j\le k} \begin{pmatrix} k \\ j \end{pmatrix} \bar\zeta^{2j-k}y^{2j-k}|\zeta|^{2k-2j}|y|^{2k-2j} + \begin{pmatrix} k \\ \frac{k}{2} \end{pmatrix} |\zeta|^k|y|^k$$

for k even, by symmetry we can simplify the expression of I as follows:

$$\begin{aligned} I &= \sum_{k=0}^{+\infty} c_k \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}}(|y|^2+|\zeta|^2-1)(y\bar\zeta+\bar y\zeta)^k}{(1+|y|^2+|\zeta|^2)^{5+k}} dy - \sum_{k=0}^{+\infty} c_k \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}}(y\bar\zeta+\bar y\zeta)^{k+1}}{(1+|y|^2+|\zeta|^2)^{5+k}} dy \\ &= \sum_{k=0}^{+\infty} c_{2k} \begin{pmatrix} 2k \\ k \end{pmatrix} |\zeta|^{2k} \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}+2k}(|y|^2+|\zeta|^2-1)}{(1+|y|^2+|\zeta|^2)^{5+2k}} dy - \sum_{k=1}^{+\infty} c_{2k-1} \begin{pmatrix} 2k \\ k \end{pmatrix} |\zeta|^{2k} \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{4+2k}} dy \end{aligned}$$

Since $I_q^p = \displaystyle\int_0^\infty \frac{\rho^p}{(1+\rho)^q} d\rho$, $q > p+1$, does satisfy the relations:

$$I_{q+1}^p = \frac{q-p-1}{q} I_q^p\,, \qquad I_q^{p+1} = \frac{p+1}{q-p-2} I_q^p, \tag{D.1}$$

through the change of variable $\rho^2 = \lambda t$, $\lambda = 1+|\zeta|^2$, in polar coordinates we have that

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{5+2k}} dy &= \pi\lambda^{\frac{n}{n+1}-4-k} I_{5+2k}^{\frac{n}{n+1}+k} = \pi \frac{3+k-\frac{n}{n+1}}{4+2k} \lambda^{\frac{n}{n+1}-4-k} I_{4+2k}^{\frac{n}{n+1}+k} \\ &= \frac{3+k-\frac{n}{n+1}}{2(2+k)(1+|\zeta|^2)} \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{4+2k}} dy \end{aligned} \tag{D.2}$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}-2+2k}}{(1+|y|^2+|\zeta|^2)^{2+2k}} dy &= \pi\lambda^{\frac{n}{n+1}-2-k} I_{2+2k}^{\frac{n}{n+1}-1+k} = \pi \frac{(2+2k)(3+2k)}{(k+\frac{n}{n+1})(2+k-\frac{n}{n+1})} \lambda^{\frac{n}{n+1}-2-k} I_{4+2k}^{\frac{n}{n+1}+k} \\ &= \frac{(2+2k)(3+2k)}{(k+\frac{n}{n+1})(2+k-\frac{n}{n+1})} (1+|\zeta|^2) \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{4+2k}} dy \end{aligned} \tag{D.3}$$

Inserting (D.2) and (D.3) into I, we get that

$$\begin{aligned} I &= \sum_{k=0}^{+\infty} c_{2k}\left(1-\frac{3+k-\frac{n}{n+1}}{(2+k)(1+|\zeta|^2)}\right)\begin{pmatrix} 2k \\ k \end{pmatrix}|\zeta|^{2k}\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{4+2k}}dy \\ &\quad -\sum_{k=1}^{+\infty} c_{2k-1}\begin{pmatrix} 2k \\ k \end{pmatrix}|\zeta|^{2k}\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{4+2k}}dy \\ &= \sum_{k=1}^{+\infty}\left[\frac{2(3+2k)c_{2k-2}}{k+\frac{n}{n+1}}\left(\frac{1+k}{2+k-\frac{n}{n+1}}-\frac{1}{1+|\zeta|^2}\right)\begin{pmatrix} 2k-2 \\ k-1 \end{pmatrix}(1+|\zeta|^2)-c_{2k-1}\begin{pmatrix} 2k \\ k \end{pmatrix}|\zeta|^2\right]\times \\ &\quad \times|\zeta|^{2k-2}\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{4+2k}}dy. \end{aligned}$$

Since $2(3+2k)c_{2k-2}\begin{pmatrix} 2k-2 \\ k-1 \end{pmatrix}=kc_{2k-1}\begin{pmatrix} 2k \\ k \end{pmatrix}$ for all $k\geq 1$, setting $\beta_k=c_{2k-1}\begin{pmatrix} 2k \\ k \end{pmatrix}|\zeta|^{2k-2}\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{4+2k}}dy$ we deduce that

$$\begin{aligned} I &= \sum_{k=1}^{+\infty}\left[\frac{k}{k+\frac{n}{n+1}}\left(\frac{1+k}{2+k-\frac{n}{n+1}}-\frac{1}{1+|\zeta|^2}\right)(1+|\zeta|^2)-|\zeta|^2\right]\beta_k \\ &= \sum_{k=1}^{+\infty}\left[\frac{k}{k+\frac{n}{n+1}}\left(\frac{|\zeta|^2}{1+|\zeta|^2}-\frac{1}{(2+k)(n+1)-n}\right)(1+|\zeta|^2)-|\zeta|^2\right]\beta_k<\sum_{k=1}^{+\infty}\left[\frac{k}{k+\frac{n}{n+1}}-1\right]|\zeta|^2\beta_k<0. \end{aligned}$$

In conclusion, we have shown that $I=f(|\zeta|)$ with $f<0$.

By the change of variable $y\to y+\zeta$ and the Taylor expansion of $(1-x)^{-5}$, arguing as before K can be re-written as

$$K = \int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}}(y-\zeta)}{(1+|y-\zeta|^2)^5}dy=\sum_{k=0}^{+\infty}c_k\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}}(y-\zeta)(y\bar{\zeta}+\bar{y}\zeta)^k}{(1+|y|^2+|\zeta|^2)^{5+k}}dy.$$

By the previous expansions of $(y\bar{\zeta}+\bar{y}\zeta)^k$ and

$$\begin{aligned} \int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}+2+2k}}{(1+|y|^2+|\zeta|^2)^{6+2k}}dy &= \pi\lambda^{\frac{n}{n+1}-4-k}I_{6+2k}^{\frac{n}{n+1}+1+k}=\pi\frac{\frac{n}{n+1}+1+k}{5+2k}\lambda^{\frac{n}{n+1}-4-k}I_{5+2k}^{\frac{n}{n+1}+k} \\ &= \frac{\frac{n}{n+1}+1+k}{5+2k}\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{5+2k}}dy, \end{aligned}$$

for symmetry K reduces to

$$K = \zeta\sum_{k=0}^{+\infty}\left[c_{2k+1}\frac{\frac{n}{n+1}+1+k}{5+2k}\begin{pmatrix} 2k+1 \\ k \end{pmatrix}-c_{2k}\begin{pmatrix} 2k \\ k \end{pmatrix}\right]|\zeta|^{2k}\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{5+2k}}dy.$$

Since $(1+k)c_{2k+1}\begin{pmatrix} 2k+1 \\ k \end{pmatrix}=(5+2k)c_{2k}\begin{pmatrix} 2k \\ k \end{pmatrix}$ for all $k\geq 0$, we get that

$$K = \zeta\sum_{k=0}^{+\infty}\frac{n}{(n+1)(1+k)}c_{2k}\begin{pmatrix} 2k \\ k \end{pmatrix}|\zeta|^{2k}\int_{\mathbb{R}^2}\frac{|y|^{\frac{2n}{n+1}+2k}}{(1+|y|^2+|\zeta|^2)^{5+2k}}dy.$$

In conclusion, we have shown that $K = g(|\zeta|)\zeta$ with $g > 0$.

In order to determine the asymptotic behavior of f and g as $|\zeta| \to +\infty$, we will use complex analysis to get some integral representation of f and g, see (D.6) and (D.9). We split I as $I = J_1 - 2J_2$, and we compute separately the constants

$$J_1 = \int_{\mathbb{R}^2} \frac{|y+\zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^4} dy \,, \quad J_2 = \int_{\mathbb{R}^2} \frac{|y+\zeta|^{\frac{2n}{n+1}}}{(1+|y|^2)^5} dy.$$

Concerning J_1, we re-write it in polar coordinates as

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^2} \frac{|y|^{\frac{2n}{n+1}}}{(1+|y-\zeta|^2)^4} dy = \int_0^{+\infty} \rho^{\frac{2n}{n+1}+1} d\rho \int_0^{2\pi} \frac{d\theta}{(1+\rho^2+|\zeta|^2 - \zeta\rho e^{-i\theta} - \overline{\zeta}\rho e^{i\theta})^4} \\ &= -i \int_0^{+\infty} \rho^{\frac{2n}{n+1}+1} d\rho \int_{\partial^+ B_1(0)} \frac{w^3}{(\overline{\zeta}\rho)^4 (w^2 - \frac{1+\rho^2+|\zeta|^2}{\overline{\zeta}\rho} w + \frac{\zeta^2}{|\zeta|^2})^4} dw. \end{aligned}$$

Since $w^2 - \dfrac{1+\rho^2+|\zeta|^2}{\overline{\zeta}\rho} w + \dfrac{\zeta^2}{|\zeta|^2}$ vanishes only at

$$w_\pm = \frac{1+\rho^2+|\zeta|^2 \pm \sqrt{(1+\rho^2+|\zeta|^2)^2 - 4\rho^2|\zeta|^2}}{2\overline{\zeta}\rho}$$

with $|w_-| < 1 < |w_+|$, by the Residue Theorem we have that

$$J_1 = -i \int_0^{+\infty} \rho^{\frac{2n}{n+1}+1} d\rho \int_{\partial^+ B_1(0)} \frac{w^3}{(\overline{\zeta}\rho)^4 (w-w_-)^4 (w-w_+)^4} dw = 2\pi \int_0^{\infty} \frac{\rho^{\frac{2n}{n+1}+1}}{6(\overline{\zeta}\rho)^4} \frac{d^3}{dw^3}\left[\frac{w^3}{(w-w_+)^4}\right](w_-) d\rho.$$

A straightforward computation shows that

$$\frac{d^3}{dw^3}\left[\frac{w^3}{(w-w_+)^4}\right] = -6 \frac{w^3 + w_+^3 + 9ww_+(w+w_+)}{(w-w_+)^7},$$

and then

$$\frac{d^3}{dw^3}\left[\frac{w^3}{(w-w_+)^4}\right](w_-) = 6(\overline{\zeta}\rho)^4 \frac{(1+\rho^2+|\zeta|^2)[(1+\rho^2+|\zeta|^2)^2 + 6\rho^2|\zeta|^2]}{[(1+\rho^2+|\zeta|^2)^2 - 4\rho^2|\zeta|^2]^{\frac{7}{2}}}.$$

Recalling that $\lambda = 1 + |\zeta|^2$, through the change of variable $\rho \to \rho^2$ we finally get for J_1 the expression

$$J_1 = \pi \int_0^{\infty} \rho^{\frac{n}{n+1}} \frac{(\lambda+\rho)[(\lambda+\rho)^2 + 6(\lambda-1)\rho]}{[(\lambda+\rho)^2 - 4(\lambda-1)\rho]^{\frac{7}{2}}} d\rho. \tag{D.4}$$

In a similar way, we first re-write J_2 as

$$J_2 = i \int_0^{+\infty} \rho^{\frac{2n}{n+1}+1} d\rho \int_{\partial^+ B_1(0)} \frac{w^4}{(\overline{\zeta}\rho)^5 (w-w_-)^5 (w-w_+)^5} dw = -2\pi \int_0^{+\infty} \frac{\rho^{\frac{2n}{n+1}+1}}{24(\overline{\zeta}\rho)^5} \frac{d^4}{dw^4}\left[\frac{w^4}{(w-w_+)^5}\right](w_-) d\rho$$

in view of the Residue Theorem. Since

$$\frac{d^4}{dw^4}\left[\frac{w^4}{(w-w_+)^5}\right] = 24 \frac{w^4 + w_+^4 + 16ww_+(w^2+w_+^2) + 36w^2w_+^2}{(w-w_+)^9},$$

we get that

$$\frac{d^4}{dw^4}\left[\frac{w^4}{(w-w_+)^5}\right](w_-)=-24(\overline{\zeta}\rho)^5\frac{(1+\rho^2+|\zeta|^2)^4+12\rho^2|\zeta|^2(1+\rho^2+|\zeta|^2)^2+42\rho^4|\zeta|^4}{[(1+\rho^2+|\zeta|^2)^2-4\rho^2|\zeta|^2]^{\frac{9}{2}}},$$

and then

$$J_2=\pi\int_0^\infty \rho^{\frac{n}{n+1}}\frac{(\lambda+\rho)^4+12(\lambda-1)\rho(\lambda+\rho)^2+42(\lambda-1)^2\rho^2}{[(\lambda+\rho)^2-4(\lambda-1)\rho]^{\frac{9}{2}}}d\rho. \tag{D.5}$$

By (D.4)-(D.5) we finally get that $f(|\zeta|)$ takes the form

$$f=\pi\int_0^\infty \rho^{\frac{n}{n+1}}\frac{(\lambda+\rho)^5-2(\lambda+\rho)^4+2(\lambda-1)\rho(\lambda+\rho)^3-24\lambda(\lambda-1)\rho(\rho+1)(\lambda+\rho)-84(\lambda-1)^2\rho^2}{[(\lambda+\rho)^2-4(\lambda-1)\rho]^{\frac{9}{2}}}d\rho \tag{D.6}$$

where $\lambda=1+|\zeta|^2$.

Observe that for $\zeta=0$ (i.e. $\lambda=1$) we simply have that

$$f(0)=J_1-2J_2=\pi[I_4^{\frac{n}{n+1}}-2I_5^{\frac{n}{n+1}}]=-\frac{2\pi}{2n+3}I_5^{\frac{n}{n+1}} \tag{D.7}$$

in view of (D.1). By the change of variable $\rho=\lambda+\sqrt{\lambda}t$ and the Lebesgue Theorem we get that

$$\lambda^{-\frac{n}{n+1}}J_1=\pi\int_{-\sqrt{\lambda}}^\infty(1+\frac{t}{\sqrt{\lambda}})^{\frac{n}{n+1}}\frac{(2+\frac{t}{\sqrt{\lambda}})^3+6\frac{\lambda-1}{\lambda}(1+\frac{t}{\sqrt{\lambda}})(2+\frac{t}{\sqrt{\lambda}})}{(t^2+4+\frac{4t}{\sqrt{\lambda}})^{\frac{7}{2}}}dt\to 20\pi\int_{\mathbb{R}}\frac{dt}{(t^2+4)^{\frac{7}{2}}}$$

and

$$\begin{aligned}\lambda^{-\frac{n}{n+1}}J_2 &= \pi\int_{-\sqrt{\lambda}}^\infty(1+\frac{t}{\sqrt{\lambda}})^{\frac{n}{n+1}}\frac{(2+\frac{t}{\sqrt{\lambda}})^4+12\frac{\lambda-1}{\lambda}(1+\frac{t}{\sqrt{\lambda}})(2+\frac{t}{\sqrt{\lambda}})^2+42(\frac{\lambda-1}{\lambda})^2(1+\frac{t}{\sqrt{\lambda}})^2}{(t^2+4+\frac{4t}{\sqrt{\lambda}})^{\frac{9}{2}}}dt\\ &\to 106\pi\int_{\mathbb{R}}\frac{dt}{(t^2+4)^{\frac{9}{2}}}\end{aligned}$$

as $|\zeta|\to+\infty$ (i.e. $\lambda\to+\infty$). Since $\int_{\mathbb{R}}\frac{dt}{(t^2+4)^{\frac{7}{2}}}=\frac{14}{3}\int_{\mathbb{R}}\frac{dt}{(t^2+4)^{\frac{9}{2}}}$, we get that

$$\frac{f(|\zeta|)}{|\zeta|^{\frac{2n}{n+1}}}\to-\frac{356}{3}\pi\int_{\mathbb{R}}\frac{dt}{(t^2+4)^{\frac{9}{2}}} \tag{D.8}$$

as $|\zeta|\to\infty$.

In a similar way, for K we have that

$$K=i\int_0^{+\infty}\rho^{\frac{2n}{n+1}+1}d\rho\int_{\partial^+B_1(0)}\frac{w^4(\rho w-\zeta)}{(\overline{\zeta}\rho)^5(w-w_-)^5(w-w_+)^5}dw=-2\pi\int_0^{+\infty}\frac{\rho^{\frac{2n}{n+1}+1}}{24(\overline{\zeta}\rho)^5}\frac{d^4}{dw^4}\left[\frac{w^4(\rho w-\zeta)}{(w-w_+)^5}\right](w_-)d\rho$$

in view of the Residue Theorem. Since

$$\frac{d^4}{dw^4}\left[\frac{w^4(\rho w-\zeta)}{(w-w_+)^5}\right]=24\frac{5\rho ww_+[w^3+w_+^3+6ww_+(w+w_+)]-\zeta[w^4+w_+^4+16ww_+(w^2+w_+^2)+36w^2w_+^2]}{(w-w_+)^9},$$

we get that

$$\frac{d^4}{dw^4}\left[\frac{w^4(\rho w-\zeta)}{(w-w_+)^5}\right](w_-)=12(\overline{\zeta}\rho)^5\zeta\frac{(\lambda+\rho^2)^4+2\rho^2(\lambda-6-5\rho^2)(\lambda+\rho^2)^2+6(\lambda-1)\rho^4(2\lambda-7-5\rho^2)}{[(\lambda+\rho^2)^2-4(\lambda-1)\rho^2]^{\frac{9}{2}}},$$

and then

$$g(|\zeta|)=-\frac{\pi}{2}\int_0^\infty \rho^{\frac{n}{n+1}}\frac{(\lambda+\rho)^4+2\rho(\lambda-6-5\rho)(\lambda+\rho)^2+6(\lambda-1)\rho^2(2\lambda-7-5\rho)}{[(\lambda+\rho)^2-4(\lambda-1)\rho]^{\frac{9}{2}}}d\rho. \tag{D.9}$$

So, we have that

$$g(0)=\frac{\pi}{2}(9I_5^{\frac{n}{n+1}}-10I_6^{\frac{n}{n+1}})=\frac{3n+1}{2(n+1)}\pi I_5^{\frac{n}{n+1}} \tag{D.10}$$

in view of (D.1), and, by the change of variable $\rho=\lambda+\sqrt{\lambda}t$ and the Lebesgue Theorem,

$$\frac{g(|\zeta|)}{|\zeta|^{\frac{2n}{n+1}}}\to 17\pi\int_{\mathbb{R}}\frac{dt}{(t^2+4)^{\frac{9}{2}}} \tag{D.11}$$

as $|\zeta|\to+\infty$, in view of

$$\int_{-\sqrt{\lambda}}^\infty (1+\frac{t}{\sqrt{\lambda}})^{\frac{n}{n+1}}\frac{(2+\frac{t}{\sqrt{\lambda}})^4-2(1+\frac{t}{\sqrt{\lambda}})(4+\frac{6+5\sqrt{\lambda}t}{\lambda})(2+\frac{t}{\sqrt{\lambda}})^2-6\frac{\lambda-1}{\lambda}(1+\frac{t}{\sqrt{\lambda}})^2(3+\frac{7+5\sqrt{\lambda}t}{\lambda})}{(t^2+4+\frac{4t}{\sqrt{\lambda}})^{\frac{9}{2}}}dt\to-\int_{\mathbb{R}}\frac{34\,dt}{(t^2+4)^{\frac{9}{2}}}$$

as $\lambda\to+\infty$.

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