A CLASSIFICATION RESULT FOR THE QUASI-LINEAR LIOUVILLE EQUATION

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Abstract. Entire solutions of the $n-$Laplace Liouville equation in $\mathbb{R}^n$ with finite mass are completely classified.

1. Introduction

We are concerned with the following Liouville equation

$$\begin{cases} -\Delta_n U = e^U & \text{in } \mathbb{R}^n \\ \int_{\mathbb{R}^n} e^U < +\infty \end{cases} \tag{1.1}$$

involving the $n-$Laplace operator $\Delta_n(\cdot) = \text{div}(|\nabla(\cdot)|^{n-2}\nabla(\cdot))$, $n \geq 2$. Here, a solution U of (1.1) stands for a function $U \in C^{1,\alpha}(\mathbb{R}^n)$ which satisfies

$$\int_{\mathbb{R}^n} |\nabla U|^{n-2}\langle \nabla U, \nabla \Phi\rangle = \int_{\mathbb{R}^n} e^U \Phi \qquad \forall\ \Phi \in H = \{\Phi \in W_0^{1,n}(\Omega) : \Omega \subset \mathbb{R}^n \text{ bounded}\}. \tag{1.2}$$

As wee will see, the regularity assumption on U is not restrictive since a solution in $W^{1,n}_{\text{loc}}(\mathbb{R}^n)$ is automatically in $C^{1,\alpha}(\mathbb{R}^n)$, for some $\alpha \in (0,1)$.

Problem (1.1) has the explicit solution

$$U(x) = \log \frac{c_n}{(1+|x|^{\frac{n}{n-1}})^n}, \quad x \in \mathbb{R}^n,$$

where $c_n = n(\frac{n^2}{n-1})^{n-1}$. Due to scaling and translation invariance, a $(n+1)-$dimensional family of explicit solutions $U_{\lambda,p}$ to (1.1) is built as

$$U_{\lambda,p}(x) = U(\lambda(x-p)) + n\log\lambda = \log \frac{c_n \lambda^n}{(1+\lambda^{\frac{n}{n-1}}|x-p|^{\frac{n}{n-1}})^n} \tag{1.3}$$

for all $\lambda > 0$ and $p \in \mathbb{R}^n$. Notice that

$$\int_{\mathbb{R}^n} e^{U_{\lambda,p}} = \int_{\mathbb{R}^n} e^U = c_n \omega_n \tag{1.4}$$

where $\omega_n = |B_1(0)|$. Our aim is the following classification result:

Theorem 1.1. *Let U be a solution of* (1.1). *Then*

$$U(x) = \log \frac{c_n \lambda^n}{(1+\lambda^{\frac{n}{n-1}}|x-p|^{\frac{n}{n-1}})^n}, \quad x \in \mathbb{R}^n \tag{1.5}$$

for some $\lambda > 0$ and $p \in \mathbb{R}^n$.

In a radial setting Theorem 1.1 has been already proved, among other things, in [19]. For the semilinear case $n = 2$ such a classification result is known since a long ago. The first proof goes back to J. Liouvillle [28] who found a formula– the so-called Liouville formula– to represent a solution U on a simply-connected domain in terms of a suitable meromorphic function. On the whole $\mathbb{R}^2$ the finite-mass condition $\int_{\mathbb{R}^2} e^U < +\infty$ completely determines such meromorphic function.

A PDE proof has been found several years later by W. Chen and C. Li [9]. The fundamental point is to represent a solution U of (1.1) in an integral form in terms of the fundamental solution and then deduce the precise asymptotic behavior of U at infinity to start the moving plane technique. Such idea has revealed very powerful and has been also applied [7, 27, 29, 39, 40] to the higher-order version of (1.1) involving the operator $(-\Delta)^{\frac{n}{2}}$. Overall, the integral equation satisfied by U can be used to derive asymptotic properties of U at infinity or can be directly studied through the method of moving planes/spheres. Since these methods are very well suited for integral equations, a research line has flourished about qualitative properties of integral equations, see [10, 18, 24, 41, 42] to quote a few.

The quasi-linear case $n > 2$ is more difficult. Very recently, the classification of positive $\mathcal{D}^{1,n}(\mathbb{R}^N)-$solutions to $-\Delta_n U = U^{\frac{nN}{N-n}-1}$, a PDE with critical Sobolev polynomial nonlinearity, has been achieved [13, 33, 38] for $n < N$, see also some previous somehow related results [14, 15, 36]. The strategy is always based on the moving plane method and

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the analytical difficulty comes from the lack of comparison/maximum principles on thin strips. Moreover for $n < N$ it is not available any Kelvin type transform, a useful tool to “gain” decay properties on a solution.

When $n = N$ the classical approach [7, 9, 27, 29, 39, 40] breaks down since an integral representation formula for a solution U of (1.1) is not available, due to the quasi-linear nature of Δ_n. It becomes a delicate issue to determine the asymptotic behavior of U at infinity and overall it is not clear how to carry out the method of moving planes/spheres. However, when $n = N$ there are some special features we aim to exploit to devise a new approach which does not make use of moving planes/spheres, providing in two dimensions an alternative proof of the result in [9]. During the completion of this work, we have discovered that such an approach has been already used in [8] for Liouville systems, where the maximum principle can possibly fail. See also [20] for a somewhat related approach to symmetry questions in a ball.

The case $n = N$ is usually referred to as the conformal situation, since Δ_n is invariant under Kelvin transform: $\hat{U}(x) = U(\frac{x}{|x|^2})$ formally satisfies

$$\Delta_n \hat{U} = \frac{1}{|x|^{2n}} (\Delta_n U)(\frac{x}{|x|^2}),$$

so that

$$\begin{cases} -\Delta_n \hat{U} = F(x) := \frac{e^{\hat{U}}}{|x|^{2n}} & \text{in } \mathbb{R}^n \setminus \{0\} \\ \int_{\mathbb{R}^n} \frac{e^{\hat{U}}}{|x|^{2n}} < +\infty. \end{cases}$$

Equation has to be interpreted in the weak sense

$$\int_{\mathbb{R}^n} |\nabla \hat{U}|^{n-2} \langle \nabla \hat{U}, \nabla \Phi \rangle = \int_{\mathbb{R}^n} \frac{e^{\hat{U}}}{|x|^{2n}} \Phi \qquad \forall\, \Phi \in \hat{H} = \{\Phi : \ \hat{\Phi} \in H\}.$$

Due to the nonlinearity of Δ_n we cannot re-absorb the factor $\frac{1}{|x|^{2n}}$ and so (1.1) still does not possess any induced invariance property of Kelvin type. The behavior near an isolated singularity has been thoroughly discussed by J. Serrin [34, 35] for very general quasi-linear equations. The case $F \in L^1(\mathbb{R}^n)$ is very delicate as it represents a limiting situation where Serrin's results do not apply. Using some ideas from [1, 4, 5], in Section 2 we first show that U is bounded from above and satisfies the following weighted Sobolev estimates at infinity:

$$\int_{\mathbb{R}^n \setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}} < +\infty \qquad \text{for all } 1 \le q < n. \tag{1.6}$$

According to Remark 3.2, estimates (1.6) seem crucial to carry out in Section 3 an isoperimetric argument, which has been originally developed in [9] thanks to the logarithmic behavior of U at infinity, to show that

$$\int_{\mathbb{R}^n} e^U \ge c_n \omega_n, \tag{1.7}$$

see also [22]. Moreover, according to [19], the Pohozaev identity leads to show that the equality in (1.7) is valid just for solutions U of the form (1.5).

Thanks to (1.7), in Section 4 we can improve the previous estimates and use Serrin's type results, see [34, 35], to show that U has a logarithmic behavior at infinity along with

$$-\Delta_n U = e^U - \gamma \delta_\infty \qquad \text{in } \mathbb{R}^n, \quad \gamma = \int_{\mathbb{R}^n} e^U.$$

Going back to an idea of Y.-Y. Li and N. Wolanski for $n = 2$, the Pohozaev identity has revealed to be a fundamental tool to derive information on the mass of a singularity when $n = N$ (see for example [3, 17, 30, 31]): applied near ∞, it finally gives in Section 5 that $\gamma = \int_{\mathbb{R}^n} e^U = c_n \omega_n$. Notice that in Sections 2 and 4 we reproduce some estimates by emphasizing the dependence of the constants. As we will explain precisely in Remark 2.4, in our argument it is crucial that all the estimates do not really depend on the structural assumption (2.1).

Problems with exponential nonlinearity on a bounded domain can exhibit non-compact solution-sequences, whose shape near a blow-up point is asymptotically described by (1.1). A concentration-compactness principle has been established [6] for $n = 2$ and [1] for $n \ge 2$. In the non-compact situation the nonlinearity concentrates at the blow-up points as a sum of Dirac measures, whose masses likely belong to $c_n \omega_n \mathbb{N}$ thanks to (1.4). Such a quantization for the concentration masses has been proved [25] for $n = 2$ and extended [17] to $n \ge 2$ by requiring an additional boundary assumption. Very refined asymptotic properties have been later established [2, 11, 23]. The classification result for (1.1) is the starting point in all these issues, which might be now investigated also for $n \ge 2$ thanks to Theorem 1.1.

2. Some estimates

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\mathbf{a} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function so that

$$|\mathbf{a}(x,p)| \le c(a(x) + |p|^{n-1}) \qquad \forall p \in \mathbb{R}^n,\ a.e.\ x \in \Omega \tag{2.1}$$

$$\langle \mathbf{a}(x,p) - \mathbf{a}(x,q), p-q\rangle \ge d|p-q|^n \qquad \forall p,q \in \mathbb{R}^n,\ a.e.\ x \in \Omega \tag{2.2}$$

for some $c,d>0$ and $a \in L^{\frac{n}{n-1}}(\Omega)$. Given $f \in L^1(\Omega)$, let $u \in W^{1,n}(\Omega)$ be a weak solution of

$$-\operatorname{div} \mathbf{a}(x,\nabla u) = f \qquad \text{in } \Omega. \tag{2.3}$$

Thanks to (2.1) equation (2.3) is interpreted in the following sense:

$$\int_\Omega \langle \mathbf{a}(x,\nabla u), \nabla \phi\rangle = \int_\Omega f\phi \qquad \forall \phi \in W_0^{1,n}(\Omega) \cap L^\infty(\Omega). \tag{2.4}$$

Since $u \in W^{1,n}(\Omega)$ let us consider the weak solution $h \in W^{1,n}(\Omega)$ of

$$\begin{cases} \operatorname{div} \mathbf{a}(x,\nabla h) = 0 & \text{in } \Omega \\ h = u & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Introduce the truncature operator T_k, $k>0$, as

$$T_k(u) = \begin{cases} u & \text{if } |u| \le k \\ k\frac{u}{|u|} & \text{if } |u| > k. \end{cases} \tag{2.6}$$

According to [1, 4, 5] we have the following estimates.

Proposition 2.1. *Let $f \in L^1(\Omega)$ and assume (2.1)-(2.2). Let u be a weak solution of (2.3) in the sense (2.4), and set*

$$\Lambda_q = \Big(\frac{S_q^{\frac{n}{q}} d}{\|f\|_1}\Big)^{\frac{1}{n-1}}$$

where S_q is the Sobolev constant for the embedding $\mathcal{D}^{1,q}(\mathbb{R}^n) \hookrightarrow L^{\frac{nq}{n-q}}(\mathbb{R}^n)$, $1 \le q < n$. Then, for every $0<\lambda<\Lambda_1$ there hold

$$\int_\Omega e^{\lambda|u-h|} \le \frac{|\Omega|}{1-\lambda\Lambda_1^{-1}}, \qquad \int_\Omega |\nabla(u-h)|^q \le \frac{2S_q}{\Lambda_q^{\frac{q(n-1)}{n}}}\left(1 + \frac{2^{\frac{n}{q(n-1)}}}{(n-1)^{\frac{1}{n-1}}\Lambda_q}\right)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}}. \tag{2.7}$$

Proof. Fix $k \ge 0$, $a>0$. Since $T_{k+a}(u-h) - T_k(u-h) \in W_0^{1,n}(\Omega) \cap L^\infty(\Omega)$, by (2.4)-(2.5) we get that

$$\int_\Omega \langle \mathbf{a}(x,\nabla u) - \mathbf{a}(x,\nabla h), \nabla\ [T_{k+a}(u-h) - T_k(u-h)]\rangle = \int_\Omega f[T_{k+a}(u-h) - T_k(u-h)], \tag{2.8}$$

yielding to

$$\frac{1}{a}\int_{\{k<|u-h|\le k+a\}} |\nabla(u-h)|^n \le \frac{\|f\|_1}{d} \tag{2.9}$$

in view of (2.2). By (2.9) and the following Lemma we deduce the validity of (2.7) and the proof of Proposition 2.1 is complete. □

Lemma 2.2. *Let w be a measurable function with $T_k(w) \in W_0^{1,n}(\Omega)$ so that for all $k \ge 0$, $a>0$*

$$\frac{1}{a}\int_{\{k<|w|\le k+a\}} |\nabla w|^n \le C_0 \tag{2.10}$$

for some $C_0>0$. Then there hold

$$\int_\Omega e^{\lambda|w|} \le \frac{|\Omega|}{1-\lambda\Lambda^{-1}}, \qquad \int_\Omega |\nabla w|^q \le 2C_0^{\frac{q}{n}}\left(1 + \Big(\frac{2^{\frac{n}{q}}C_0}{(n-1)S_q^{\frac{n}{q}}}\Big)^{\frac{1}{n-1}}\right)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}} \tag{2.11}$$

for every $0<\lambda<\Lambda = (\frac{S_1^n}{C_0})^{\frac{1}{n-1}}$ and $1 \le q < n$, where k_0 is given in (2.15).

Proof. Let $\Phi(k) = |\{x \in \Omega : |w(x)| > k\}|$ be the distribution function of $|w|$. We have that

$$\begin{aligned} \Phi(k+a)^{\frac{n-1}{n}} &\le \frac{1}{a}\left(\int_\Omega |T_{k+a}(w) - T_k(w)|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \le \frac{1}{aS_1}\int_\Omega |\nabla T_{k+a}(w) - \nabla T_k(w)| \\ &= \frac{1}{aS_1}\int_{\{k<|w|\le k+a\}} |\nabla w| \end{aligned}$$

where S_1 is the Sobolev constant of the embedding $\mathcal{D}^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$. By the Hölder's inequality and (2.10) we deduce that

$$\Phi(k+a) \le \frac{\Phi(k) - \Phi(k+a)}{a\Lambda}$$

and, as $a \to 0^+$,

$$\Phi(k) \le -\frac{1}{\Lambda}\Phi'(k) \tag{2.12}$$

for a.e. $k > 0$. Since Φ is a monotone decreasing function, an integration of (2.12)

$$\ln \frac{\Phi(k)}{\Phi(0)} \le \int_0^k \frac{\Phi'}{\Phi} ds \le -\Lambda k$$

provides that

$$\Phi(k) \le |\Omega| e^{-\Lambda k}$$

for all $k > 0$, and then

$$\begin{aligned} \int_\Omega e^{\lambda |w|} &= |\Omega| + \lambda \int_\Omega dx \int_0^{|w(x)|} e^{\lambda k} dk = |\Omega| + \lambda \int_0^\infty e^{\lambda k} \Phi(k) dk \\ &\le |\Omega| + \lambda |\Omega| \int_0^\infty e^{-(\Lambda - \lambda)k} dk = \frac{|\Omega|}{1 - \lambda \Lambda^{-1}} \end{aligned}$$

for all $0 < \lambda < \Lambda$. Given $k_0 \in \mathbb{N}$ introduce the sets

$$\Omega_{k_0} = \{x \in \Omega : \ |w(x)| \le k_0\}, \qquad \Omega_k = \{x \in \Omega : \ k-1 < |w(x)| \le k\} \ (k > k_0),$$

and by the Hölder's inequality write for $1 \le q < n$

$$\int_{\Omega_{k_0}} |\nabla w|^q \le (C_0 k_0)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}}, \quad \int_{\Omega_k} |\nabla w|^q \le C_0^{\frac{q}{n}} |\Omega_k|^{\frac{n-q}{n}} \le \frac{C_0^{\frac{q}{n}}}{(k-1)^q} \left(\int_{\Omega_k} |w|^{\frac{nq}{n-q}} \right)^{\frac{n-q}{n}}$$

thanks to (2.10). For $N \in \mathbb{N}$ let us sum up to get by the Hölder's inequality

$$\begin{aligned} \int_\Omega |\nabla T_{k_0+N}(w)|^q &= \sum_{k=k_0}^{k_0+N} \int_{\Omega_k} |\nabla w|^q \le (C_0 k_0)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}} + C_0^{\frac{q}{n}} \left(\sum_{k=k_0+1}^{k_0+N} \frac{1}{(k-1)^n} \right)^{\frac{q}{n}} \left(\sum_{k=k_0+1}^{k_0+N} \int_{\Omega_k} |w|^{\frac{nq}{n-q}} \right)^{\frac{n-q}{n}} \\ &\le (C_0 k_0)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}} + C_0^{\frac{q}{n}} \left(\sum_{k=k_0+1}^{k_0+N} \frac{1}{(k-1)^n} \right)^{\frac{q}{n}} \left(\int_\Omega |T_{k_0+N}(w)|^{\frac{nq}{n-q}} \right)^{\frac{n-q}{n}} . \end{aligned} \tag{2.13}$$

Letting

$$k_0 = 1 + \Big(\frac{2^{\frac{n}{q}} C_0}{(n-1) S_q^{\frac{n}{q}}}\Big)^{\frac{1}{n-1}}, \tag{2.14}$$

we have that

$$\sum_{k \ge k_0} \frac{1}{k^n} \le \int_{k_0 - 1}^\infty \frac{dt}{t^n} = \frac{(k_0 - 1)^{-(n-1)}}{n-1} = \frac{1}{C_0} \Big(\frac{S_q}{2}\Big)^{\frac{n}{q}}. \tag{2.15}$$

By using the Sobolev embedding $\mathcal{D}^{1,q}(\mathbb{R}^n) \hookrightarrow L^{\frac{nq}{n-q}}(\mathbb{R}^n)$ on the L.H.S. of (2.13) and by (2.15) we deduce that

$$S_q \left(\int_\Omega |T_{k_0+N}(w)|^{\frac{nq}{n-q}} \right)^{\frac{n-q}{n}} \le 2 (C_0 k_0)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}},$$

which inserted into (2.13) gives in turn

$$\int_\Omega |\nabla T_{k_0+N}(w)|^q \le 2 (C_0 k_0)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}}.$$

Letting $N \to +\infty$ we finally deduce that

$$\int_\Omega |\nabla w|^q \le 2 (C_0 k_0)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}} = 2 C_0^{\frac{q}{n}} \left(1 + \Big(\frac{2^{\frac{n}{q}} C_0}{(n-1) S_q^{\frac{n}{q}}}\Big)^{\frac{1}{n-1}} \right)^{\frac{q}{n}} |\Omega|^{\frac{n-q}{n}}$$

in view of (2.14) and the proof is complete. $\square$

As a first by-product of Proposition 2.1 we have that

Theorem 2.3. *Let $U \in W^{1,n}_{loc}(\mathbb{R}^n)$ be a weak solution of* (1.1). *Then* $\sup\limits_{\mathbb{R}^n} U < +\infty$ *and* $U \in C^{1,\alpha}(\mathbb{R}^n)$, $\alpha \in (0,1)$.

Proof. Assume that for $0 < \epsilon \le 1$

$$\int_{B_\epsilon(x)} e^U \le \frac{S_1^n d}{3^{n-1}}. \tag{2.16}$$

Thanks to Proposition 2.1 by (2.16) we deduce that

$$\int_{B_\epsilon(x)} e^{2|U-H|} \le 3\omega_n, \tag{2.17}$$

where H is a $n-$harmonic function in $B_\epsilon(x)$ with $H = U$ on $\partial B_\epsilon(x)$. Since $H \le U$ on $B_\epsilon(x)$ by the comparison principle, we have that

$$\int_{B_\epsilon(x)} H_+^n \le \int_{B_\epsilon(x)} U_+^n \le n! \int_{\mathbb{R}^n} e^U \tag{2.18}$$

where u_+ denotes the positive part of u. Since Theorem 2 in [34] is easily seen to be valid for H^+ too (simply by replacing $|H|$ with H^+ in the proof), by (2.18) we have that

$$\sup_{B_{\frac{\epsilon}{2}}(x)} H_+ \le C_0(\epsilon) \tag{2.19}$$

for some $C_0(\epsilon) > 0$ independent on x. By (2.17) and (2.19) we deduce that

$$\int_{B_{\frac{\epsilon}{2}}(x)} e^{2U} = \int_{B_{\frac{\epsilon}{2}}(x)} e^{2|U-H|} e^{2H} \le 3e^{2C_0(\epsilon)}\omega_n. \tag{2.20}$$

Still thanks to the elliptic estimates in [34] on U^+, by (2.18) and (2.20) we have that

$$\sup_{B_{\frac{\epsilon}{4}}(x)} U_+ \le C_1(\epsilon) \tag{2.21}$$

for some $C_1(\epsilon) > 0$ independent on x. To complete the proof, we argue as follows. Since $\int_{\mathbb{R}^n} e^U < +\infty$ we can find $R > 0$ so that

$$\int_{\mathbb{R}^n \setminus B_R(0)} e^U \le \frac{S_1^n d}{3^{n-1}}. \tag{2.22}$$

Given $|x| > R+1$, by (2.22) we have the validity of (2.16) with $\epsilon = 1$. For all $|x| \le R+1$ we can find $\epsilon_x > 0$ small so that (2.16) holds. By the compactness of the set $\{|x| \le R+1\}$ we can find points $x_1, \dots, x_L$ so that

$$\{|x| \le R+1\} \subset \bigcup_{i=1}^L B_{\frac{\epsilon_{x_i}}{4}}(x_i). \tag{2.23}$$

Therefore, by (2.21) we deduce that

$$\sup_{\mathbb{R}^n} U \le \max\{C_1(1), C_1(\epsilon_{x_1}), \dots, C_1(\epsilon_{x_L})\} < +\infty$$

in view of (2.23). Since $e^U \in L^\infty(\mathbb{R}^n)$ and $U \in L^n_{\text{loc}}(\mathbb{R}^n)$, we can use the elliptic estimates in [16, 34, 37] to show that $U \in C^{1,\alpha}(\mathbb{R}^n)$, for some $\alpha \in (0,1)$. $\square$

We aim now to establish some bounds on U at infinity. Let us recall that the Kelvin transform $\hat{U}(x) = U(\frac{x}{|x|^2})$ of U satisfies

$$\begin{cases} -\Delta_n \hat{U} = \frac{e^{\hat{U}}}{|x|^{2n}} & \text{in } \mathbb{R}^n \setminus \{0\} \\ \int_{\mathbb{R}^n} \frac{e^{\hat{U}}}{|x|^{2n}} < +\infty, \end{cases} \tag{2.24}$$

where the equation is meant in the weak sense

$$\int_{\mathbb{R}^n} |\nabla \hat{U}|^{n-2} \langle \nabla \hat{U}, \nabla \Phi \rangle = \int_{\mathbb{R}^n} \frac{e^{\hat{U}}}{|x|^{2n}} \Phi \qquad \forall \Phi \in \hat{H} = \{\Phi : \ \hat{\Phi} \in H\} \tag{2.25}$$

with H given in (1.2). By Theorem 2.3 we know that $\hat{U} \in C^{1,\alpha}(\mathbb{R}^n \setminus \{0\})$. Here and in the sequel, $\alpha \in (0,1)$ will denote an Hölder exponent which can varies from line to line.

In order to understand the behavior of $\hat{U}$ at 0, we fix $r > 0$ small and, for all $0 < \epsilon < r$, let $H_\epsilon \in W^{1,n}(A_\epsilon)$ satisfy

$$\begin{cases} \Delta_n H_\epsilon = 0 & \text{in } A_\epsilon := B_r(0) \setminus B_\epsilon(0) \\ H_\epsilon = \hat{U} & \text{on } \partial A_\epsilon. \end{cases} \tag{2.26}$$

Regularity issues for quasi-linear PDEs involving Δ_n are well established since the works of DiBenedetto, Evans, Lewis, Serrin, Tolksdorf, Uhlenbeck, Uraltseva. For example, local Hölder estimates on H_ϵ can be found in [34] and then it follows by [16, 37] that $H_\epsilon \in C^{1,\alpha}(A_\epsilon)$. Thanks to [26] such regularity can be pushed up to the boundary to deduce that $H_\epsilon \in C^{1,\alpha}(\overline{A_\epsilon})$. By (2.26) the function $U_\epsilon = \hat{U} - H_\epsilon \in C^{1,\alpha}(\overline{A_\epsilon})$ satisfies

$$\begin{cases} \Delta_n(\hat{U} - U_\epsilon) = 0 & \text{in } A_\epsilon \\ U_\epsilon = 0 & \text{on } \partial A_\epsilon. \end{cases} \tag{2.27}$$

We aim to derive estimates on H_ϵ and U_ϵ on the whole A_ϵ by using Proposition 2.1 with

$$\mathbf{a}(x,p) = |\nabla \hat{U}(x)|^{n-2} \nabla \hat{U}(x) - |\nabla \hat{U}(x) - p|^{n-2} (\nabla \hat{U}(x) - p). \tag{2.28}$$

Remark 2.4. *Let us notice that* $\mathbf{a}(x,p)$ *in* (2.28) *satisfies* (2.1) *with* $a = |\nabla \hat{U}|^{n-1}$. *Since* $\hat{U}$ *is expected to be singular at* 0*, it is likely true that* $\|a\|_{\frac{n}{n-1},A_\epsilon} \to +\infty$ *as* $\epsilon \to 0$. *In order to get uniform estimates in* ϵ*, it is crucial that the estimates in Propositions 2.1 do not depend on* $\|a\|_{L^{\frac{n}{n-1}}(\Omega)}$. *Assumption* (2.1) *is just necessary to make meaningful the notion of* $W^{1,n}-$*weak solution for* (2.3)*. The same remark is in order for Proposition 4.1, when we will use it in Section 4 to show the logarithmic behavior of* $\hat{U}$ *at* 0.

As a second by-product of Proposition 2.1 we have that

Theorem 2.5. *There holds*

$$\hat{U} \in W^{1,q}_{loc}(\mathbb{R}^n) \tag{2.29}$$

for all $1 \le q < n$.

Proof. Since (2.24) does hold in A_ϵ, (2.27) can be re-written as

$$\begin{cases} \Delta_n(\hat{U} - U_\epsilon) - \Delta_n \hat{U} = \frac{e^{\hat{U}}}{|x|^{2n}} & \text{in } A_\epsilon \\ U_\epsilon = 0 & \text{on } \partial A_\epsilon. \end{cases} \tag{2.30}$$

Since

$$d = \inf_{v \ne w} \frac{\langle |v|^{n-2}v - |w|^{n-2}w, v-w\rangle}{|v-w|^n} > 0, \tag{2.31}$$

we can apply Proposition 2.1 to $\mathbf{a}(x,p)$ in (2.28). Since $|A_\epsilon| \le \omega_n r^n$ and $\mathbf{a}(x,0)=0$, we deduce that

$$\int_{A_\epsilon} |\nabla U_\epsilon|^q + \int_{A_\epsilon} e^{pU_\epsilon} \le C \tag{2.32}$$

for all $1 \le q < n$ and all $p \ge 1$ if r is sufficiently small, where C is uniform in ϵ. Notice that

$$\int_{B_r(0)} \frac{e^{\hat{U}}}{|x|^{2n}} = \int_{\mathbb{R}^n \setminus B_{\frac{1}{r}}(0)} e^U \to 0$$

as $r \to 0$. By the Sobolev embedding $\mathcal{D}^{1,\frac{n}{2}}(\mathbb{R}^n) \hookrightarrow L^n(\mathbb{R}^n)$ estimate (2.32) yields that

$$\int_{A_\epsilon} |U_\epsilon|^n \le C \tag{2.33}$$

for some C uniform in ϵ. Since $H_\epsilon = \hat{U} - U_\epsilon$ with $\hat{U} \in C^{1,\alpha}(\mathbb{R}^n \setminus \{0\})$, by (2.33) we deduce that

$$\|H_\epsilon\|_{L^n(A)} \le C(A) \qquad \forall\ A \subset\subset \overline{B_r(0)} \setminus \{0\}$$

for all ϵ sufficently small. Arguing as before, by [16, 26, 34, 37] it follows that

$$\|H_\epsilon\|_{C^{1,\alpha}(A)} \le C(A) \qquad \forall\ A \subset\subset \overline{B_r(0)} \setminus \{0\}$$

for ϵ small. By the Ascoli-Arzelá's Theorem and a diagonal process, we can find a sequence $\epsilon \to 0$ so that $H_\epsilon \to H_0$ in $C^1_{\text{loc}}(\overline{B_r(0)} \setminus \{0\})$, where H_0 satisfies

$$\begin{cases} \Delta_n H_0 = 0 & \text{in } B_r(0) \setminus \{0\} \\ H_0 = \hat{U} & \text{on } \partial B_r(0). \end{cases}$$

Since $H_\epsilon \le \hat{U}$ in A_ϵ by the comparison principle, we have that $U_\epsilon \to U_0 := \hat{U} - H_0$ in $C^1_{\text{loc}}(\overline{B_r(0)} \setminus \{0\})$, where U_0 satisfies

$$U_0 \ge 0 \text{ in } B_r(0) \setminus \{0\}, \qquad \partial_\nu U_0 \le 0 \text{ on } \partial B_r(0).$$

Moreover, by (2.32) we get that

$$U_0 \in W^{1,q}_0(B_r(0)), \qquad e^{U_0} \in L^p(B_r(0)) \tag{2.34}$$

for all $1 \le q < n$ and all $p \ge 1$ if r is sufficiently small.

Since H_0 is a continuous $n-$harmonic function in $B_r(0) \setminus \{0\}$ with

$$H_0 \le \sup_{\mathbb{R}^n \setminus \{0\}} \hat{U} = \sup_{\mathbb{R}^n} U < \infty$$

in view of Theorem 2.3, we can apply the result in [34] about isolated singularities: either H_0 has a removable singularity at 0 or

$$\frac{1}{C} \le \frac{H_0(x)}{\ln|x|} \le C$$

near 0 for some $C > 1$. According to [35], in both situations we have that

$$H_0 \in W^{1,q}(B_r(0)) \tag{2.35}$$

for all $1 \le q < n$. The combination of (2.34) and (2.35) establishes the validity of (2.29) for $\hat{U} = U_0 + H_0$. □

In terms of U, Theorem 2.5 simply gives that

Corollary 2.6. *There holds*

$$\int_{\mathbb{R}^n\setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}} < +\infty \tag{2.36}$$

for all $1 \le q < n$.

Proof. Since

$$\left|\det D\frac{x}{|x|^2}\right| = \frac{1}{|x|^{2n}}$$

and

$$|\nabla \hat{U}|(x) = \frac{1}{|x|^2}|\nabla U|(\frac{x}{|x|^2}),$$

we have that

$$\int_{B_r(0)} |\nabla \hat{U}|^q = \int_{\mathbb{R}^n\setminus B_{\frac{1}{r}}(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}}.$$

By Theorems 2.3 and 2.5 we then deduce that

$$\int_{\mathbb{R}^n\setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}} < +\infty$$

for all $1 \le q < n$, as desired. □

3. An isoperimetric argument

The aim is to classify all the solutions U of (1.1) with small "mass". The following isoperimetric approach leads to:

Theorem 3.1. *Let U be a solution of* (1.1) *with $\int_{\mathbb{R}^n} e^U \le c_n\omega_n$. Then U is given by* (1.3).

Proof. Since $U \in C^{1,\alpha}(\mathbb{R}^n)$, we can use Theorem 3.1 in [32] to get that $Z_k = \{x \in B_k(0) : \nabla U(x) = 0\}$ is a null set for all $k \in \mathbb{N}$. By the Lipschitz continuity of U on $B_k(0)$, we deduce that

$$\{t \in \mathbb{R} : \exists\, x \in \mathbb{R}^n \text{ s.t. } U(x) = t,\ \nabla U(x) = 0\} = \bigcup_{k\in\mathbb{N}} U(Z_k)$$

is a null set in $\mathbb{R}$. Therefore $\Omega_t = \{U > t\}$ is a smooth set for a.e. $t \le t_0$, $t_0 = \sup_{\mathbb{R}^n} U$, and has bounded Lebesgue measure in view of $\displaystyle\int_{\mathbb{R}^n} e^U < +\infty$.

Let $t \le t_0$ and $r > 0$. Given $\delta, \eta > 0$, let us define the following functions:

$$\chi_\delta(s) = \begin{cases} 0 & \text{if } s \le t \\ \frac{s-t}{\delta} & \text{if } t \le s \le t+\delta \\ 1 & \text{if } s \ge t+\delta \end{cases}$$

and

$$\chi_\eta(x) = \begin{cases} 1 & \text{if } x \in B_r(0) \\ \frac{r+\eta-|x|}{\eta} & \text{if } x \in B_{r+\eta}(0)\setminus B_r(0) \\ 0 & \text{if } x \notin B_{r+\eta}. \end{cases}$$

We can use $\chi_\delta(U)\chi_\eta(x)$ as a test function in (1.2) to get

$$\int_{\mathbb{R}^n} e^U \chi_\delta(U)\chi_\eta(x) = \frac{1}{\delta}\int_{\Omega_t\setminus\Omega_{t+\delta}} \chi_\eta |\nabla U|^n - \frac{1}{\eta}\int_{B_{r+\eta}(0)\setminus B_r(0)} \chi_\delta(U)|\nabla U|^{n-2}\langle \nabla U, \frac{x}{|x|}\rangle. \tag{3.1}$$

By the Lebesgue's monotone convergence theorem for the first term in the R.H.S. of (3.1) we have that

$$\frac{1}{\delta}\int_{\Omega_t\setminus\Omega_{t+\delta}} \chi_\eta|\nabla U|^n \to \frac{1}{\delta}\int_{(\Omega_t\setminus\Omega_{t+\delta})\cap B_r(0)} |\nabla U|^n$$

as $\eta \to 0$. Since by the co-area formula we can write

$$\int_{(\Omega_t\setminus\Omega_{t+\delta})\cap B_r(0)} |\nabla U|^n = \int_t^{t+\delta} ds \int_{\partial\Omega_s\cap B_r(0)} |\nabla U|^{n-1} d\sigma,$$

it results that the function $t \to \int_{\partial\Omega_t\cap B_r(0)} |\nabla U|^{n-1} d\sigma$ is in $L^1_{\text{loc}}(\mathbb{R})$, and as $\delta \to 0$ by the Lebesgue's differentiation Theorem we conclude that for a.e. $t \le t_0$

$$\frac{1}{\delta}\int_{\Omega_t\setminus\Omega_{t+\delta}} \chi_\eta|\nabla U|^n \to \int_{\partial\Omega_t\cap B_r(0)} |\nabla U|^{n-1} d\sigma \tag{3.2}$$

as $\eta \to 0$ and $\delta \to 0$. The second term in the R.H.S. of (3.1) writes in radial coordinates as

$$\frac{1}{\eta}\int_{B_{r+\eta}(0)\setminus B_r(0)} \chi_\delta(U)|\nabla U|^{n-2}\langle\nabla U, \frac{x}{|x|}\rangle = \frac{1}{\eta}\int_r^{r+\eta} ds \int_{\partial B_s(0)} \chi_\delta(U)|\nabla U|^{n-2}\langle \nabla U, \frac{x}{|x|}\rangle d\sigma,$$

and by the fundamental Theorem of calculus we get that for all $r>0$

$$\frac{1}{\eta}\int_{B_{r+\eta}(0)\setminus B_r(0)} \chi_\delta(U)|\nabla U|^{n-2}\langle \nabla U, \frac{x}{|x|}\rangle \to \int_{\partial B_r(0)} \chi_\delta(U)|\nabla U|^{n-2}\langle \nabla U, \frac{x}{|x|}\rangle d\sigma$$

as $\eta\to 0$. By the Lebesgue's monotone convergence theorem we deduce that for all $r>0$

$$\frac{1}{\eta}\int_{B_{r+\eta}(0)\setminus B_r(0)} \chi_\delta(U)|\nabla U|^{n-2}\langle \nabla U, \frac{x}{|x|}\rangle \to \int_{\Omega_t\cap \partial B_r(0)} |\nabla U|^{n-2}\langle \nabla U, \frac{x}{|x|}\rangle d\sigma \tag{3.3}$$

as $\eta\to 0$ and $\delta\to 0$. Letting $\eta\to 0$ and $\delta\to 0$ in (3.1), by (3.2)-(3.3) we finally get that

$$\int_{\Omega_t\cap B_r(0)} e^U = \int_{\partial\Omega_t\cap B_r(0)} |\nabla U|^{n-1}d\sigma - \int_{\Omega_t\cap \partial B_r(0)} |\nabla U|^{n-2}\langle \nabla U, \frac{x}{|x|}\rangle d\sigma. \tag{3.4}$$

for all $r>0$ and a.e. $t\le t_0$ (possibly depending on r) in view of the Lebesgue's monotone convergence theorem.

Remark 3.2. *We aim to let $r\to+\infty$ in (3.4). In [9] no special care is required since for $n=2$ U has a logarithmic behavior at infinity and then Ω_t is a bounded set. When $n>2$ we still don't know that U behaves logarithmically at infinity and the validity of Theorem 3.1 is crucial in the next Section to establish such a property. Our argument relies instead on (2.36) and on the finite measure property of Ω_t, compare with [22].*

In radial coordinates we can write

$$|\Omega_t| = \int_0^\infty dr \int_{\Omega_t\cap\partial B_r(0)} d\sigma<+\infty, \qquad \int_{\mathbb{R}^n\setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}} = \int_1^\infty \frac{dr}{r^{2(n-q)}}\int_{\partial B_r(0)} |\nabla U|^q d\sigma<+\infty \tag{3.5}$$

in view of (2.36). We claim that for all $M\ge 1$ there exists $r\ge M$ so that

$$\int_{\Omega_t\cap\partial B_r(0)} d\sigma \le \frac{1}{r} \quad \text{and} \quad \frac{1}{r^{2(n-q)}}\int_{\partial B_r(0)} |\nabla U|^q d\sigma\le \frac{1}{r}.$$

Indeed, if the claim were not true, we would find $M\ge 1$ so that for all $r\ge M$ there holds either

$$\int_{\Omega_t\cap\partial B_r(0)} d\sigma > \frac{1}{r} \tag{3.6}$$

or

$$\frac{1}{r^{2(n-q)}}\int_{\partial B_r(0)} |\nabla U|^q d\sigma > \frac{1}{r}. \tag{3.7}$$

Setting $I=\{r\ge M:\ (3.6)\text{ holds}\}$ and $II=[M,\infty)\setminus I$, we have that

$$\int_I \frac{dr}{r} < \int_M^\infty dr\int_{\Omega_t\cap\partial B_r(0)} d\sigma \le |\Omega_t| \tag{3.8}$$

and

$$\int_{II}\frac{dr}{r} < \int_M^\infty \frac{dr}{r^{2(n-q)}}\int_{\partial B_r(0)}|\nabla U|^q d\sigma \le \int_{\mathbb{R}^n\setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}} \tag{3.9}$$

since (3.7) does hold for all $r\in II$. Summing up (3.8)-(3.9) we get that

$$\infty = \int_M^\infty \frac{dr}{r}\le |\Omega_t| + \int_{\mathbb{R}^n\setminus B_1(0)} \frac{|\nabla U|^q}{|x|^{2(n-q)}}$$

in contradiction with (3.5), and the claim is established.

Thanks to the claim we can construct a sequence $r_k\to+\infty$ so that

$$\int_{\Omega_t\cap\partial B_{r_k}(0)} d\sigma\le\frac{1}{r_k},\quad \frac{1}{r_k^{2(n-q)}}\int_{\partial B_{r_k}(0)}|\nabla U|^q d\sigma \le \frac{1}{r_k}. \tag{3.10}$$

By (3.10) and the Hölder's inequality we deduce the crucial estimate

$$\int_{\Omega_t\cap\partial B_{r_k}(0)} |\nabla U|^{n-1}d\sigma \le \left(\int_{\Omega_t\cap\partial B_{r_k}(0)}|\nabla U|^q d\sigma\right)^{\frac{n-1}{q}}\left(\int_{\Omega_t\cap\partial B_{r_k}(0)} d\sigma\right)^{\frac{q-(n-1)}{q}} \le \frac{1}{r_k^{1-2\frac{(n-q)(n-1)}{q}}}\to 0 \tag{3.11}$$

by choosing $q\in(n-1,n)$ sufficiently close to n.

Choosing $r=r_k$ in (3.4) and letting $k\to+\infty$ we get that

$$\int_{\Omega_t} e^U = \int_{\partial\Omega_t}|\nabla U|^{n-1}d\sigma \tag{3.12}$$

for a.e. $t\le t_0$ in view of (3.11). Arguing as previously, by the co-area formula and the Lebesgue's differentiation theorem we have that

$$|\Omega_t| = \lim_{r\to+\infty}|\Omega_t\cap B_r(0)| = \lim_{r\to+\infty}\int_t^\infty ds\int_{\partial\Omega_s\cap B_r(0)}\frac{d\sigma}{|\nabla U|} = \int_t^\infty ds\int_{\partial\Omega_s}\frac{d\sigma}{|\nabla U|},$$

and then

$$-\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} \frac{d\sigma}{|\nabla U|} \tag{3.13}$$

for a.e. $t \le t_0$. Thanks to (3.12)-(3.13), by the Hölder's and the isoperimetric inequalities we can now compute

$$\begin{aligned} -\frac{d}{dt}\left(\int_{\Omega_t} e^U dx\right)^{\frac{n}{n-1}} &= -\frac{n}{n-1}\left(\int_{\Omega_t} e^U dx\right)^{\frac{1}{n-1}} e^t \frac{d}{dt}|\Omega_t| \\ &= \frac{n}{n-1}\left(\int_{\partial\Omega_t} |\nabla U|^{n-1} d\sigma\right)^{\frac{1}{n-1}} e^t \int_{\partial\Omega_t} \frac{d\sigma}{|\nabla U|} \\ &\ge \frac{n}{n-1} e^t |\partial\Omega_t|^{\frac{n}{n-1}} \ge (c_n\omega_n)^{\frac{1}{n-1}} e^t |\Omega_t| \end{aligned} \tag{3.14}$$

for a.e. $t \le t_0$. Since $t \to \int_{\Omega_t} e^U dx$ is a monotone decreasing function, we get that

$$\left(\int_{\mathbb{R}^n} e^U dx\right)^{\frac{n}{n-1}} \ge \int_{-\infty}^{t_0} -\frac{d}{dt}\left(\int_{\Omega_t} e^U dx\right)^{\frac{n}{n-1}} dt \ge (c_n\omega_n)^{\frac{1}{n-1}} \int_{\mathbb{R}^n} e^U dx. \tag{3.15}$$

Since by assumption $\int_{\mathbb{R}^n} e^U dx \le c_n\omega_n$, we get that

$$\int_{\mathbb{R}^n} e^U dx = c_n\omega_n$$

and the inequalities in (3.14)-(3.15) are actually equalities. We have that for a.e. $t \le t_0$

- $\Omega_t = B_{R(t)}(x(t))$ for some $R(t) > 0$ and $x(t) \in \mathbb{R}^n$, since Ω_t in an extremal of the isoperimetric inequality
- $|\nabla U|^{n-1}$ is a multiple of $\frac{1}{|\nabla U|}$ on $\partial\Omega_t$,
- the function $M(t) = \int_{\Omega_t} e^U dx$ is absolutely continuous in $(-\infty, t_0)$ with

$$\frac{1}{n-1} M^{\frac{1}{n-1}}(t) M'(t) = \frac{1}{n}\frac{d}{dt} M^{\frac{n}{n-1}}(t) = -(c_n\omega_n)^{\frac{1}{n-1}} \frac{\omega_n}{n} e^t R^n(t). \tag{3.16}$$

The aim now is to derive an equation for $M(t)$ by means of some Pohozaev identity. Let us emphasize that $U \in C^{1,\alpha}(\mathbb{R}^n)$ and the classical Pohozaev identities usually require more regularity. In [12] a self-contained proof is provided in the quasilinear case, which reads in our case as

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a smooth bounded domain and f be a locally Lipschitz continuous function. Then, there holds*

$$n\int_\Omega F(U) = \int_{\partial\Omega} \left[F(U)\langle x-y,\nu\rangle + |\nabla U|^{n-2}\langle x-y,\nabla U\rangle \partial_\nu U - \frac{|\nabla U|^n}{n}\langle x-y,\nu\rangle\right]$$

for all $y \in \mathbb{R}^n$ and all weak solution $U \in C^{1,\alpha}(\Omega)$ of $-\Delta_n U = f(U)$ in Ω, where $F(t) = \int_0^t f(s)ds$ and ν is the unit outward normal vector at $\partial\Omega$.

Let us re-write (3.12) as

$$M(t) = n\omega_n |\nabla U|^{n-1} R^{n-1}(t) \tag{3.17}$$

and use Lemma 3.3 on $\Omega_t = B_{R(t)}(x(t))$ with $y = x(t)$ to deduce

$$M(t) = \omega_n e^t R^n(t) + \frac{n-1}{n}\omega_n |\nabla U|^n R^n(t) \tag{3.18}$$

in view of $U = t$ and $|\nabla U| = -\partial_\nu U$ constant on $\partial\Omega_t$. By (3.17)-(3.18) we have that

$$\omega_n e^t R^n(t) = M(t) - (c_n\omega_n)^{-\frac{1}{n-1}} M^{\frac{n}{n-1}}(t), \tag{3.19}$$

which, inserted into (3.16), gives rise to

$$M'(t) = -\frac{n-1}{n}(c_n\omega_n)^{\frac{1}{n-1}} M^{\frac{n-2}{n-1}}(t) + \frac{n-1}{n} M(t) \tag{3.20}$$

for a.e. $t \le t_0$. Since M is absolutely continuous in $\mathbb{R}$ and

$$\frac{1}{n-1}\int \frac{dM}{M - (c_n\omega_n)^{\frac{1}{n-1}} M^{\frac{n-2}{n-1}}} = \ln |M^{\frac{1}{n-1}} - (c_n\omega_n)^{\frac{1}{n-1}}|,$$

we can integrate (3.20) to get

$$M(t) = c_n\omega_n \left[1 - e^{\frac{t-t_0}{n}}\right]^{n-1} \tag{3.21}$$

for all $t \le t_0$, in view of $M(t_0) = 0$. Inserting (3.21) into (3.19) we deduce that

$$R^n(t) = c_n \left[1 - e^{\frac{t-t_0}{n}}\right]^{n-1} e^{-\frac{(n-1)t}{n} - \frac{t_0}{n}} \tag{3.22}$$

for a.e. $t \le t_0$. Since $R(t)$ is monotone, notice that (3.22) is valid for all $t \le t_0$ and can be re-written as

$$e^t = \frac{c_n \lambda^n}{(1+\lambda^{\frac{n}{n-1}} R^{\frac{n}{n-1}}(t))^n} \tag{3.23}$$

where $\lambda = (\frac{e^{t_0}}{c_n})^{\frac{1}{n}}$. To conclude, we just need to show that $x(t) = x_0$. First notice that a.e. $t_1, t_2 \le t_0$ either $x(t_1) = x(t_2)$ or, assuming for example $t_2 < t_1$, $B_{R(t_1)}(x(t_1)) \subset\subset B_{R(t_2)}(x(t_2))$ and $x(t_2) - R(t_2)\frac{x(t_2)-x(t_1)}{|x(t_2)-x(t_1)|} \in \partial B_{R(t_2)}(x(t_2))$ implies

$$R(t_2) - |x(t_1) - x(t_2)| = \big||x(t_2) - x(t_1)| - R(t_2)\big| = |x(t_2) - R(t_2)\frac{x(t_2)-x(t_1)}{|x(t_2)-x(t_1)|} - x(t_1)| > R(t_1).$$

In both cases, we have that $|x(t_2) - x(t_1)| \le |R(t_2) - R(t_1)|$ for a.e. $t_1, t_2 \le t_0$. Since $R \in C(-\infty, t_0] \cap C^1(-\infty, t_0)$, $x(t)$ can be uniquely extended as a map $\tilde{x}(t)$ which is continuous in $(-\infty, t_0]$ and locally Lipschitz in $(-\infty, t_0)$. Given $t < t_0$ we can alway find $t_n \downarrow t$ so that $\Omega_{t_n} = B_{R(t_n)}(x(t_n))$, $x(t_n) = \tilde{x}(t_n)$, and then there holds

$$\Omega_t = \bigcup_{n\in\mathbb{N}} \Omega_{t_n} = \bigcup_{n\in\mathbb{N}} B_{R(t_n)}(x(t_n)) = B_{R(t)}(\tilde{x}(t))$$

by the continuity of $R(t)$ and $\tilde{x}(t)$. Identifying x and $\tilde{x}$, we can assume that $x \in C(-\infty, t_0] \cap Lip_{\rm loc}(-\infty, t_0)$ and $\Omega_t = B_{R(t)}(x(t))$ for all $t \le t_0$. Use now the property $t = U(x(t) + R(t)\omega)$, $\omega \in \mathbb{S}^n$, to deduce

$$\begin{aligned} h &= U(x(t+h) + R(t+h)\omega) - U(x(t) + R(t)\omega) = \langle \nabla U(x(t) + R(t)\omega), x(t+h) - x(t)\rangle \\ &\quad + [R(t+h) - R(t)]\langle \nabla U(x(t) + R(t)\omega), \omega\rangle + o(|x(t+h) - x(t)| + |R(t+h) - R(t)|) \end{aligned}$$

as $h \to 0$, uniformly in $\omega \in \mathbb{S}^n$. Since $|\nabla U|$ is a non-zero constant on $\partial\Omega_t$ for a.e. $t \le t_0$ and $\Omega_t = B_{R(t)}(x(t))$, we have that

$$\nabla U(x(t) + R(t)\omega) = -|\nabla U|\omega,$$

and then, applied to $-\omega$ and ω, it yields that

$$\begin{aligned} h &= |\nabla U|\langle x(t+h) - x(t), \omega\rangle - [R(t+h) - R(t)]|\nabla U| + o(|x(t+h) - x(t)| + |R(t+h) - R(t)|) \\ h &= -|\nabla U|\langle x(t+h) - x(t), \omega\rangle - [R(t+h) - R(t)]|\nabla U| + o(|x(t+h) - x(t)| + |R(t+h) - R(t)|). \end{aligned}$$

Since $|\nabla U| \ne 0$, the difference then gives

$$\langle x(t+h) - x(t), \omega\rangle = o(|x(t+h) - x(t)| + |R(t+h) - R(t)|)$$

as $h \to 0$, uniformly in $\omega \in \mathbb{S}^n$. If $x(t+h) \ne x(t)$, the choice $\omega = \frac{x(t+h)-x(t)}{|x(t+h)-x(t)|}$ leads to

$$|\frac{x(t+h) - x(t)}{h}| \le o(|\frac{R(t+h) - R(t)}{h}|) \to 0$$

as $h \to 0$. So we have shown that $x'(t) = \lim_{h\to 0} \dfrac{x(t+h) - x(t)}{h} = 0$ for a.e. $t \le t_0$. Since $x \in Lip_{\rm loc}(-\infty, t_0)$, by integration we deduce that $x(t)$ is constant for all $t \le t_0$, say $x(t) = x_0$.

Given $x \in \mathbb{R}^n \setminus \{x_0\}$, by (3.22) we can find a unique $t < t_0$ so that $R(t) = |x - x_0|$ and then

$$e^{U(x)} = \frac{c_n \lambda^n}{(1 + \lambda^{\frac{n}{n-1}} |x - x_0|^{\frac{n}{n-1}})^n}$$

in view of (3.23) and $U = t$ on $\partial B_{R(t)}(x_0)$. The proof is complete since we have shown that $U = U_{\lambda, x_0}$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.

□

4. Behavior of U at infinity

The estimates in Proposition 2.1 are not sufficient to establish the logarithmic behavior of U at infinity but are essentially optimal in the limiting case $f \in L^1(\Omega)$. According to [34, 35], a bit more regularity on f gives L^∞-bounds as stated in

Proposition 4.1. *Let $f \in L^p(\Omega)$, $p > 1$, and assume (2.1)-(2.2). Let $u \in W_0^{1,n}(\Omega)$ be a weak solution of $-div\, \mathbf{a}(x, \nabla u) = f$. Then*

$$\|u\|_\infty \le C(\frac{\|f\|_p}{d} + 1)^{\alpha_0} (|\Omega| + 1)^{\beta_0} \|u\|^{\bar{q}}_{\frac{npq_1}{p-1}}$$

for some constants $C, \alpha_0, \beta_0, \bar{q} > 0$ just depending on n, p and $q_1 \ge 1$.

Proof. Given $q \ge 1$ and $k > 0$ set

$$F(s) = \begin{cases} s^q & \text{if } 0 \le s \le k \\ qk^{q-1}s - (q-1)k^q & \text{if } s \ge k \end{cases}$$

and $G(s) = F(s)[F'(s)]^{n-1}$. Notice that G is a piecewise C^1-function with a corner just at $s = k$ so that

$$[F'(s)]^n \le G'(s), \qquad G(s) \le q^{n-1} F^{\frac{n(q-1)+1}{q}}(s). \tag{4.1}$$

Since $G(|u|) \in W_0^{1,n}(\Omega)$ for G is linear at infinity, use $\mathrm{sign}(u)G(|u|)$ as a test function in the equation of u to get

$$\int_\Omega |\nabla F(|u|)|^n \leq \frac{1}{d}\int_\Omega G'(|u|)\langle \mathbf{a}(x,\nabla u),\nabla u\rangle = \frac{1}{d}\int_\Omega f\ \mathrm{sign}(u)G(|u|) \tag{4.2}$$

in view of (2.2) and (4.1). Setting $m=\frac{p}{p-1}$ in view of $p>1$, by (4.1) and the Hölder's inequality we deduce that

$$|\int_\Omega f\ \mathrm{sign}(u)G(|u|)| \leq q^{n-1}\int_\Omega |f|F^{\frac{n(q-1)+1}{q}}(|u|) \leq q^{n-1}|\Omega|^{\frac{n-1}{mnq}}\|f\|_p\left(\int_\Omega F^{mn}(|u|)\right)^{\frac{n(q-1)+1}{mnq}}. \tag{4.3}$$

The Sobolev embedding Theorem applied on $F(|u|)\in W_0^{1,n}(\Omega)$ now implies that

$$\left(\int_\Omega F^{2mn}(|u|)\right)^{\frac{1}{2m}} \leq C\int_\Omega |\nabla F(|u|)|^n \leq \frac{C}{d}q^{n-1}|\Omega|^{\frac{n-1}{mnq}}\|f\|_p\left(\int_\Omega F^{mn}(|u|)\right)^{\frac{n(q-1)+1}{mnq}}$$

for some $C\geq 1$ in view of (4.2)-(4.3). Since $F(s)\to s^q$ in a monotone way as $k\to+\infty$, we have that

$$\left(\int_\Omega |u|^{2mnq}\right)^{\frac{1}{2mq}} \leq \exp\left[\frac{1}{q}\ln\frac{C\|f\|_p}{d}+\frac{(n-1)\ln|\Omega|}{mnq^2}+(n-1)\frac{\ln q}{q}\right]\left(\int_\Omega |u|^{mnq}\right)^{\frac{1}{mq}[1-\frac{n-1}{nq}]}. \tag{4.4}$$

Assume now that $u\in L^{mnq_1}(\Omega)$ for some $q_1\geq 1$. Setting $q_j=2^{j-1}q_1$, $j\in\mathbb{N}$, by iterating (4.4) we deduce that

$$\begin{aligned}
\left(\int_\Omega |u|^{mnq_{j+1}}\right)^{\frac{1}{mq_{j+1}}} &\leq \exp\left[\frac{1}{q_j}\ln\frac{C\|f\|_p}{d}+\frac{(n-1)\ln|\Omega|}{mnq_j^2}+(n-1)\frac{\ln q_j}{q_j}\right]\left[\left(\int_\Omega |u|^{mnq_j}\right)^{\frac{1}{mq_j}}\right]^{1-\frac{n-1}{nq_j}}\\
&\leq \exp\left[\ln\frac{C\|f\|_p}{d}\sum_{k=j-1}^{j}\frac{a_k^j}{q_k}+\frac{(n-1)\ln|\Omega|}{mn}\sum_{k=j-1}^{j}\frac{a_k^j}{q_k^2}+(n-1)\sum_{k=j-1}^{j}\frac{a_k^j\ln q_k}{q_k}\right]\left[\left(\int_\Omega |u|^{mnq_{j-1}}\right)^{\frac{1}{mq_{j-1}}}\right]^{a_{j-2}^j}\\
\cdots &\leq \exp\left[\ln\frac{C\|f\|_p}{d}\sum_{k=1}^{j}\frac{a_k^j}{q_k}+\frac{(n-1)\ln|\Omega|}{mn}\sum_{k=1}^{j}\frac{a_k^j}{q_k^2}+(n-1)\sum_{k=1}^{j}\frac{a_k^j\ln q_k}{q_k}\right]\left(\int_\Omega |u|^{mnq_1}\right)^{\frac{a_0^j}{mq_1}}
\end{aligned}$$

where

$$a_k^j=\begin{cases}[1-\frac{n-1}{nq_{k+1}}]\times\cdots\times[1-\frac{n-1}{nq_j}] & \text{if } 0\leq k<j\\ 1 & \text{if } k=j.\end{cases}$$

Since $a_k^j\leq 1$ for all $k=0,\dots,j$, we have that

$$\begin{aligned}
\alpha_0 &= \frac{1}{n}\sup_{j\in\mathbb{N}}\sum_{k=1}^{j}\frac{a_k^j}{q_k}\leq\frac{1}{n}\sup_{j\in\mathbb{N}}\sum_{k=1}^{j}\frac{1}{q_k}=\frac{2}{n}\sum_{k=1}^{\infty}\frac{1}{q_1 2^k}<\infty\\
\beta_0 &= \frac{n-1}{mn^2}\sup_{j\in\mathbb{N}}\sum_{k=1}^{j}\frac{a_k^j}{q_k^2}\leq\frac{4(n-1)}{mn^2}\sum_{k=1}^{\infty}\frac{1}{q_1^2 4^k}<+\infty\\
\gamma_0 &= \frac{n-1}{n}\sup_{j\in\mathbb{N}}\sum_{k=1}^{j}\frac{a_k^j\ln q_k}{q_k}\leq 2\frac{n-1}{n}\sum_{k=1}^{\infty}\frac{(k-1)\ln 2+\ln q_1}{q_1 2^k}<+\infty,
\end{aligned}$$

and then it follows that

$$\left(\int_\Omega |u|^{mnq_{j+1}}\right)^{\frac{1}{mnq_{j+1}}} \leq \exp\left[\alpha_0\ln C(\frac{\|f\|_p}{d}+1)+\beta_0\ln(|\Omega|+1)+\gamma_0\right]\left(\int_\Omega |u|^{mnq_1}\right)^{\frac{a_0^j}{mnq_1}}. \tag{4.5}$$

Since

$$\bar q=\lim_{j\to+\infty}a_0^j=\prod_{k=1}^{\infty}(1-\frac{n-1}{nq_k})<\infty,$$

letting $j\to+\infty$ in (4.5) we finally deduce that

$$\|u\|_\infty\leq e^{\alpha_0\ln C+\gamma_0}(\frac{\|f\|_p}{d}+1)^{\alpha_0}(|\Omega|+1)^{\beta_0}\|u\|_{mnq_1}^{\bar q}$$

and the proof is complete. □

Thanks to Theorem 3.1 we are just concerned with the range

$$\int_{\mathbb{R}^n} e^U\geq c_n\omega_n. \tag{4.6}$$

By Proposition 4.1 we can improve the estimates in Section 2 to get

Theorem 4.2. *Let U be a solution of* (1.1) *which satisfies* (4.6)*. Then $\hat{U}(x) = U(\frac{x}{|x|^2})$ satisfies*

$$\hat{U}(x) - (\frac{\gamma_0}{n\omega_n})^{\frac{1}{n-1}} \ln|x| \in L^\infty_{loc}(\mathbb{R}^n) \tag{4.7}$$

and

$$\sup_{|x|=r} |x||\nabla \left(\hat{U}(x) - (\frac{\gamma_0}{n\omega_n})^{\frac{1}{n-1}} \ln|x|\right)| \to 0 \tag{4.8}$$

for a sequence $r \to 0$, where $\gamma_0 = \int_{\mathbb{R}^n} e^U$.

Proof. We adopt the same notations as in Theorem 2.5, and we try to push more the analysis thanks to (4.6). Given $r > 0$, recall that $\hat{U}$ has been decomposed in $B_r(0)$ as $\hat{U} = U_0 + H_0$, $U_0, H_0 \in C^1_{\text{loc}}(\overline{B_r(0)} \setminus \{0\})$, where H_0 is a $n-$harmonic function in $B_r(0) \setminus \{0\}$ with $\sup_{B_r(0)\setminus\{0\}} H_0 < +\infty$ and $U_0 \geq 0$ satisfies (2.34) with

$$U_0 = 0, \qquad \partial_\nu U_0 \leq 0 \text{ on } \partial B_r(0).$$

The desciption of the behavior of H_0 at 0, as established in [34, 35], has been later improved in [21] to show that there exists $\gamma \geq 0$ with

$$H_0(x) - (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln|x| \in L^\infty(B_r(0)), \qquad \Delta_n H_0 = \gamma\delta_0 \text{ in } \mathcal{D}'(B_r(0)). \tag{4.9}$$

Since $\hat{U} \in W^{1,n-1}(B_r(0))$ according to Theorem 2.5, we can extend (2.24) at 0 as

$$-\Delta_n \hat{U} = \frac{e^{\hat{U}}}{|x|^{2n}} - \gamma_0\delta_0 \tag{4.10}$$

in the sense

$$\int_{\mathbb{R}^n} |\nabla\hat{U}|^{n-2}\langle\nabla\hat{U}, \nabla\Phi\rangle = \int_{\mathbb{R}^n} \frac{e^{\hat{U}}}{|x|^{2n}}\Phi - \gamma_0\Phi(0) \tag{4.11}$$

for all $\Phi \in C^1(\mathbb{R}^n)$ so that $\hat{\Phi} \in W^{1,n}_{\text{loc}}(\mathbb{R}^n)$. Indeed, let us consider a smooth function η so that $\eta = 0$ for $|x| \leq \delta$, $\eta = 1$ for $|x| \geq 2\delta$ and $|\nabla\eta| \leq \frac{2}{\delta}$. Use $\eta[\Phi - \Phi(0)] \in \hat{H}$ as a test function in (2.25) to provide

$$\int_{\mathbb{R}^n} \eta|\nabla\hat{U}|^{n-2}\langle\nabla\hat{U}, \nabla\Phi\rangle + O(\int_{\mathbb{R}^n} |\nabla\hat{U}|^{n-1}|\nabla\eta||\Phi - \Phi(0)|) = \int_{\mathbb{R}^n} \eta\frac{e^{\hat{U}}}{|x|^{2n}}(\Phi - \Phi(0)). \tag{4.12}$$

Since

$$\int_{\mathbb{R}^n} |\nabla\hat{U}|^{n-1}|\nabla\eta||\Phi - \Phi(0)| \leq C\int_{B_{2\delta}(0)} |\nabla\hat{U}|^{n-1} \to 0$$

as $\delta \to 0$, we can let $\delta \to 0$ in (4.12) and get the validity of (4.11) in view of $\gamma_0 = \int_{\mathbb{R}^n} \frac{e^{\hat{U}}}{|x|^{2n}} = \int_{\mathbb{R}^n} e^U$

Since $U_0 \geq 0$, the singularity of $\hat{U} = U_0 + H_0$ at 0 should be weaker than that of H_0. Via an approximation procedure, it is easily seen that equations (4.9)-(4.10) can be re-written as

$$\gamma\Phi(0) = \int_{\partial B_r(0)} |\nabla H_0|^{n-2}\partial_\nu H_0\Phi - \int_{B_r(0)} |\nabla H_0|^{n-2}\langle\nabla H_0, \nabla\Phi\rangle \tag{4.13}$$

$$\gamma_0\Phi(0) = \int_{B_r(0)} \frac{e^{\hat{U}}}{|x|^{2n}}\Phi + \int_{\partial B_r(0)} |\nabla\hat{U}|^{n-2}\partial_\nu\hat{U}\Phi - \int_{B_r(0)} |\nabla\hat{U}|^{n-2}\langle\nabla\hat{U}, \nabla\Phi\rangle \tag{4.14}$$

for all $\Phi \in C^1(B_r(0))$. We claim that

$$|\nabla H_0|^{n-2}\partial_\nu H_0 \geq |\nabla\hat{U}|^{n-2}\partial_\nu\hat{U} \qquad \text{on } \partial B_r(0) \tag{4.15}$$

and then, by taking $\Phi = 1$ in (4.13)-(4.14), we deduce that

$$\gamma = \int_{\partial B_r(0)} |\nabla H_0|^{n-2}\partial_\nu H_0 \geq \int_{\partial B_r(0)} |\nabla\hat{U}|^{n-2}\partial_\nu\hat{U} = \gamma_0 - \int_{B_r(0)} \frac{e^{\hat{U}}}{|x|^{2n}}. \tag{4.16}$$

To establish the claim (4.15), we write $H_0 = \hat{U} - U_0$ and recall that $\nabla U_0 = (\partial_\nu U_0)\nu$ with $\partial_\nu U_0 \leq 0$ on $\partial B_r(0)$. Since

$$|\nabla H_0|^{n-2} = \left[|\nabla\hat{U}|^2 + (\partial_\nu U_0)^2 - 2\partial_\nu\hat{U}\partial_\nu U_0\right]^{\frac{n-2}{2}},$$

when $\partial_\nu\hat{U} \geq 0$ we have that

$$|\nabla H_0|^{n-2} \geq |\nabla\hat{U}|^{n-2}, \quad \partial_\nu H_0 \geq \partial_\nu\hat{U} \geq 0$$

and then (4.15) does hold. When $\partial_\nu U_0 \leq \partial_\nu\hat{U} < 0$ there holds $\partial_\nu H_0 \geq 0$ and then

$$|\nabla H_0|^{n-2}\partial_\nu H_0 \geq 0 > |\nabla\hat{U}|^{n-2}\partial_\nu\hat{U}.$$

When $\partial_\nu\hat{U} < \partial_\nu U_0$ we have that

$$|\nabla H_0|^{n-2} \leq |\nabla\hat{U}|^{n-2}, \quad 0 > \partial_\nu H_0 \geq \partial_\nu\hat{U}$$

and then (4.15) does hold.

Since $(\frac{\gamma_0}{n\omega_n})^{\frac{1}{n-1}} \geq \frac{n^2}{n-1}$ in view of (4.6), by (4.9) and (4.16) we have that

$$\frac{e^{H_0}}{|x|^{2n}} \in L^q(B_r(0)) \tag{4.17}$$

for all $1 \leq q < \frac{n-1}{n-2}$ if r is sufficiently small. By (2.34) and (4.17) it follows that

$$\frac{e^{\hat{U}}}{|x|^{2n}} = e^{U_0}\frac{e^{H_0}}{|x|^{2n}} \in L^q(B_r(0)) \tag{4.18}$$

for all $1 \leq q < \frac{n-1}{n-2}$ if $r > 0$ is sufficiently small. Thanks to (4.18) we can apply Proposition 4.1 to U_ϵ on A_ϵ (see (2.26)-(2.27)) with $\mathbf{a}(x,p)$ given by (2.28) to get

$$\|U_\epsilon\|_{\infty,A_\epsilon} \leq C$$

for some uniform $C > 0$. We have used that

$$\sup_\epsilon \|U_\epsilon\|_{p,A_\epsilon} < +\infty$$

for all $p \geq 1$ in view of (2.32) and the Sobolev embedding Theorem. Letting $\epsilon \to 0$ we get that $\|U_0\|_{\infty,B_r(0)} < +\infty$ and then

$$\hat{U} = U_0 + H_0 = (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln|x| + H(x), \quad H \in L^\infty_{\text{loc}}(\mathbb{R}^n) \tag{4.19}$$

in view of (4.9). Notice that now γ does not depend on r and then satisfies

$$\gamma \geq c_n\omega_n$$

in view of (4.6) and (4.16). Given $r > 0$ small, let us define the function

$$V_r(y) = \hat{U}(ry) - (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln r = (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln|y| + H(ry).$$

Since

$$\Delta_n V_r = -\frac{e^{\hat{U}(ry)}}{r^n|y|^{2n}} = -\frac{r^{\frac{n}{n-1}+\alpha}e^{H(ry)}}{|y|^{\frac{n(n-2)}{n-1}-\alpha}}$$

in view of (4.19) with $\alpha = (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} - \frac{n^2}{n-1} \geq 0$, we have that V_r and $\Delta_n V_r$ are bounded in $L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, uniformly in r. By [16, 34, 37] we deduce that V_r is bounded in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, uniformly in r. By the Ascoli-Arzelá's Theorem and a diagonal process we can find a sequence $r \to 0$ so that $V_r \to V_0$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where V_0 is a n-harmonic function in $\mathbb{R}^n \setminus \{0\}$. Setting $H_r(y) = H(ry)$, we deduce that $H_r \to H_0$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where $H_0 \in L^\infty(\mathbb{R}^n)$ in view of (4.19). Since $V_0 = (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}} \ln|y| + H_0$ with $H_0 \in L^\infty(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$, we can apply Lemma 4.3 below to show that H_0 is a constant function. In particular we get that

$$\sup_{|x|=r} |x|\Big|\nabla\left(\hat{U}(x) - (\frac{\gamma}{n\omega_n})^{\frac{1}{n-1}}\ln|x|\right)\Big| = \sup_{|y|=1}|\nabla H_r(y)| \to \sup_{|y|=1}|\nabla H_0(y)| = 0 \tag{4.20}$$

along the sequence $r \to 0$. The proof of (4.7)-(4.8) now follows by (4.19)-(4.20) once we show that $\gamma = \gamma_0$. Indeed, by (4.14) we have that

$$\gamma_0 = \int_{B_r(0)} \frac{e^{\hat{U}}}{|x|^{2n}} + \int_{\partial B_r(0)} |\nabla\hat{U}|^{n-2}\partial_\nu\hat{U} = o(1) + \frac{\gamma}{n\omega_n}\int_{\partial B_r(0)} \frac{1}{|x|^{n-1}}(1+o(1)) \to \gamma$$

where $r \to 0$ is any sequence with property (4.20). The proof is complete. □

We have used the following simple result:

Lemma 4.3. *Let $\gamma \ln|x| + H$ be a $n-$harmonic function in $\mathbb{R}^n \setminus \{0\}$ with $H \in C^1(\mathbb{R}^n \setminus \{0\})$. If $H \in L^\infty(\mathbb{R}^n)$, then H is a constant function.*

Proof. Let η be a cut-off function with compact support in $\mathbb{R}^n \setminus \{0\}$. Since

$$-\Delta_n(\gamma\ln|x| + H) = -\Delta_n(\gamma\ln|x| + H) + \Delta_n(\gamma\ln|x|) = 0 \qquad \text{in } \mathbb{R}^n \setminus \{0\},$$

we can use $\eta^n H$ as a test function to get

$$\begin{aligned} d\int_{\mathbb{R}^n} \eta^n|\nabla H|^n &\leq \int_{\mathbb{R}^n} \eta^n\langle|\nabla(\gamma\ln|x|+H)|^{n-2}\nabla(\gamma\ln|x|+H) - |\nabla(\gamma\ln|x|)|^{n-2}\nabla(\gamma\ln|x|), \nabla H\rangle \\ &= -n\int_{\mathbb{R}^n} \eta^{n-1}H\langle|\nabla(\gamma\ln|x|+H)|^{n-2}\nabla(\gamma\ln|x|+H) - |\nabla(\gamma\ln|x|)|^{n-2}\nabla(\gamma\ln|x|), \nabla\eta\rangle \end{aligned}$$

in view of (2.31). Since $H \in L^\infty(\mathbb{R}^n)$, by the Young's inequality we get that

$$d\int_{\mathbb{R}^n} \eta^n |\nabla H|^n \le Cn\|H\|_\infty \int_{\mathbb{R}^n} \eta^{n-1}\left[|\nabla H|^{n-1} + \frac{|\nabla H|}{|x|^{n-2}}\right]|\nabla \eta| \le \frac{d}{2}\int_{\mathbb{R}^n} \eta^n|\nabla H|^n + C\left[\int_{\mathbb{R}^n}|\nabla\eta|^n + \int_{\mathbb{R}^n}\frac{|\nabla\eta|^{\frac{n}{n-1}}}{|x|^{\frac{n(n-2)}{n-1}}}\right]$$

in view of $\eta \le 1$ and

$$||v+w|^{n-2}(v+w) - |w|^{n-2}w| \le C(|v|^{n-1} + |v||w|^{n-2}).$$

Hence, we have found that

$$\int_{\mathbb{R}^n} \eta^n|\nabla H|^n \le C\left[\int_{\mathbb{R}^n}|\nabla\eta|^n + \int_{\mathbb{R}^n}\frac{|\nabla\eta|^{\frac{n}{n-1}}}{|x|^{\frac{n(n-2)}{n-1}}}\right]. \tag{4.21}$$

Given $\delta \in (0,1)$, we make the following choice for η:

$$\eta(x) = \begin{cases} 0 & \text{if } |x| \le \delta^2 \\ -\frac{\ln|x| - 2\ln\delta}{\ln\delta} & \text{if } \delta^2 \le |x| \le \delta \\ 1 & \text{if } \delta \le |x| \le \frac{1}{\delta} \\ \frac{\ln|x| + 2\ln\delta}{\ln\delta} & \text{if } \frac{1}{\delta} \le |x| \le \frac{1}{\delta^2} \\ 0 & \text{if } |x| \ge \frac{1}{\delta^2}. \end{cases}$$

Since

$$\int_{\mathbb{R}^n}|\nabla\eta|^n = \frac{2}{|\ln\delta|^n}\int_{\{\delta^2\le|x|\le\delta\}}\frac{1}{|x|^n} = \frac{2\omega_{n-1}}{|\ln\delta|^{n-1}} \to 0$$

and

$$\int_{\mathbb{R}^n}\frac{|\nabla\eta|^{\frac{n}{n-1}}}{|x|^{\frac{n(n-2)}{n-1}}} = \frac{2}{|\ln\delta|^{\frac{n}{n-1}}}\int_{\{\delta^2\le|x|\le\delta\}}\frac{1}{|x|^n} = \frac{2\omega_{n-1}}{|\ln\delta|^{\frac{1}{n-1}}} \to 0$$

as $\delta \to 0$, we deduce that

$$\int_{\mathbb{R}^n}|\nabla H|^n = 0$$

by letting $\delta \to 0$ in (4.21). Then H is a constant function. □

5. Pohozaev identity

Thanks to Theorem 4.2, we aim to apply the Pohozaev identity of Lemma 3.3 to show that (4.6) automatically implies $\int_{\mathbb{R}^n} e^U = c_n\omega_n$. Combined with Theorem 3.1, it completes the proof of the classification result in Theorem 1.1.

To this aim, we show the following:

Theorem 5.1. *Let U be a solution of* (1.1) *which satisfies* (4.6)*. Then, there holds*

$$\int_{\mathbb{R}^n} e^U = c_n\omega_n.$$

Proof. Since

$$\partial_i U(x) = \sum_{k=1}^n \frac{1}{|x|^2}\left(\delta_{ik} - 2\frac{x_i x_k}{|x|^2}\right)(\partial_k\hat{U})(\frac{x}{|x|^2}),$$

we have that

$$|\nabla U|(x) = \frac{1}{|x|^2}|\nabla\hat{U}|(\frac{x}{|x|^2}), \qquad \langle x, \nabla U(x)\rangle = -\langle\frac{x}{|x|^2}, \nabla\hat{U}(\frac{x}{|x|^2})\rangle.$$

We can apply Theorem 4.2 and deduce by (4.8) that

$$|\nabla U|(x) = \frac{1}{|x|}[(\frac{\gamma_0}{n\omega_n})^{\frac{1}{n-1}} + o(1)], \qquad \langle x, \nabla U(x)\rangle = -(\frac{\gamma_0}{n\omega_n})^{\frac{1}{n-1}} + o(1) \tag{5.1}$$

uniformly for $x \in \partial B_R(0)$, for a sequence $R = \frac{1}{r} \to +\infty$ and $\gamma_0 = \int_{\mathbb{R}^n} e^U$. By (5.1) we have that

$$\int_{\partial B_R(0)}\left[|\nabla U|^{n-2}\langle x, \nabla U\rangle\partial_\nu U - \frac{|\nabla U|^n}{n}\langle x,\nu\rangle\right] \to \omega_{n-1}(1-\frac{1}{n})(\frac{\gamma_0}{n\omega_n})^{\frac{n}{n-1}} \tag{5.2}$$

as $R \to +\infty$. Since by (4.7)

$$|x|^{(\frac{\gamma_0}{n\omega_n})^{\frac{1}{n-1}}} e^U \in L^\infty(\mathbb{R}^n \setminus B_1(0))$$

with $(\frac{\gamma_0}{n\omega_n})^{\frac{1}{n-1}} \ge \frac{n^2}{n-1}$ in view of (4.6), we also get that

$$\int_{\partial B_R(0)} e^U\langle x,\nu\rangle \to 0 \tag{5.3}$$

as $R \to +\infty$. We apply Lemma 3.3 to U on $B_R(0)$ with $y = 0$ and let $R \to +\infty$ to get

$$n\gamma_0 = \omega_n(n-1)(\frac{\gamma_0}{n\omega_n})^{\frac{n}{n-1}}$$

in view of (5.2)-(5.3). It results that

$$\gamma_0 = \int_{\mathbb{R}^n} e^U = c_n \omega_n.$$

□

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