ISOLATED SINGULARITIES FOR THE $n-$LIOUVILLE EQUATION

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Abstract. In dimension n isolated singularities – at a finite point or at infinity– for solutions of finite total mass to the $n-$Liouville equation are of logarithmic type. As a consequence, we simplify the classification argument in [11] and establish a quantization result for entire solutions of the singular $n-$Liouville equation.

1. Introduction

The behavior near an isolated singularity has been discussed by Serrin in [19, 20] for a very general class of second-order quasi-linear equations. The simplest example is given by the prototypical equation $-\Delta_n u = f$, where $\Delta_n(\cdot) = \operatorname{div}(|\nabla(\cdot)|^{n-2}\nabla(\cdot))$, $n \geq 2$, is the $n-$Laplace operator. In dimension n, the case $f \in L^1$ is very delicate as it represents a limiting situation where Serrin's results do not apply. We will be interested in the $n-$Liouville equation, where f is taken as an exponential function of u according to Liouville's seminal paper [16], and the singularity might be at a finite point or at infinity.

To this aim, it is enough to consider the generalized $n-$Liouville equation

$$-\Delta_n u = |x|^{n\alpha} e^u \text{ in } \Omega \setminus \{0\}, \int_\Omega |x|^{n\alpha} e^u < +\infty \tag{1.1}$$

on an open set $\Omega \subset \mathbb{R}^n$ with $0 \in \Omega$, and we will be concerned with describing the behavior of u at 0. A solution u of (1.1) stands for a function $u \in C^{1,\eta}_{loc}(\Omega \setminus \{0\})$ which satisfies

$$\int_\Omega |\nabla u|^{n-2}\langle \nabla u, \nabla \varphi\rangle = \int_\Omega |x|^{n\alpha} e^u \varphi \qquad \forall\, \varphi \in C^1_0(\Omega \setminus \{0\}).$$

The regularity assumption on u is not restrictive since a solution in $W^{1,n}_{\text{loc}}(\Omega \setminus \{0\})$ is automatically in $C^{1,\eta}_{loc}(\Omega \setminus \{0\})$, for some $\eta \in (0, 1)$, thanks to [9, 19, 22], see Theorem 2.3 in [11].

Concerning the behavior near an isolated singularity, our main result is

Theorem 1.1. *Let u be a solution of (1.1). Then there exists $\gamma > -n^n|\alpha + 1|^{n-2}(\alpha + 1)\omega_n$, $\omega_n = |B_1(0)|$, so that*

$$-\Delta_n u = |x|^{n\alpha} e^u - \gamma\delta_0 \text{ in } \Omega \tag{1.2}$$

with

$$u - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log|x| \in L^\infty_{loc}(\Omega) \tag{1.3}$$

and

$$\lim_{x\to 0} \left[|x|\nabla u(x) - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \frac{x}{|x|} \right] = 0. \tag{1.4}$$

The case $\alpha = -2$ is relevant for the asymptotic behavior at infinity for solutions u of

$$-\Delta_n u = e^u \text{ in } \Omega, \int_\Omega e^u < +\infty, \tag{1.5}$$

where Ω is an unbounded open set so that $B_R(0)^c \subset \Omega$ for some $R > 0$. Indeed, let us recall that Δ_n is invariant under Kelvin transform: if u solves (1.1), then $\hat{u}(x) = u(\frac{x}{|x|^2})$ does satisfy

$$-\Delta_n \hat{u} = |x|^{-2n}(-\Delta_n u)(\frac{x}{|x|^2}) = |x|^{-n(\alpha+2)} e^{\hat{u}} \text{ in } \hat{\Omega} = \{x \neq 0 : \frac{x}{|x|^2} \in \Omega\}. \tag{1.6}$$

By Theorem 1.1 applied with $\alpha = -2$ to $\hat{u}$ at 0 we find:

Corollary 1.2. *Let u be a solution of (1.5) on an unbounded open set Ω with $B_R(0)^c \subset \Omega$ for some $R > 0$. Then there holds*

$$u = -\left(\frac{\gamma_\infty}{n\omega_n}\right)^{\frac{1}{n-1}} \log|x| + O(1) \tag{1.7}$$

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as $|x| \to \infty$ for some $\gamma_\infty > n^n \omega_n$. In particular, when $\Omega = \mathbb{R}^n$ there holds

$$u = -\left(\frac{1}{n\omega_n} \int_{\mathbb{R}^n} e^u \right)^{\frac{1}{n-1}} \log |x| + O(1) \tag{1.8}$$

as $|x| \to \infty$.

When $n = 2$ the asymptotic expansion (1.8) is a well known property established in [6] by means of the Green representation formula – unfortunately not available in the quasi-linear case– and of the growth properties of entire harmonic functions. Notice that

$$\gamma_\infty = \int_{\mathbb{R}^n} |x|^{-2n} e^{\hat{u}} = \int_{\mathbb{R}^n} e^u$$

follows by integrating (1.2) written for $\hat{u}$ on $\mathbb{R}^n$. Property (1.8) has been already proved in [11] under the assumption $\gamma_\infty > n^n \omega_n$ and the present full generality allows to simplify the classification argument in [11]: a Pohozaev identity leads for γ_∞ to the quantization property

$$\int_{\mathbb{R}^n} e^u = n(\frac{n^2}{n-1})^{n-1} \omega_n \tag{1.9}$$

and an isoperimetric argument concludes the classification result thanks to (1.9).

In the punctured plane $\Omega = \mathbb{R}^n \setminus \{0\}$ the isoperimetric argument fails and in general the classification result is no longer available. The two-dimensional case $n = 2$ has been treated via complex analysis in [7, 17]: solutions u to

$$-\Delta u = e^u - \gamma \delta_0 \text{ in } \mathbb{R}^2, \ \int_{\mathbb{R}^2} e^u < +\infty, \tag{1.10}$$

have been classified for $\gamma > -4\pi$ of the form

$$u(x) = \log \frac{8(\alpha+1)^2 \lambda^2 |x|^{2\alpha}}{(1 + \lambda^2 |x^{\alpha+1} + c|^2)^2}, \ \alpha = \frac{\gamma}{4\pi},$$

with $\lambda > 0$ and a complex number $c = 0$ if $\alpha \notin \mathbb{N}$, and in particular satisfy

$$\int_{\mathbb{R}^n} e^u = 8\pi(\alpha + 1). \tag{1.11}$$

The structure of entire solutions u to (1.10) changes drastically passing from radial solutions when $\alpha \notin \mathbb{N}$ to non-radial solutions when $\alpha \in \mathbb{N}$ (and $c \neq 0$). Unfortunately a PDE approach is not available for $n = 2$ and a classification result is completely out of reach when $n \geq 3$. However, quantization properties are still in order as it follows by Theorem 1.1 and the Pohozaev identities:

Theorem 1.3. *Let u be a solution of*

$$-\Delta_n u = e^u - \gamma \delta_0 \ \text{in } \mathbb{R}^n, \ \int_{\mathbb{R}^n} e^u < +\infty. \tag{1.12}$$

Then $\gamma > -n^n \omega_n$ and

$$\int_{\mathbb{R}^n} e^u = \gamma + \gamma_\infty \tag{1.13}$$

with γ_∞ the unique solution in $(n^n \omega_n, +\infty)$ of

$$\frac{n-1}{n} (n\omega_n)^{-\frac{1}{n-1}} \gamma_\infty^{\frac{n}{n-1}} - n\gamma_\infty = n\gamma + \frac{n-1}{n} (n\omega_n)^{-\frac{1}{n-1}} |\gamma|^{\frac{n}{n-1}}. \tag{1.14}$$

When $n = 2$ notice that for $\gamma > -4\pi$ the unique solution $\gamma_\infty > 4\pi$ of (1.14) is given explicitly as $\gamma_\infty = \gamma + 8\pi$ and then $\int_{\mathbb{R}^n} e^u = 2\gamma + 8\pi = 8\pi(\alpha + 1)$ in accordance with (1.11). To have Theorem 1.3 meaningful, in Section 4 we will show the existence of a family of radial solutions u to (1.12) but we don't know whether other solutions might exist or not, depending on the value of γ. Notice that (1.10) is equivalent to

$$-\Delta v = |x|^{2\alpha} e^v \text{ in } \mathbb{R}^2, \ \int_{\mathbb{R}^2} |x|^{2\alpha} e^v < +\infty,$$

in terms of $v = u - 2\alpha \log |x|$. For $n \geq 3$ such equivalence breaks down and the problem

$$-\Delta_n v = |x|^{n\alpha} e^v \text{ in } \mathbb{R}^n, \ \int_{\mathbb{R}^n} |x|^{n\alpha} e^v < +\infty, \tag{1.15}$$

has its own interest, independently of (1.12). As in Theorem 1.3, for (1.15) we have the following quantization result:

Theorem 1.4. *Let v be a solution of* (1.15)*. Then $\alpha > -1$ and*

$$\int_{\mathbb{R}^n} |x|^{n\alpha} e^v = n(\frac{n^2}{n-1})^{n-1} (\alpha+1)^{n-1} \omega_n.$$

Radial solutions v of (1.15) can be easily obtained as $v = n\log(\alpha+1) + u(|x|^{\alpha+1})$ in terms of a radial entire solution u to (1.5). Thanks to the classification result in [11], for (1.15) we can therefore exhibit the following family of radial solutions:

$$v_\lambda = \log \frac{c_n(\alpha+1)^n \lambda^n}{(1+\lambda^{\frac{n}{n-1}}|x|^{\frac{n(\alpha+1)}{n-1}})^n},\ c_n = n(\frac{n^2}{n-1})^{n-1}.$$

Problems with exponential nonlinearities on a bounded domain with given singular sources can exhibit non-compact solution-sequences, whose shape near a blow-up point is asymptotically described by the limiting problem (1.12). In the regular case (i.e. in absence of singular sources) a concentration-compactness principle has been established [5] for $n = 2$ and [1] for $n \geq 2$. In the non-compact situation the exponential nonlinearity concentrates at the blow-up points as a sum of Dirac measures. Theorem 1.3 gives information on the concentration mass of such Dirac measures at a singular blow-up point, which is expected bo te a super-position of several masses $c_n\omega_n$ carried by multiple sharp collapsing peaks governed by $(1.12)_{\gamma=0}$ and possibly the mass (1.13) of a sharp peak described by (1.12). In the regular case such quantization property on the concentration masses has been proved [14] for $n = 2$ and extended [12] to $n \geq 2$ by requiring an additional boundary assumption, while the singular case has been addressed in [2, 21] for $n = 2$. For Theorem 1.4 a similar comment is in order.

Let us briefly explain the main ideas behind Theorem 1.1. We can re-adapt the argument in [11] to show that $u \in \bigcap_{1\leq q<n} W^{1,q}_{\text{loc}}(\Omega)$ and then u satisfies (1.2) for some $\gamma \in \mathbb{R}$. On a radial ball $B \subset\subset \Omega$ decompose u as $u = u_0 + h$, where h is given by

$$\Delta_n h = \gamma\delta_0 \text{ in } B, \quad h = u \text{ on } \partial B$$

and satisfies (1.3) thanks to [13, 19, 20]. The key property stems from a simple observation: $|x|^{n\alpha}e^h \in L^1$ near 0 implies $|x|^{n\alpha}e^h \in L^p$ near 0 for some $p > 1$ whenever h has a logarithmic singularity at 0. Back to [19, 20] thanks to such improved integrability, one aims to show that $u_0 \in L^\infty(B)$ and then u has the same logarithmic behavior (1.3) as h. In order to develop a regularity theory for the solution u_0 of

$$-\Delta_n(u_0+h) + \Delta_n h = |x|^{n\alpha}e^{u_0+h} \text{ in } B,\ u_0 = 0 \text{ on } \partial B, \tag{1.16}$$

the crucial point is to establish several integral inequalities involving u_0 paralleling the estimates available for entropy solutions in [1, 3] and for $W^{1,n}-$solutions in [19]. To this aim, we make use of the deep uniqueness result [10] to show that u can be regarded as a Solution Obtained as Limit of Approximations (the so-called SOLA, see for example [4]).

The paper is organized as follows. In Section 2 we develop the above argument to prove Theorem 1.1. Section 3 is devoted to establish Theorems 1.3-1.4 via Pohozaev identities: going back to an idea of Y.Y. Li and N. Wolanski for $n = 2$, the Pohozaev identities have revealed to be a fundamental tool to derive information on the mass of a singularity (see for example [2, 12, 18]). In Section 4 a family of radial solutions u to (1.12) is constructed.

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2. Proof of Theorem 1.1

Assume $B_1(0) \subset\subset \Omega$. Let us first establish the following property on u.

Proposition 2.1. *Let u be a solution of (1.1). There exists $C > 0$ so that*

$$u(x) \leq C - n(\alpha+1)\log|x| \qquad \text{in } B_1(0)\setminus\{0\}. \tag{2.1}$$

Proof. Letting $U_r(y) = \hat u(\frac{y}{r}) + n(\alpha+1)\log r = u(\frac{ry}{|y|^2}) + n(\alpha+1)\log r$ for $0 < r \leq \frac12$, we have that U_r solves

$$-\Delta_n U_r = |y|^{-n(\alpha+2)}e^{U_r} \qquad \text{in } \mathbb{R}^n\setminus B_{\frac12}(0), \quad \int_{\mathbb{R}^n\setminus B_{\frac12}(0)} |y|^{-n(\alpha+2)}e^{U_r} = \int_{B_{2r}(0)} |x|^{n\alpha}e^u \tag{2.2}$$

in view of (1.6). Given a ball $B_{\frac12}(x_0)$ for $x_0 \in \mathbb{S}^{n-1}$, let us consider the $n-$harmonic function H_r in $B_{\frac12}(x_0)$ so that $H_r = U_r$ on $\partial B_{\frac12}(x_0)$. By the weak maximum principle we deduce that $H_r \leq U_r$ in $B_{\frac12}(x_0)$ and then

$$\int_{B_{\frac12}(x_0)} (H_r)_+^n \leq \int_{B_{\frac12}(x_0)} (U_r)_+^n \leq n! \int_{B_{\frac12}(x_0)} e^{U_r} \leq C\int_{\mathbb{R}^n\setminus B_{\frac12}(0)} |y|^{-n(\alpha+2)}e^{U_r} \leq C\int_\Omega |x|^{n\alpha}e^u < +\infty \tag{2.3}$$

for all $0 < r \leq \frac12$. By the estimates in [19] we have that there exists $C > 0$ so that

$$\|H_r\|_{\infty, B_{\frac14}(x_0)} \leq C \tag{2.4}$$

for all $0 < r \leq \frac12$. At the same time, by the exponential estimate in [1] we have that there exist $0 < r_0 \leq \frac12$ and $C > 0$ so that

$$\int_{B_{\frac12}(x_0)} e^{2|U_r - H_r|} \leq C \tag{2.5}$$

for all $0<r\leq r_0$ in view of $\lim_{r\to 0^+}\int_{B_{\frac{1}{2}}(x_0)}|y|^{-n(\alpha+2)}e^{U_r}=0$ thanks to (1.1) and (2.2). Since $|y|^{-n(\alpha+2)}e^{U_r}\leq Ce^{|U_r-H_r|}$ on $B_{\frac{1}{4}}(x_0)$ for all $0<r\leq \frac{1}{2}$ in view of (2.4), we deduce that $|y|^{-n(\alpha+2)}e^{U_r}$ and $(U_r)_+^{\frac{n}{2}}$ are uniformly bounded in $L^2(B_{\frac{1}{4}}(x_0))$ for all $0<r\leq r_0$ in view of (2.3) and (2.5). Thanks again to the estimates in [19], we finally deduce that

$$\|U_r^+\|_{\infty,B_{\frac{1}{8}}(x_0)}\leq C \tag{2.6}$$

for all $0<r\leq r_0$. Since $\mathbb{S}^{n-1}$ can be covered by a finite number of balls $B_{\frac{1}{8}}(x_0)$, $x_0\in\mathbb{S}^n$, going back to u from (2.6) one deduces that

$$u(x)\leq C-n(\alpha+1)\log|x|$$

for all $|x|=r\leq r_0$. Since this estimate does hold in $B_1(0)\setminus B_{r_0}(0)$ too, we have established the validity of (2.1). □

From now on, set $B=B_r(0)$ for $0<r\leq 1$. We are now ready to establish the starting point for the argument we will develop in the sequel. There holds

Proposition 2.2. *Let u be a solution of* (1.1). *Then*

$$u\in\bigcap_{1\leq q<n}W^{1,q}(\Omega). \tag{2.7}$$

Proof. Let us go through the argument in [11] to obtain $W^{1,q}-$estimates on u. For $0<\epsilon<r<1$ let us introduce $h_{\epsilon,r}\in W^{1,n}(A_{\epsilon,r})$, $A_{\epsilon,r}:=B\setminus\overline{B_\epsilon(0)}$, as the solution of

$$\Delta_n h_{\epsilon,r}=0 \text{ in } A_{\epsilon,r},\ h_{\epsilon,r}=u \text{ on } \partial A_{\epsilon,r}.$$

Regularity issues for quasi-linear PDEs involving Δ_n are well established since the works of DiBenedetto, Evans, Lewis, Serrin, Tolksdorf, Uhlenbeck, Uraltseva. For example, by [9, 15, 19, 22] we deduce that $h_{\epsilon,r}$, $u_{\epsilon,r}=u-h_{\epsilon,r}\in C^{1,\eta}(\overline{A_{\epsilon,r}})$ and $u_{\epsilon,r}$ satisfies

$$-\Delta_n(u_{\epsilon,r}+h_{\epsilon,r})+\Delta_n h_{\epsilon,r}=|x|^{n\alpha}e^u \text{ in } A_{\epsilon,r},\ u_{\epsilon,r}=0 \text{ on } \partial A_{\epsilon,r}. \tag{2.8}$$

By the techniques in [1, 3, 4] we have the following estimates, see Proposition 2.1 in [11]: for all $1\leq q<n$ and all $p\geq 1$ there exist $0<r_0<1$ and $C>0$ so that

$$\int_{A_{\epsilon,r}}|\nabla u_{\epsilon,r}|^q+\int_{A_{\epsilon,r}}e^{pu_{\epsilon,r}}\leq C \tag{2.9}$$

for all $0<\epsilon<r\leq r_0$ thanks to (2.8) and $\lim_{r\to 0^+}\int_B|x|^{n\alpha}e^u=0$. Since by the Sobolev embedding $W_0^{1,\frac{n}{2}}(B_1(0))\hookrightarrow L^n(B_1(0))$ there holds $\int_{A_{\epsilon,r}}|u_{\epsilon,r}|^n\leq C$ for all $0<\epsilon<r\leq r_0$ in view of (2.9) with $q=\frac{n}{2}$ and $A_{\epsilon,r}\subset B_1(0)$, we have that

$$\|h_{\epsilon,r}\|_{L^n(A)}\leq C(A)\qquad \forall\ A\subset\subset \overline{B}\setminus\{0\}\,,\ \forall\ 0<\epsilon<r\leq r_0$$

in view of $u\in C^{1,\eta}_{loc}(B_1(0)\setminus\{0\})$ and then

$$\|h_{\epsilon,r}\|_{C^{1,\eta}(A)}\leq C(A)\qquad \forall\ A\subset\subset \overline{B}\setminus\{0\}\,,\ \forall\ 0<\epsilon<r\leq r_0$$

thanks to [9, 15, 19, 22]. By the Ascoli-Arzelá's Theorem and a diagonal process we can find a sequence $\epsilon\to 0$ so that $h_{\epsilon,r}\to h_r$ and $u_{\epsilon,r}\to u_r:=u-h_r$ in $C^1_{\rm loc}(\overline{B}\setminus\{0\})$ as $\epsilon\to 0$, where $h_r\leq u$ is a $n-$harmonic function in $B\setminus\{0\}$ and u_r satisfies

$$u_r\in W_0^{1,q}(B),\qquad e^{u_r}\in L^p(B) \tag{2.10}$$

for all $1\leq q<n$ and all $p\geq 1$ if r is sufficiently small in view of (2.9). Since

$$h_r(x)\leq C-n(\alpha+1)\log|x|\qquad \text{in } B$$

in view of $h_r\leq u$ and (2.1), we have that $H^\lambda(y)=-\frac{h_r(\lambda y)}{\log\lambda}$ is a $n-$harmonic function in $B_{\frac{r}{\lambda}}(0)$ so that $H^\lambda\leq n(\alpha+1)+1$ in $B_2(0)\setminus B_{\frac{1}{2}}(0)$ for all $0<\lambda\leq\lambda_0$, where $\lambda_0\in(0,\frac{r}{2}]$ is a suitable small number. By the Harnack inequality in Theorem 7-[19] applied to $n(\alpha+1)+1-H^\lambda\geq 0$ we deduce that

$$\max_{|y|=1}H^\lambda\leq C\left[\frac{n|\alpha+1|+1}{C}+\min_{|y|=1}H^\lambda\right] \tag{2.11}$$

for all $0<\lambda\leq\lambda_0$, for a suitable $C\in(0,1)$. There are two possibilities:

- either $\min_{|y|=1}H^\lambda\geq -\frac{n|\alpha+1|+1}{C}$ for all $0<\lambda\leq\lambda_1$ and some $\lambda_1\in(0,\lambda_0]$, which implies $\max_{|y|=1}|H^\lambda|\leq \frac{n|\alpha+1|+1}{C}$ for all $0<\lambda\leq\lambda_1$ and in particular

$$|h_r|\leq -C_0\log|x|\qquad \text{in } B_{\lambda_1}(0) \tag{2.12}$$

for some $C_0>0$;

- or $\min_{|y|=1} H^{\lambda_n} \leq -\frac{n|\alpha+1|+1}{C}$ for a sequence $\lambda_n \downarrow 0$, which implies $\max_{|y|=1} H^{\lambda_n} \leq 0$ in view of (2.11) and in turn $h_r \leq 0$ on $|x| = \lambda_n$ for all $n \in \mathbb{N}$, leading to
$$h_r \leq 0 \qquad \text{in } B_{\lambda_1}(0) \tag{2.13}$$
by the weak maximum principle.

Notice that (2.13) implies the validity of (2.12) for some $C_0 > 0$ in view of Theorem 12-[19]. Thanks to (2.12) one can apply Theorem 1.1-[13] to show that
$$h_r \in W^{1,q}(B) \tag{2.14}$$
for all $1 \leq q < n$ and there exists $\gamma_r \in \mathbb{R}$ so that
$$h_r - \gamma_r (n\omega_n |\gamma_r|^{n-2})^{-\frac{1}{n-1}} \log |x| \in L^\infty(B), \qquad \Delta_n h_r = \gamma_r \delta_0 \text{ in } B. \tag{2.15}$$
In particular, $u \in \bigcap_{1\leq q<n} W^{1,q}(\Omega)$ in view of (2.10) and (2.14), and (2.7) is established. $\square$

Even if $\gamma_r > -n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$, at this stage we cannot exclude that $\lim_{r\to 0} \gamma_r = -n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$. Therefore, we are not able to use (2.10) and (2.15) for improving the exponential integrability on u to reach $|x|^{n\alpha}e^u = |x|^{n\alpha}e^{h_r}e^{u_r} \in L^p$ near 0 for some $p > 1$ and r sufficiently small, as it would be necessary to prove $L^\infty-$bounds on u_r via (2.8) on $u_{\epsilon,r}$.

We need to argue in a different way. Since $u \in W^{1,n-1}(\Omega)$ in view of (2.7), we can extend (1.1) at 0 as
$$-\Delta_n u = |x|^{n\alpha}e^u - \gamma\delta_0 \quad \text{in } \Omega. \tag{2.16}$$
To see it, let $\varphi \in C_0^1(\Omega)$ and consider a function $\chi_\epsilon \in C^\infty(\Omega)$ with $0 \leq \chi_\epsilon \leq 1$, $\chi_\epsilon = 0$ in $B_{\frac{\epsilon}{2}}(0)$, $\chi_\epsilon = 1$ in $\Omega \setminus B_\epsilon(0)$ and $\epsilon|\nabla\chi_\epsilon| \leq C$. Taking $\chi_\epsilon\varphi \in C_0^1(\Omega \setminus \{0\})$ as a test function in (1.1) we have that
$$\int_\Omega |\nabla u|^{n-2}\langle \nabla u, \varphi\nabla\chi_\epsilon + \chi_\epsilon\nabla\varphi\rangle = \int_\Omega \chi_\epsilon |x|^{n\alpha}e^u\varphi. \tag{2.17}$$
Since $u \in W^{1,n-1}(\Omega)$ and $|x|^{n\alpha}e^u \in L^1(\Omega)$ it is easily seen that
$$\int_\Omega \chi_\epsilon|\nabla u|^{n-2}\langle\nabla u, \nabla\varphi\rangle \to \int_\Omega |\nabla u|^{n-2}\langle\nabla u,\nabla\varphi\rangle, \qquad \int_\Omega \chi_\epsilon|x|^{n\alpha}e^u\varphi \to \int_\Omega |x|^{n\alpha}e^u\varphi \tag{2.18}$$
as $\epsilon \to 0$. Since
$$\int_\Omega |\nabla u|^{n-1}|\varphi - \varphi(0)||\nabla\chi_\epsilon| \leq C\int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)} |\nabla u|^{n-1} \to 0$$
as $\epsilon \to 0$ in view of $|\varphi - \varphi(0)| \leq C\epsilon$ in $B_\epsilon(0) \setminus B_{\frac{\epsilon}{2}}(0)$ and $u \in W^{1,n-1}(\Omega)$, the remaining term in (2.17) can be re-written as follows:
$$\int_\Omega |\nabla u|^{n-2}\varphi\langle\nabla u, \nabla\chi_\epsilon\rangle = \varphi(0)\int_\Omega |\nabla u|^{n-2}\langle\nabla u,\nabla\chi_\epsilon\rangle + o(1) \tag{2.19}$$
as $\epsilon \to 0$. By inserting (2.18)-(2.19) into (2.17) we deduce the existence of
$$\gamma = \lim_{\epsilon\to 0}\int_\Omega |\nabla u|^{n-2}\langle\nabla u,\nabla\chi_\epsilon\rangle$$
and the validity of (2.16) for u. Moreover, if we assume $u \in C^1(\overline{\Omega}\setminus\{0\})$, we can interpret γ as
$$\gamma = \lim_{\epsilon\to 0}[\int_\Omega |x|^{n\alpha}e^u\chi_\epsilon + \int_{\partial\Omega}|\nabla u|^{n-2}\partial_n u] = \int_\Omega |x|^{n\alpha}e^u + \int_{\partial\Omega}|\nabla u|^{n-2}\partial_n u.$$
Since $\gamma_r \geq \gamma + o(1)$ as $r \to 0$ according to (4.16)-[11], we find that h_r is possibly much lower than u and then needs to be compensated by an unbounded function $u_r \geq 0$ in order to keep the validity of $u = u_r + h_r$. Instead, thanks to Theorem 2.1-[13] introduce $h \in \bigcap_{1\leq q<n} W^{1,q}(B)$ as the solution of
$$\Delta_n h = \gamma\delta_0 \text{ in } B, \quad h = u \text{ on } \partial B$$
so that
$$h - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}}\log|x| \in L^\infty(B). \tag{2.20}$$
Decomposing u as $u = u_0 + h$, the solution u_0 of (1.16) on B is very likely a bounded function, as we will prove below.

In order to establish some crucial integral inequalities involving u_0, let us introduce the following approximation scheme. By convolution with mollifiers consider sequences $f_j, g_j \in C_0^\infty(B)$ so that $f_j \rightharpoonup |x|^{n\alpha}e^u - \gamma\delta_0$ weakly in the sense of measures and $0 \leq f_j - g_j \to |x|^{n\alpha}e^u$ in $L^1(B)$ as $j \to +\infty$. Since $u \in C^{1,\eta}(\partial B)$, let $\varphi \in C^{1,\eta}(B)$ be the $n-$harmonic extension of $u\mid_{\partial B}$ in B. Let $v_j, w_j \in W_0^{1,n}(B)$ be the weak solutions of $-\text{div } \mathbf{a}(x,\nabla v_j) = f_j$ and $-\text{div } \mathbf{a}(x,\nabla w_j) = g_j$ in B, where $\mathbf{a}(x,p) = |p+\nabla\varphi|^{n-2}(p+\nabla\varphi) - |\nabla\varphi|^{n-2}\nabla\varphi$. In this way, $u_j = v_j + \varphi$ and $h_j = w_j + \varphi$ do solve
$$-\Delta_n u_j = f_j \text{ and } -\Delta_n h_j = g_j \text{ in } B,\ u_j = h_j = u \text{ on } \partial B.$$

Since f_j, g_j are uniformly bounded in $L^1(B)$, by (21) in [4] we can assume that $v_j \to v$ and $w_j \to w$ in $W_0^{1,q}(B)$ for all $1 \le q < n$ as $j \to +\infty$, where v and w do satisfy

$$-\text{ div } \mathbf{a}(x, \nabla v) = |x|^{n\alpha} e^u - \gamma\delta_0 \text{ and } -\text{ div } \mathbf{a}(x, \nabla w) = -\gamma\delta_0 \text{ in } B \tag{2.21}$$

in view of $g_j \rightharpoonup -\gamma\delta_0$ weakly in the sense of measures as $j \to +\infty$. Since $u - \varphi, h - \varphi \in \bigcap_{1\le q<n} W_0^{1,q}(B)$ do solve the first and the second equation in (2.21), respectively, by the uniqueness result in [10] (see Theorems 1.2 and 4.2 in [10]) we have that $v = u - \varphi$ and $w = h - \varphi$, i.e.

$$u_j \to u \text{ and } h_j \to h \text{ in } W^{1,q}(B) \text{ for all } 1 \le q < n \text{ as } j \to +\infty.$$

Thanks to the approximation given by the u_j's and h_j's, we can now derive some crucial integral inequalities on u_0.

Proposition 2.3. *Let u_0 be a solution of* (1.16)*. Then $u_0 \ge 0$ and we have:*

$$\int_{\{k<|u_0|<k+a\}} |\nabla u_0|^n \le \frac{a}{d} \int_B |x|^{n\alpha} e^u \qquad \forall\, k, a > 0 \tag{2.22}$$

and, if $|x|^{n\alpha} e^u \in L^p(B)$ for some $p > 1$,

$$\left(\int_B u_0^{2mnq}\right)^{\frac{1}{2m}} \le \frac{Cq^{n-1}}{d} |B|^{\frac{n-1}{mnq}} \left(\int_B |x|^{np\alpha} e^{pu}\right)^{\frac{1}{p}} \left(\int_B u_0^{mnq}\right)^{\frac{n(q-1)+1}{mnq}}, \tag{2.23}$$

where $m = \frac{p}{p-1}$ and

$$d = \inf_{X \ne Y} \frac{\langle |X|^{n-2}X - |Y|^{n-2}Y, X - Y\rangle}{|X - Y|^n} > 0. \tag{2.24}$$

Proof. First use $-(v_j - w_j)_- \in W_0^{1,n}(B)$ as a test function for $-\text{div } \mathbf{a}(x, \nabla v_j) + \text{div } \mathbf{a}(x, \nabla w_j)$ to get

$$d \int_{\{v_j - w_j < 0\}} |\nabla(v_j - w_j)|^n \le -\int_B \langle \mathbf{a}(x, \nabla v_j) - \mathbf{a}(x, \nabla w_j), \nabla(v_j - w_j)_-\rangle = -\int_B (f_j - g_j)(v_j - w_j)_- \le 0$$

in view of (2.24) and $f_j - g_j \ge 0$. Hence, $v_j - w_j \ge 0$ and then $u_0 \ge 0$ in view of $v_j - w_j \to u - h = u_0$ in $W_0^{1,q}(B)$ for all $1 \le q < n$ as $j \to +\infty$. Now, introduce the truncature operator $T_{k,a}$, for $k, a > 0$, as

$$T_{k,a}(s) = \begin{cases} s - k \text{ sign}(s) & \text{if } k < |s| < k + a, \\ a \text{ sign}(s) & \text{if } |s| \ge k + a, \\ 0 & \text{if } |s| \le k, \end{cases}$$

and use $T_{k,a}(v_j - w_j) \in W_0^{1,n}(B)$ as a test function for $-\text{div } \mathbf{a}(x, \nabla v_j) + \text{div } \mathbf{a}(x, \nabla w_j)$ to get

$$d \int_{\{k<|v_j - w_j|<k+a\}} |\nabla(v_j - w_j)|^n \le \int_B \langle \mathbf{a}(x, \nabla v_j) - \mathbf{a}(x, \nabla w_j), \nabla T_{k,a}(v_j - w_j)\rangle = \int_B (f_j - g_j) T_{k,a}(v_j - w_j) \tag{2.25}$$

in view of (2.24). Since $v_j - w_j \to u_0$ in $W_0^{1,q}(B)$ for all $1 \le q < n$ and $f_j - g_j \to |x|^{n\alpha} e^u$ in $L^1(B)$ as $j \to +\infty$, we can let $j \to +\infty$ in (2.25) and get by Fatou's Lemma that

$$d \int_{\{k<|u_0|<k+a\}} |\nabla u_0|^n \le \int_B |x|^{n\alpha} e^u T_{k,a}(u_0) \le a \int_B |x|^{n\alpha} e^u$$

yielding the validity of (2.22). Finally, if $|x|^{n\alpha} e^u \in L^p(B)$ for some $p > 1$, we can assume that $f_j - g_j \to |x|^{n\alpha} e^u$ in $L^p(B)$ as $j \to +\infty$ and use $T_a[|v_j - w_j|^{n(q-1)}(v_j - w_j)] \in W_0^{1,n}(B)$, where $T_a = T_{0,a}$ and $a > 0, q \ge 1$, as a test function for $-\text{div } \mathbf{a}(x, \nabla v_j) + \text{div } \mathbf{a}(x, \nabla w_j)$ to get by Hölder's inequality

$$d\frac{n(q-1)+1}{q^n} \int_{\{|v_j - w_j|^{n(q-1)+1} < a\}} |\nabla |v_j - w_j|^q|^n \le \int_B |f_j - g_j||v_j - w_j|^{n(q-1)+1} \le |B|^{\frac{n-1}{mnq}} \|f_j - g_j\|_p \left(\int_B |v_j - w_j|^{mnq}\right)^{\frac{n(q-1)+1}{mnq}}$$

in view of $|T_a(s)| \le |s|$ and (2.24). We have used that $v_j - w_j \in W_0^{1,n}(B) \subset \bigcap_{q\ge1} L^q(B)$ by the Sobolev embedding Theorem. Letting $a \to +\infty$, by Fatou's Lemma we get that

$$\int_B |\nabla |v_j - w_j|^q|^n \le \frac{q^n}{d[n(q-1)+1]} |B|^{\frac{n-1}{mnq}} \|f_j - g_j\|_p \left(\int_B |v_j - w_j|^{mnq}\right)^{\frac{n(q-1)+1}{mnq}}.$$

In particular, $|v_j - w_j|^q \in W_0^{1,n}(B)$ and by the Sobolev embedding $W_0^{1,n}(B) \subset L^{2mn}(B)$ we have that

$$\left(\int_B |v_j - w_j|^{2mnq}\right)^{\frac{1}{2m}} \le \frac{Cq^n}{d[n(q-1)+1]} |B|^{\frac{n-1}{mnq}} \|f_j - g_j\|_p \left(\int_B |v_j - w_j|^{mnq}\right)^{\frac{n(q-1)+1}{mnq}}.$$

Letting $j \to +\infty$, we finally deduce the validity of (2.23):

$$\left(\int_B u_0^{2mnq}\right)^{\frac{1}{2m}} \le \frac{Cq^{n-1}}{d} |B|^{\frac{n-1}{mnq}} \left(\int_B |x|^{np\alpha} e^{pu}\right)^{\frac{1}{p}} \left(\int_B u_0^{mnq}\right)^{\frac{n(q-1)+1}{mnq}}$$

in view of $v_j - w_j \to u_0$ in $W_0^{1,q}(B)$ for all $1 \leq q < n$ and $f_j - g_j \to |x|^{n\alpha} e^u$ in $L^p(B)$ as $j \to +\infty$. □

We are now ready to complete the proof of Theorem 1.1.

Proof (of Theorem 1.1). Since $u_0 \geq 0$ by Proposition 2.3, we have that $h \leq u$. By (1.1) and (2.20) we have that

$$\int_B |x|^{n\alpha+\gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}}} \leq C \int_B |x|^{n\alpha} e^h \leq C \int_\Omega |x|^{n\alpha} e^u < +\infty,$$

which implies

$$n\alpha + \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} > -n \tag{2.26}$$

or equivalently

$$\gamma > -n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n. \tag{2.27}$$

Since $|x|^{n\alpha} e^h \in L^1$ near 0 and h has a logarithmic singularity at 0, then, as already observed in the Introduction, a stronger integrability follows:

$$|x|^{n\alpha} e^h \in L^p(B) \tag{2.28}$$

for some $p > 1$. Inequality (2.22) is used in [1] to deduce exponential estimates on u_0 like

$$\int_B e^{\frac{\delta u_0}{\|f\|_1}} \leq C_r \tag{2.29}$$

for some $\delta > 0$ where $f = |x|^{n\alpha} e^u$. Since $\lim\limits_{r\to 0} \int_B |x|^{n\alpha} e^u = 0$, by (2.29) we deduce that $e^{u_0} \in L^p(B)$ for all $p \geq 1$ if r is sufficiently small and then by (2.28)

$$|x|^{n\alpha} e^u = |x|^{n\alpha} e^h e^{u_0} \in L^p(B)$$

for some $p > 1$. Inequality (2.23) is used in Proposition 4.1-[11] (compare with (4.4) in [11]) to get $u_0 \in L^\infty(B)$ and then (2.20) does hold for u too, yielding the validity of (1.3). In order to prove (1.4), set $H = u - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log|x|$ and introduce the function

$$U_r(y) = u(ry) - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log r = \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log|y| + H(ry)$$

for a given sequence $r \to 0$. Since

$$-\Delta_n U_r = r^{n(1+\alpha)}|y|^{n\alpha} e^{u(ry)} = r^{n(1+\alpha)+\gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}}} |y|^{n\alpha+\gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}}} e^{H(ry)},$$

by (1.3) and (2.26)-(2.27) we have that U_r and $\Delta_n U_r$ are bounded in $L^\infty_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, uniformly in r. By [9, 19, 22] we deduce that U_r is bounded in $C^{1,\eta}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, uniformly in r. By the Ascoli-Arzelá's Theorem and a diagonal process, up to a sub-sequence we have that $U_r \to U_0$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where U_0 is a n-harmonic function in $\mathbb{R}^n \setminus \{0\}$. Setting $H_r(y) = H(ry)$, we deduce that $H_r \to H_0$ in $C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$, where $H_0 \in L^\infty(\mathbb{R}^n)$ in view of (1.3). Since $U_0 = \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log|y| + H_0$ with $H_0 \in L^\infty(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$, it is well known that H_0 is a constant function, as shown in Corollary 2.2-[13] (see also [11] for a direct proof). In particular we get that

$$\sup_{|x|=r} |x| \Big|\nabla[u - \gamma(n\omega_n|\gamma|^{n-2})^{-\frac{1}{n-1}} \log|x|]\Big| = \sup_{|y|=1} |\nabla H_r| \to \sup_{|y|=1} |\nabla H_0| = 0.$$

Since this is true for any sequence $r \to 0$ up to extracting a sub-sequence, we have established the validity of (1.4). The proof of Theorem 1.1 is concluded.

□

3. Quantization results

In this section we will make crucial use of the following integral identity: for any solution u of

$$-\Delta_n u = |x|^{n\alpha} e^u \text{ in } \mathbb{R}^n \setminus \{0\} \tag{3.1}$$

there holds

$$n(\alpha+1)\int_A |x|^{n\alpha} e^u = \int_{\partial A} \left[|x|^{n\alpha} e^u \langle x, \nu\rangle + |\nabla u|^{n-2}\partial_\nu u \ \langle \nabla u, x\rangle - \frac{|\nabla u|^n}{n}\langle x, \nu\rangle\right], \tag{3.2}$$

where A is the annulus $A = B_R(0) \setminus B_\epsilon(0)$, $0 < \epsilon < R < +\infty$, and ν is the unit outward normal vector at ∂A. Notice that (3.2) is simply a special case of the well-known Pohozaev identities associated to (3.1). Even though the classical Pohozaev identities require more regularity than simply $u \in C^{1,\eta}(\mathbb{R}^n \setminus \{0\})$, (3.2) is still valid in the quasilinear case and we refer to [8] for a justification. Thanks to (3.2) we are able to show the following general result.

Proposition 3.1. *Let u be a solution of* (3.1) *so that* (1.3)-(1.4) *do hold at* 0 *and* ∞ *with* γ *and* $-\gamma_\infty$, *respectively, so that* $\gamma > -n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$ *and* $\gamma_\infty > n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$. *Then* $\int_{\mathbb{R}^n} |x|^{n\alpha} e^u = \gamma + \gamma_\infty$ *satisfies*

$$n(\alpha+1)(\gamma+\gamma_\infty) = \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}}\left[|\gamma_\infty|^{\frac{n}{n-1}} - |\gamma|^{\frac{n}{n-1}}\right]. \tag{3.3}$$

Proof. By (1.3)-(1.4) at 0 with $\gamma > -n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$ we deduce that

$$|\nabla u| = \frac{1}{|x|}\left[(\frac{|\gamma|}{n\omega_n})^{\frac{1}{n-1}} + o(1) \right], \quad \langle \nabla u, x\rangle = \gamma (n\omega_n |\gamma|^{n-2})^{-\frac{1}{n-1}} + o(1), \quad |x|^{n\alpha}e^u = o(\frac{1}{|x|^n}) \tag{3.4}$$

as $x \to 0$ thanks to the equivalence between (2.26) and (2.27). By (3.4) we have that

$$\int_{\partial B_\epsilon(0)} |x| \left[|x|^{n\alpha}e^u + |\nabla u|^{n-2}\langle \nabla u, \frac{x}{|x|}\rangle^2 - \frac{|\nabla u|^n}{n} \right] \to \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}}|\gamma|^{\frac{n}{n-1}} \tag{3.5}$$

as $\epsilon \to 0^+$ in view of Area$(\mathbb{S}^{n-1}) = n\omega_n$. Similarly, by (1.3)-(1.4) at ∞ with $-\gamma_\infty$ so that $\gamma_\infty > n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$ we deduce that

$$|\nabla u| = \frac{1}{|x|}\left[(\frac{|\gamma_\infty|}{n\omega_n})^{\frac{1}{n-1}} + o(1) \right], \quad \langle \nabla u, x\rangle = -\gamma_\infty (n\omega_n |\gamma_\infty|^{n-2})^{-\frac{1}{n-1}} + o(1), \quad |x|^{n\alpha}e^u = o(\frac{1}{|x|^n})$$

as $|x| \to \infty$ and then

$$\int_{\partial B_R(0)} |x| \left[|x|^{n\alpha}e^u + |\nabla u|^{n-2}\langle \nabla u, \frac{x}{|x|}\rangle^2 - \frac{|\nabla u|^n}{n} \right] \to \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}}|\gamma_\infty|^{\frac{n}{n-1}} \tag{3.6}$$

as $R \to +\infty$. In view of (1.4) at 0 and ∞ we easily get that

$$-\Delta_n u = |x|^{n\alpha}e^u - \gamma\delta_0 - \gamma_\infty\delta_\infty \text{ in } \mathbb{R}^n$$

in the sense

$$\int_{\mathbb{R}^n} |\nabla u|^{n-2}\langle \nabla u, \nabla\varphi\rangle = \int_{\mathbb{R}^n} |x|^{n\alpha}e^u\varphi - \gamma\varphi(0) - \gamma_\infty\varphi(\infty)$$

for all $\varphi \in C^1(\mathbb{R}^n)$ so that $\varphi(\infty) := \lim\limits_{|x|\to\infty}\varphi(x)$ does exist. Choosing $\varphi = 1$ we deduce that

$$\int_{\mathbb{R}^n} |x|^{n\alpha}e^u = \gamma + \gamma_\infty. \tag{3.7}$$

By inserting (3.5)-(3.7) into (3.2) and letting $\epsilon \to 0^+$, $R \to +\infty$ we deduce the validity of (3.3).

□

Let us now apply Proposition 3.1 to problems (1.12) and (1.15).

Proof (of Theorem 1.3). Let u be a solution of (1.12). By Theorem 1.1 (1.3)-(1.4) do hold for u at 0 with $\gamma > -n^n\omega_n$. By (1.6) the Kelvin transform $\hat{u}$ satisfies

$$-\Delta_n \hat{u} = |x|^{-2n}e^{\hat{u}} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

Let us apply Theorem 1.1 to deduce the validity of (1.3)-(1.4) for $\hat{u}$ at 0 with $\gamma_\infty > n^n\omega_n$. Back to u, (1.3)-(1.4) do hold for u at ∞ with $-\gamma_\infty$ so that $\gamma_\infty > n^n\omega_n$. Let us apply Proposition 3.1 with $\alpha = 0$ to get $\int_{\mathbb{R}^n} e^u = \gamma + \gamma_\infty$ with γ_∞ satisfying (1.14). Notice that the function $f(s) = ns + \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}}|s|^{\frac{n}{n-1}}$ is increasing in $(-n^n\omega_n, +\infty)$ and then $f(s) > f(-n^n\omega_n) = -n^n\omega_n$ for all $s \in (-n^n\omega_n, +\infty)$. At the same time the function $g(s) = \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}}s^{\frac{n}{n-1}} - ns$ is increasing in $(n^n\omega_n, +\infty)$ and then $g(s) > g(n^n\omega_n) = -n^n\omega_n$ for all $s \in (n^n\omega_n, +\infty)$. Therefore, for any $\gamma > -n^n\omega_n$ equation (1.14) has a unique solution $\gamma_\infty > n^n\omega_n$. The proof of Theorem 1.3 is concluded.

□

Remark 3.2. *Concerning Corollary 1.2, observe that in the argument above we have established* (1.7) *for problem* (1.5) *on* $\Omega = \mathbb{R}^n$ *and a similar proof is in order for a general unbounded open set* Ω*. Since* $\gamma = 0$*, we deduce the validity of* (1.8) *in view of* (1.13).

Proof (of Theorem 1.4). Let v be a solution of (1.15). Applying Theorem 1.1 to the Kelvin transform $\hat{v}$, solution of

$$-\Delta_n \hat{v} = |x|^{-n(\alpha+2)}e^{\hat{v}} \text{ in } \mathbb{R}^n \setminus \{0\},$$

we deduce the validity of (1.3)-(1.4) for v at ∞ with $-\gamma_\infty$ so that $\gamma_\infty > n^n|\alpha+1|^{n-2}(\alpha+1)\omega_n$. By Proposition 3.1 with $\gamma = 0$ we deduce that $\gamma_\infty = \int_{\mathbb{R}^n}|x|^{n\alpha}e^v$ satisfies

$$n(\alpha+1)\gamma_\infty = \frac{n-1}{n}(n\omega_n)^{-\frac{1}{n-1}}\gamma_\infty^{\frac{n}{n-1}}.$$

Therefore, $\alpha > -1$ and

$$\int_{\mathbb{R}^n} |x|^{n\alpha}e^v = n(\frac{n^2}{n-1})^{n-1}(\alpha+1)^{n-1}\omega_n,$$

concluding the proof of Theorem 1.4.

□

4. Radial solutions for (1.12)

Fix $M>1$ and assume that

$$\frac{1}{M}\le r_0\le M,\quad \alpha_0\le M,\quad \frac{1}{M}\le |\alpha_1|\le M. \tag{4.1}$$

Let us first discuss the local existence theory for the following Cauchy problem:

$$\begin{cases} -\frac{1}{r^{n-1}}(r^{n-1}|U'|^{n-2}U')'=e^U\\ U(r_0)=\alpha_0,\quad U'(r_0)=\alpha_1.\end{cases} \tag{4.2}$$

Given $0<\delta<\frac{1}{2M}$, define $I=[r_0-\delta,r_0+\delta]$ and $E=\{U\in C(I,[\alpha_0-1,\alpha_0+1]):\ U(r_0)=\alpha_0\}$, which is a Banach space endowed with $\|\cdot\|_\infty$ as a norm. We can re-formulate (4.2) as $U=TU$, where

$$TU(r)=\alpha_0+\int_{r_0}^r\frac{ds}{s}\Big|r_0^{n-1}|\alpha_1|^{n-2}\alpha_1-\int_{r_0}^s t^{n-1}e^{U(t)}dt\Big|^{-\frac{n-2}{n-1}}\left(r_0^{n-1}|\alpha_1|^{n-2}\alpha_1-\int_{r_0}^s t^{n-1}e^{U(t)}dt\right).$$

In view of

$$|s^n-r_0^n|\le n(M+1)^{n-1}\delta\qquad\forall\, s\in I \tag{4.3}$$

we have that $\max\limits_I U\le M+1$ and $\max\limits_I|\int_{r_0}^s t^{n-1}e^{U(t)}dt|\le e^{M+1}(M+1)^{n-1}\delta$ for all $U\in E$, and then for $0<\delta<\frac{e^{-M-1}}{2(M+1)^{3n-3}}$ we have that

$$r_0^{n-1}|\alpha_1|^{n-2}\alpha_1-\int_{r_0}^s t^{n-1}e^{U(t)}dt \text{ has the same sign as } \alpha_1\ \ \forall s\in I \tag{4.4}$$

and

$$\frac{1}{2M^{2n-2}}\le\frac12 r_0^{n-1}|\alpha_1|^{n-1}\le|r_0^{n-1}|\alpha_1|^{n-2}\alpha_1-\int_{r_0}^s t^{n-1}e^{U(t)}dt|\le\frac32 r_0^{n-1}|\alpha_1|^{n-1}\le\frac32 M^{2n-2} \tag{4.5}$$

for all $U\in E$. Since $\log\frac{r_0+\delta}{r_0}\le\log\frac{r_0}{r_0-\delta}\le\frac{\delta}{r_0-\delta}\le 2M\delta$ in view of $\delta<\frac{r_0}{2}$ and

$$||x|^{-\frac{n-2}{n-1}}x-|y|^{-\frac{n-2}{n-1}}y|\le C_M|x-y|\quad\forall\, x,y\in\mathbb{R}:\ xy\ge0,\ \min\{|x|,|y|\}\ge\frac{1}{2M^{2n-2}}$$

(for example, take $C_M=(1+\frac{n-2}{n-1})(4M^{2n-2})^{\frac{n-2}{n-1}}$), by (4.4)-(4.5) we have that

$$\|TU-\alpha_0\|_{\infty,I}\le\sup_{r\in I}|\int_{r_0}^r\frac{ds}{s}\Big|r_0^{n-1}|\alpha_1|^{n-2}\alpha_1-\int_{r_0}^s t^{n-1}e^{U(t)}dt\Big|^{\frac{1}{n-1}}|\le 2(\frac32)^{\frac{1}{n-1}}M^3\delta\le 3M^3\delta$$

and

$$\|TU-TV\|_{\infty,I}\le C_M\sup_{r\in I}|\int_{r_0}^r\frac{ds}{s}|\int_{r_0}^s t^{n-1}[e^{U(t)}-e^{V(t)}]dt||\le 2C_M(M+1)^ne^{M+1}\delta\|U-V\|_{\infty,I}$$

for all $U,V\in C^1(I)$ in view of $\delta<1$ and (4.3). In conclusion, if

$$0<\delta<\min\{\frac{1}{3M^3},\frac{e^{-M-1}}{2(M+1)^{3n-3}},\frac{e^{-M-1}}{2C_M(M+1)^n}\}, \tag{4.6}$$

then T is a contraction map from E into itself and a unique fixed point $U\in E$ of T is found by the Contraction Mapping Theorem, providing a solution U of (4.2) in $I=[r_0-\delta,r_0+\delta]$.

Once a local existence result has been established for (4.2), we can turn the attention to global issues. Given $r_0>0$, α_0 and $\alpha_1\neq0$, let $I=(r_1,r_2)$, $0\le r_1<r_0<r_2\le+\infty$, be the maximal interval of existence for the solution U of (4.2). We claim that $r_1=0$ when $\alpha_1>0$ and $r_2=+\infty$ when $\alpha_1<0$.

Consider first the case $\alpha_1>0$ and assume by contradiction $r_1>0$. Since

$$U'(r)=\frac1r\left(r_0^{n-1}\alpha_1^{n-1}+\int_r^{r_0}t^{n-1}e^{U(t)}dt\right)^{\frac{1}{n-1}}\ge\frac{r_0\alpha_1}{r}>0 \tag{4.7}$$

for all $r\in(r_1,r_0]$, one would have that

$$U(r)\le\alpha_0,\quad \alpha_1\le U'(r)\le\frac{1}{r_1}[r_0^{n-1}\alpha_1^{n-1}+\frac{r_0^n}{n}e^{\alpha_0}]^{\frac{1}{n-1}}$$

for all $r\in(r_1,r_0]$ and then (4.1) would hold for initial conditions $\alpha_0'=U(r_0')$, $\alpha_1'=U'(r_0')$ in (4.2) at r_0' approaching r_1 from the right. Since this would allow to continue the solution U on the left of r_1 in view of the estimate (4.6) on the time for local existence, we would reach a contradiction and then the property $r_1=0$ has been established.

In the case $\alpha_1<0$ assume by contradiction $r_2<+\infty$. Since

$$U'(r)=-\frac1r\left(r_0^{n-1}|\alpha_1|^{n-1}+\int_{r_0}^r t^{n-1}e^{U(t)}dt\right)^{\frac{1}{n-1}}\le-\frac{r_0|\alpha_1|}{r}<0 \tag{4.8}$$

for all $r \in [r_0, r_2)$, one would have that

$$U(r) \le \alpha_0, \quad -\frac{1}{r_0}[r_0^{n-1}|\alpha_1|^{n-1} + \frac{r_2^n}{n} e^{\alpha_0}]^{\frac{1}{n-1}} \le U'(r) \le -\frac{r_0|\alpha_1|}{r_2}$$

for all $r \in [r_0, r_2)$ and then (4.1) would hold for initial conditions $\alpha_0' = U(r_0')$, $\alpha_1' = U'(r_0')$ in (4.2) at r_0' approaching r_2 from the left. Since one could continue the solution U past r_2 thanks to (4.6), a contradiction would arise. Then, we have shown that $r_2 = +\infty$.

Given $\epsilon > 0$, let now $U_\epsilon^\pm$ be the maximal solution of

$$\begin{cases} -\frac{1}{r^{n-1}}(r^{n-1}|U'|^{n-2}U')' = e^U \\ U(1) = \alpha_0, \quad U'(1) = \pm\epsilon. \end{cases}$$

By the discussion above we have that U_ϵ^+ and U_ϵ^- are well defined in $(0,1]$ and $[1,+\infty)$, respectively. According to (4.7)-(4.8) one has

$$(U_\epsilon^+)' = \frac{1}{r}\left(\epsilon^{n-1} + \int_r^1 t^{n-1} e^{U_\epsilon^+(t)} dt\right)^{\frac{1}{n-1}} \text{ in } (0,1], \quad (U_\epsilon^-)' = -\frac{1}{r}\left(\epsilon^{n-1} + \int_1^r t^{n-1} e^{U_\epsilon^-(t)} dt\right)^{\frac{1}{n-1}} \text{ in } [1,+\infty) \tag{4.9}$$

and then U_ϵ^+, U_ϵ^- are uniformly bounded in $C^{1,\gamma}_{loc}(0,1]$, $C^{1,\gamma}_{loc}[1,+\infty)$, respectively, in view of $U_\epsilon^+, U_\epsilon^- \le \alpha_0$. Up to a subsequence and a diagonal argument, we can assume that $U_\epsilon^+ \to U^+$ in $C^1_{loc}(0,1]$ and $U_\epsilon^- \to U^-$ in $C^1_{loc}[1,+\infty)$ as $\epsilon \to 0^+$, where

$$(U^+)' = \frac{1}{r}\left(\int_r^1 t^{n-1} e^{U^+(t)} dt\right)^{\frac{1}{n-1}} \text{ in } (0,1], \quad (U^-)' = -\frac{1}{r}\left(\int_1^r t^{n-1} e^{U_-(t)} dt\right)^{\frac{1}{n-1}} \text{ in } [1,+\infty) \tag{4.10}$$

thanks to (4.9). Since $U^+(1) = U^-(1) = \alpha_0$ and $(U^+)'(1) = (U^-)'(1) = 0$ in view of (4.10), we have that

$$U = \begin{cases} U^+ & \text{in } (0,1] \\ U^- & \text{in } [1,+\infty) \end{cases}$$

is in $C^1(0,+\infty)$ with $U \le U(1) = \alpha_0$, $U'(1) = 0$ and

$$U'(r) = \frac{1}{r}\Big|\int_r^1 t^{n-1} e^{U(t)} dt\Big|^{-\frac{n-2}{n-1}} \int_r^1 t^{n-1} e^{U(t)} dt \text{ in } (0,+\infty). \tag{4.11}$$

It is not difficult to check that U satisfies $-\Delta_n U = e^U$ in $\mathbb{R}^n \setminus \{0\}$ and

$$\lim_{r\to 0} \frac{U(r)}{\log r} = \lim_{r\to 0} rU'(r) = (\int_0^1 t^{n-1} e^{U(t)} dt)^{\frac{1}{n-1}} = \left(\frac{1}{n\omega_n}\int_{B_1(0)} e^U\right)^{\frac{1}{n-1}} \tag{4.12}$$

in view of (4.11). By Theorem 1.1 and (4.12) we deduce that U is a radial solution of

$$-\Delta_n U = e^U - \gamma\delta_0 \text{ in } \mathbb{R}^n, \quad U \le U(1) = \alpha_0,$$

with $\gamma = \int_{B_1(0)} e^U$ depending on the choice of α_0. By the Pohozaev identity (3.2) on $A = B_1(0) \setminus B_\epsilon(0)$, $\epsilon \in (0,1)$, we have that

$$\omega_n[e^{\alpha_0} - \epsilon^n e^{U(\epsilon)}] = \int_{B_1(0)\setminus B_\epsilon(0)} e^U + \frac{n-1}{n}\omega_n[\epsilon U'(\epsilon)]^n$$

in view of $U(1) = \alpha_0$ and $U'(1) = 0$, and letting $\epsilon \to 0^+$ one deduces that

$$\omega_n e^{\alpha_0} = \gamma + \frac{n-1}{n}\omega_n\left(\frac{\gamma}{n\omega_n}\right)^{\frac{n}{n-1}}$$

in view of (4.12). Since $\gamma \in (0,+\infty) \to \gamma + \frac{n-1}{n}\omega_n\left(\frac{\gamma}{n\omega_n}\right)^{\frac{n}{n-1}} \in (0,+\infty)$ is a bijection, for any given $\gamma > 0$ let $\alpha_0 = \log[\frac{\gamma}{\omega_n} + \frac{n-1}{n}(\frac{\gamma}{n\omega_n})^{\frac{n}{n-1}}]$ and the corresponding U is the solution of (1.12) we were searching for. Notice that $\int_{\mathbb{R}^n} e^U < +\infty$ in view of $\int_1^\infty t^{n-1} e^{U(t)} dt < +\infty$, as it can be deduced by

$$\lim_{r\to+\infty} \frac{U(r)}{\log r} = \lim_{r\to+\infty} rU'(r) = -\left(\int_1^\infty t^{n-1} e^{U(t)} dt\right)^{\frac{1}{n-1}}$$

due to (4.11). We have established the following result:

Theorem 4.1. *For any $\gamma > 0$ there exists a $1-$parameter family of distinct solutions U_λ, $\lambda > 0$, to* (1.12) *given by $U_\lambda(x) = U(\lambda x) + n\log\lambda$ such that U_λ takes its unique absolute maximum point at $\frac{1}{\lambda}$.*

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