HARNACK INEQUALITIES AND QUANTIZATION PROPERTIES FOR THE $n-$LIOUVILLE EQUATION

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Abstract. We consider a quasilinear equation involving the $n-$Laplacian and an exponential nonlinearity, a problem that includes the celebrated Liouville equation in the plane as a special case. For a non-compact sequence of solutions it is known that the exponential nonlinearity converges, up to a subsequence, to a sum of Dirac measures. By performing a precise local asymptotic analysis we complete such a result by showing that the corresponding Dirac masses are quantized as multiples of a given one, related to the mass of limiting profiles after rescaling according to the classification result obtained by the first author in [9]. A fundamental tool is provided here by some Harnack inequality of "sup+inf" type, a question of independent interest that we prove in the quasilinear context through a new and simple blow-up approach.

1. Introduction

In the present paper we are concerned with solutions to

$$-\Delta_n u = h(x)e^u \text{ in } \Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded open set and $\Delta_n u = \operatorname{div}(|\nabla u|^{n-1}\nabla u)$ stands for the n-Laplace operator. Solutions are meant in a weak sense and by elliptic estimates [7, 20, 22] such solutions are in $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$.

When $n = 2$ problem (1.1) reduces to the so-called Liouville equation, see [15], that represents the simplest case of "Gauss curvature equation" on a two-dimensional surface arising in differential geometry. In the higher dimensional case similar geometrical problems have led to different type of curvature equations. Recently, it has been observed that the n-Laplace operator comes into play when expressing the Ricci curvature after a conformal change of the metric [18], leading to another class of curvature equations that are of relevance. Moreover, the $n-$Liouville equation (1.1) represents a simplified version of a quasilinear fourth-order problem arising [10] in the theory of log-determinant functionals, that are relevant in the study of the conformal geometry of a $4-$dimensional closed manifold. In order to understand some of the bubbling phenomena that may occur in such geometrical contexts, we are naturally led to study the simplest situation given by (1.1).

Starting from the seminal work of Brezis and Merle [3] in dimension two, the asymptotic behavior of a sequence u_k of solutions to

$$-\Delta_n u_k = h_k(x)e^{u_k} \text{ in } \Omega, \tag{1.2}$$

with

$$\sup_k \int_\Omega e^{u_k} < +\infty \tag{1.3}$$

and h_k in the class

$$\Lambda_{a,b} = \{h \in C(\Omega) :\ a \leq h \leq b \text{ in } \Omega\}, \tag{1.4}$$

can be generally described by a "concentration-compactness" alternative. Extended [1] to the quasi-linear case, it reads as follows.

Concentration-Compactness Principle: *Consider a sequence of functions u_k such that* (1.2)-(1.3) *hold with $h_k \in \Lambda_{0,b}$. Then, up to a subsequence, the following alternative holds:*

(i) *u_k is bounded in $L^\infty_{loc}(\Omega)$;*
(ii) *$u_k \to -\infty$ locally uniformly in Ω as $k \to +\infty$;*

(iii) *the blow-up set $\mathcal{S}$ of the sequence u_k, defined as*

$$\mathcal{S} = \{p \in \Omega : \text{ there exists } x_k \in \Omega \text{ s.t. } x_k \to p, u_k(x_k) \to \infty \text{ as } k \to +\infty\},$$

is finite, $u_k \to -\infty$ locally uniformly in $\Omega \setminus S$ and

$$h_k e^{u_k} \rightharpoonup \sum_{p \in \mathcal{S}} \beta_p \delta_p \tag{1.5}$$

weakly in the sense of measures as $k \to +\infty$ for some coefficients $\beta_p \geq n^n \omega_n$, where ω_n stands for the volume of the unit ball in $\mathbb{R}^n$.

The compact case, in which the sequence e^{u_k} does converge locally uniformly in Ω, is expressed by alternatives (i) and (ii), thanks to elliptic estimates [7, 22]; alternative (iii) describes the non-compact case and the characterization of the possible values for the Dirac masses β_p becomes crucial towards an accurate description of the blow-up mechanism.

When a boundary control on u_k is assumed, the answer is generally very simple. If one assumes that the oscillation of u_k on $\partial B_\delta(p)$, $p \in \mathcal{S}$, is uniformly bounded for some $\delta > 0$ small, using a Pohozaev identity it has been shown [11] that $\beta_p = c_n \omega_n$, $c_n = n(\frac{n^2}{n-1})^{n-1}$, provided h_k is in the class

$$\Lambda'_{a,b} = \{h \in C^1(\Omega) : \ a \leq h \leq b, \ |\nabla h| \leq b \text{ in } \Omega\} \tag{1.6}$$

with $a > 0$. Moreover, in the two-dimensional situation and under the condition

$$0 \leq h_k \to h \text{ in } C_{loc}(\Omega) \text{ as } k \to +\infty, \tag{1.7}$$

a general answer has been found by Y.Y. Li and Shafrir [17] showing that, for any $p \in \mathcal{S}$, $h(p) > 0$ and the concentration mass β_p is quantized as follows:

$$\beta_p \in 8\pi\mathbb{N}. \tag{1.8}$$

The meaning of the value 8π in (1.8) can be roughly understood as the sequence u_k was developing several sharp peaks collapsing in p, each of them looking like, after a proper rescaling, as a solution U of

$$-\Delta U = h(p)e^U \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U < \infty, \tag{1.9}$$

with $h(p) > 0$. Using the complex representation formula obtained by Liouville [15] or the more recent PDE approach by Chen-Li [5], the solutions of (1.9) are explicitly known and they all have the same mass: $\int_{\mathbb{R}^2} h(p)e^U = 8\pi$. Therefore the value of β_p in (1.8) just represents the sum of the masses 8π carried by each of such sharp peaks collapsing in p.

When $n \geq 3$ a similar classification result for solutions U of

$$-\Delta_n U = h(p)e^U \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} e^U < \infty, \tag{1.10}$$

with $h(p) > 0$, has been recently provided by the first author in [9]. For later convenience, observe in particular that the unique solution to

$$-\Delta_n U = h(p)e^U \quad \text{in } \mathbb{R}^n, \quad U \leq U(0) = 0, \quad \int_{\mathbb{R}^n} e^U < +\infty, \tag{1.11}$$

is given by

$$U(y) = -n \log\left(1 + c_n^{-\frac{1}{n-1}} h(p)^{\frac{1}{n-1}} |y|^{\frac{n}{n-1}}\right) \tag{1.12}$$

and satisfies

$$\int_{\mathbb{R}^n} h(p)e^U = c_n \omega_n, \quad c_n = n(\frac{n^2}{n-1})^{n-1}. \tag{1.13}$$

Due to the invariance of (1.10) under translations and scalings, all solutions to (1.10) are given by the $(n+1)-$parameter family

$$U_{a,\lambda}(y) = U(\lambda(y-a)) + n\log\lambda = \log \frac{\lambda^n}{(1 + c_n^{-\frac{1}{n-1}} h(p)^{\frac{1}{n-1}} \lambda^{\frac{n}{n-1}} |y-a|^{\frac{n}{n-1}})^n}, \quad (a,\lambda) \in \mathbb{R}^n \times (0,\infty),$$

and satisfy $\int_{\mathbb{R}^n} h(p)e^U = c_n\omega_n$. As a by-product, under the condition (1.7) we necessarily have in (1.5) that

$$\beta_p \geq c_n\omega_n, \tag{1.14}$$

a bigger value than the one appearing in the alternative (iii) of the Concentration-Compactness Principle.

The blow-up mechanism that leads to the quantization result (1.8) relies on an almost scaling-invariance property of the corresponding PDE, which guarantees that all the involved sharp peaks carry the same mass. Since it is also shared by the n-Liouville equation, a similar quantization property is expected to hold for the quasilinear case too:

$$\beta_p \in c_n\omega_n\mathbb{N}. \tag{1.15}$$

However, the main point in proving (1.8) is the limiting vanishing of the mass contribution coming from the neck regions between the sharp peaks. In the two-dimensional situation such crucial property follows by a Harnack inequality of $\sup + \inf$ type, established first in [21] through an isoperimetric argument and an analysis of the mean average for a solution u to $(1.1)_{n=2}$. A different proof can be given according to [19] through Green's representation formula, see Remark 2.3 for more details, and a sharp form of such inequality has been later established in [2, 4, 6] via isoperimetric arguments or moving planes/spheres techniques. However, all such approaches are not operating for the $n-$Liouville equation due to the nonlinearity of the differential operator; for instance, the Green representation formula is not anymore available in the quasilinear context and in the nonlinear potential theory an alternative has been found [13] in terms of the Wolff potential, which however fails to provide sharp constants as needed to derive precise asymptotic estimates on blowing-up solutions to (1.2)-(1.3). We refer the interested reader to [12, 14] for an overview on the nonlinear potential theory.

In order to establish the validity of (1.15), the first main contribution of our paper is represented by a new and very simple blow-up approach to $\sup + \inf$ inequalities. Since the limiting profiles have the form (1.12), near a blow-up point we are able to compare in an effective way a blowing-up sequence u_k with the radial situation, in which sharp constants are readily available. Using the notations (1.4) and (1.6) our first main result reads as follows:

Theorem 1.1. *Given $0 < a \leq b < \infty$, let $\Lambda \subset \Lambda_{a,b}$ be a set which is equicontinous at each point of Ω and consider*

$$\mathcal{U} := \{u \in C^{1,\alpha}(\Omega) \,:\, u \text{ solves } (1.1) \text{ with } h \in \Lambda\}.$$

Given a compact set $K \subset \Omega$ and $C_1 > n-1$, then there exists $C_2 = C_2(\Lambda, K, C_1) > 0$ so that

$$\max_K u + C_1 \inf_\Omega u \leq C_2, \quad \forall u \in \mathcal{U}. \tag{1.16}$$

In particular, the inequality (1.16) *holds for the solutions u of* (1.1) *with $h \in \Lambda'_{a,b}$.*

By combining the $\sup + \inf -$inequality with a careful blow-up analysis, we are able to prove our second main result:

Theorem 1.2. *Let u_k be a sequence of solutions to* (1.2) *so that* (1.3)-(1.5) *hold. If one assumes* (1.7), *then $h(p) > 0$ and β_p satisfies* (1.15) *for any $p \in \mathcal{S}$.*

Our paper is structured as follows. Section 2 is devoted to establish the $\sup + \inf$ inequality. Starting from a basic description of the blow-up mechanism, reported in the appendix for reader's convenience, a refined asymptotic analysis is carried over in Section 3 to establish Theorem 1.2 when the blow-up point is "isolated", according to some well established terminology as for instance in [16]. The quantization result in its full generality will be the object of Section 4.

2. The sup + inf inequality

When $n = 2$ the so-called "sup+inf" inequality has been first derived by Shafrir [21]: given $a, b > 0$ and $K \subset \Omega$ a non-empty compact set, there exist constants $C_1, C_2 > 0$ so that

$$\sup_K u + C_1 \inf_\Omega u \le C_2 \tag{2.1}$$

does hold for any solution u of $(1.1)_{n=2}$ with $0 < a \le h \le b$ in Ω; moreover one can take $C_1 = 1$ if $h \equiv 1$. Later on, Brezis, Li and Shafrir showed [2] the validity of (2.1) in its sharp form with $C_1 = 1$ for any $h \in \Lambda'_{a,b}$, $a > 0$.

To reach this goal, a first tool needed is a general Harnack inequality that holds for solutions u of $-\Delta_n u = f \ge 0$ in Ω. By means of the so-called nonlinear Wolff potential in [13] it is proved that there exists a constant $c_1 > 0$ such that

$$u(x) - \inf_\Omega u \ge c_1 \int_0^\delta \Big[\int_{B_t(x)} f \Big]^{\frac{1}{n-1}} \frac{dt}{t}$$

holds for each ball $B_{2\delta}(x) \subset \Omega$. Since $f \ge 0$, note that the above inequality implies that

$$u(x) - \inf_\Omega u \ge c_1 \Big[\int_{B_r(x)} f \Big]^{\frac{1}{n-1}} \log \frac{\delta}{r} \tag{2.2}$$

for all $0 < r < \delta$. The constant c_1 is not explicit and our argument could be significantly simplified if we knew $c_1 = (n\omega_n)^{-\frac{1}{n-1}}$, see Remark 2.3 for a thourough discussion. However, in the class of radial functions, the following lemma shows that indeed (2.2) holds with the sharp constant $c_1 = (n\omega_n)^{-\frac{1}{n-1}}$:

Lemma 2.1. *Let $u \in C^1(B_{R_2}(a))$ and $0 \le f \in C(\overline{B_{R_2}(a)})$ a radial function with respect to $a \in \mathbb{R}^n$ so that*

$$-\Delta_n u \ge f \quad \text{in } B_{R_2}(a).$$

Then

$$u(a) - \inf_{B_{R_2}(a)} u \ge (n\omega_n)^{-\frac{1}{n-1}} \int_0^{R_2} \left(\int_{B_t(a)} f \right)^{\frac{1}{n-1}} \frac{dt}{t}. \tag{2.3}$$

In particular, for each $0 < R_1 < R_2$ there holds

$$u(a) - \inf_{B_{R_2}(a)} u \ge (n\omega_n)^{-\frac{1}{n-1}} \left(\int_{B_{R_1}(a)} f \right)^{\frac{1}{n-1}} \log \frac{R_2}{R_1}.$$

Proof. Consider the radial solution u_0 solving

$$-\Delta_n u_0 = f \text{ in } B_{R_2}(a), \quad u_0 = 0 \text{ on } \partial B_{R_2}(a).$$

Since

$$-\Delta_n u \ge -\Delta_n u_0 \text{ in } B_{R_2}(a), \quad u - \inf_{B_{R_2}(a)} u \ge u_0 \text{ on } \partial B_{R_2}(a),$$

by comparison principle there holds

$$u - \inf_{B_{R_2}(a)} \ge u_0 \qquad \text{in } B_{R_2}(a). \tag{2.4}$$

Furthermore, u_0 is radial with respect to a and can be explicitly written as ($r = |x - a|$):

$$u_0(r) = \int_r^{R_2} \left(\int_0^t s^{n-1} f(s) ds \right)^{\frac{1}{n-1}} \frac{dt}{t} = \int_r^{R_2} \left(\frac{1}{n\omega_n} \int_{B_t(a)} f \right)^{\frac{1}{n-1}} \frac{dt}{t}. \tag{2.5}$$

By (2.4)-(2.5) we deduce the validity of (2.3). Since the function $t \to \int_{B_t(a)} f$ is non decreasing in view of $f \geq 0$, we have for each $0 < R_1 < R_2$ that

$$u_0(a) \geq (n\omega_n)^{-\frac{1}{n-1}} \int_{R_1}^{R_2} \left(\int_{B_t(a)} f \right)^{\frac{1}{n-1}} \frac{dt}{t} \geq (n\omega_n)^{-\frac{1}{n-1}} \left(\int_{B_{R_1}(a)} f \right)^{\frac{1}{n-1}} \log \frac{R_2}{R_1}$$

and the proof is complete thanks to (2.4). □

This lemma is helpful to extend (2.1) to the quasilinear case for all $C_1 > n-1$ and this will be enough to establish the quantization result (1.15). It is an interesting open question to know whether or not the sharp inequality with $C_1 = n-1$ is valid for a reasonable class of weights h, as when $n = 2$ [2]. The sup + inf inequality in Theorem 1.1 will be an immediate consequence of the following result.

Theorem 2.2. *Let $0 < a \leq b < \infty$, consider the sets Λ and $\mathcal{U}$ defined in Theorem 1.1. Then given $K \subset \Omega$ a nonempty compact set and $C_1 > n-1$, there exists a constant $C_3 > 0$ such that $\max_K u \leq C_3$ holds for all $u \in \mathcal{U}$ satisfying $\max_K u + C_1 \inf_\Omega u \geq 0$ (Theorem 1.1 follows by taking $C_2 = C_1 + C_3$).*

Proof. Choose $\delta > 0$ so that $K_\delta = \{x \in \Omega : \operatorname{dist}(x, K) \leq 2\delta\} \subset \Omega$. Let u be a solution to (1.1) with $h \geq 0$ so that

$$\max_K u + C_1 \inf_\Omega u \geq 0. \tag{2.6}$$

Denote by $\bar{x} \in K$ a maximum point of u in K: $u(\bar{x}) = \max_K u$. It follows from (2.2) that

$$u(\bar{x}) - \inf_\Omega u \geq c_1 \Big[\int_{B_r(\bar{x})} h e^u \Big]^{\frac{1}{n-1}} \log \frac{\delta}{r} \tag{2.7}$$

for all $0 < r < \delta$ in view of $B_{2\delta}(\bar{x}) \subset \Omega$. Therefore, we deduce that

$$c_1 \Big[\int_{B_r(\bar{x})} h e^u \Big]^{\frac{1}{n-1}} \leq \left\{ \frac{u(\bar{x}) - \inf_\Omega u}{\log \frac{\delta}{r}} \right\} \leq (1 + \frac{1}{C_1}) \frac{u(\bar{x})}{\log \frac{\delta}{r}}$$

for all $0 < r < \delta$ in view of (2.6)-(2.7).

Arguing by contradiction, if the conclusion of the theorem is wrong, we can find a sequence $u_k \in \mathcal{U}$ satisfying (2.6) such that, as $k \to \infty$, we have

$$\max_K u_k \to +\infty. \tag{2.8}$$

Letting $\bar{x}_k \in K$ so that $u_k(\bar{x}_k) = \max_K u_k$ and $\bar{\mu}_k = e^{-\frac{u_k(\bar{x}_k)}{n}}$, we have that $\bar{\mu}_k \to 0$ as $k \to +\infty$ in view of (2.8). Since for each $R > 0$ we can find $k_0 \in \mathbb{N}$ so that $R\bar{\mu}_k < \delta$ for all $k \geq k_0$, by (2.7) we deduce that

$$c_1 \limsup_{k \to \infty} \Big[\int_{B_{R\bar{\mu}_k}(\bar{x}_k)} h_k e^{u_k} \Big]^{\frac{1}{n-1}} \leq n \left(1 + \frac{1}{C_1} \right). \tag{2.9}$$

By applying Ascoli-Arzela, we can further assume, up to a subsequence, that

$$\bar{x}_k \to p \in K, \quad h_k \to h \geq a > 0 \text{ in } C_{loc}(\Omega) \tag{2.10}$$

as $k \to +\infty$.

Once (2.9) is established, in order to reach a contradiction we aim to replace $\bar{x}_k$ by a nearby local maximum point $x_k \in \Omega$ of u_k with $u_k(x_k) \geq u_k(\bar{x}_k) = \max_K u_k$. We can argue as follows: the function $\bar{U}_k(y) = u_k(\bar{\mu}_k y + \bar{x}_k) + n \log \bar{\mu}_k$ satisfies

$$-\Delta_n \bar{U}_k = h_k(\bar{\mu}_k y + \bar{x}_k) e^{\bar{U}_k} \text{ in } \Omega_k = \frac{\Omega - \bar{x}_k}{\bar{\mu}_k}$$

and

$$\bar U_k \le \bar U_k(0)=0 \ \text{ in } \frac{K-\bar x_k}{\bar\mu_k}, \qquad \limsup_{k\to+\infty}\int_{B_R(0)} e^{\bar U_k} \le \frac{1}{a}\Big(\frac{n}{c_1}\Big)^{n-1}\Big(1+\frac{1}{C_1}\Big)^{n-1} \tag{2.11}$$

in view of (2.9)-(2.10). From (2.11), the Concentration-Compactness Principle and $\bar U_k(0)=0$ we deduce, up to a subsequence, that:

(i) either, $\bar U_k$ is bounded in $L^\infty_{loc}(\mathbb{R}^n)$

(ii) or, $h_k(\bar\mu_k y+\bar x_k)e^{\bar U_k} \rightharpoonup \beta_0\delta_0+\sum_{i=1}^I \beta_i\delta_{p_i}$ weakly in the sense of measures in $\mathbb{R}^n$, for some $\beta_i\ge n^n\omega_n$, $i\in\{0,\dots,I\}$, and distinct points $p_1,\dots,p_I\in\mathbb{R}^n\setminus\{0\}$, and $\bar U_k\to-\infty$ locally uniformly in $\mathbb{R}^n\setminus\{0,p_1,\dots,p_I\}$.

Case (i): $\bar U_k$ is bounded in $L^\infty_{loc}(\mathbb{R}^n)$

By elliptic estimates [7, 22] we deduce that $\bar U_k\to\bar U$ in $C^1_{loc}(\mathbb{R}^n)$ as $k\to+\infty$, where $\bar U$ satisfies (1.10) with $h(p)>0$ and $\bar U(0)=0$ in view of (2.10)-(2.11). Since in general $\frac{K-\bar x_k}{\bar\mu_k}$ does not tend to $\mathbb{R}^n$ as $k\to+\infty$, by (2.11) we cannot guarantee that $\bar U$ achieves the maximum value at 0. However, by the classification result in [9] we have that $\bar U=U_{a,\lambda}$ for some $(a,\lambda)\in\mathbb{R}^n\times(0,\infty)$. Since $\bar U$ is a radially strictly decreasing function with respect to a, we can find a sequence $a_k\to a$ such that as $k\to+\infty$

$$\bar U_k(a_k)=\max_{B_R(a_k)}\bar U_k,\quad \bar U_k(a_k)\to\bar U(a)=\max_{\mathbb{R}^n}\bar U \tag{2.12}$$

for all $R>0$ and k large (depending on R). Setting $x_k=\bar\mu_k a_k+\bar x_k$ and $\mu_k=e^{-\frac{u_k(x_k)}{n}}$, we have that

$$u_k(x_k)=\bar U_k(a_k)-n\log\bar\mu_k\ge \bar U_k(0)-n\log\bar\mu_k=u_k(\bar x_k)$$

and

$$1\le\frac{\bar\mu_k}{\mu_k}=e^{\frac{u_k(x_k)-u_k(\bar x_k)}{n}}=e^{\frac{\bar U_k(a_k)}{n}}\xrightarrow{k\to\infty} e^{\frac{\max_{\mathbb{R}^n}\bar U}{n}} \tag{2.13}$$

in view of (2.12). Let us now rescale u_k with respect to x_k by setting

$$U_k(y)=u_k(\mu_k y+x_k)+n\log\mu_k.$$

Since (2.11)-(2.12) re-write in terms of U_k as

$$\limsup_{k\to+\infty}\int_{B_{R\frac{\bar\mu_k}{\mu_k}}(-\frac{\bar\mu_k}{\mu_k}a_k)} e^{U_k}\le\frac{1}{a}\Big(\frac{n}{c_1}\Big)^{n-1}\Big(1+\frac{1}{C_1}\Big)^{n-1} \tag{2.14}$$

$$U_k(0)=\max_{B_{R\frac{\bar\mu_k}{\mu_k}}(0)}U_k=0 \tag{2.15}$$

for all $R>0$, thanks to the uniform convergence (2.10), by (2.13)-(2.15) and elliptic estimates [7, 22] we have that $U_k\to U$ in $C^1_{loc}(\mathbb{R}^n)$ as $k\to+\infty$, where U satisfies (1.11) with $h(p)>0$. Then U takes precisely the form (1.12) and satisfies (1.13).

Therefore for each $R>0$ and $\epsilon\in(0,1)$, there exists $k_0=k_0(R,\varepsilon)>0$ so that for all $k\ge k_0$ there hold $B_{R\mu_k}(x_k)\subset B_\delta(x_k)\subset B_{2\delta}(\bar x_k)$ and

$$h_k(x)\ge\sqrt{1-\epsilon}\,h(p),\quad u_k(x)\ge U_{x_k,\mu_k^{-1}}+\log\sqrt{1-\epsilon}\qquad\text{in } B_{R\mu_k}(x_k) \tag{2.16}$$

in view of (2.10) and $U_k\ge U+\log\sqrt{1-\epsilon}$ in $B_R(0)$. Setting $f_k(t)=(1-\epsilon)h(p)e^{U_{x_k,\mu_k^{-1}}}\chi_{B_{R\mu_k}(x_k)}$, by (2.16) we have that $h_ke^{u_k}\ge f_k$ in $B_\delta(x_k)$ and then Lemma 2.1 implies the following lower bound for all $k\ge k_0$:

$$u_k(x_k)-\inf_{B_\delta(x_k)}u_k\ge\left(\frac{1-\epsilon}{n\omega_n}\int_{B_R(0)}h(p)e^U\right)^{\frac{1}{n-1}}\log\frac{\delta}{R\mu_k} \tag{2.17}$$

in view of $\int_{B_{R\mu_k}(x_k)} f_k = (1-\epsilon)\int_{B_R(0)} h(p)e^U$. Recalling that $\mu_k = e^{-\frac{u_k(x_k)}{n}}$, by (2.17) we deduce that

$$\left(\frac{1-\epsilon}{n\omega_n}\int_{B_R(0)} h(p)e^U\right)^{\frac{1}{n-1}} \le \frac{u_k(x_k) - \inf_{B_\delta(x_k)} u_k}{\log\frac{\delta}{R} + \frac{u_k(x_k)}{n}}.$$

Since

$$u_k(x_k) + C_1 \inf_{B_\delta(x_k)} u_k \ge u_k(\bar{x}_k) + C_1 \inf_\Omega u_k = \max_K u_k + C_1 \inf_\Omega u_k \ge 0$$

in view of (2.6), letting $k\to\infty$ we deduce

$$\left(\frac{1-\epsilon}{n\omega_n}\int_{B_R(0)} h(p)e^U\right)^{\frac{1}{n-1}} \le n\limsup_{k\to\infty}\left\{1 - \frac{\inf_{B_\delta(x_k)} u_k}{u_k(x_k)}\right\} \le n\left(1+\frac{1}{C_1}\right).$$

Since this holds for each $R,\varepsilon>0$ we deduce that

$$\frac{1}{n\omega_n}\int_{\mathbb{R}^n} h(p)e^U \le \left[n(1+\frac{1}{C_1})\right]^{n-1} < \left(\frac{n^2}{n-1}\right)^{n-1}$$

in view of the assumption $C_1 > n-1$. On the other hand, by (1.13) the left hand side is precisely $\left(\frac{n^2}{n-1}\right)^{n-1}$ and this is a contradiction.

Case (ii): $h_k(\bar{\mu}_k y + \bar{x}_k)e^{\bar{U}_k} \rightharpoonup \beta_0\delta_0 + \sum_{i=1}^I \beta_i\delta_{p_i}$ weakly in the sense of measures in $\mathbb{R}^n$, for some $\beta_i \ge n^n\omega_n$, $i\in\{0,\dots,I\}$, and distinct points $p_1,\dots,p_I \in \mathbb{R}^n\setminus\{0\}$, and $\bar{U}_k \to -\infty$ locally uniformly in $\mathbb{R}^n\setminus\{0,p_1,\dots,p_I\}$

If $I\ge 1$, w.l.o.g. assume that $p_1,\dots,p_I \notin \overline{B_1(0)}$. Since $\bar{U}_k\to-\infty$ locally uniformly in $\overline{B_1(0)}\setminus\{0\}$ and $\max_{B_1(0)} \bar{U}_k \to +\infty$ as $k\to+\infty$, we can find $a_k\to 0$ so that

$$\bar{U}_k(a_k) = \max_{B_1(0)} \bar{U}_k \to +\infty \tag{2.18}$$

as $k\to+\infty$. We now argue in a similar way as in case (i). Setting $x_k = \bar{\mu}_k a_k + \bar{x}_k$ and $\mu_k = e^{-\frac{u_k(x_k)}{n}}$, we have that $u_k(x_k) = \bar{U}_k(a_k) - n\log\bar{\mu}_k \ge \bar{U}_k(0) - n\log\bar{\mu}_k = u_k(\bar{x}_k)$ and

$$\frac{\bar{\mu}_k}{\mu_k} = e^{\frac{\bar{U}_k(a_k)}{n}} \to +\infty \tag{2.19}$$

as $k\to+\infty$ in view of (2.18). Setting

$$U_k(y) = u_k(\mu_k y + x_k) + n\log\mu_k,$$

by (2.11) and (2.18) we have that

$$\limsup_{k\to+\infty}\int_{B_{R\frac{\bar{\mu}_k}{\mu_k}}(-\frac{\bar{\mu}_k}{\mu_k}a_k)} e^{U_k} \le \frac{1}{a}(\frac{n}{c_1})^{n-1}\left(1+\frac{1}{C_1}\right)^{n-1} \tag{2.20}$$

$$U_k(0) = \max_{B_{\frac{\bar{\mu}_k}{\mu_k}}(0)} U_k = 0 \tag{2.21}$$

for all $R>0$. Since $B_R(0)\subset B_{R\frac{\bar{\mu}_k}{\mu_k}}(-\frac{\bar{\mu}_k}{\mu_k}a_k)$ for all k large in view of (2.19) and $\lim_{k\to+\infty} a_k = 0$, by (2.19)-(2.21) and elliptic estimates [7, 22] we have that $U_k\to U$ in $C^1_{loc}(\mathbb{R}^n)$ as $k\to+\infty$, where U satisfies (1.11)-(1.13). We now proceed exactly as in case (i) to reach a contradiction. The proof is complete. □

Remark 2.3. *When $n=2$, the "sup+inf" inequality was first derived by Shafrir [21] through an isoperimetric argument. It becomes clear in [19], when dealing with a fourth-order exponential PDE in $\mathbb{R}^4$, that the main point comes from the linear theory, which allows there to avoid the extra work needed in our framework. For instance, in the two dimensional case, inequality (2.2) is an easy consequence of the Green representation formula: given a solution u to $-\Delta u = f$ in a domain containing $B_1(0)$, we can use the fundamental solution of the Laplacian to obtain*

$$u(x) - \inf_{B_1(0)} u \geq -\frac{1}{2\pi}\int_{B_1(0)} \log \frac{|x-y|}{||x|y - \frac{x}{|x|}|} f(y) \qquad \forall\, x \in B_1(0),$$

which through an integration by parts gives

$$u(0) - \inf_{\Omega} u \geq -\frac{1}{2\pi}\int_{B_1(0)} \log|y| f(y) = \frac{1}{2\pi}\int_0^1 [\int_{B_t(0)} f]\frac{dt}{t}.$$

This linear argument also provides the optimal constant $c_1 = \frac{1}{2\pi}$, which can be exploited to simplify the proof of Theorem 2.2 as follows. The estimates (2.9) re-writes as

$$\begin{aligned}\limsup_{k\to+\infty}\int_{B_R(0)} h_k(\bar\mu_k y + \bar x_k) e^{\bar U_k} &= \limsup_{k\to+\infty}\int_{B_{R\bar\mu_k}(\bar x_k)} h_k e^{u_k} \leq \Big[\frac{n}{c_1}(1+\frac{1}{C_1})\Big]^{n-1} \\ &< \Big[\frac{n^2}{c_1(n-1)}\Big]^{n-1}\end{aligned} \tag{2.22}$$

for all $R>0$ when $C_1 > n-1$. Since $c_1 = \frac{1}{2\pi}$ and $[\frac{n^2}{c_1(n-1)}]^{n-1} = 8\pi$ when $n=2$, in case (i) of the above proof we deduce that $\int_{\mathbb{R}^2} h(p)e^{\bar U} < 8\pi$, in contrast with the quantization property $\int_{\mathbb{R}^2} h(p)e^U = 8\pi$ for every solution U of (1.9). Assuming w.l.o.g. $p_1,\dots,p_I \notin \overline{B_1(0)}$ if $I\geq 1$, in case (ii) of the above proof we deduce from (2.22) with $R=1$ that $\beta_0 < 8\pi$, in contrast with the lower estimate $\beta_0 \geq 8\pi$ coming from (1.7) and (1.14) when $n=2$. Therefore, the proof of Theorem 2.2 in dimension two becomes considerably simpler.

When $n\geq 3$ Green's representation formula is not available for Δ_n and (2.2) does hold [13] with some constant $0 < c_1 \leq (n\omega_n)^{-\frac{1}{n-1}}$. Since c_1 is in general strictly below the optimal one $(n\omega_n)^{-\frac{1}{n-1}}$, we need to fill the gap thanks to the exponential form of the nonlinearity through a blow-up approach. With this strategy a comparison argument with the radial case is exploited, since in the radial context inequality (2.2) does hold with optimal constant $c_1 = (n\omega_n)^{-\frac{1}{n-1}}$ thanks to Lemma 2.1.

As a consequence of the sup + inf estimates in Theorem 2.2, we deduce the following useful decay estimate:

Corollary 2.4. *Let u_k be a sequence of solutions to (1.2), satisfying (1.7) with $h_k \geq \epsilon_0 > 0$ in $B_{4r_0}(x_k) \subset \Omega$ and*

$$|x-x_k|^n e^{u_k} \leq C \qquad \text{in } B_{2b_k}(x_k)\setminus B_{a_k}(x_k) \tag{2.23}$$

for $0 < 2a_k < b_k \leq 2r_0$. Then, there exist $\alpha, C > 0$ such that

$$u_k \leq C - \frac{\alpha}{n} u_k(x_k) - (n+\alpha)\log|x-x_k| \tag{2.24}$$

for all $2a_k \leq |x-x_k| \leq b_k$. In particular, if $e^{-\frac{u_k(x_k)}{n}} = o(a_k)$ as $k\to+\infty$ we have that

$$\lim_{k\to+\infty}\int_{B_{b_k}(x_k)\setminus B_{2a_k}(x_k)} h_k e^{u_k} = 0. \tag{2.25}$$

Proof. Letting $V_k(y) = u_k(ry+x_k) + n\log r$ for any $0 < r \leq b_k$, we have that $-\Delta_n V_k = h_k(ry+x_k)e^{V_k}$ does hold in $\Omega_k = \frac{\Omega - x_k}{r}$ and (2.23) implies that

$$\sup_{B_2(0)\setminus B_{\frac12}(0)} |y|^n e^{V_k} \leq C < +\infty \tag{2.26}$$

for all $2a_k \le r \le b_k$. Since V_k is uniformly bounded from above in $B_2(0) \setminus B_{\frac12}(0)$ in view of (2.26), by the Harnack inequality [20, 23] it follows that there exist $C>0$ and $C_0 \in (0,1]$ so that

$$C_0 \sup_{|y|=1} V_k \le \inf_{|y|=1} V_k + C \tag{2.27}$$

for all $2a_k \le r \le b_k$.

Up to a subsequence, assume that $\lim\limits_{k\to+\infty} x_k = x_0$. By assumption we have that $h_k(ry+x_k) \to h(ry+x_0) \ge \epsilon_0 > 0$ in $C_{loc}(B_1(0))$ as $k \to +\infty$ for all $0<r\le 2r_0$. For any given $C_1 > n-1$, by Theorem 1.1 applied to V_k in $B_1(0)$ with $K=\{0\}$ we obtain the existence of $C_2>0$ so that

$$V_k(0) + C_1 \inf_{B_1(0)} V_k = V_k(0) + C_1 \inf_{|y|=1} V_k \le C_2 \tag{2.28}$$

does hold for all k and all $0<r\le 2r_0$. Inserting (2.28) into (2.27) we deduce that

$$\sup_{|y|=1} V_k \le C - \frac{\alpha}{n} V_k(0)$$

for all $2a_k \le r \le b_k$, with $\alpha = \frac{n}{C_0 C_1} > 0$ and some $C>0$, which re-writes in terms of u_k as (2.24). In particular, by (2.24) we deduce that

$$0 \le \int_{2a_k \le |x-x_k| \le b_k} h_k e^{u_k} \le C e^{-\frac{\alpha}{n} u_k(x_k)} \int_{2a_k \le |x-x_k| \le b_k} \frac{dx}{|x-x_k|^{n+\alpha}} = \frac{C n \omega_n}{\alpha 2^\alpha} [a_k e^{\frac{u_k(x_k)}{n}}]^{-\alpha} \to 0$$

provided $e^{-\frac{u_k(x_k)}{n}} = o(a_k)$ as $k\to+\infty$, in view of (1.7) and $B_{4r_0}(x_k) \subset \Omega$. □

3. The case of isolated blow-up

The following basic description of the blow-up mechanism is very well known, see [17] in the two-dimensional case and for example [8] in a related higher-dimensional context, and is the starting point for performing a more refined asymptotic analysis. For reader's convenience its proof is reported in the appendix.

Theorem 3.1. *Let u_k be a sequence of solutions to* (1.2) *which satisfies* (1.3) *and*

$$h_k e^{u_k} \rightharpoonup \beta \delta_0 \text{ weakly in the sense of measures in } B_{3\delta}(0) \subset \Omega \tag{3.1}$$

for some $\beta>0$ as $k\to\infty$. Assuming (1.7)*, then $h(0)>0$ and, up to a subsequence, we can find a finite number of points $x_k^1,\dots,x_k^N$ so that for all $i\ne j$*

$$|x_k^i| + \mu_k^i + \frac{\mu_k^i + \mu_k^j}{|x_k^i - x_k^j|} \to 0 \tag{3.2}$$

$$u_k(\mu_k^i y + x_k^i) + n \log \mu_k^i \to U(y) \text{ in } C^1_{loc}(\mathbb{R}^n) \tag{3.3}$$

as $k\to+\infty$ and

$$\min\{|x-x_k^1|^n,\dots,|x-x_k^N|^n\} e^{u_k} \le C \text{ in } B_{2\delta}(0) \tag{3.4}$$

for all k and some $C>0$, where U is given by (1.12) *with $p=0$ and*

$$u_k(x_k^i) = \max_{B_{\mu_k^i}(x_k^i)} u_k, \quad \mu_k^i = e^{-\frac{u_k(x_k^i)}{n}}. \tag{3.5}$$

In this section we consider the case of an "isolated" blow-up point corresponding to have $N=1$ in Theorem 3.1, namely

$$|x-x_k|^n e^{u_k} \le C \text{ in } B_{2\delta}(0) \tag{3.6}$$

for all k and some $C>0$, where x_k simply denotes x_k^1. The following result, corresponding to Theorem 1.2 for the case of an isolated blow-up, extends the analogous two-dimensional one [17, Prop. 2] to $n\ge 2$.

Theorem 3.2. *Let u_k be a sequence of solutions to* (1.2) *which satisfies* (1.3), (1.7), (3.1) *and* (3.6). *Then*

$$\beta = c_n \omega_n.$$

Proof. First, notice that $x_k \to 0$ as $k \to +\infty$ and $h(0) > 0$ in view of Theorem 3.1. Since $h \in C(\Omega)$ take $0 < r_0 \leq \frac{\delta}{2}$ and $\epsilon_0 > 0$ so that $h \geq 2\epsilon_0$ for all $y \in B_{5r_0}(0)$. By (1.7) we then deduce that $h_k \geq \epsilon_0 > 0$ in $B_{4r_0}(x_k) \subset \Omega$. Letting $\mu_k = e^{-\frac{u_k(x_k)}{n}}$ and $U_k = u_k(\mu_k y + x_k) + n \log \mu_k$, there holds

$$\lim_{k\to+\infty} \int_{B_{R\mu_k}(x_k)} h_k e^{u_k} dx = \lim_{k\to+\infty} \int_{B_R(0)} h_k(\mu_k y + x_k) e^{U_k} dy = \int_{B_R(0)} h(0) e^U dy$$

in view of (1.7) and (3.3). Therefore we can construct $R_k \to +\infty$ so that $R_k \mu_k \leq r_0$ and

$$\lim_{k\to+\infty} \int_{B_{R_k\mu_k}(x_k)} h_k e^{u_k} dx = c_n \omega_n \tag{3.7}$$

in view of (1.13) with $p = 0$. Since (3.6) implies the validity of (2.23) with $b_k = r_0$ and $a_k = \frac{R_k \mu_k}{2}$, we can apply Corollary 2.4 to deduce by (2.25) that

$$\lim_{k\to+\infty} \int_{B_{r_0}(x_k)\setminus B_{R_k\mu_k}(x_k)} h_k e^{u_k} = 0 \tag{3.8}$$

in view of $\mu_k = e^{-\frac{u_k(x_k)}{n}} = o(a_k)$ as $k \to +\infty$. Since by the Concentration-Compactness Principle we have that $u_k \to -\infty$ locally uniformly in $B_{3\delta}(0) \setminus \{0\}$ as $k \to +\infty$, we finally deduce that β in (3.1) satisfies

$$\beta = \lim_{k\to+\infty} \int_{B_{r_0}(x_k)} h_k e^{u_k} = c_n \omega_n$$

in view of (3.7)-(3.8), and the proof is complete. □

4. General quantization result

In order to address quantization issues in the general case where $N \geq 2$ in Theorem 3.1, in the following result let us consider a more general situation.

Theorem 4.1. *Let u_k be a sequence of solutions to* (1.2) *which satisfies* (1.3) *and* (3.1). *Assume* (1.7) *and the existence of a finite number of points $x_k^1, \dots, x_k^N$ and radii $r_k^1, \dots, r_k^N$ so that for all $i \neq j$*

$$|x_k^i| + \frac{\mu_k^i}{r_k^i} + \frac{r_k^i + r_k^j}{|x_k^i - x_k^j|} \to 0 \tag{4.1}$$

as $k \to +\infty$, where $\mu_k^i = e^{-\frac{u_k(x_k^i)}{n}}$, and

$$\min\{|x - x_k^1|^n, \dots, |x - x_k^N|^n\} e^{u_k} \leq C \text{ in } B_{2\delta}(0) \setminus \bigcup_{i=1}^N B_{r_k^i}(x_k^i) \tag{4.2}$$

for all k and some $C > 0$. If $\lim_{k\to+\infty} \int_{B_{2r_k^i}(x_k^i)} h_k e^{u_k} = \beta_i$ for all $i = 1, \dots, N$, then

$$\lim_{k\to+\infty} \int_{B_{\frac{\delta}{2}}(0)} h_k e^{u_k} = \sum_{i=1}^N \beta_i. \tag{4.3}$$

Proof. First of all, by applying the Concentration-Compactness Principle to $u_k(r_k^i y + x_k^i) + n \log r_k^i$ we obtain that $\beta_i > 0$, $i = 1, \dots, N$, in view of $\frac{\mu_k^i}{r_k^i} \to 0$ as $k \to +\infty$. Since $h(0) > 0$ by Theorem 3.1 and $h \in C(\Omega)$, we can find $0 < r_0 \leq \frac{\delta}{2}$ so that $h_k \geq \epsilon_0 > 0$ in $B_{4r_0}(x_k) \subset \Omega$ in view of (1.7). The case $N = 1$ follows the same lines as in Theorem 3.2: since (4.1)-(4.2) imply the validity of (2.23) with $b_k = r_0$ and $a_k = r_k$, by Corollary 2.4 we get that

$$\lim_{k\to+\infty} \int_{B_{r_0}(x_k)\setminus B_{2r_k}(x_k)} h_k e^{u_k} = 0$$

in view of $\mu_k = o(r_k)$. Since $u_k \to -\infty$ locally uniformly in $B_{3\delta}(0) \setminus \{0\}$ as $k \to +\infty$ in view of the Concentration-Compactness Principle, (4.3) is then established when $N = 1$.

We proceed by strong induction in N and assume the validity of Theorem 4.1 for a number of points $\le N-1$. Given $x_k^1, \dots, x_k^N$, define their minimal distance as $d_k = \min\{|x_k^i - x_k^j| : i, j = 1, \dots, N, i \neq j\}$. Since $B_{\frac{d_k}{2}}(x_k^i) \cap B_{\frac{d_k}{2}}(x_k^j) = \emptyset$ for $i \neq j$, we deduce that $|x - x_k^i| \le |x - x_k^j|$ in $B_{\frac{d_k}{2}}(x_k^i)$ for all $i \neq j$ and then (4.2) gets rewritten as $|x - x_k^i|^n e^{u_k} \le C$ in $B_{\frac{d_k}{2}}(x_k^i) \setminus B_{r_k^i}(x_k^i)$ for all $i = 1, \dots, N$. By (4.1) and Corollary 2.4 with $b_k = \frac{d_k}{4}$ and $a_k = r_k^i$ we deduce that $\int_{B_{\frac{d_k}{4}}(x_k^i)\setminus B_{2r_k^i}(x_k^i)} h_k e^{u_k} \to 0$ as $k \to +\infty$ and then

$$\lim_{k\to+\infty} \int_{B_{\frac{d_k}{4}}(x_k^i)} h_k e^{u_k} = \beta_i \qquad \forall\, i = 1, \dots, N. \tag{4.4}$$

Up to relabelling, assume that $d_k = |x_k^1 - x_k^2|$ and consider the following set of indices

$$I = \{i = 1, \dots, N : \ |x_k^i - x_k^1| \le C d_k \text{ for some } C > 0\}$$

of cardinality $N_0 \in [2, N]$ since $1, 2 \in I$ by construction. Up to a subsequence, we can assume that

$$\frac{|x_k^j - x_k^i|}{d_k} \to +\infty \text{ as } k \to +\infty \tag{4.5}$$

for all $i \in I$ and $j \notin I$. Letting $\tilde{u}_k(y) = u_k(d_k y + x_k^1) + n \log d_k$, notice that

$$\tilde{u}_k\Big(\frac{x_k^i - x_k^1}{d_k}\Big) = u_k(x_k^i) + n \log d_k = n \log \frac{d_k}{\mu_k^i} \to +\infty \tag{4.6}$$

as $k \to +\infty$ in view of (4.1), and (4.2) re-writes as

$$\min\{|y - \frac{x_k^i - x_k^1}{d_k}|^n : i \in I\} e^{\tilde{u}_k} \le C_R \text{ uniformly in } B_R(0) \setminus \bigcup_{i\in I} B_{\frac{r_k^i}{d_k}}\Big(\frac{x_k^i - x_k^1}{d_k}\Big) \tag{4.7}$$

for any $R > 0$ thanks to (4.5). Since $\frac{r_k^i}{d_k} \to 0$ as $k \to +\infty$ in view of (4.1), by (4.6)-(4.7) and the Concentration-Compactness Principle we deduce that

$$\tilde{u}_k \to -\infty \text{ uniformly on } B_R(0) \setminus \bigcup_{i\in I} B_{\frac{1}{4}}\Big(\frac{x_k^i - x_k^1}{d_k}\Big)$$

as $k \to +\infty$ and then

$$\lim_{k\to+\infty} \int_{B_{Rd_k}(x_k^1)\setminus \bigcup_{i\in I} B_{\frac{d_k}{4}}(x_k^i)} h_k e^{u_k} = 0. \tag{4.8}$$

By (4.4) and (4.8) we finally deduce that

$$\lim_{k\to+\infty} \int_{B_{Rd_k}(x_k^1)} h_k e^{u_k} = \sum_{i\in I} \beta_i$$

since the balls $B_{\frac{d_k}{4}}(x_k^i)$, $i \in I$, are disjoint.

Set $x'_k = x^1_k$, $r'_k = \frac{Rd_k}{2}$ and $\beta' = \sum_{i \in I} \beta_i$. We apply the inductive assumption with the $N - N_0 + 1$ points x'_k and $\{x^j_k\}_{j \notin I}$, radii r'_k and $\{r^j_k\}_{j \notin I}$, masses β' and $\{\beta_j\}_{j \notin I}$ thanks to the following reduced form of assumption (4.2):

$$\min\{|x - x'_k|^n, \, |x - x^j_k|^n : \, j \notin I\} e^{u_k} \leq C \text{ in } B_{2\delta}(0) \setminus [B_{r'_k}(x'_k) \cup \bigcup_{j \notin I} B_{r^j_k}(x^j_k)]$$

provided R is taken sufficiently large. It finally shows the validity of (4.3) for the index N, and the proof is achieved by induction. □

We are now in position to establish Theorem 1.2 in full generality.

Proof. We first apply Theorem 3.1 to have a first blow-up description of u_k. We have that $\beta_p = N c_n \omega_n$ for all $p \in \mathcal{S}$ in view of Theorem 4.1, provided we can construct radii r^i_k, $i = 1, \dots, N$, satisfying (4.1) and

$$\lim_{k \to +\infty} \int_{B_{2r^i_k}(x^i_k)} h_k e^{u_k} = c_n \omega_n. \tag{4.9}$$

Since by (1.7) and (3.3) we deduce that

$$\int_{B_{R\mu^i_k}(x^i_k)} h_k e^{u_k} \to \int_{B_R(0)} h(p) e^U \tag{4.10}$$

as $k \to +\infty$, by (1.13) for all $i = 1, \dots N$ we can find $R^i_k \to +\infty$ so that $R^i_k \mu^i_k \leq \delta$ and

$$\int_{B_{2R^i_k \mu^i_k}(x^i_k)} h_k e^{u_k} \to c_n \omega_n. \tag{4.11}$$

If $N = 1$ we simply set $r_k = R_k \mu_k$ (omitting the index $i = 1$). When $N \geq 2$, by (3.2) we deduce that $\mu^i_k = o(d^i_k)$, where $d^i_k = \min\{|x^j_k - x^i_k| : \, j \neq i\}$, and we can set $r^i_k = \min\{\sqrt{d^i_k \mu^i_k}, R^i_k \mu^i_k\}$ in this case. By construction the radii r^i_k satisfy (4.1) and (4.9) easily follows by (1.13) and (4.10) and (4.11), in view of the chain of inequalities

$$\int_{B_{R\mu^i_k}(x^i_k)} h_k e^{u_k} \leq \int_{B_{2r^i_k}(x^i_k)} h_k e^{u_k} \leq \int_{B_{2R^i_k \mu^i_k}(x^i_k)} h_k e^{u_k}$$

for all $R > 0$ and k large (depending on R). □

5. Appendix

For the sake of completeness, we give below the proof of Theorem 3.1.

Proof. By the Concentration-Compactness Principle and (3.1) we know that

$$\max_{\overline{B_{2\delta}(0)}} u_k \to +\infty, \qquad u_k \to -\infty \text{ locally uniformly in } \overline{B_{2\delta}(0)} \setminus \{0\}. \tag{5.1}$$

Let $x_k = x^1_k$ be the sequence of maximum points of u_k in $\overline{B_{2\delta}(0)}$: $u_k(x_k) = \max_{\overline{B_{2\delta}(0)}} u_k$. If (3.4) does already hold, the result is established by simply taking $k = 1$ and $\mu_k = \mu^1_k$ according to (3.5), since (3.2) follows by (5.1) and the proof of (3.3) is classical and indipendent on the validity of (3.4). Indeed, $U_k(y) = u_k(\mu_k y + x_k) + n \log \mu_k$ satisfies $U_k(y) \leq U_k(0) = 0$ in $B_{\frac{2\delta}{\mu_k}}(0)$ and

$$-\Delta_n U_k = h_k(\mu_k y + x_k) e^{U_k} \text{ in } \frac{\Omega - x_k}{\mu_k}, \qquad \int_{\frac{\Omega - x_k}{\mu_k}} e^{U_k} = \int_\Omega e^{u_k}. \tag{5.2}$$

Since $\frac{\Omega - x_k}{\mu_k} \to \mathbb{R}^n$ as $k \to +\infty$ in view of (3.2) and $B_{3\delta}(0) \subset \Omega$, by (1.3), (1.7) and elliptic estimates [7, 22] we deduce that, up to a subsequence, $U_k \to U$ in $C^1_{loc}(\mathbb{R}^n)$, where U solves (1.11) with $p = 0$.

Notice that $h(0)=0$ would imply that U is an upper-bounded $n-$harmonic function in $\mathbb{R}^n$ and therefore a constant function (see for instance Corollary 6.11 in [12]), contradicting $\int_{\mathbb{R}^n} e^U<\infty$. As a consequence, we deduce that $h(0)>0$ and U is the unique solution of (1.11) given by (1.12) with $p=0$.

Assume that (3.4) does not hold with $x_k=x_k^1$ and proceed by induction. Suppose to have found $x_k^1,\dots,x_k^l$ so that (3.2)-(3.3) and (3.5) do hold. If (3.4) is not valid for $x_k^1,\dots,x_k^l$, in view of (5.1) we construct $\bar x_k\in B_{2\delta}(0)$ as

$$u_k(\bar x_k)+n\log \min_{i=1,\dots,l}|\bar x_k-x_k^i|=\max_{B_{2\delta}(0)}[u_k+n\log \min_{i=1,\dots,l}|x-x_k^i|]\to+\infty \tag{5.3}$$

and have that (3.2) is still valid for $x_k^1,\dots,x_k^l,\bar x_k$ with $\bar\mu_k=e^{-\frac{u_k(\bar x_k)}{n}}$ as it follows by (3.3) for $i=1,\dots,l$ and (5.3).

Let us argue in a similar way as in the proof of Theorem 2.2. Observe that $\min\limits_{i=1,\dots,l}|\bar x_k+\bar\mu_k y-x_k^i|\geq \frac{1}{2}\min\limits_{i=1,\dots,l}|\bar x_k-x_k^i|$ and

$$u_k(\bar\mu_k y+\bar x_k)+n\log\bar\mu_k\leq n\log \min_{i=1,\dots,l}|\bar x_k-x_k^i|-n\log\min_{i=1,\dots,l}|\bar\mu_k y+\bar x_k-x_k^i|\leq n\log 2$$

for $|y|\leq R_k=\frac{1}{2\bar\mu_k}\min\limits_{i=1,\dots,l}|\bar x_k-x_k^i|$ in view of (5.3). Hence $\bar U_k(y)=u_k(\bar\mu_k y+\bar x_k)+n\log\bar\mu_k$ satisfies the analogue of (5.2) with $\bar U_k\leq n\log 2$ in $B_{R_k}(0)$. Since $R_k\to+\infty$ in view of (3.2) for $x_k^1,\dots,x_k^l,\bar x_k$, up to a subsequence, by elliptic estimates [7, 22] $\bar U_k\to\bar U$ in $C^1_{loc}(\mathbb{R}^n)$, where $\bar U$ is a solution of (1.10) with $p=0$. By the classification result [9] we know that $\bar U=U_{a,\lambda}$ for some $(a,\lambda)\in\mathbb{R}^n\times(0,\infty)$. Since $\bar U$ is a radially strictly decreasing function with respect to a, there exists a sequence $a_k\to a$ as $k\to+\infty$ so that

$$\bar U_k(a_k)=\max_{B_R(a_k)}\bar U_k \tag{5.4}$$

for all $R>0$ and k large (depending on R). Setting $x_k^{l+1}=\bar\mu_k a_k+\bar x_k$, since $\mu_k^{l+1}=e^{-\frac{u_k(x_k^{l+1})}{n}}$ satisfies

$$\frac{\bar\mu_k}{\mu_k^{l+1}}=e^{\frac{\bar U_k(a_k)}{n}}\to e^{\frac{\max_{\mathbb{R}^n}\bar U}{n}} \tag{5.5}$$

as $k\to+\infty$, we deduce that (3.2) is valid for $x_k^1,\dots,x_k^{l+1}$ and (3.5) follows by (5.4) with some $R>e^{-\frac{\max_{\mathbb{R}^n}\bar U}{n}}$. Since $U_k^{l+1}=u_k(\mu_k^{l+1}y+x_k^{l+1})+n\log\mu_k^{l+1}$ satisfies $U_k^{l+1}(y)\leq U_k^{l+1}(0)=0$ in $B_{R\frac{\bar\mu_k}{\mu_k^{l+1}}}(0)$ in view of (5.4), by (1.3), (1.7), (5.5) and elliptic estimates [7, 22] we deduce that, up to a subsequence, $U_k^{l+1}\to U$ in $C^1_{loc}(\mathbb{R}^n)$, where U is the unique solution of (1.11) given by (1.12) with $p=0$, establishing the validity of (3.3) for $i=l+1$ too.

Since (3.2)-(3.3) and (3.5) on $x_k^1,\dots,x_k^l$ imply

$$\lim_{k\to+\infty}\int_{B_{3\delta}(0)}h_k e^{u_k}\geq\lim_{R\to+\infty}\lim_{k\to+\infty}\sum_{i=1}^{l}\int_{B_{R\mu_k^i}(x_k^i)}h_k e^{u_k}=lc_n\omega_n$$

thanks to (1.7), (1.13) and (3.3), in view of (3.1) the inductive process must stop after a finite number of iterations, say N, yielding the validity of Theorem 3.1 with $x_k^1,\dots,x_k^N$. □

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