THE GREEN FUNCTION FOR $p-$LAPLACE OPERATORS

SABINA ANGELONI AND PIERPAOLO ESPOSITO

Abstract. On a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, we consider existence, uniqueness and "regularity" issues for the Green function G_λ of the quasi-linear operator $u \to -\Delta_p u - \lambda |u|^{p-2}u$ with $1 < p \leq N$, homogeneous Dirichlet boundary condition and $\lambda < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta_p$.

1. Introduction

Given $1 < p \leq N$ and a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, for $x_0 \in \Omega$ we are interested to nonnegative solutions G_λ of

$$-\Delta_p G - \lambda G^{p-1} = 0 \qquad \text{in } \Omega \setminus \{x_0\},$$

where $\Delta_p(\cdot) = \operatorname{div}\big(|\nabla(\cdot)|^{p-2}\nabla(\cdot)\big)$ is the p-Laplace operator and $\lambda < \lambda_1$. Here $G_\lambda \in W^{1,p}_{\text{loc}}(\Omega \setminus \{x_0\})$ and λ_1 is the first eigenvalue of $-\Delta_p$ given by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

When $\lambda = 0$, by elliptic regularity theory a nonnegative $p-$harmonic function G_0 in $\Omega \setminus \{x_0\}$ belongs to $C^{1,\alpha}_{\text{loc}}(\Omega \setminus \{x_0\})$ for some $\alpha \in (0,1)$ and, according to [32], behaves - if singular - like the fundamental solution

$$\Gamma(x) = \begin{cases} \frac{C_0}{|x-x_0|^{\frac{N-p}{p-1}}} & \text{if } 1 < p < N \\ -(N\omega_N)^{-\frac{1}{N-1}} \log|x-x_0| & \text{if } p = N \end{cases}$$

of $-\Delta_p \Gamma = \delta_{x_0}$ in $\mathbb{R}^N$, where $C_0 = \frac{p-1}{N-p}(N\omega_N)^{-\frac{1}{p-1}}$ and ω_N is the measure of the unit ball in $\mathbb{R}^N$. By a combination of scaling arguments and regularity estimates, Kichenassamy and Veron [24] showed that, in the singular situation, up to a re-normalization, G_0 is a solution of

$$-\Delta_p G = \delta_{x_0} \qquad \text{in } \Omega \tag{1.1}$$

and differs from Γ by a locally bounded function $H_0 = G_0 - \Gamma$ in Ω. Given $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, a solution $G_0 \in W^{1,p}_{\text{loc}}(\Omega \setminus \{x_0\}) \cap W^{1,p-1}(\Omega)$ to (1.1) with $G_0\Big|_{\partial\Omega} = g$ can be found in many different ways (see for example [24, 32]) and turns out to be unique thanks to the property $\nabla H_0 = o(|\nabla\Gamma|)$ as $x \to x_0$. As noticed in [24], the same approach via scaling arguments leads to a continuity property of H_0 at x_0.

The aim of the present paper is to establish the Hölder continuity of $H_\lambda = G_\lambda - \Gamma$ at x_0 when $\lambda = 0$ and to include the case $\lambda < \lambda_1$. Notice that such Hölder property is new already when $\lambda = 0$ and is relevant since Green's functions naturally arise in the description of concentration phenomena for quasi-linear PDE's, see for example [2], even if representation formulas are no-longer available in a quasi-linear context. Since the seminal works [26, 32, 33] in the sixties, the regularity theory for quasi-linear elliptic problems has been first refined in [18, 27] in the $p-$harmonic setting, see also [35], and then in [11, 28, 34] for general $p-$Laplace type equations. To treat the case of a Radon measure as right hand side, a general existence and uniqueness theory has been developed, both in the scalar and vectorial case, through different approaches: renormalized solutions, see for instance [9, 30]; entropy solutions or SOLA (solutions obtained as limit of approximations) in [1, 3, 4, 5]; in weak Lebesgue spaces [12, 13, 14]; in grand Sobolev spaces [21]. A powerful and general approach has also been developed through a potential theory in nonlinear form, see for example [23, 25] for an overview on old and recent achievements. Also in the simplest case $\lambda = 0$ the problem we are interested in does not fit into these general theories and a different approach, based on a new but rather simple idea, is necessary. The main point is to consider H_λ as a solution of

$$-\Delta_p(\Gamma + H_\lambda) + \Delta_p \Gamma = \lambda G_\lambda^{p-1} \qquad \text{in } \Omega \setminus \{x_0\} \tag{1.2}$$

for any $G_\lambda = \Gamma + H_\lambda$ solving (1.5) below and to apply the Moser iterative scheme in [32] to derive Hölder estimates on H_λ thanks to the coercivity of the difference operator, as expressed by the estimate

$$\inf_{X \neq Y} \frac{\langle |X+Y|^{p-2}(X+Y) - |X|^{p-2}X, Y\rangle}{(|X|+|Y|)^{p-2}|Y|^2} > 0. \tag{1.3}$$

When $p \geq 2$ gradient L^p-estimates on H_λ can be derived for the difference equation (1.2) as in the pure $p-$Laplace case and the only difficulty, when performing local estimates, comes from the failure of good upper estimates on $|\nabla\Gamma + \nabla H_\lambda|^{p-2}(\nabla\Gamma + \nabla H_\lambda) - |\nabla\Gamma|^{p-2}\nabla\Gamma$, caused by the singular behavior of $\nabla\Gamma$ at x_0. Since the inequality $(|X|+|Y|)^{p-2}|Y|^2 \geq \delta|Y|^p$, $\delta > 0$, is no longer true for $1 < p < 2$, one realizes that the difference equation (1.2) differs from the pure $p-$Laplace case and weighted gradient L^2-estimates on H_λ are the natural ones one can hope for.

Let us first discuss the case $\lambda = 0$, which is the most relevant since it concerns the behavior of $p-$harmonic functions at isolated singularities. In the two-dimensional situation a very precise description has been provided in [29], whereas for $N \geq 2$ the only available result concerns the continuity of H_0 and has been given in [24], as already discussed. A special attention is paid here to avoid any restrictions on p and our first main result below improves in full generality what was previously known:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $x_0 \in \Omega$ and $1 < p \leq N$. The unique nonnegative solution G_0 to*

$$\begin{cases} -\Delta_p G = \delta_{x_0} & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\nabla(G_0 - \Gamma) \in L^{\bar{q}}(\Omega), \quad \bar{q} = \frac{N(p-1)}{N-1}, \tag{1.4}$$

and the regular part $H_0 = G_0 - \Gamma$ is Hölder continuous at x_0.

Let us stress that the integrability condition (1.4) can be improved into $\nabla H_0 \in L^p(\Omega)$ if $p \geq 2$. Since $\nabla\Gamma \in L^q(\Omega)$ for all $q < \bar{q}$, the exponent $\bar{q}$ represents the threshold gradient-integrability which distinguishes the singular situation from the non-singular one and the property (1.4) is crucial, when running the Moser iterative scheme, to use appropriate test functions $\Psi(H_\lambda)$ into (1.2) as the equation were valid in the whole Ω. The validity of higher regularity properties for H_0 represents a challenging open question in this context.

Let us now address the case $\lambda \neq 0$ and consider the problem

$$\begin{cases} -\Delta_p G - \lambda G^{p-1} = \delta_{x_0} & \text{in } \Omega \\ G \geq 0 & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Our second main result is the following:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $x_0 \in \Omega$ and $2 \leq p \leq N$. If $\lambda < \lambda_1$ with $\lambda \neq 0$, problem* (1.5) *has a solution G_λ with*

$$\nabla(G_\lambda - \Gamma) \in L^{\bar{q}}(\Omega), \quad \bar{q} = \frac{N(p-1)}{N-1}, \tag{1.6}$$

which is unique in the class of solutions satisfying (1.6)*. Moreover, the regular part $H_\lambda = G_\lambda - \Gamma$ is Hölder continuous at x_0 if $p > \frac{N}{2}$.*

Some comments are in order. While (1.4) is proved to be true for G_0, for $\lambda \neq 0$ we cannot guarantee the validity of (1.6) for any solution G_λ. However, since (1.6) is generally valid for all solutions obtained through an approximation scheme, assumption (1.6) in Theorem 1.2 is a rather natural request which - at the same time - allows us to show uniqueness of G_λ when $p \geq 2$ and Hölder continuity of H_λ when $p > \frac{N}{2}$. In view of $H_\lambda \in L^\infty(\Omega)$ and

$$\Gamma \in L^q(\Omega) \quad \text{for } 1 \leq q < \bar{q}^*,\ \bar{q}^* = \begin{cases} \frac{N(p-1)}{N-p} & \text{if } 1 < p < N \\ +\infty & \text{if } p = N, \end{cases}$$

notice that condition $p > \frac{N}{2}$ ensures $G_\lambda^{p-1} \in L^q(\Omega)$ for some $q > \frac{N}{p}$ in (1.2), a natural condition arising in [32] to prove $L^\infty-$bounds. In this respect, observe that also in the semilinear case $p = 2$ the function H_λ is no longer regular at x_0 when $2 = p \leq \frac{N}{2}$.

The paper is organized as follows. Section 2 is devoted to establish the existence part in Theorems 1.1 and 1.2 along with some $L^\infty-$estimates, while uniqueness issues are addressed in Section 3. Harnack inequalities and Hölder estimates for H_λ are established in Section 4. For easy of notations, we will just consider the case $x_0 = 0$.
The results of the present paper are crucial in [2] to discuss existence results for a quasi-linear elliptic equation of critical Sobolev growth [6, 22] in the low-dimensional case as in [15, 16].

2. Existence of Green's functions

Given $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, set $W_g^{1,q}(\Omega) = g + W_0^{1,q}(\Omega)$ for all $q \geq 1$ and consider

$$\lambda_{1,g} = \inf_{u \in W_g^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

Since the minimizer $\tilde{g}$ of $\int_\Omega |\nabla u|^p$ in $W_g^{1,p}(\Omega)$ is a $p-$harmonic function in Ω so that $\|\tilde{g}\|_\infty \leq \|g\|_\infty$, we assume that either $g = 0$ or $g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$ is a $p-$harmonic and non-constant function in Ω so to guarantee $\lambda_{1,g} > 0$.

For $g \geq 0$ and $\lambda < \lambda_{1,g}$ let us discuss the problem

$$\begin{cases} -\Delta_p G - \lambda G^{p-1} = \delta_0 & \text{in } \Omega \\ G \geq 0 & \text{in } \Omega \\ G = g & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

with

$$g \in L^\infty(\Omega) \cap W^{1,p}(\Omega)\ p-\text{harmonic in } \Omega,\ g \text{ non-constant unless } g = 0. \tag{2.2}$$

Solutions of (2.1) are found by an approximation procedure based either on removing small balls $B_\epsilon(0)$ when $\lambda = 0$ as in [24] or on approximating δ_0 by smooth functions when $\lambda \neq 0$ as in [1, 3, 4, 5]. We have the following existence result.

Theorem 2.1. *Let $1 < p \leq N$, $g \geq 0$ satisfying* (2.2)*, $\lambda < \lambda_{1,g}$ and assume $p \geq 2$ only when $\lambda \neq 0$. Then there exists a solution G_λ of problem* (2.1) *so that $H_\lambda = G_\lambda - \Gamma$ satisfies* (1.6)*. Moroever, there holds $H_\lambda \in L^\infty(\Omega)$ whenever either $\lambda = 0$ or $\lambda \neq 0$, $p > \frac{N}{2}$.*

Proof. Consider first the case $\lambda = 0$. We repeat the argument in [24] and the only point is to establish suitable bounds on $H_0 = G_0 - \Gamma$. Let G_ϵ be the $p-$harmonic function in $\Omega_\epsilon = \Omega \setminus B_\epsilon(0)$ so that $G_\epsilon = g$ on $\partial\Omega$ and $G_\epsilon = \Gamma$ on $\partial B_\epsilon(0)$. Since Γ is a positive $p-$harmonic function in $\Omega \setminus \{0\}$, by comparison principle we deduce that $G_\epsilon \geq 0$ and $|G_\epsilon - \Gamma| \leq C_0$ in Ω_ϵ, with $C_0 = \|g\|_\infty + \|\Gamma\|_{\infty,\partial\Omega}$. By elliptic estimates [18, 27, 35] for $p-$harmonic functions we deduce that G_ϵ is uniformly bounded in $C^{1,\alpha}_{\text{loc}}(\Omega \setminus \{0\})$. By Ascoli-Arzelá Theorem we can find a sequence $\epsilon_n \to 0$ so that $G_n := G_{\epsilon_n} \to G_0$ in $C^1_{\text{loc}}(\Omega \setminus \{0\})$ as $n \to +\infty$, where $G_0 \geq 0$ is a $p-$harmonic function in $\Omega \setminus \{0\}$ so that

$$H_0 = G_0 - \Gamma \in L^\infty(\Omega). \tag{2.3}$$

Letting η be a cut-off function with $\eta = 1$ near $\partial\Omega$ and $\eta = 0$ near 0, use $\eta^p(G_\epsilon - g) \in W_0^{1,p}(\Omega_\epsilon)$ as a test function for $\Delta_p G_\epsilon = 0$ in Ω_ϵ to get

$$\int_{\Omega_\epsilon} \eta^p \langle |\nabla G_\epsilon|^{p-2}\nabla G_\epsilon, \nabla(G_\epsilon - g)\rangle = -p\int_{\Omega_\epsilon} \eta^{p-1}(G_\epsilon - g)\langle |\nabla G_\epsilon|^{p-2}\nabla G_\epsilon, \nabla \eta\rangle \le C \tag{2.4}$$

in view of $\nabla\eta = 0$ near $\partial\Omega$ and 0. Since

$$\int_{\Omega_\epsilon} \eta^p |\nabla G_\epsilon|^{p-1}|\nabla g| \le \frac{1}{2}\int_{\Omega_\epsilon} \eta^p |\nabla G_\epsilon|^p + C\int_{\Omega_\epsilon} \eta^p |\nabla g|^p$$

for some $C > 0$ in view of the Young inequality, by (2.4) we deduce that G_ϵ is uniformly bounded in $W^{1,p}$ near $\partial\Omega$. Then $G_0 = g$ on $\partial\Omega$ and G_0 solves (2.1) with $\lambda = 0$ in view of (2.3) and [24, 32].

Moreover, use $(1-\eta)(G_\epsilon - \Gamma) \in W_0^{1,p}(\Omega_\epsilon)$ as a test function for $-\Delta_p G_\epsilon + \Delta_p \Gamma = 0$ in Ω_ϵ to get

$$\int_{\Omega_\epsilon} (1-\eta)\langle |\nabla G_\epsilon|^{p-2}\nabla G_\epsilon - |\nabla \Gamma|^{p-2}\nabla\Gamma, \nabla(G_\epsilon - \Gamma)\rangle \le C \tag{2.5}$$

in view of $\nabla\eta = 0$ near $\partial\Omega$ and 0. By the coercivity estimate (1.3) and the uniform $W^{1,p}-$bound on G_ϵ and Γ away from 0 we deduce that (2.5) implies

$$\int_{\Omega_\epsilon} (|\nabla\Gamma| + |\nabla H_\epsilon|)^{p-2}|\nabla H_\epsilon|^2 \le C \tag{2.6}$$

for some uniform constant $C > 0$, where $H_\epsilon = G_\epsilon - \Gamma$. When $p \ge 2$ estimate (2.6) implies

$$\nabla H_0 \in L^p(\Omega)$$

thanks to the Fatou convergence Theorem along the sequence ϵ_n. For $1 < p < 2$ by (2.6) and the Hölder inequality we get

$$\begin{aligned}
\int_{\Omega_\epsilon} |\nabla H_\epsilon|^{\bar q} &= \int_{\Omega_\epsilon} (|\nabla\Gamma| + |\nabla H_\epsilon|)^{\frac{(p-2)\bar q}{2}}|\nabla H_\epsilon|^{\bar q}(|\nabla\Gamma| + |\nabla H_\epsilon|)^{\frac{(2-p)\bar q}{2}} \le C(\|\nabla\Gamma\|_{s,\Omega_\epsilon}^{\frac{(2-p)\bar q}{2}} + \|\nabla H_\epsilon\|_{s,\Omega_\epsilon}^{\frac{(2-p)\bar q}{2}})\\
&\le C(\|\nabla\Gamma\|_{s,\Omega_\epsilon}^{\frac{(2-p)\bar q}{2}} + |\Omega_\epsilon|^{\frac{(2-p)(\bar q - s)}{2s}}\|\nabla H_\epsilon\|_{\bar q,\Omega_\epsilon}^{\frac{(2-p)\bar q}{2}})
\end{aligned}$$

for some $C > 0$ and $s = \frac{N(p-1)(2-p)}{3N-2-Np}$, thanks to $s < \bar q$ in view of $p < 2 \le N$. By $\nabla\Gamma \in L^q(\Omega)$ for all $q < \bar q$ and the Young inequality we finally obtain $\int_{\Omega_\epsilon} |\nabla H_\epsilon|^{\bar q} \le C$ for some uniform constant $C > 0$ and then

$$\nabla H_0 \in L^{\bar q}(\Omega)$$

does hold in the case $1 < p < 2$ thanks to the Fatou convergence Theorem.

Once the case $\lambda = 0$ has been treated, assume $p \ge 2$ and follow the approach in [1, 3, 4, 5]. Notice that for $\lambda = 0$ we provide below an efficient approximation scheme which is different from the previous one. Consider a sequence $0 \le f_n \in C_0^\infty(\Omega)$ so that $f_n \rightharpoonup \delta_0$ weakly in the sense of measures in Ω with $\sup_n \|f_n\|_1 < +\infty$ and $f_n \to 0$ locally uniformly in $\Omega \setminus \{0\}$ as $n \to +\infty$. Since $\lambda < \lambda_{1,g}$ and $g, f_n \ge 0$, the minimization of

$$\frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_\Omega |u|^p - \int_\Omega f_n u, \quad u \in W_g^{1,p}(\Omega),$$

provides a nonnegative solution $G_n \in W_g^{1,p}(\Omega)$ to

$$-\Delta_p G_n - \lambda G_n^{p-1} = f_n \quad \text{in } \Omega. \tag{2.7}$$

We use here Lemmas 2.2 and 2.3 below to show first that G_n^{p-1} is uniformly bounded in $L^1(\Omega)$ and then, up to a subsequence, $G_n \to G_\lambda$ in $W_g^{1,q}(\Omega)$ as $n \to +\infty$ for some G_λ and for all $1 \le q < \bar q$. By the Sobolev embedding Theorem we have that $G_n \to G_\lambda$ in $L^q(\Omega)$ as $n \to +\infty$ for all $1 \le q < \bar q^*$ and in particular in $L^{p-1}(\Omega)$. Therefore one can pass to the limit in (2.7) and get that $G_\lambda \ge 0$ solves (2.1) in view of $\bar q > p - 1$.

In order to establish suitable bounds on $H_\lambda = G_\lambda - \Gamma$, let $0 \le \tilde G_n \in W^{1,p}_g(\Omega)$ be the solution of

$$-\Delta_p \tilde G_n = f_n \quad \text{in } \Omega,$$

obtained as a minimizer of $\frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega f_n u$ in $W^{1,p}_g(\Omega)$ in view of $\lambda_{1,g} > 0$. Arguing as for (2.7), we deduce that, up to a subsequence, $\tilde G_n \to \tilde G$ in $W^{1,q}_g(\Omega)$ as $n \to +\infty$ for all $1 \le q < \bar q$, where $\tilde G \ge 0$ solves $-\Delta_p \tilde G = \delta_0$ in Ω. By [32] and the uniqueness result in [24] we have that $\tilde G = G_0$ and $\tilde H = \tilde G - \Gamma = H_0$. Since $-\Delta_p G_n + \Delta_p \tilde G_n = \lambda G_n^{p-1}$ in Ω with $G_n = \tilde G_n$ on $\partial\Omega$, by Lemma 2.3 we deduce that $\sup\limits_n \|\nabla(G_n - \tilde G_n)\|_{\bar q} < +\infty$ in view of $\sup\limits_n \|G_n^{p-1}\|_m < +\infty$ for all $1 \le m < \frac{\bar q^*}{p-1}$. Since $\nabla(G_n - \tilde G_n) \to \nabla(H_\lambda - H_0)$ a.e. in Ω as $n \to +\infty$ and ∇H_0 satisfies (1.4), by the Fatou convergence Theorem we obtain that ∇H_λ satisfies (1.6). If either $\lambda = 0$ or $\lambda \neq 0$, $p > \frac{N}{2}$ a $L^\infty-$bound on H_λ follows by Theorem 2.6 below and the proof is complete. $\square$

The following result has been crucially used in the proof of Theorem 2.1 and in its proof we closely follow a tricky idea in [31] combined with some apriori estimates given in Lemma 2.3 below.

Lemma 2.2. *Let $2 \le p \le N$. Assume that $a_n \in L^\infty(\Omega)$, $f_n \in L^1(\Omega)$, g_n satisfy (2.2) and*

$$\lim_{n\to+\infty} \|a_n - a\|_\infty = 0, \quad \sup_\Omega a < \lambda_1, \quad \sup_{n\in\mathbb{N}} [\|f_n\|_1 + \|g_n\|_\infty] < +\infty. \tag{2.8}$$

If $u_n \in W^{1,p}_{g_n}(\Omega)$ is a sequence of solutions to

$$-\Delta_p u_n - a_n |u_n|^{p-2} u_n = f_n \quad \text{in } \Omega,$$

then $\sup\limits_{n\in\mathbb{N}} \|u_n\|_{p-1} < +\infty$.

Proof. Assume by contradiction that

$$\|u_n\|_{p-1} \to +\infty \qquad \text{as } n \to +\infty. \tag{2.9}$$

Setting $\hat u_n = \frac{u_n}{\|u_n\|_{p-1}}$, $\hat f_n = \frac{f_n}{\|u_n\|_{p-1}^{p-1}}$ and $\hat g_n = \frac{g_n}{\|u_n\|_{p-1}}$, we have that $\hat u_n$ solves

$$\begin{cases} -\Delta_p \hat u_n - a_n |\hat u_n|^{p-2} \hat u_n = \hat f_n & \text{in } \Omega \\ \hat u_n = \hat g_n & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

with

$$\|\hat u_n\|_{p-1} = 1,\ \sup_{n\in\mathbb{N}} \|a_n\|_\infty < \infty,\ \|\hat f_n\|_{L^1(\Omega)} + \|\hat g_n\|_\infty \to 0 \text{ as } n \to +\infty \tag{2.11}$$

in view of (2.8)-(2.9). Fix $p - 1 < p_0 < \bar q$ and define $p_j = \frac{N^2(p-1)p_{j-1}}{(N+1)[N(p-1)-p_{j-1}]}$ in a recursive way for $j \ge 1$. Notice that $\frac{N(p-1)}{N+1} < p_j < p_{j+1}$ by induction and there exists a unique $J \ge 0$ so that $p_0, \dots, p_{J-1} \le \frac{Np(p-1)}{Np-N+p} < p_J$. Since $\Delta_p \hat g_n = 0$ in Ω, by Lemma 2.3 with $m = 1$ we get that $\hat u_n - \hat g_n$ is uniformly bounded in $W^{1,q}_0(\Omega)$ for all $1 \le q < \bar q$ in view of (2.10)-(2.11) and then, up to a subsequence, $\hat u_n - \hat g_n \rightharpoonup v^0$ in $W^{1,p_0}(\Omega)$ as $n \to +\infty$. Define $v_n^0 = \hat u_n$ and $v_n^j \in W^{1,p}_{\hat g_n}$ as the solution of $-\Delta_p v_n^j = a_n |v_n^{j-1}|^{p-2} v_n^{j-1}$ in Ω in view of $\lambda_{1,\hat g_n} = \lambda_{1,g_n} > 0$. Lemma 2.3, applied to $v_n^1 - \hat g_n$ with $m = \frac{p_0}{p-1} \le \frac{Np}{Np-N+p}$, $q = \frac{N}{N+1}\frac{mN(p-1)}{N-m}$ and to $v_n^1 - v_n^0$ with $m = 1$, $q = p_0$ in view of (2.10)-(2.11), provides that, up to a subsequence, $v_n^1 - \hat g_n \rightharpoonup v^1$ in $W^{1,p_1}_0(\Omega)$ and $v_n^1 - v_n^0 \to 0$ in $W^{1,p_0}_0(\Omega)$ as $n \to +\infty$. By iterating we deduce that, up to a subsequence, $v_n^j - \hat g_n \rightharpoonup v^j$ in $W^{1,p_j}_0(\Omega)$ and $v_n^j - v_n^{j-1} \to 0$ in $W^{1,p_{j-1}}_0(\Omega)$ as $n \to +\infty$ for all $j = 1, \dots, J$. Since $a_n |v_n^J|^{p-2} v_n^J$ is uniformly bounded in $L^m(\Omega)$ with $m = \frac{p_J}{p-1} > \frac{Np}{Np-N+p}$, by Lemma 2.3 we deduce that, up to a subsequence, $v_n^{J+1} - \hat g_n \rightharpoonup v^{J+1}$ in $W^{1,p}_0(\Omega)$ as $n \to +\infty$. At the same time, by Lemma 2.3 $v_n^{J+1} - v_n^J \to 0$ in $W^{1,p_J}_0(\Omega)$ as $n \to +\infty$. Since $v_n^j - v_n^{j-1} \to 0$ in $W^{1,p_0}_0(\Omega)$ and $v_n^j - v_n^{j-1} = (v_n^j - \hat g_n) - (v_n^{j-1} - \hat g_n) \rightharpoonup v^j - v^{j-1}$ weakly in $W^{1,p_0}_0(\Omega)$ as $n \to +\infty$ for all $j = 1, \dots, J+1$, we deduce that $v^0 = \ldots = v^{J+1}$ and then $\hat u_n - \hat g_n \rightharpoonup v^0$ in $W^{1,p_0}_0(\Omega)$ as $n \to +\infty$ with $v^0 = v^{J+1} \in W^{1,p}_0(\Omega)$.

Let us compare $\hat{u}_n$ with $z_n \in W_0^{1,p}(\Omega)$, solution to

$$-\Delta_p z_n = a_n|\hat{u}_n|^{p-2}\hat{u}_n + \hat{f}_n \quad \text{in } \Omega. \tag{2.12}$$

Since $|\hat{u}_n - z_n| \le \|\hat{g}_n\|_\infty$ on $\partial\Omega$, by the weak maximum principle we deduce that $\|\hat{u}_n - z_n\|_\infty \le \|\hat{g}_n\|_\infty$. By (2.11)-(2.12) and Lemma 2.3 we deduce that, up to a subsequence and for some z^0, there holds

$$z_n \to z^0 \quad \text{in } W_0^{1,q}(\Omega),\ 1 \le q < \bar{q}. \tag{2.13}$$

By testing $-\Delta_p \hat{u}_n + \Delta_p z_n = 0$ in Ω against $\eta^p(\hat{u}_n - z_n)$, $0 \le \eta \in C_0^\infty(\Omega)$, one gets

$$\begin{aligned}\int_\Omega \eta^p|\nabla(\hat{u}_n - z_n)|^p &\le C'\int_\Omega \eta^{p-1}|\nabla\eta|(|\nabla(\hat{u}_n - z_n)|^{p-2} + |\nabla z_n|^{p-2})|\nabla(\hat{u}_n - z_n)||\hat{u}_n - z_n| \\ &\le \frac{1}{2}\int_\Omega \eta^p|\nabla(\hat{u}_n - z_n)|^p + C\left(\|\hat{g}_n\|_\infty^p + \|\hat{g}_n\|_\infty^{\frac{p}{p-1}}\|\nabla z_n\|_{\frac{p(p-2)}{p-1}}^{p-2}\right) \to 0\end{aligned}$$

as $n \to +\infty$ in view of the Young's inequality and (2.11). We have used that $\sup_n \|\nabla z_n\|_{\frac{p(p-2)}{p-1}} < +\infty$ thanks to (2.13) and $\frac{p(p-2)}{p-1} < \bar{q}$. Since $\nabla(\hat{u}_n - z_n) \to 0$ locally in L^p-norm as $n \to +\infty$, by (2.13) we deduce that

$$\hat{u}_n \to v^0 \quad \text{in } L^{p-1}(\Omega) \text{ and } W^{1,q}(\Omega'), \quad \forall\, \Omega' \subset\subset \Omega, \quad \forall\, 1 \le q < \bar{q}, \tag{2.14}$$

in view of $\|\hat{u}_n - z_n\|_\infty \le \|\hat{g}_n\|_\infty \to 0$ and $\hat{u}_n - \hat{g}_n \rightharpoonup v^0$ in $W_0^{1,p_0}(\Omega)$ as $n \to +\infty$ for $p_0 \ge p-1$.

By (2.10) and (2.14) we have that $v^0 \in W_0^{1,p}(\Omega)$ solves

$$-\Delta_p v^0 - a|v^0|^{p-2}v^0 = 0 \qquad \text{in } \Omega \tag{2.15}$$

in view of (2.8) and (2.11). Since

$$\int_\Omega |\nabla v^0|^p - \int_\Omega a|v^0|^p = 0$$

by integration of (2.15) against $v^0 \in W_0^{1,p}(\Omega)$, by $\sup_\Omega a < \lambda_1$ one finally deduces that $v^0 = 0$ and then $\hat{u}_n \to 0$ in $L^{p-1}(\Omega)$, in contradiction with $\|\hat{u}_n\|_{p-1} = 1$. $\square$

The results in [1, 4, 5], valid for homogeneous boundary values, can be easily extended to non-homogeneous ones when $p \ge 2$, as discussed for instance in the Appendix of [1] when $p = N$. For the sake of completeness, we reproduce it here in the following simplest form, sufficient for our purposes:

Lemma 2.3. *Let* $2 \le p \le N$. *Assume* $\|f_1 - f_2\|_m \le C_0$ *for some* $C_0 > 0$ *and either* $1 \le m \le \frac{Np}{Np-N+p}$, $1 \le q < \frac{mN(p-1)}{N-m}$ *or* $m > \frac{Np}{Np-N+p}$, $1 \le q \le p$. *Then there exists* $C > 0$ *so that* $\|\nabla(u_1 - u_2)\|_q \le C\|f_1 - f_2\|_m^{\frac{1}{p}}$ *for all solutions* $u_1, u_2 \in W^{1,p}(\Omega)$ *of* $-\Delta_p u_i = f_i$, $i = 1,2$, *in* Ω *with* $u_1 = u_2$ *on* $\partial\Omega$. *Moreover, given* g *satisying* (2.2) *the set of solutions* $u \in W_g^{1,p}(\Omega)$ *of* $-\Delta_p u = f$ *in* Ω *with* $\|f\|_1 \le C_0$ *is relatively compact in* $W^{1,q}(\Omega)$ *for all* $1 \le q < \bar{q}$.

Proof. Let $u_1, u_2 \in W^{1,p}(\Omega)$ be solutions of $-\Delta_p u_i = f_i$, $i = 1,2$, in Ω with $u_1 = u_2$ on $\partial\Omega$. Take $T_{k,l}$, $0 \le k \le l$, as the odd function so that

$$T_{k,l}(s) = \min\{\max\{s-k, 0\}, l-k\} \quad \text{in } [0, +\infty) \tag{2.16}$$

and use $T_{k,k+1}(u_1 - u_2)$ as a test function to get

$$\int_{\{k \le |u_1 - u_2| < k+1\}} \langle |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla(u_1 - u_2)\rangle = \int_\Omega (f_1 - f_2)T_{k,k+1}(u_1 - u_2),$$

which implies

$$\int_{\{k \le |u_1 - u_2| < k+1\}} |\nabla(u_1 - u_2)|^p \le C\|f_1 - f_2\|_m |\{|u_1 - u_2| \ge k\}|^{\frac{m-1}{m}} \tag{2.17}$$

in view of (1.3) and $p \geq 2$. By (2.17) the function $v = u_1 - u_2 \in W_0^{1,p}(\Omega)$ satisfies

$$\int_{B_k} |\nabla v|^p \leq c_0 |E_k|^{\frac{m-1}{m}},\ k \geq 0, \tag{2.18}$$

with $c_0 = C\|f_1 - f_2\|_m$, where $E_k = \{|v| \geq k\}$ and $B_k = E_k \setminus E_{k+1}$.

Consider first the case $1 \leq m \leq \frac{Np}{Np-N+p}$, $1 \leq q < \frac{mN(p-1)}{N-m}$ and set $q^* = \frac{Nq}{N-q}$. Since $q < \frac{mN(p-1)}{N-m} \leq p$ thanks to $m \leq \frac{Np}{Np-N+p}$ and

$$\int_{B_k} |\nabla v|^q \leq (\int_{B_k} |\nabla v|^p)^{\frac{q}{p}} |B_k|^{\frac{p-q}{p}} \tag{2.19}$$

in view of the Hölder inequality, by (2.18) we obtain that

$$\int_{B_k} |\nabla v|^q \leq c_0^{\frac{q}{p}} \|v\|_{q^*}^{\frac{qq^*(m-1)}{pm}} (\int_{B_k} |v|^{q^*})^{\frac{p-q}{p}} \frac{1}{k^{\frac{q^*(pm-q)}{pm}}}$$

for all $k \geq 1$ thanks to

$$|B_k| \leq k^{-q^*} \int_{B_k} |v|^{q^*}, \quad |E_k| \leq k^{-q^*} \int_\Omega |v|^{q^*}.$$

Summing up and still by Hölder's inequality one deduces

$$\int_{\{|v| \geq k_0\}} |\nabla v|^q \leq c_0^{\frac{q}{p}} \|v\|_{q^*}^{\frac{qq^*(m-1)}{pm}} (\sum_{k=k_0}^{\infty} \int_{B_k} |v|^{q^*})^{\frac{p-q}{p}} (\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{q^*(pm-q)}{mq}}})^{\frac{q}{p}}$$

and then

$$\int_\Omega |\nabla v|^q \leq k_0 c_0^{\frac{q}{p}} |\Omega|^{\frac{pm-q}{pm}} + c_0^{\frac{q}{p}} \|v\|_{q^*}^{\frac{q^*(pm-q)}{pm}} (\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{q^*(pm-q)}{mq}}})^{\frac{q}{p}} \tag{2.20}$$

for a given $k_0 \in \mathbb{N}$ in view of (2.18)-(2.19) for $k = 0, \ldots, k_0 - 1$. Since $\frac{q^*(pm-q)}{pm} \leq q$, by Young's inequality (2.20) implies in turn that

$$\int_\Omega |\nabla v|^q \leq k_0 c_0^{\frac{q}{p}} |\Omega|^{\frac{pm-q}{pm}} + C c_0^{\frac{q}{p}} (\|v\|_{q^*}^q + 1) (\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{q^*(pm-q)}{mq}}})^{\frac{q}{p}}. \tag{2.21}$$

Since $\frac{q^*(pm-q)}{mq} > 1$ thanks to $q < \frac{mN(p-1)}{N-m}$, the series in (2.21) is convergent and we can choose k_0 sufficienty large (depending on C_0) so that $\|v\|_{q^*} \leq C' c_0^{\frac{1}{p}}$ and then $\|\nabla v\|_q \leq C c_0^{\frac{1}{p}}$ in view of the Sobolev embedding Theorem, where the last estimate gets rewritten as

$$\|\nabla(u_1 - u_2)\|_q \leq C \|f_1 - f_2\|_m^{\frac{1}{p}}. \tag{2.22}$$

Consider now the case $m > \frac{Np}{Np-N+p}$, $1 \leq q \leq p$. Use $u_1 - u_2$ as a test function to get

$$\|\nabla(u_1 - u_2)\|_p^p \leq C \|u_1 - u_2\|_{\frac{m}{m-1}} \|f_1 - f_2\|_m$$

in view of the Hölder inequality and then $\|\nabla(u_1 - u_2)\|_p \leq C \|f_1 - f_2\|_m^{\frac{1}{p-1}}$ by the Sobolev embedding Theorem in view of $\frac{m}{m-1} < p^*$. Notice that such last argument works as well as $m = \frac{Np}{Np-N+p}$ for $p < N$ since $\frac{Np}{Np-N+p} > 1$ in this case.

Fix now $m = 1$ and let $u_1, u_2 \in W_g^{1,p}(\Omega)$ be solutions of $-\Delta_p u_i = f_i$, $i = 1, 2$, in Ω with $\|f_i\|_1 \leq C_0$. Use $T_{0,\epsilon}(u_1 - u_2)$, $T_{k,l}$ given by (2.16), as a test function to get

$$\int_{\{|u_1 - u_2| \leq \epsilon\}} |\nabla(u_1 - u_2)|^p \leq C\epsilon \|f_1 - f_2\|_1 \leq 2CC_0\epsilon \tag{2.23}$$

in view of (1.3) and $p \geq 2$. Given $1 \leq q < \bar{q}$, by (2.22) and Hölder's inequality (2.23) implies

$$\begin{aligned}\int_\Omega |\nabla(u_1-u_2)|^q &\leq C'\epsilon^{\frac{q}{p}} + (\int_{\{|u_1-u_2|>\epsilon\}} |\nabla(u_1-u_2)|^s)^{\frac{q}{s}} |\{|u_1-u_2|>\epsilon\}|^{\frac{s-q}{s}} \\ &\leq C(\epsilon^{\frac{q}{p}} + \{|u_1-u_2|>\epsilon\}|^{\frac{s-q}{s}}) \end{aligned} \tag{2.24}$$

for some $q < s < \bar{q}$ in view of $\bar{q} < p$. Since g is $p-$harmonic in Ω, taking now a sequence of solutions $u_n \in W^{1,p}_g(\Omega)$ to $-\Delta_p u_n = f_n$ in Ω with $\sup_n \|f_n\|_1 < +\infty$, by the first part we know that $u_n - g$ is bounded in $W^{1,q}_0(\Omega)$ and then, up to a subsequence, we have that $u_n \rightharpoonup u$ in $W^{1,q}_g(\Omega)$ for all $1 \leq q < \bar{q}$ and strongly in $L^s(\Omega)$ for all $1 \leq s < \bar{q}^*$. Applying (2.24) to $u_n - u_m$ it is easily seen that u_n is a Cauchy sequence in $W^{1,q}_g(\Omega)$ and then converges to u in $W^{1,q}_g(\Omega)$ for all $1 \leq q < \bar{q}$. The proof is complete. □

Let us push further the analysis in Lemma 2.2 towards an L^∞-estimate when $p > \frac{N}{2}$.

Proposition 2.4. *Let $2 \leq p \leq N$ with $p > \frac{N}{2}$ and $M > 0$. Then there exists $C > 0$ so that $\|u_1 - u_2\|_\infty \leq C$ for any pair $u_i \in W^{1,p}_{g_i}(\Omega)$, $i = 1, 2$, of solutions to*

$$-\Delta_p u_i - \lambda^i |u_i|^{p-2} u_i = f \quad in\ \Omega, \tag{2.25}$$

where $\|f\|_1 + \sup_{i=1,2} \left[\frac{1}{(\lambda_1 - \lambda^i)_+} + \|g_i\|_\infty\right] \leq M$ *and g_1, g_2 satisfy (2.2).*

Proof. By Lemma 2.2 we get an universal bound on $\|f + \lambda^i |u_i|^{p-2} u_i\|_1$. Since g_i is $p-$harmonic function in Ω, Lemma 2.3 and the Sobolev embedding Theorem provide an universal bound on $u_i - g_i$ in $W^{1,q}_0(\Omega)$ for all $1 \leq q < \bar{q}$ and u_i in $L^q(\Omega)$ for all $1 \leq q < \bar{q}^*$. Since $\frac{\bar{q}^*}{p-1} > \frac{N}{p}$ thanks to $p > \frac{N}{2}$, we can find $q_0 > \frac{N}{p}$ so that $\hat{f} = \lambda^1 |u_1|^{p-2} u_1 - \lambda^2 |u_2|^{p-2} u_2$ satisfies

$$\|\hat{f}\|_{q_0} \leq C \tag{2.26}$$

for some universal $C > 0$. Thanks to (2.25) we can write

$$\begin{cases} -\Delta_p u_1 + \Delta_p u_2 = \hat{f} & \text{in } \Omega \\ u_1 - u_2 = g_1 - g_2 & \text{on } \partial\Omega. \end{cases} \tag{2.27}$$

Since $q_0 > \frac{N}{p}$ let us fix $\beta_0 > 0$ sufficiently small so that $p_0 := \frac{q_0(\beta_0 - 1 + p)}{q_0 - 1} < \bar{q}^*$. Set $u = u_1 - u_2$, $C_0 = \|g_1\|_\infty + \|g_2\|_\infty$ and define $\Psi(s) = [T_{0,l}(s \mp C_0)_\pm + \epsilon]^\beta - \epsilon^\beta$, with $l, \epsilon > 0$ and $\beta \geq \beta_0$, where $T_{k,l}$ is given by (2.16). Notice that $l < +\infty$ and $\epsilon > 0$ guarantee the boundedness and the differentiability of Ψ in $\mathbb{R}$, respectively. Use $\Psi(u) \in W^{1,p}_0(\Omega)$ as a test function in (2.27) to get

$$\beta \int_{\{(u \mp C_0)_\pm \leq l\}} [T_{0,l}(u \mp C_0)_\pm + \epsilon]^{\beta-1} (|\nabla u_2| + |\nabla u|)^{p-2} |\nabla u|^2 \leq C \int_\Omega |\hat{f}| [T_{0,l}(u \mp C_0)_\pm + \epsilon]^\beta \tag{2.28}$$

in view of (1.3). Since $p \geq 2$, by Hölder's inequality with exponents $\frac{q_0(\beta-1+p)}{(q_0-1)(p-1)}$, q_0 and $\frac{q_0(\beta-1+p)}{(q_0-1)\beta}$ estimate (2.28) implies the following estimate:

$$\frac{\delta p^p \beta}{(\beta - 1 + p)^p} \int_\Omega |\nabla w_{l,\epsilon}|^p \leq |\Omega|^{\frac{(q_0-1)(p-1)}{q_0(\beta-1+p)}} \| |\hat{f}\|_{q_0} \|w_{l,\epsilon}\|^{\frac{\beta p}{\beta - 1 + p}}_{\frac{pq_0}{q_0-1}} \leq C \|w_\epsilon\|^{\frac{\beta p}{\beta-1+p}}_{\frac{pq_0}{q_0-1}}$$

for some $C > 0$, where

$$w_{l,\epsilon} = [T_{0,l}(u \mp C_0)_\pm + \epsilon]^{\frac{\beta-1+p}{p}}, \quad w_\epsilon = [(u \mp C_0)_\pm + \epsilon]^{\frac{\beta-1+p}{p}}, \quad w = (u \mp C_0)_\pm^{\frac{\beta-1+p}{p}}.$$

By the Sobolev embedding Theorem on $w_{l,\epsilon} - \epsilon^{\frac{\beta-1+p}{p}} \in W^{1,p}_0(\Omega)$ and the Fatou convergence Theorem as $l \to +\infty$ we deduce that

$$\|w_\epsilon - \epsilon^{\frac{\beta-1+p}{p}}\|_{p^*} \leq C(\beta - 1 + p) \|w_\epsilon\|^{\frac{\beta}{\beta-1+p}}_{\frac{pq_0}{q_0-1}} \tag{2.29}$$

for some $C>0$ provided the R.H.S. is finite, where $p^*=\frac{Np}{N-p}$ if $p<N$ and $p^*\in(\frac{pq_0}{q_0-1},+\infty)$ if $p=N$. By using again the Fatou convergence Theorem on the L.H.S. and the Lebesgue convergence Theorem on the R.H.S. in (2.29), as $\epsilon\to 0$ we deduce that

$$\|w\|_{p^*}\le C(\beta-1+p)\|w\|_{\frac{pq_0}{q_0-1}}^{\frac{\beta}{\beta-1+p}}$$

for some $C>0$, provided $\|w\|_{\frac{pq_0}{q_0-1}}<+\infty$. By the definition of w and taking the $\frac{p}{\beta-1+p}-$power we then deduce that

$$\|(u\mp C_0)_\pm\|_{\frac{(\beta-1+p)p^*}{p}}\le [C(\beta-1+p)]^{\frac{p}{\beta-1+p}}\|(u\mp C_0)_\pm\|_{\frac{q_0(\beta-1+p)}{q_0-1}}^{\frac{\beta}{\beta-1+p}},$$

or equivalently

$$\|(u\mp C_0)_\pm\|_{\kappa\mu}\le [C\frac{q_0-1}{q_0}\mu]^{\frac{pq_0}{\mu(q_0-1)}}\|(u\mp C_0)_\pm\|_{\mu}^{1-\frac{(p-1)q_0}{\mu(q_0-1)}},\tag{2.30}$$

where $\mu=\frac{q_0(\beta-1+p)}{q_0-1}$ and $\kappa=\frac{(q_0-1)p^*}{pq_0}>1$ in view of $q_0>\frac{N}{p}$. Setting $\mu_j=\kappa^j p_0$, we can perform $j+1$ iterations of (2.30) to get

$$\begin{aligned}\|(u\mp C_0)_\pm\|_{\mu_{j+1}} &\le [C(\beta_0-1+p)\kappa^j]^{\frac{p}{(\beta_0-1+p)\kappa^j}}\|(u\mp C_0)_\pm\|_{\mu_j}^{1-\frac{p-1}{(\beta_0-1+p)\kappa^j}}\le\dots\\ &\le [C(\beta_0-1+p)+1]^{\sum_{s=0}^{j}\frac{1}{\kappa^s}}\kappa^{\sum_{s=0}^{j}\frac{s}{\kappa^s}}\|(u\mp C_0)_\pm\|_{p_0}^{\prod_{s=0}^{j}(1-\frac{p-1}{(\beta_0-1+p)\kappa^s})}\end{aligned}$$

in view of $[C(\beta_0-1+p)+1]\kappa^j\ge 1$ and $1-\frac{p-1}{(\beta_0-1+p)\kappa^s}\le 1$. By letting $j\to+\infty$ we deduce that

$$\|(u\mp C_0)_\pm\|_\infty\le C'\|(u\mp C_0)_\pm\|_{p_0}^{\theta_0}\le C'_M$$

in view of

$$\theta_0:=\prod_{s=0}^{\infty}(1-\frac{p-1}{(\beta_0-1+p)\kappa^s})<+\infty,\quad \sum_{s=0}^{\infty}\frac{1}{\kappa^s}+\sum_{s=0}^{\infty}\frac{s}{\kappa^s}<+\infty.$$

In conclusion, $\|u_1-u_2\|_\infty\le C'_M+C_0\le C_M$ and the proof is complete. □

The aim now is to extend Proposition 2.4 to H_λ as a solution of (1.2) (to be compared with (2.27)) and to include the case $1<p<2$. Since it is no longer a matter of universal estimates, the argument is potentially simpler but the singular character of equation (1.2) has to be controlled thanks to the assumption $\nabla H_\lambda\in L^{\bar q}(\Omega)$. For later convenience, let us write the following result in a sufficiently general way.

Lemma 2.5. *Let $1<p\le N$ and $u\in W^{1,p}_{loc}(\Omega\setminus\{0\})$ be a solution of*

$$-\Delta_p(\Gamma+u)+\Delta_p\Gamma=f\quad \text{in }\Omega\setminus\{0\}\tag{2.31}$$

with $f\in L^1(\Omega)$, $\nabla u\in L^{\bar q}(\Omega)$ and

$$\begin{aligned}\frac{1}{C}|\nabla\Gamma|\le|\nabla\Gamma|\le C|\nabla\Gamma| &\quad \text{if } 1<p<2\\ |\nabla\Gamma|\le C|\nabla\Gamma| &\quad \text{if } p\ge 2\end{aligned}\tag{2.32}$$

in Ω for some $C>1$. Let $\eta\in C^1(\bar\Omega)$ and $\Psi:\mathbb{R}\to\mathbb{R}$ be a bounded monotone Lipschitz function. Assuming either $\eta=0$ or $\Psi(u)=0$ on $\partial\Omega$, then there holds

$$\int_\Omega \eta^2|\Psi'(u)|(|\nabla\Gamma|+|\nabla u|)^{p-2}|\nabla u|^2\le C\Big(\int_\Omega|\eta||\nabla\eta||\Psi(u)|(|\nabla\Gamma|+|\nabla u|)^{p-2}|\nabla u|+\int_\Omega\eta^2|f||\Psi(u)|\Big)$$

for some $C>0$.

Proof. Consider a sequence $\eta_\epsilon \in C^1(\bar\Omega)$ so that

$$\eta_\epsilon = \eta \text{ in } \Omega \setminus B_\epsilon(0), \quad \eta_\epsilon = 0 \text{ in } B_{\frac{\epsilon}{2}}(0), \quad |\eta_\epsilon| + \epsilon |\nabla \eta_\epsilon| \le C \text{ in } B_\epsilon(0) \setminus B_{\frac{\epsilon}{2}}(0) \tag{2.33}$$

for some $C > 0$. Since $\eta_\epsilon^2 \Psi(u)$ vanishes in $B_{\frac{\epsilon}{2}}(0)$ and on $\partial\Omega$, it can be used a test function in (2.31):

$$\int_\Omega \eta_\epsilon^2 |\Psi'(u)| (|\nabla \mathit{\Gamma}| + |\nabla u|)^{p-2} |\nabla u|^2 \le C \int_\Omega \Big[|\eta_\epsilon| |\nabla \eta_\epsilon| (|\nabla \mathit{\Gamma}| + |\nabla u|)^{p-2} |\nabla u| + \eta_\epsilon^2 |f| \Big] |\Psi(u)| \tag{2.34}$$

for some $C > 0$ since Ψ' has given sign. We have used here (1.3) and the estimate

$$\Big| |x+y|^{p-2}(x+y) - |x|^{p-2}x \Big| = (|x| + |y|)^{p-2} O(|y|).$$

Since $(|\nabla \mathit{\Gamma}| + |\nabla u|)^{p-2} = O(|\nabla \Gamma|^{p-2} + |\nabla u|^{p-2})$ in view of (2.32), by the Hölder inequality we have that

$$\begin{aligned} \int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)} |\eta_\epsilon| |\nabla \eta_\epsilon| |\Psi(u)| (|\nabla \mathit{\Gamma}| + |\nabla u|)^{p-2} |\nabla u| &\le C \int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)} \Big(\frac{|\nabla u|}{\epsilon^{\frac{N(p-1)-(N-1)}{p-1}}} + \frac{|\nabla u|^{p-1}}{\epsilon} \Big) \\ &\le C \left[\Big(\int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)} |\nabla u|^{\bar q} \Big)^{\frac{1}{\bar q}} + \Big(\int_{B_\epsilon(0)\setminus B_{\frac{\epsilon}{2}}(0)} |\nabla u|^{\bar q} \Big)^{\frac{N-1}{N}} \right] \to 0 \end{aligned} \tag{2.35}$$

as $\epsilon \to 0$, in view of $\|\Psi\|_\infty < +\infty$ and $\nabla u \in L^{\bar q}(\Omega)$. By inserting (2.35) into (2.34) and by using the Lebesgue convergence Theorem for $\int_\Omega \eta_\epsilon^2 |f||\Psi(u)|$ we get the validity of Lemma 2.5 in view of the monotone convergence Theorem. □

We are now ready to complete the proof of Theorem 2.1 by establishing $L^\infty-$bounds on H_λ.

Theorem 2.6. *Let* $1 < p \le N$ *and assume either* $\lambda = 0$ *or* $\lambda \ne 0$ *and* $p \ge 2$ *with* $p > \frac{N}{2}$. *Then* $H_\lambda = G_\lambda - \Gamma \in L^\infty(\Omega)$, *where* G_λ *is any solution to* (2.1) *satisfying* (1.6).

Proof. By (2.1) the function $u = H_\lambda$ solves (2.31) with $\mathit{\Gamma} = \Gamma$ and $f = \lambda G_\lambda^{p-1}$. Given $0 < \beta_0 < 1$ to be fixed later, by Lemma 2.5 with $\eta = 1$ and $\Psi(s) = [T_{0,l}(s \mp C_0)_\pm + \epsilon]^\beta - \epsilon^\beta$, with $l, \epsilon > 0$, $\beta \ge \beta_0$, $C_0 = \|g\|_\infty + \|\Gamma\|_{\infty,\partial\Omega}$ and $T_{k,l}$ given by (2.16), we get that

$$\beta \int_{\{(u \mp C_0)_\pm \le l\}} [T_{0,l}(u \mp C_0)_\pm + \epsilon]^{\beta-1} (|\nabla \Gamma| + |\nabla u|)^{p-2} |\nabla u|^2 \le C \int_\Omega |f| [T_{0,l}(u \mp C_0)_\pm + \epsilon]^\beta \tag{2.36}$$

in view of $\Psi(u) = 0$ on $\partial\Omega$ thanks to $H_\lambda = g - \Gamma$ on $\partial\Omega$.

Let us first consider the case $\lambda = 0$. Then $f = 0$ and the choice $\beta = 1$ in (2.36) gives

$$\int_\Omega (|\nabla \Gamma| + |\nabla u|)^{p-2} |\nabla T_{0,l}(u \mp C_0)_\pm|^2 \le 0.$$

Then $T_{0,l}(u \mp C_0)_\pm = 0$ a.e. in Ω for any $l > 0$, which implies $|H_0| \le C_0$ a.e. in Ω.

Consider now the case $\lambda \ne 0$ and assume $p \ge 2$ with $p > \frac{N}{2}$. Since $\nabla G_\lambda = \nabla \Gamma + \nabla H_\lambda \in L^q(\Omega)$ for all $1 \le q < \bar q$ in view of (1.6), by the Sobolev embedding Theorem $G_\lambda \in L^q(\Omega)$ for all $1 \le q < \bar q^*$ and in particular f satisfies

$$\|f\|_{q_0} < \infty \tag{2.37}$$

for some $q_0 > \frac{N}{p}$ in view of $p > \frac{N}{2}$.

Notice that (2.36)-(2.37) are the analogue of (2.26) and (2.28), and then the argument now goes exactly as in the proof of Proposition 2.4. □

For the case $g = 0$ let us collect here some useful facts which will be used in the next two sections. Given $1 < p < N$, an important ingredient is given by the estimate

$$|\nabla H_\lambda| = O(|\nabla \Gamma|) \quad \text{in } \Omega \tag{2.38}$$

for any solution $G_\lambda=\Gamma+H_\lambda$ of $(2.1)_{g=0}$. Indeed, by [33] any solution G_λ of $(2.1)_{g=0}$ satisfies

$$\frac{\Gamma}{C}\le G_\lambda\le C\Gamma \quad \text{in } B_{2R_0}(0) \tag{2.39}$$

for some $C>1$, where $R_0=\frac14\mathrm{dist}(0,\partial\Omega)$. For $0<R\le R_0$ consider the scaling $G_{\lambda,R}(y)=R^{\frac{N-p}{p-1}}G_\lambda(Ry)$ of G_λ in $\Omega_R=\frac{\Omega}{R}$ which satisfies

$$\begin{cases}-\Delta_p G_{\lambda,R}-\lambda R^p G_{\lambda,R}^{p-1}=\delta_0 & \text{in } \Omega_R\\ G_{\lambda,R}\ge 0 & \text{in } \Omega_R\\ G_{\lambda,R}=0 & \text{on } \partial\Omega_R.\end{cases} \tag{2.40}$$

Since $\Gamma_R(y)=R^{\frac{N-p}{p-1}}\Gamma(Ry)=\Gamma(y)$ in view of $1<p<N$, we have that condition (2.39) is scaling invariant:

$$\frac{\Gamma}{C}\le G_{\lambda,R}\le C\Gamma \quad \text{in } B_{\frac{2R_0}{R}}(0). \tag{2.41}$$

Since $G_{\lambda,R}$ is uniformly bounded in $L^\infty_{\rm loc}(B_2(0)\setminus\{0\})$ thanks to (2.41), elliptic estimates [11, 34] for (2.40) imply that

$$G_{\lambda,R} \text{ uniformly bounded in } C^{1,\alpha}_{\rm loc}(B_2(0)\setminus\{0\})$$

for some $\alpha\in(0,1)$. Since in particular $\|\nabla G_{\lambda,R}\|_{\infty,\partial B_1(0)}\le C$, setting $H_{\lambda,R}(y)=R^{\frac{N-p}{p-1}}H_\lambda(Ry)$ we deduce that $\|\nabla H_{\lambda,R}\|_{\infty,\partial B_1(0)}\le C'$ in view of $\nabla G_{\lambda,R}=\nabla\Gamma+\nabla H_{\lambda,R}$, which can be re-written as

$$|\nabla H_\lambda|\le \frac{C'}{|x|^{\frac{N-1}{p-1}}}=C|\nabla\Gamma| \quad \text{on } \partial B_R(0) \tag{2.42}$$

for all $0<R\le\frac14\mathrm{dist}(0,\partial\Omega)$. Away from the origin ∇H_λ is bounded thanks to [11, 28, 34] and $|\nabla\Gamma|$ is bounded from below, and then estimate (2.38) follows by (2.42). Moreover, notice that for $1<p\le N$ there holds

$$\|H_\lambda\|_\infty<+\infty \quad\Rightarrow\quad |\nabla H_\lambda(x)|=o(|\nabla\Gamma(x)|) \quad \text{as } x\to 0. \tag{2.43}$$

Indeed, for $1<p<N$ we have that $\|H_{\lambda,R}\|_{\infty,\Omega_R}\to 0$ and then $\|\nabla H_{\lambda,R}\|_{\infty,\partial B_1(0)}\to 0$ as $R\to 0$, which provides the validity of (2.43). When $p=N$ the function $G_{\lambda,R}(y)=G_\lambda(Ry)+(N\omega_N)^{-\frac{1}{N-1}}\log R=\Gamma(y)+H_\lambda(Ry)$ is uniformly bounded in $L^\infty_{\rm loc}(\mathbb{R}^N\setminus\{0\})$ and satisfies

$$-\Delta_N G_{\lambda,R}-\lambda R^N\Big[G_{\lambda,R}-(N\omega_N)^{-\frac{1}{N-1}}\log R\Big]^{N-1}=\delta_0 \quad \text{in } \Omega_R.$$

We argue as above to show that, up to a subsequence, $H_{\lambda,R}(y)=H_\lambda(Ry)\to H_0$ in $C^1_{\rm loc}(\mathbb{R}^N\setminus\{0\})$ as $R\to 0$, where $\|H_0\|_\infty<+\infty$ and $\Gamma+H_0$ is a $N-$harmonic function in $\mathbb{R}^N\setminus\{0\}$. It follows that H_0 is a constant function, see for example Lemma 4.3 in [17]. Since this is true along any such subsequence, then $\nabla H_{\lambda,R}\to 0$ in $C_{\rm loc}(\mathbb{R}^N\setminus\{0\})$ as $R\to 0$ and (2.43) does hold also in the case $p=N$.

Once we have $\delta|\nabla\Gamma|^{p-2}\le(|\nabla\Gamma|+|\nabla H_\lambda|)^{p-2}$ for $1<p<2$ in view of (2.38), it becomes clear the usefulness of the following weigthed Sobolev inequalities of Caffarelli-Kohn-Nirenberg type [7]: given $1<p<2$, there exists $C>0$ so that

$$\left(\int_{\mathbb{R}^N}|\nabla\Gamma|^{p-2}|u|^{\frac{2(N-2+p)}{N-p}}\right)^{\frac{N-p}{N-2+p}}\le C\int_{\mathbb{R}^N}|\nabla\Gamma|^{p-2}|\nabla u|^2 \tag{2.44}$$

for any compactly supported $u\in L^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N}|\nabla\Gamma|^{p-2}|\nabla u|^2<+\infty$. Valid in $C_0^\infty(\mathbb{R}^N)$, (2.44) can be first extended to $W^{1,2}-$functions with compact support in view of $|\nabla\Gamma|^{p-2}\in L^\infty_{\rm loc}(\mathbb{R}^N)$ and then to compactly supported $u \in L^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |\nabla \Gamma|^{p-2}|\nabla u|^2 < +\infty$ through the sequence $\eta_\epsilon u \in W^{1,2}(\mathbb{R}^N)$, η_ϵ being given by (2.33) with $\eta = 1$ in $\mathbb{R}^N$, since

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^N} |\nabla \Gamma|^{p-2}|\nabla \eta_\epsilon|^2 u^2 \to 0.$$

For later convenience, when either $2 \le p < N$ or $p = N \ge 3$ observe also the validity of the following inequality

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{2N(p-1)}{N(p-1)-p}}\right)^{\frac{N(p-1)-p}{N(p-1)}} \le C \int_{\mathbb{R}^N} |x|^{\frac{p-2}{p-1}}|\nabla u|^2 \tag{2.45}$$

for any compactly supported $u \in L^\infty(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |x|^{\frac{p-2}{p-1}}|\nabla u|^2 < +\infty$.

3. Weak comparison principle and uniqueness results

This section is devoted to discuss the uniqueness part in Theorem 1.2 when $2 \le p \le N$ among solutions satisfying the natural condition (1.6). When $\lambda = 0$ maximum and comparison principle in weak or strong form are well known, see for example [36], and have been extended in various forms to the case $\lambda < \lambda_1$ in connection with existence and uniqueness results, see [8, 10, 19, 20] just to quote a few.

To extend the previous uniqueness results to the singular situation, the crucial property is given by the convexity of the functional

$$I(w) = \begin{cases} \int_\Omega |\nabla w^{\frac{1}{p}}|^p & \text{if } w \ge 0 \text{ and } \nabla(w^{\frac{1}{p}}) \in L^p(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Proved in [10] for $p > 1$, a quantitative form is established here giving a positive lower bound for I'' when $2 \le p \le N$, crucial to be applied on $\Omega_\epsilon = \Omega \setminus B_\epsilon(0)$ as $\epsilon \to 0$.

Lemma 3.1. *Let $w \ge 0$ a.e. in Ω so that $\nabla(w^{\frac{1}{p}}) \in L^p(\Omega)$. Let ϕ be a direction so that $w_t = w + t\phi \ge 0$ a.e. in Ω and $\nabla(w_t^{\frac{1}{p}}) \in L^p(\Omega)$ for $t \ge 0$ small. Letting $\rho(w,\phi)$ be given in (3.7), there hold*

$$I'(w)[\phi] = \int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\langle \nabla w^{\frac{1}{p}}, \nabla(w^{\frac{1-p}{p}}\phi)\rangle, \quad I''(w)[\phi,\phi] = \int_\Omega \rho(w,\phi) \tag{3.1}$$

with

$$\begin{aligned} \rho(w,\phi) \quad \ge \quad & \frac{p-1}{p}(p^3-3p^2+5p-2)|\nabla w^{\frac{1}{p}}|^p \left(\frac{\phi}{w} - \frac{p(p^2-2p+2)\langle \nabla w, \nabla \phi\rangle}{(p^3-3p^2+5p-2)|\nabla w|^2}\right)^2 \\ & + \frac{(p-1)(p-2)}{p(p^3-3p^2+5p-2)} w^{\frac{2(1-p)}{p}} |\nabla w^{\frac{1}{p}}|^{p-2}|\nabla \phi|^2, \end{aligned} \tag{3.2}$$

where $I'(w)[\phi] = \frac{d}{dt} I(w_t)\Big|_{t=0^+}$ and $I''(w)[\phi,\phi] = \frac{d}{dt} I'(w_t)[\phi]\Big|_{t=0^+}$.

Proof. Since $\frac{d}{dt} w_t^{\frac{1}{p}} = \frac{1}{p} w_t^{\frac{1-p}{p}}\phi$, we have that

$$I'(w_t)[\phi] = \int_\Omega |\nabla w_t^{\frac{1}{p}}|^{p-2}\langle \nabla w_t^{\frac{1}{p}}, \nabla(w_t^{\frac{1-p}{p}}\phi)\rangle,$$

providing, when evaluated at $t = 0$, the validity of the first in formula (3.1). Differentiating once more in t at 0^+, we have that

$$\begin{aligned} I''(w)[\phi,\phi] \quad = \quad & (p-2)\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-4}\langle \nabla w^{\frac{1}{p}}, \nabla(w^{\frac{1-p}{p}}\phi)\rangle^2 + \frac{1}{p}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}|\nabla(w^{\frac{1-p}{p}}\phi)|^2 \\ & - \frac{p-1}{p}\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\langle \nabla w^{\frac{1}{p}}, \nabla(w^{\frac{1-2p}{p}}\phi^2)\rangle. \end{aligned} \tag{3.3}$$

Writing $\langle \nabla w, \nabla \phi\rangle = \cos\alpha |\nabla w||\nabla \phi|$ the first, second and third term in (3.3) produce, respectively,

$$\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-4}\langle \nabla w^{\frac{1}{p}}, \nabla (w^{\frac{1-p}{p}}\phi)\rangle^2 = \int_\Omega \frac{|\nabla w^{\frac{1}{p}}|^{p-2}}{w^{\frac{2(p-1)}{p}}}\Big[\frac{(p-1)^2}{p^2}\frac{|\nabla w|^2}{w^2}\phi^2 + \cos^2\alpha|\nabla\phi|^2 - \frac{2(p-1)}{p}\cos\alpha\frac{|\nabla w|}{w}\phi|\nabla\phi|\Big], \tag{3.4}$$

$$\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}|\nabla (w^{\frac{1-p}{p}}\phi)|^2 = \int_\Omega \frac{|\nabla w^{\frac{1}{p}}|^{p-2}}{w^{\frac{2(p-1)}{p}}}\Big[\frac{(p-1)^2}{p^2}\frac{|\nabla w|^2}{w^2}\phi^2 + |\nabla\phi|^2 - \frac{2(p-1)}{p}\cos\alpha\frac{|\nabla w|}{w}\phi|\nabla\phi|\Big], \tag{3.5}$$

$$\int_\Omega |\nabla w^{\frac{1}{p}}|^{p-2}\langle \nabla w^{\frac{1}{p}}, \nabla (w^{\frac{1-2p}{p}}\phi^2)\rangle = \int_\Omega \frac{|\nabla w^{\frac{1}{p}}|^{p-2}}{w^{\frac{2(p-1)}{p}}}\Big[-\frac{2p-1}{p^2}\frac{|\nabla w|^2}{w^2}\phi^2 + \frac{2}{p}\cos\alpha\frac{|\nabla w|}{w}\phi|\nabla\phi|\Big]. \tag{3.6}$$

Collecting (3.4)-(3.6), the expression of (3.3) becomes $I''(w)[\phi,\phi] = \int_\Omega \rho(w,\phi)$, with

$$\begin{aligned} \rho(w,\phi) &= w^{\frac{2(1-p)}{p}}|\nabla w^{\frac{1}{p}}|^{p-2}\Big[C_1\frac{|\nabla w|^2}{w^2}\phi^2 - C_2\cos\alpha\frac{|\nabla w|}{w}\phi|\nabla\phi| + C_3|\nabla\phi|^2\Big] \\ &= w^{\frac{2(1-p)}{p}}|\nabla w^{\frac{1}{p}}|^{p-2}\Big[C_1(\frac{|\nabla w|}{w}\phi - \frac{C_2}{2C_1}\cos\alpha|\nabla\phi|)^2 + \frac{4C_1C_3 - C_2^2\cos^2\alpha}{4C_1}|\nabla\phi|^2\Big] \end{aligned} \tag{3.7}$$

by a square completion in view of $C_1 > 0$, where

$$C_1 = \frac{p-1}{p^3}(p^3 - 3p^2 + 5p - 2), \quad C_2 = \frac{2(p-1)}{p^2}(p^2 - 2p + 2), \quad C_3 = \frac{1}{p} + (p-2)\cos^2\alpha.$$

Since

$$4\frac{p-1}{p^3}(p^3 - 3p^2 + 5p - 2)(p-2) - \frac{4(p-1)^2}{p^4}(p^2 - 2p + 2)^2 = -4\frac{p-1}{p^4}(p^3 - 4p^2 + 8p - 4) < 0,$$

then $4C_1C_3 - C_2^2\cos^2\alpha \geq 4\frac{(p-1)^2(p-2)}{p^4}$ and (3.2) follows by (3.7). □

As a first application, we deduce the validity of a weak comparison principle for positive solutions.

Proposition 3.2. *Let $2 \leq p \leq N$ and $a, f_1, f_2 \in L^\infty(\Omega)$. Let $u_i \in C^1(\bar\Omega)$, $i = 1,2$, be solutions to*

$$-\Delta_p u_i - a u_i^{p-1} = f_i \quad \text{in } \Omega \tag{3.8}$$

so that

$$u_i > 0 \text{ in } \Omega, \quad \frac{u_1}{u_2} \leq C \text{ near } \partial\Omega \tag{3.9}$$

for some $C > 0$. If $f_1 \leq f_2$ with $f_2 \geq 0$ in Ω and $u_1 \leq u_2$ on $\partial\Omega$, then $u_1 \leq u_2$ in Ω.

Proof. Setting $w_1 = u_1^p$, $w_2 = u_2^p$ and $\phi = (w_1 - w_2)_+$, consider $w_s = sw_1 + (1-s)w_2$ for $s \in [0,1]$. Since

$$w_s + t\phi = u_2^p\Big[s(\frac{u_1}{u_2})^p + (1-s) + t\left((\frac{u_1}{u_2})^p - 1\right)_+\Big],$$

by (3.9) there exists $t_0 > 0$ small so that $w_s + t\phi \geq 0$ in Ω and $\nabla(w_s + t\phi)^{\frac{1}{p}} \in L^p(\Omega)$ for each $s \in [0,1]$ and $|t| \leq t_0$. Then we can apply (3.1) at $s = 0,1$ to get

$$\begin{aligned} I'(w_1)[\phi] - I'(w_2)[\phi] &= \int_\Omega |\nabla w_1^{\frac{1}{p}}|^{p-2}\langle \nabla w_1^{\frac{1}{p}}, \nabla(w_1^{\frac{1-p}{p}}\phi)\rangle - \int_\Omega |\nabla w_2^{\frac{1}{p}}|^{p-2}\langle \nabla w_2^{\frac{1}{p}}, \nabla(w_2^{\frac{1-p}{p}}\phi)\rangle \\ &= \int_\Omega |\nabla u_1|^{p-2}\langle \nabla u_1, \nabla\frac{\phi}{u_1^{p-1}}\rangle - \int_\Omega |\nabla u_2|^{p-2}\langle \nabla u_2, \nabla\frac{\phi}{u_2^{p-1}}\rangle. \end{aligned}$$

Since $\phi \in W_0^{1,p}(\Omega)$ we deduce that

$$I'(w_1)[\phi] - I'(w_2)[\phi] = \int_\Omega \left(\frac{f_1}{u_1^{p-1}} - \frac{f_2}{u_2^{p-1}} \right) (u_1^p - u_2^p)^+ \leq 0$$

in view of (3.8) and $f_1 \leq f_2$ with $f_2 \geq 0$. Since

$$I'(w_1)[\phi] - I'(w_2)[\phi] = \int_0^1 I''(w_s)[w_1 - w_2, \phi] ds = \int_0^1 I''(w_s)[\phi, \phi] ds$$

in view of $I''(w_s)[w_1 - w_2, \phi] = I''(w_s)[\phi, \phi]$, by Lemma 3.1 $I''(w_s)[\phi, \phi] = \int_\Omega \rho(w_s, \phi)$ with $\rho(w_s, \phi) \geq 0$ thanks to (3.2) when $p \geq 2$. Then, we deduce that $\rho(w_s, \phi) = 0$ for all $s \in [0, 1]$ and then

- $\nabla \phi = 0$ in Ω if $p > 2$
- $\langle \nabla w_s, \nabla \phi \rangle = \phi \frac{|\nabla w_s|^2}{w_s}$ if $p = 2$, which implies $\langle \nabla(w_1 - w_2), \nabla \phi \rangle = s\phi \frac{|\nabla(w_1 - w_2)|^2}{w_s}$ for all $0 \leq s \leq 1$.

In both cases $\nabla \phi = 0$ in Ω and then $w_1 \leq w_2$ in Ω, or equivalently $u_1 \leq u_2$ in Ω. □

Finally, we use Lemma 3.1 to show the uniqueness part in Theorem 1.2.

Theorem 3.3. *Let $2 \leq p \leq N$. If $\lambda < \lambda_1$ with $\lambda \neq 0$ and $p > \frac{N}{2}$, problem $(2.1)_{g=0}$ has exactly one solution G_λ so that $H_\lambda = G_\lambda - \Gamma$ satisfies (1.6). Moreover, if $H_\lambda \in C(\Omega)$ for all $\lambda < \lambda_1$, then the map $\lambda \in (-\infty, \lambda_1) \to H_\lambda(x)$ is strictly increasing at any given $x \in \Omega$.*

Proof. We follow the same argument as in the proof of Proposition 3.2. Letting G_1 and G_2 be two solutions of $(2.1)_{g=0}$ satisfying (1.6), by elliptic regularity theory [11, 28, 32, 34] we know that $G_i \in C^{1,\alpha}(\bar{\Omega} \setminus \{0\})$, $i = 1, 2$, for some $\alpha > 0$. By [33] we know that G_i, $i = 1, 2$, satisfies (2.39) and by the strong maximum principle [36] $\partial_\nu G_i < 0$, $i = 1, 2$, on $\partial\Omega$, where ν denotes the outward unit normal vector. Set $w_1 = G_1^p$, $w_2 = G_2^p$, $\phi = w_1 - w_2$ and $w_s = sw_1 + (1 - s)w_2$ for $s \in [0, 1]$. We have that for each $s \in [0, 1]$ there hold $w_s + t\phi \geq 0$ in Ω and $\nabla(w_s + t\phi)^{\frac{1}{p}} \in L^p(\Omega)$ for t small, in view of the properties of G_1 and G_2. Letting I_ϵ be the functional I defined on $\Omega_\epsilon = \Omega \setminus B_\epsilon(0)$, by (3.1) at $s = 0, 1$ we have that

$$\begin{aligned} I'_\epsilon(w_1)[\phi] - I'_\epsilon(w_2)[\phi] &= \int_{\Omega_\epsilon} |\nabla G_1|^{p-2} \langle \nabla G_1, \nabla \frac{\phi}{G_1^{p-1}} \rangle - \int_\Omega |\nabla G_2|^{p-2} \langle \nabla G_2, \nabla \frac{\phi}{G_2^{p-1}} \rangle \\ &= \int_{\partial B_\epsilon(0)} \left(\frac{|\nabla G_2|^{p-2} \partial_\nu G_2}{G_2^{p-1}} - \frac{|\nabla G_1|^{p-2} \partial_\nu G_1}{G_1^{p-1}} \right) (G_1^p - G_2^p) \end{aligned}$$

in view of $\phi = 0$ on $\partial\Omega$ and the equation $(2.1)_{g=0}$ satisfied by G_1, G_2. Notice that

$$I'_\epsilon(w_1)[\phi] - I'_\epsilon(w_2)[\phi] = \int_0^1 I''_\epsilon(w_s)[\phi, \phi] ds$$

with $I''_\epsilon(w_s)[\phi, \phi] = \int_{\Omega_\epsilon} \rho(w_s, \phi)$ in view of Lemma 3.1. Since $\rho(w_s, \phi) \geq 0$ when $p \geq 2$ in view of (3.2), by the Fatou convergence Theorem we deduce that

$$\int_0^1 ds \int_\Omega \rho(w_s, \phi) \leq \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(0)} \left(\frac{|\nabla G_2|^{p-2} \partial_\nu G_2}{G_2^{p-1}} - \frac{|\nabla G_1|^{p-2} \partial_\nu G_1}{G_1^{p-1}} \right) (G_1^p - G_2^p). \tag{3.10}$$

We claim that the R.H.S. in (3.10) vanishes and then $\rho(w_s, \phi) = 0$ for all $s \in [0, 1]$, which implies, as already discussed in the proof of Proposition 3.2, $\nabla \phi = 0$ in Ω and then $G_1 = G_2$ in Ω.

In order to prove the previous claim, for $i = 1, 2$ notice that $H_i = G_i - \Gamma \in L^\infty(\Omega)$ follows by Theorem 2.6 in view of the assumption (1.6) for G_i. Once $H_i \in L^\infty(\Omega)$, we have that H_i satisfies (2.43) and then

$$G_i^q = \Gamma^q + O(\Gamma^{q-1}), \quad |\nabla G_i|^{p-2} \partial_\nu G_i = |\nabla \Gamma|^{p-2} \partial_\nu \Gamma + o(|\nabla \Gamma|^{p-1}) \tag{3.11}$$

as $x \to 0$ for $q > 0$. By (3.11) we deduce that $G_1^p - G_2^p = O(\Gamma^{p-1})$ and

$$\frac{|\nabla G_i|^{p-2}\partial_\nu G_i}{G_i^{p-1}} = \frac{|\nabla \Gamma|^{p-2}\partial_\nu \Gamma}{\Gamma^{p-1}} + o(\frac{|\nabla \Gamma|^{p-1}}{\Gamma^{p-1}}),$$

which imply

$$\Big|\int_{\partial B_\epsilon(0)} (\frac{|\nabla G_2|^{p-2}\partial_\nu G_2}{G_2^{p-1}} - \frac{|\nabla G_1|^{p-2}\partial_\nu G_1}{G_1^{p-1}})(G_1^p - G_2^p)\Big| = o(\int_{\partial B_\epsilon(0)} |\nabla \Gamma|^{p-1}) = o(1)$$

as $\epsilon \to 0$, as claimed.

Finally, assume $H_\lambda \in C(\Omega)$ for all $\lambda < \lambda_1$ to have well defined values $H_\lambda(x)$ for all $x \in \Omega$ (at $x = 0$ too) and take $\mu_1 < \mu_2$. Letting $0 \le G_n^1, G_n^2 \in W_0^{1,p}(\Omega)$ be the solutions of (2.7) corresponding to $\lambda = \mu_1$ and $\lambda = \mu_2$, respectively, by the proof of Theorem 2.1 recall that $G_{\mu_1} = \lim_{n\to+\infty} G_n^1$ and $G_{\mu_2} = \lim_{n\to+\infty} G_n^2$ a.e. in Ω, where $f_n \ge 0$ is a suitable smooth approximating sequence for the measure δ_0. Since $G_n^i > 0$ in Ω and $\partial_\nu G_n^i < 0$ on $\partial\Omega$ by the strong maximum principle [36], we can apply Proposition 3.2 to get $G_n^1 \le G_n^2$ in view of $0 \le f_n \le f_n + (\mu_2 - \mu_1)(G_n^2)^{p-1}$ with $f_n, G_n^2 \in L^\infty(\Omega)$, and then $G_{\mu_1} \le G_{\mu_2}$ in Ω as $n \to +\infty$. Since

$$-\Delta_p G_{\mu_1} = \mu_1 (G_{\mu_1})^{p-1} < \mu_2 (G_{\mu_2})^{p-1} = -\Delta_p G_{\mu_2} \quad \text{in } \Omega \setminus B_\epsilon(0),$$

apply once again the strong maximum principle [36] to deduce $G_{\mu_1} < G_{\mu_2}$ in $\Omega \setminus B_\epsilon(0)$ for all $\epsilon > 0$, and the strict monotonicity is established in $\Omega \setminus \{0\}$. Given $0 < \epsilon <$ dist $(0, \partial\Omega)$, we can find $\eta \in C_0^1(\Omega)$ with $\eta = 1$ in $B_\epsilon(0)$ and $\delta > 0$ so that $H_{\mu_1} - H_{\mu_2} + \delta \le 0$ on supp$(\eta) \setminus B_\epsilon(0)$. Observe that $u = H_{\mu_1} - H_{\mu_2}$ and $\mathit{\Gamma} = \Gamma + H_{\mu_2}$ satisfy $\nabla u \in L^{\bar q}(\Omega)$, (2.32) and

$$-\Delta_p(\mathit{\Gamma} + u) + \Delta_p(\mathit{\Gamma}) = f \quad \text{in } \Omega \setminus \{0\}$$

with $f = \mu_1 (G_{\mu_1})^{p-1} - \mu_2 (G_{\mu_2})^{p-1} \le 0$. We can apply a variant of Lemma 2.5 with η and $\Psi(u) = (u + \delta)_+$ to get

$$\int_\Omega \eta^2 |\nabla (u+\delta)_+|^p \le C \int_\Omega |\eta||\nabla \eta|(u+\delta)_+ (|\nabla \mathit{\Gamma}| + |\nabla u|)^{p-2}|\nabla u| + \int_\Omega \eta^2 f (u+\delta)_+ \le 0$$

and then $(u + \delta)_+ = 0$ in $B_\epsilon(0)$, providing $H_{\mu_1} - H_{\mu_2} \le -\delta < 0$ in $B_\epsilon(0)$ too. The proof is complete. □

4. Harnack inequalities and Hölder continuity of H_λ at the pole

In this section we will use the Moser iterative scheme in [32] to establish local estimates for the solution H_λ of (1.2) at 0, leading to an Harnack inequality for $H_\lambda + c$ which is the crucial tool to show Hölder estimates at 0. The function $\mathcal{H}(x) = R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx) + c)$, $0 < R < \frac{1}{2}$dist $(0, \partial\Omega)$, satisfies

$$-\Delta_p(\mathit{\Gamma} + \mathcal{H}) + \Delta_p \mathit{\Gamma} = \mathcal{G} \quad \text{in } B_2(0) \setminus \{0\} \tag{4.1}$$

in view of (1.2), where $\mathit{\Gamma} = \pm R^{\frac{N-p}{p-1}} \Gamma(Rx)$ with $\nabla \mathit{\Gamma} = \pm \nabla \Gamma$ and $\mathcal{G} = \pm \lambda R^N G_\lambda^{p-1}(Rx)$. Differently from Proposition 2.4 and Theorem 2.6, we need to perform homogeneuos estimates on $\mathcal{H}$ and to this aim for $2 \le p \le N$ assume

$$\Lambda = \|\mathcal{G}\|_{q_0, B_2(0)}^{\frac{1}{p-1}} < +\infty \tag{4.2}$$

for some $q_0 > \frac{N}{p}$. Consider the weight function $\rho = |\nabla \Gamma|^{p-2}$ when $1 < p < 2$, $\mathcal{G} = 0$ and $\rho = 1$ otherwise, and introduce the weighted integrals $\Phi_\rho(s, h) = \left(\int_{B_h(0)} \rho |u|^s\right)^{\frac{1}{s}}$, $h, s > 0$. Define κ as

$$\kappa = \begin{cases} \frac{N-2+p}{N-p} & \text{if } 1 < p < 2 \text{ and } \mathcal{G} = 0 \\ \frac{N(p-1)}{N(p-1)-p} & \text{if either } 2 \le p < N \text{ or } p = N \ge 3 \\ 2 & \text{if } p = N = 2. \end{cases} \tag{4.3}$$

We are now ready to establish the main estimates in the section.

Proposition 4.1. *Let $\mathcal{H} \in L^\infty(B_2(0))$ be a solution of (4.1) so that $\nabla \mathcal{H} \in L^{\bar q}(B_2(0))$, Γ satisfies (2.32) and (4.2) holds. Assume $\mathcal{G}=0$, $|\nabla \mathcal{H}| \leq M|\nabla \Gamma|$ in $B_2(0)$ when $1<p<2$ and $\|\mathcal{H}\|_\infty+\Lambda \leq M$, $|x|^{\frac{1}{p-1}} \leq M|\nabla \Gamma|$ in $B_2(0)$ when $2 \leq p \leq N$, for some $M>0$. Given $\mu \in \mathbb{R}\setminus\{0\}$, there exist $\nu,\beta \geq 0$ and $C>0$ so that the function $u=|\mathcal{H}|+\Lambda+\epsilon$ satisfies*

$$\pm\, \Phi_\rho(\kappa\mu,h_1) \leq \pm[C|\mu|^\nu (h_2-h_1)^{-\beta}]^{\frac{1}{\mu}} \Phi_\rho(\mu,h_2) \tag{4.4}$$

for all $1 \leq h_1 < h_2 \leq 2$ and $0<\epsilon \leq 1$, uniformly for μ away from $2-p$, 0 and 1, where $\kappa>1$ is given in (4.3) and $\pm$ simply denotes the sign of μ.

Remark 4.2. *The assumption $|x|^{\frac{1}{p-1}} \leq M|\nabla \Gamma|$ when $2 \leq p \leq N$ is sufficiently general in order to establish the validity of Corollary 4.5, which will be used in a crucial way in [2].*

Proof. Given $T_{k,l}$ in (2.16), introduce the bounded monotone Lipschitz function

$$\Psi(s)=\text{sign } s\left([T_{0,l}(|s|+\Lambda+\epsilon)]^\beta-[T_{0,l}(\Lambda+\epsilon)]^\beta\right), \beta \in \mathbb{R}\setminus\{0\}.$$

Let $\eta \in C_0^\infty(B_{h_2}(0))$ be a cut-off function so that $0 \leq \eta \leq 1$, $\eta=1$ in $B_{h_1}(0)$ and $|\nabla \eta| \leq \frac{2}{h_2-h_1}$. Since $\eta=0$ on $\partial B_2(0)$ and $\nabla \mathcal{H} \in L^{\bar q}(B_2(0))$ we can apply Lemma 2.5 to $\mathcal{H}$, solution of (4.1), to get

$$\begin{aligned}\int \eta^2 |\Psi'(\mathcal{H})|(|\nabla \Gamma|+|\nabla \mathcal{H}|)^{p-2}|\nabla \mathcal{H}|^2 \quad &\leq \quad C\int \eta|\nabla \eta||\Psi(\mathcal{H})|(|\nabla \Gamma|+|\nabla \mathcal{H}|)^{p-2}|\nabla \mathcal{H}| \\ &\quad +C\int \eta^2|\mathcal{G}||\Psi(\mathcal{H})|\end{aligned} \tag{4.5}$$

for some $C>0$. Define $v=u^{\frac{\beta+1}{2}}$ and $w=u^{\frac{\beta-1+p}{p}}$ with $u=|\mathcal{H}|+\Lambda+\epsilon$. Since $\Psi'(\mathcal{H})=\beta u^{\beta-1}$ and $|\Psi(\mathcal{H})| \leq u^\beta$ for $l>M+1$, by (4.5) we deduce that

$$|\beta| \int \eta^2 u^{\beta-1}(|\nabla \Gamma|+|\nabla u|)^{p-2}|\nabla u|^2 \leq C\left(\int \eta|\nabla \eta| u^\beta (|\nabla \Gamma|+|\nabla u|)^{p-2}|\nabla u|+\int \eta^2|\mathcal{G}|u^\beta\right) \tag{4.6}$$

in view of $|\nabla \mathcal{H}|=|\nabla u|$.

Consider first the case $1<p<2$, for which (4.6) implies

$$\int \eta^2 |\nabla \Gamma|^{p-2}|\nabla v|^2 \leq C\int \eta|\nabla \eta||\nabla \Gamma|^{p-2} v|\nabla v| \tag{4.7}$$

uniformly for β away from 0 in view of $|\nabla u| \leq M|\nabla \Gamma|$ in $B_2(0)$. Since

$$C\int \eta|\nabla \eta||\nabla \Gamma|^{p-2}v|\nabla v| \leq \frac{1}{2}\int \eta^2|\nabla \Gamma|^{p-2}|\nabla v|^2+C'\int |\nabla \eta|^2|\nabla \Gamma|^{p-2}v^2$$

thanks to the Young inequality, we can re-write (4.7) as

$$\int |\nabla \Gamma|^{p-2}|\nabla(\eta v)|^2 \leq C\int |\nabla \eta|^2|\nabla \Gamma|^{p-2}v^2. \tag{4.8}$$

Thanks to (2.32) and making use of (2.44), by (4.8) we deduce for $\mu=\beta+1$ that

$$\pm\Phi_\rho(\kappa\mu,h_1) \leq \pm(\frac{C}{(h_2-h_1)^2})^{\frac{1}{\mu}}\Phi_\rho(\mu,h_2)$$

does hold uniformly for μ away from 1, where κ is given by (4.3). Observe that the $(\beta+1)-$th root of (4.8) for $\beta<-1$ reverses the inequality causing the presence of $\pm$ in (4.4).

Consider now the case $2 \le p \le N$. Since

$$
\begin{aligned}
&C\int \eta^{\frac{p}{2}}|\nabla \eta^{\frac{p}{2}}|u^\beta(|\nabla \Gamma|+|\nabla u|)^{p-2}|\nabla u|\\
&\le \frac{|\beta|}{4}\int \eta^p u^{\beta-1}(|\nabla \Gamma|+|\nabla u|)^{p-2}|\nabla u|^2+\frac{C'}{|\beta|}\int |\nabla\eta|^2u^{\beta+1}|\nabla\Gamma|^{p-2}+\frac{C'}{|\beta|}\int \eta^{p-2}|\nabla \eta|^2u^{\beta+1}|\nabla u|^{p-2}\\
&\le \frac{|\beta|}{2}\int \eta^p u^{\beta-1}(|\nabla \Gamma|+|\nabla u|)^{p-2}|\nabla u|^2+\frac{C}{|\beta|}\int |\nabla\eta|^2v^2+\frac{C}{|\beta|^{p-1}}\int|\nabla \eta|^p w^p
\end{aligned}
$$

in view of the Young inequality, (2.32) and $\sup\limits_{B_2\setminus B_1}|\nabla \Gamma|^{p-2}<+\infty$, by replacing η with $\eta^{\frac{p}{2}}$ (4.6) implies

$$
\int \eta^p|\nabla\Gamma|^{p-2}|\nabla v|^2+\frac{1}{|\beta|^{p-2}}\int \eta^p|\nabla w|^p\le C(\int |\nabla \eta|^2v^2+\frac{1}{|\beta|^{p-2}}\int|\nabla\eta|^pw^p+|\beta|\int \eta^p|\mathcal{G}|u^\beta) \tag{4.9}
$$

uniformly for β away from $1-p$ and 0. Since $q_0>\frac{N}{p}$, fix α and γ so that $\alpha\in(\frac{q_0}{q_0-1},\frac{pq_0}{N-p})$ and $\frac{1}{\alpha}+\frac{1}{\gamma}=\frac{q_0-1}{q_0}$. By the Hölder inequality with exponents q_0, γ and α we have that

$$
\int \eta^p|\mathcal{G}|u^\beta\le\frac{1}{\Lambda^{p-1}}\int|\mathcal{G}|(\eta w)^{\frac{p}{\gamma}+\frac{p(q_0+\alpha)}{\alpha q_0}}\le\frac{1}{\Lambda^{p-1}}\|\mathcal{G}\|_{q_0,B_2(0)}\|\eta w\|_p^{\frac{p}{\gamma}}\|\eta w\|_{\frac{p(q_0+\alpha)}{q_0}}^{\frac{p(q_0+\alpha)}{\alpha q_0}}=\|\eta w\|_p^{\frac{p}{\gamma}}\|\eta w\|_{\frac{p(q_0+\alpha)}{q_0}}^{\frac{p(q_0+\alpha)}{\alpha q_0}}
$$

in view of (4.2) and then

$$
\begin{aligned}
C|\beta|\int \eta^p|\mathcal{G}|u^\beta &\le C'|\beta|\|\eta w\|_p^{\frac{p}{\gamma}}(\|\eta\nabla w\|_p^{\frac{p(q_0+\alpha)}{\alpha q_0}}+\|w\nabla\eta\|_p^{\frac{p(q_0+\alpha)}{\alpha q_0}}) \\
&\le \frac{1}{2|\beta|^{p-2}}\|\eta\nabla w\|_p^p+C''|\beta|^{\frac{\alpha q_0+(p-2)(q_0+\alpha)}{\alpha q_0-\alpha-q_0}}\|\eta w\|_p^p+\frac{1}{|\beta|^{p-2}}\|w\nabla\eta\|_p^p
\end{aligned}\tag{4.10}
$$

by the Sobolev embedding Theorem in view of $(N-p)(q_0+\alpha)<Nq_0$ and the Young inequality. Inserting (4.10) into (4.9) we get that

$$
\int |x|^{\frac{p-2}{p-1}}|\nabla(\eta^{\frac{p}{2}}v)|^2\le C\left(\int|\nabla\eta|^2v^2+|\beta|^{\frac{\alpha q_0+(p-2)(q_0+\alpha)}{\alpha q_0-\alpha-q_0}}\int\eta^p|w|^p+\frac{1}{|\beta|^{p-2}}\int|\nabla \eta|^p|w|^p\right) \tag{4.11}
$$

in view of $|x|^{\frac{1}{p-1}}\le M|\nabla\Gamma|$ in $B_2(0)$. Since $\|\mathcal{H}\|_\infty+\Lambda\le M$ if $p\ge 2$, we have that $\|u\|_\infty\le M+1$ when $0<\epsilon\le 1$ and then $w^p=u^{\beta+1}u^{p-2}\le (M+1)^{p-2}v^2$. By using the Sobolev embedding Theorem when $p=N=2$ or (2.45) otherwise, for $\mu=\beta+1$ estimate (4.11) gives that

$$
\pm\Phi_1(\kappa\mu,h_1)\le\pm[C\frac{|\mu|^{\frac{\alpha q_0+(p-2)(q_0+\alpha)}{\alpha q_0-\alpha-q_0}}}{(h_2-h_1)^p}]^{\frac{1}{\mu}}\Phi_1(\mu,h_2)
$$

does hold uniformly for μ away from $2-p$ and 1, where κ is given by (4.3). Estimate (4.4) is then established in all the cases and the proof is complete. □

Hereafter we specialize the argument to $\mathcal{H}=R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx)+c)$, $R>0$. Let us consider now the case $\beta=-1$ in the proof of Proposition 4.1 when $\mathcal{H}\ge 0$ and the result we have is the following.

Proposition 4.3. *Let $1<p\le N$ if $\lambda=0$ and $p\ge 2$ with $p>\frac{N}{2}$ if $\lambda\neq 0$. Assume $\frac{N}{p}<q_0<\frac{N}{N-p}$ if $\lambda\neq 0$ and $\mathcal{H}=R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx)+c)\ge 0$. There exist $R_0>0$ and $C>0$ so that $v=\log u$, where $u=\mathcal{H}+\Lambda+\epsilon$ and $\epsilon>0$, satisfies*

$$
⨍_B|v-\bar v|\le C
$$

for all open ball $B\subset B_1(0)$, $0<R\le R_0$ and $0<\epsilon\le 1$, where $⨍$ denotes an integral mean and $\bar v=⨍_B v$.

Proof. First of all, observe that $p \ge 2$ and $p > \frac{N}{2}$ imply $p^2 \ge 2p > N$. Let $B = B_h(x_0) \subset B_1(0)$. Since $|x_0| + h < 1$ implies $|x| \le |x - x_0| + |x_0| < \frac{3}{2}h + |x_0| < 2$ for all $x \in B_{\frac{3}{2}h}(x_0)$, we have that $B_{\frac{3}{2}h}(x_0) \subset B_2$. Let $\eta \in C_0^\infty(B_{\frac{3}{2}h}(x_0))$ be a cut-off function with $0 \le \eta \le 1$, $\eta = 1$ in $B_h(x_0)$ and $|\nabla \eta| \le \frac{4}{h}$. Since $\mathcal{H}$ solves (4.1) with $\nabla \mathit{\Gamma} = \pm \nabla \Gamma$ and $\mathcal{G} = \pm \lambda R^N G^{p-1}(Rx)$, we can apply Lemma 2.5 with the bounded monotone Lipschitz function $\Psi(s) = \text{sign } s \left([T_{0,l}(|s| + \Lambda + \epsilon)]^{-1} - [T_{0,l}(\Lambda + \epsilon)]^{-1} \right)$, for $l > \|\mathcal{H}\|_\infty + \Lambda + 1$ and $T_{k,l}$ given by (2.16), and a cut-off function $\eta_\delta = \eta(\delta + |x|^2)^{\frac{(N-1)(p-2)}{4(p-1)} - 1} |x|^{\frac{5}{2}}$, $\delta > 0$, to get

$$\int \eta_\delta^2 |\nabla \Gamma|^{p-2} |\nabla v|^2 \le C \left(\int \eta_\delta |\nabla \eta_\delta| |\nabla \Gamma|^{p-2} |\nabla v| + \int \eta_\delta^2 \frac{|\mathcal{G}|}{u} \right)$$

in view of (2.38) (which follows by (2.43) and $\|H_\lambda\|_\infty < +\infty$ when $p = N$) and then by the Young inequality

$$\begin{aligned} \int \eta_\delta^2 |\nabla \Gamma|^{p-2} |\nabla v|^2 \quad & \le C' \left(\int |\nabla \eta_\delta|^2 |\nabla \Gamma|^{p-2} + \int \eta_\delta^2 \frac{|\mathcal{G}|}{u} \right) \le C \left(\int |x| (\frac{|x|^2}{\delta + |x|^2})^{-\frac{(N-1)(p-2)}{2(p-1)}} |\nabla \eta|^2 \right. \\ & \left. + \int (\frac{|x|^2}{\delta + |x|^2})^{2 - \frac{(N-1)(p-2)}{2(p-1)}} \frac{\eta^2}{|x|} + \int |x| (\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}} \eta^2 \frac{|\mathcal{G}|}{u} \right) \end{aligned} \tag{4.12}$$

for universal constants in R, δ and c. Since $(\frac{|x|^2}{\delta + |x|^2})^\alpha \le C|x|^{-\max\{-2\alpha, 0\}}$, we have that

$$\begin{aligned} & |x| (\frac{|x|^2}{\delta + |x|^2})^{-\frac{(N-1)(p-2)}{2(p-1)}} \le C|x|^{-\max\{\frac{(N-1)(p-2)}{p-1} - 1, -1\}} \in L^1_{\text{loc}}(\mathbb{R}^N) \\ & (\frac{|x|^2}{\delta + |x|^2})^{2 - \frac{(N-1)(p-2)}{2(p-1)}} \frac{1}{|x|} \le C|x|^{-\max\{\frac{(N-1)(p-2)}{p-1} - 3, 1\}} \in L^1_{\text{loc}}(\mathbb{R}^N) \end{aligned} \tag{4.13}$$

in view of $\frac{(N-1)(p-2)}{p-1} < N$. Since $\mathcal{G} = \pm \lambda R^p \Gamma^{p-1}(x)[1 + O(R^{\frac{N-p}{p-1}})]$ when $2 \le p < N$ in view of $\|H_\lambda\|_\infty < +\infty$, for $\lambda \ne 0$ there holds $\Lambda \ge CR^{\frac{p}{p-1}}$ for some $C > 0$ and all R small in view of $q_0 < \frac{N}{N-p}$, where Λ is given by (4.2), and then

$$\begin{aligned} \int |x| (\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}} \eta^2 \frac{|\mathcal{G}|}{u} \quad & \le \quad \frac{1}{\Lambda} \int |x| (\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}} \eta^2 |\mathcal{G}| \\ & \le \quad C \int |x|^{p+1-N} (\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}} \eta^2. \end{aligned} \tag{4.14}$$

On the other hand, since $\mathcal{G} = \pm \lambda R^N |\log R|^{N-1}[1 + O(\frac{\log |x|}{\log R})]$ in $B_2(0)$ when $p = N$ thanks to $\|H_\lambda\|_\infty < +\infty$, for $\lambda \ne 0$ there holds $\Lambda \ge CR^{\frac{N}{N-1}} |\log R|$ for some $C > 0$ and for all R small and then

$$\int |x| (\delta + |x|^2)^{\frac{N-2}{2}} \eta^2 \frac{|\mathcal{G}|}{u} \le \frac{1}{\Lambda} \int |x| (\delta + |x|^2)^{\frac{N-2}{2}} \eta^2 |\mathcal{G}| \le C \int |x| |\log |x|| (\delta + |x|^2)^{\frac{N-2}{2}} \eta^2. \tag{4.15}$$

Since

$$|x|^{p+1-N} |\log |x|| (\delta + |x|^2)^{\frac{(N-1)(p-2)}{2(p-1)}} \le C|x|^{p+1-N} |\log |x|| \in L^1_{\text{loc}}(\mathbb{R}^N) \tag{4.16}$$

when $\lambda \ne 0$ in view of $p \ge 2$, we can use (4.13), (4.16) and the Lebesgue convergence Theorem in (4.12) and (4.14)-(4.15) to get

$$\int \eta^2 |x| |\nabla v|^2 \le C \left(\int |x| |\nabla \eta|^2 + \int \frac{\eta^2}{|x|} + \underbrace{\int |x|^{p - \frac{N-1}{p-1}} |\log |x|| \eta^2}_{\lambda \ne 0} \right) \tag{4.17}$$

thanks to the Fatou convergence Theorem. Since $p-\frac{N-1}{p-1}>-1$ if $\lambda\neq 0$ and

$$\begin{aligned}\int_B |v-\bar v| &\leq C'h\int_B |\nabla v|\leq C'h(\int_B \frac{1}{|x|})^{\frac12}(\int_B |x||\nabla v|^2)^{\frac12}\\ &\leq Ch(\int_B \frac{1}{|x|})^{\frac12}\left(\int |x||\nabla \eta|^2+\int \frac{\eta^2}{|x|}+\underbrace{\int |x|^{p-\frac{N-1}{p-1}}|\log|x||\eta^2}_{\lambda\neq 0}\right)^{\frac12}\end{aligned}$$

in view of (4.17), for $|x_0|<3h$ one has that

$$\int_B |v-\bar v|\leq Ch^{\frac{N+1}{2}}\left(h^{N-1}+\underbrace{h^{p-\frac{N-1}{p-1}+N}|\log h|}_{\lambda\neq 0}\right)^{\frac12}=O(h^N)$$

in view of $B_{\frac32 h}(x_0)\subset B_{5h}(0)$, while for $|x_0|\geq 3h$ there holds

$$\begin{aligned}\int_B |v-\bar v| &\leq C'\left[h^2(\int_B \frac{1}{|x|})(\int |x||\nabla \eta|^2)+h^{N+1}(h^{N-1}+|\log h|h^{\min\{p-\frac{N-1}{p-1}+N,N\}})\right]^{\frac12}\\ &\leq C\left[h^2(\frac{h^N}{|x_0|})(|x_0|h^{N-2})+h^{2N}\right]^{\frac12}=O(h^N)\end{aligned}$$

in view of $\frac{3h}{2}\leq \frac{|x_0|}{2}\leq |x|\leq \frac32|x_0|$ for all $x\in B_{\frac32 h}(x_0)$. The proof is complete. □

We are now ready to establish an Harnack inequality for $\mathcal{H}=R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx)+c)$ when $\mathcal{H}\geq 0$, a crucial tool to establish the Hölder continuity of H_λ at 0.

Theorem 4.4. *Let $1<p\leq N$ if $\lambda=0$ and $p\geq 2$ with $p>\frac N2$ if $\lambda\neq 0$. Assume that $\mathcal{H}=R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx)+c)\geq 0$ in $B_2(0)$. Then there exist $R_0>0$ and $C>0$ so that*

$$\sup_{B_1(0)}\mathcal{H}\leq C(\inf_{B_1(0)}\mathcal{H}+\Lambda) \tag{4.18}$$

for all $0<R\leq R_0$, where Λ is given in (4.2) in terms of $\mathcal{G}=\pm\lambda R^N G_\lambda^{p-1}(Rx)$

Proof. Given $p_0>0$ to be specified below, let us fix $0<p_1<p_0$ so that $\kappa^j p_1\neq 2-p,1$ for all $j\geq 0$. Consider first the case $\mu>0$ in Proposition 4.1 to get

$$\Phi_\rho(\kappa\mu,h_1)\leq [\tilde C\mu^\nu(h_2-h_1)^{-\beta}]^{\frac1\mu}\Phi_\rho(\mu,h_2) \tag{4.19}$$

for all $\mu\neq 2-p,1$ and for suitable $\nu,\beta\geq 0$, where $u=|\mathcal{H}|+\Lambda+\epsilon\geq 0$. Starting from p_1 along $\mu_j=\kappa^j p_1$ estimate (4.19) with $1\leq h_1^j=1+2^{-(j+1)}<h_2^j=1+2^{-j}\leq 2$ gives

$$\Phi_\rho(\mu_{j+1},h_1^j)\leq [C(2^\beta\kappa^\nu)^j]^{\frac{1}{\kappa^j p_1}}\Phi_\rho(\mu_j,h_2^j)$$

and then

$$\sup_{B_1(0)} u\leq \lim_{j\to+\infty}\Phi_\rho(\mu_{j+1},h_1^j)\leq C_1\Phi_\rho(p_1,2),\quad C_1=C^{\frac{\kappa}{p_1(\kappa-1)}}(2^\beta\kappa^\nu)^{\frac{1}{p_1}\sum_j\frac{j}{\kappa^j}} \tag{4.20}$$

via an iteration argument as in the proof of Proposition 2.4. Since $\rho>0$ in $B_1(0)\setminus\{0\}$, notice that

$$\|u\|_{\infty,B_1(0)\setminus B_\epsilon(0)}\leq \liminf_{\mu\to+\infty}\Phi_\rho(\mu,1)\leq \limsup_{\mu\to+\infty}\Phi_\rho(\mu,1)\leq \|u\|_{\infty,B_1(0)}$$

and then as $\epsilon\to 0$

$$\lim_{\mu\to+\infty}\Phi_\rho(\mu,1)=\|u\|_{\infty,B_1(0)}=\sup_{B_1(0)} u. \tag{4.21}$$

Consider the case $\mu<0$ in Proposition 4.1 to get

$$\Phi_\rho(\kappa\mu,h_1)\geq [\tilde C|\mu|^\nu (h_2-h_1)^{-\beta}]^{\frac{1}{\mu}}\Phi_\rho(\mu,h_2) \tag{4.22}$$

for all $\mu\neq 2-p$. Starting from $-p_1$ along $\mu_j=\kappa^j(-p_1)$, one can use estimate (4.22) with h_1^j and h_2^j to get

$$\Phi_\rho(\mu_{j+1},h_1^j)\geq [C(2^\beta\kappa^\nu)^j]^{-\frac{1}{\kappa^j p_1}}\Phi_\rho(\mu_j,h_2^j)$$

and then, arguing as we did to show (4.21), one deduces that

$$\inf_{B_1(0)} u\geq \lim_{j\to+\infty}\Phi_\rho(\mu_{j+1},h_1^j)\geq C_2\Phi_\rho(-p_1,2),\quad C_2=C^{-\frac{\kappa}{p_1(\kappa-1)}}(2^\beta\kappa^\nu)^{-\frac{1}{p_1}\sum_j\frac{j}{\kappa^j}}, \tag{4.23}$$

in view of $\mu_j\to-\infty$ as $j\to+\infty$.

Assume now $\mathcal{H}\geq 0$ in $B_2(0)$. Let us finally use Proposition 4.3 to compare (4.20) and (4.23). Indeed, as a consequence of John-Nirenberg Lemma (see Lemma 7 in [32]), Proposition 4.3 shows the existence of $p_0>0$ so that

$$\left(\int_{B_2(0)}\rho e^{p_0 v}\int_{B_2(0)}\rho e^{-p_0 v}\right)^{\frac{1}{p_0}}\leq \|\rho\|_{\infty,B_2(0)}^{\frac{2}{p_0}}\left(\int_{B_2(0)}e^{p_0 v}\int_{B_2(0)}e^{-p_0 v}\right)^{\frac{1}{p_0}}\leq C_3$$

for some universal $C_3>0$, or equivalently

$$\Phi_\rho(p_0,2)\leq C_3\Phi_\rho(-p_0,2) \tag{4.24}$$

in terms of $u=e^v=\mathcal{H}+\Lambda+\epsilon$. The use of (4.24) along with (4.20) and (4.23) gives

$$\sup_{B_1(0)} u\leq C_1\Phi_\rho(p_1,2)\leq C_1'\Phi_\rho(p_0,2)\leq C_1'C_3\Phi_\rho(-p_0,2)\leq C_1'C_3'\Phi_\rho(-p_1,2)\leq \frac{C_1'C_3'}{C_2}\inf_{B_1(0)}u$$

thanks to the Hölder estimate in view of $p_1<p_0$ and $\rho\in L^\infty(B_2(0))$. Since $u=\mathcal{H}+\Lambda+\epsilon$, one then deduces

$$\sup_{B_1(0)}\mathcal{H}\leq C(\inf_{B_1(0)}\mathcal{H}+\Lambda+\epsilon)$$

for some $C>0$ and (4.18) follows by letting $\epsilon\to 0$. ☐

In particular, for $p\geq 2$ we have the following a-priori $L^\infty-$estimate.

Corollary 4.5. *Let $2\leq p\leq N$. Given $M>0$ and $p_0\geq 1$ there exists $C>0$ so that*

$$\|h+c\|_{\infty,B_R(0)}\leq C(R^{-\frac{N}{p_0}}\|h+c\|_{p_0,B_{2R}(0)}+R^{\frac{pq_0-N}{q_0(p-1)}}\|f\|_{q_0,B_{2R}(0)}^{\frac{1}{p-1}}) \tag{4.25}$$

for all $\epsilon^{p-1}\leq R\leq R_0=\frac{1}{4}dist(0,\partial\Omega)$ and all solution h to

$$-\Delta_p(u+h)+\Delta_p u=f\qquad in\ \Omega\setminus\{0\}$$

so that $\nabla h\in L^{\bar q}(\Omega)$, $\frac{|x|^{\frac{1}{p-1}}}{M(\epsilon^p+|x|^{\frac{p}{p-1}})^{\frac{N}{p}}}\leq|\nabla u|\leq M|\nabla\Gamma|$ for some $\epsilon>0$ and $|c|+\|h\|_\infty+\|f\|_{q_0}^{\frac{1}{p-1}}\leq M$ for some $q_0>\frac{N}{p}$.

Proof. Set $\mathcal{H}(x)=R^{\frac{N-p}{p-1}}(h(Rx)+c)$, $0<R<2R_0$. We have that $\mathcal{H}\in L^\infty(B_2(0))$ solves (4.1) with $\Gamma=R^{\frac{N-p}{p-1}}u(Rx)$, $\mathcal{G}=R^N f(Rx)$ and satisfies $\nabla\mathcal{H}\in L^{\bar q}(B_2(0))$. Since $\|\mathcal{H}\|_{\infty,B_2(0)}\leq 2MR^{\frac{N-p}{p-1}}$ and

$$\|\mathcal{G}\|_{q_0,B_2(0)}^{\frac{1}{p-1}}=R^{\frac{N(q_0-1)}{q_0(p-1)}}\|f\|_{q_0,B_{2R}(0)}^{\frac{1}{p-1}}\leq MR^{\frac{N(q_0-1)}{q_0(p-1)}}, \tag{4.26}$$

we have that

$$\|\mathcal{H}\|_{\infty,B_2(0)}+\|\mathcal{G}\|_{q_0,B_2(0)}^{\frac{1}{p-1}}\leq\tilde M$$

for some $\tilde{M}$ and all $0 < R \le R_0$. Since

$$\frac{|x|^{\frac{1}{p-1}}}{M2^{\frac{N}{p-1}+\frac{N}{p}}} \le \frac{|x|^{\frac{1}{p-1}}}{M((\epsilon^{p-1}R^{-1})^{\frac{p}{p-1}}+|x|^{\frac{p}{p-1}})^{\frac{N}{p}}} \le |\nabla \Gamma| \le M|\nabla \Gamma|$$

in $B_2(0)$ for $\epsilon^{p-1} \le R \le R_0$, Proposition 4.1 gives the validity of (4.4) for all $\mu \neq 0$ and we can argue as in (4.20) to get

$$\sup_{B_1(0)} u \le C_1 \Phi_1(p_1, 2) \tag{4.27}$$

for a given $0 < p_1 < p_0$ so that $\kappa^j p_1 \neq 1$ for all $j \in \mathbb{N}$, where $u = |\mathcal{H}| + \|\mathcal{G}\|_{q_0, B_2(0)}^{\frac{1}{p-1}} + \epsilon'$. Since $\Phi_1(p_1, 2) \le |B_2(0)|^{\frac{p_0-p_1}{p_0 p_1}} \Phi_1(p_0, 2)$ by Hölder estimate, by (4.27) we deduce that

$$\begin{aligned} \|h+c\|_{\infty, B_R(0)} &= R^{-\frac{N-p}{p-1}} \sup_{B_1(0)} |\mathcal{H}| \le C' R^{-\frac{N-p}{p-1}} (\|\mathcal{H}\|_{p_0, B_2(0)} + \|\mathcal{G}\|_{q_0, B_2(0)}^{\frac{1}{p-1}} + \epsilon') \\ &\le C \left(R^{-\frac{N}{p_0}} \|h+c\|_{p_0, B_{2R}(0)} + R^{\frac{pq_0 - N}{q_0(p-1)}} \|f\|_{q_0, B_{2R}(0)}^{\frac{1}{p-1}} + \epsilon' R^{-\frac{N-p}{p-1}} \right) \end{aligned} \tag{4.28}$$

in view of (4.26) and

$$\|\mathcal{H}\|_{p_0, B_2(0)} = R^{\frac{N-p}{p-1}} R^{-\frac{N}{p_0}} \|h+c\|_{p_0, B_{2R}(0)}.$$

Letting $\epsilon' \to 0$ in (4.28) we deduce the validity of (4.25) and the proof is complete. □

Finally, let us discuss the Hölder regularity of H_λ at the pole 0. Given Λ in (4.2) in terms of $\mathcal{G} = \pm \lambda R^N G_\lambda^{p-1}(Rx)$, let us re-write the Harnack inequality (4.18) for $\mathcal{H} = R^{\frac{N-p}{p-1}}(\pm H_\lambda(Rx) + c) \ge 0$ in $B_{2R}(0)$ as

$$\sup_{B_R(0)} (\pm H_\lambda + c) \le C \left(\inf_{B_R(0)} (\pm H_\lambda + c) + R^\sigma \right) \tag{4.29}$$

for all $0 < R \le R_0$, in view of (4.26) with $f = \pm \lambda G_\lambda^{p-1}$. Since we assume $p \ge 2$ with $p > \frac{N}{2}$ if $\lambda \neq 0$, notice that $\sigma = \frac{pq_0 - N}{q_0(p-1)} > 0$ when $\lambda \neq 0$ in view of (2.37) with $q_0 > \frac{N}{p}$, while the term R^σ is not present when $\lambda = 0$. In this second case, we can assume $\sigma \in (0, +\infty)$.

We are now in position to follow the argument in [32] and establish the following Hölder property.

Theorem 4.6. *Let $1 < p \le N$ if $\lambda = 0$ and $p \ge 2$ with $p > \frac{N}{2}$ if $\lambda \neq 0$. Then $H_\lambda \in C(\bar{\Omega})$ and there exists $C > 0$ such that*

$$|H_\lambda(x) - H_\lambda(0)| \le C|x|^\alpha \quad \forall\, x \in \Omega \tag{4.30}$$

for some $\alpha \in (0,1)$.

Proof. Setting $M(R) = \sup_{B_R(0)} H_\lambda$ and $\mu(R) = \inf_{B_R(0)} H_\lambda$ for $R > 0$, we claim that the oscillation $\omega(R) = M(R) - \mu(R)$ of H in $B_R(0)$ satisfies

$$\omega(R) \le C_0 R^\alpha \tag{4.31}$$

for all $0 < R \le R_0$, for some $\alpha, C_0, R_0 > 0$.

Indeed, apply (4.29) on $B_{\frac{R}{2}}(0)$ either with $c = M(R)$ and the $-$ sign or with $c = -\mu(R)$ and the $+$ sign to get

$$M(R) - \mu'(R) \le C[M(R) - M'(R)] + CR^\sigma, \quad M'(R) - \mu(R) \le C[\mu'(R) - \mu(R)] + CR^\sigma \tag{4.32}$$

for all $0 < R \le 2R_0$, where $M'(R) = M(\frac{R}{2})$ and $\mu'(R) = \mu(\frac{R}{2})$. By adding the two inequalities in (4.32) we get that

$$\omega(\frac{R}{2}) \le \theta \omega(R) + C_0 R^\sigma \tag{4.33}$$

for all $0 < R \le 2R_0$, where $\theta = \frac{C-1}{C+1} < 1$ and $C_0 = \frac{2C}{C+1}$. If $\theta \le 0$, then (4.33) implies the validity of (4.31) with $\alpha = \sigma > 0$ for all $0 < R \le R_0$ and some $C_0 > 0$. In the case $\theta > 0$, for $S \ge 2$ (4.33) gives that

$$\omega(\frac{R}{S}) \le \omega(\frac{R}{2}) \le \theta(\omega(R) + \tau R^\sigma), \quad 0 < R \le R_0,$$

for some $\tau > 0$ and an iteration starting from $r = R_0$ leads to

$$\omega(\frac{R_0}{S^j}) \le \theta^j[\omega(R_0) + \tau R_0^\sigma \sum_{k=0}^{j-1} (\theta S^\sigma)^{-k}]. \tag{4.34}$$

Since $\theta \in (0,1)$ and $\sigma > 0$ in (4.33) can be taken smaller than 1, the choice $S = (\frac{2}{\theta})^{\frac{1}{\sigma}} \ge 2$ is admissible in (4.34) yielding

$$\omega(\frac{R_0}{S^j}) \le \theta^j(\omega(R_0) + 2\tau R_0^\sigma). \tag{4.35}$$

Given $0 < R \le \frac{R_0}{S}$, let $j_0 \ge 1$ be so that $\frac{R_0}{S^{j_0+1}} < R \le \frac{R_0}{S^{j_0}}$ and by (4.35) we have

$$\omega(R) \le \omega(\frac{R_0}{S^{j_0}}) \le \theta^{j_0}(\omega(R_0) + 2\tau R_0^\sigma) \le C\theta^{j_0} \tag{4.36}$$

with $C = \omega(R_0) + 2\tau R_0^\sigma$. Setting $\gamma = -\frac{\log \theta}{\log 2} > 0$, then $\theta = 2^{-\gamma} = S^{-\alpha}$ with $\alpha = \frac{\sigma\gamma}{\gamma+1} \in (0,1)$ and (4.36) implies

$$\omega(R) \le C(\frac{S}{R_0})^\alpha R^\alpha$$

for all $0 < R \le R_0 2^{-\frac{\gamma+1}{\sigma}}$, and (4.31) is established in this case too.

Since (4.31) gives that $\lim_{R\to 0} \omega(R) = 0$, we deduce that $H_\lambda \in C(\bar\Omega)$ in view of $G_\lambda \in C^{1,\beta}(\bar\Omega \setminus \{0\})$ by elliptic regularity theory [11, 28, 32, 34]. Setting $R = |x|$, (4.31) implies

$$|H_\lambda(x) - H_\lambda(0)| \le \omega(R) \le C_0|x|^\alpha$$

for all $x \in B_{R_0}(0)$. Since (4.30) clearly holds in $\Omega \setminus B_{R_0}(0)$ in view of the boundedness of H_λ, we get the validity of (4.30) in the whole Ω and the proof is complete. □

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Sabina Angeloni, Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre, Largo S. Leonardo Murialdo 1, Roma 00146, Italy.
Email address: sabina.angeloni@uniroma3.it

Pierpaolo Esposito, Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre, Largo S. Leonardo Murialdo 1, Roma 00146, Italy.
Email address: esposito@mat.uniroma3.it