# Interior estimates for some semilinear elliptic problem with critical nonlinearity 

# Estimations à l'intérieur pour un problème elliptique semi-linéaire avec non-linéarité critique 

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#### Abstract

We study compactness properties for solutions of a semilinear elliptic equation with critical nonlinearity. For high dimensions, we are able to show that any solutions sequence with uniformly bounded energy is uniformly bounded in the interior of the domain. In particular, singularly perturbed Neumann equations admit pointwise concentration phenomena only at the boundary. © 2006 Elsevier Masson SAS. All rights reserved.


## Résumé

On étudie les propriétés de compacité pour solutions d'une équation elliptique semi-linéaire avec non-linéarité critique. En hautes dimensions, on démontre qu'une suite de solutions avec énergie uniformément bornée est uniformément bornée dans l'intérieur du domaine. En particulier, les équations de Neumann perturbées singulièrement peuvent avoir des phénomènes de concentration seulement sur la frontière.
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## 1. Introduction and statement of the results

The starting point in our investigation has been the study of asymptotic properties for the problem:

$$
\begin{cases}-\Delta u+\lambda u=u^{p} & \text { in } \Omega,  \tag{1}\\ u>0 & \text { in } \Omega, \\ \frac{\partial u}{\partial n}=0 & \text { on } \partial \Omega,\end{cases}
$$

[^0]where $\Omega$ is a smooth bounded domain in $\mathbb{R}^{N}, N \geqslant 3, p=\frac{N+2}{N-2}$ is the critical exponent from the Sobolev viewpoint and $\lambda>0$ is a large parameter. Here, $n(x)$ is the unit outward normal of $\Omega$ at $x \in \partial \Omega$.

Under the transformation $v(x)=\lambda^{-\frac{1}{p-1}} u(x), d^{2}=\frac{1}{\lambda}$, problem (1) reads equivalently as a singularly perturbed Neumann problem:

$$
\begin{cases}-d^{2} \Delta v+v=v^{p} & \text { in } \Omega  \tag{2}\\ v>0 & \text { in } \Omega \\ \frac{\partial v}{\partial n}=0 & \text { on } \partial \Omega\end{cases}
$$

where $p=\frac{N+2}{N-2}$. For general exponent $p>1$, problem (2) is related to the study of stationary solutions for a chemotaxis system (see [17]) proposed by Keller, Segel and Gierer, Meinhardt (see [18]).

Problem (1) for $\lambda$ large has been widely studied in the subcritical case $p<\frac{N+2}{N-2}$. The asymptotic behaviour and the construction of blowing up solutions have been considered by several authors. In particular, there exist peak solutions which blow up at many finitely boundary and/or interior points of $\Omega$.

The critical case $p=\frac{N+2}{N-2}$ has different features. Starting from the pioneering works of Adimurthi, Mancini and Yadava [3] (see also [1,2]), asymptotic analysis (see [13,15] for low energy solutions) and construction of solutions concentrating at boundary points of $\Omega$ have been considered by several authors (see for example [21]). We refer to [20] for an extensive list of references about subcritical and critical case.

As far as interior concentration, the situation is quite different since in literature no results are available and it is expected that in general such solutions should not exist. A first partial result in this direction is due to Cao, Noussair and Yan [6] for $N \geqslant 5$ and for isolated blow-up points. They show that any concentrating solutions sequence with bounded energy cannot have only interior peaks and so at least one blow-up point must lie on $\partial \Omega$. At the same time, Rey in [20] gets the same result for $N=3$ by removing any assumption on the nature of interior blow-up points.

Using some techniques developed by Druet, Hebey and Vaugon in [12] for related problems on Riemannian manifolds, Castorina and Mancini in [7] were able to show, among other things, that the conclusion of previous papers holds without any restriction on the dimension. Namely, for $N \geqslant 3$ at least one blow-up point lies on $\partial \Omega$.

However, all these papers do not answer to the full question: do there exist blowing up solutions for (1) with bounded energy which do not remain bounded in the interior of $\Omega$ as $\lambda \rightarrow+\infty$ ? For $N>6$ the answer is negative since we will show that ALL the blow-up points have to lie on $\partial \Omega$ :

Theorem 1.1. Let $N>6$. Let $\lambda_{n} \rightarrow+\infty$ and $u_{n}$ be a solutions sequence of

$$
\begin{cases}-\Delta u_{n}+\lambda_{n} u_{n}=N(N-2) u_{n}^{\frac{N+2}{N-2}} & \text { in } \Omega  \tag{3}\\ u_{n}>0 & \text { in } \Omega \\ \frac{\partial u_{n}}{\partial n}=0 & \text { on } \partial \Omega\end{cases}
$$

with uniformly bounded energy:

$$
\sup _{n \in \mathbb{N}} \int_{\Omega} u_{n}^{\frac{2 N}{N-2}}<+\infty
$$

Then, for any $K$ compact set in $\Omega$ there exists $C_{K}$ such that:

$$
\max _{x \in K} u_{n}(x) \leqslant C_{K}
$$

for any $n \in \mathbb{N}$.

Theorem 1.1 is based on a local description of possible compactness loss and does not need any boundary condition. In fact, we realized that Theorem 1.1 is a particular case of a more general interior compactness result, which is still more interesting that our initial question about singularly perturbed Neumann equations and becomes the main content of this paper. There holds:

Theorem 1.2. Let $N>6$. Let $K$ be a compact set in $\Omega$ and $\Lambda>0$. There exists a constant $C$, depending on $K$ and $\Lambda$, such that any solution $u$ of the problem:

$$
\begin{cases}-\Delta u+u=N(N-2) u^{\frac{N+2}{N-2}} & \text { in } \Omega, \\ u>0 & \text { in } \Omega, \\ \int_{\Omega}|\nabla u|^{2}+u^{2} \leqslant \Lambda, & \end{cases}
$$

satisfies the bound:

$$
\max _{x \in K} u(x) \leqslant C .
$$

Compactness properties of the type we are considering appear in a Riemannian context in $[8,9]$ where a careful analysis based on the $C^{0}$-theory developed in $[10,11]$ for Riemannian manifolds gives the Schoen compactness result in low dimensions and provides also in high dimensions results as in Theorem 1.2. However, in this context (without homogeneous Dirichlet boundary condition) the $C^{0}$-theory developed by Druet, Hebey and Robert is not available.

The paper is organized in the following way. In Section 2 we introduce the notion of (geometrical) blow-up set, we give a description of this set and we show by a rescaling argument that Theorem 1.1 is a particular case of Theorem 1.2. In Section 3, we provide the proof of Theorem 1.2: based on a technical result contained in [10,11] due to Druet, Hebey and Robert (which we report in Appendix A for the sake of completeness), for any interior blowing up solutions sequence we are able to prove an upper estimate (in terms of standard bubbles) which contradicts a related local Pohozaev identity.

## 2. The blow-up set

Let $u_{n}$ be a solutions sequence of

$$
\begin{cases}-\Delta u_{n}+\mu_{n} u_{n}=N(N-2) u_{n}^{\frac{N+2}{N-2}} & \text { in } \Omega,  \tag{4}\\ u_{n}>0 & \text { in } \Omega, \\ \sup _{n \in \mathbb{N}} \int_{\Omega}\left(\left|\nabla u_{n}\right|^{2}+u_{n}^{2}\right)<+\infty, & \end{cases}
$$

where $\Omega$ is a domain in $\mathbb{R}^{N}, N \geqslant 3$, and $0 \leqslant \mu_{n} \rightarrow \mu \in[0,+\infty]$.
We define the (geometrical) blow-up set of $u_{n}$ in $\Omega$ as

$$
S=\left\{x \in \Omega: \exists x_{n} \rightarrow x \text { s.t. } \limsup _{n \rightarrow+\infty} u_{n}\left(x_{n}\right)=+\infty\right\},
$$

and, by definition of $S$, clearly $u_{n}$ is uniformly bounded in $C_{\mathrm{loc}}^{0}(\Omega \backslash S)$.
Further, define the set

$$
\Sigma_{c}=\left\{x \in \Omega: \limsup _{n \rightarrow+\infty} \int_{B_{r}(x)} u_{n}^{\frac{2 N}{N-2}} \geqslant c \forall r>0\right\},
$$

where $c>0$. Let $S_{N}$ be the best constant related to the immersion of $H_{0}^{1}(\Omega)$ into $L^{\frac{2 N}{N-2}}(\Omega)$ :

$$
\begin{equation*}
S_{N}=\inf _{u \in H_{0}^{1}(\Omega) \backslash\{0\}} \frac{\int_{\Omega}|\nabla u|^{2}}{\left(\int_{\Omega}|u|^{\frac{2 N}{N-2}}\right)^{\frac{N-2}{N}}} . \tag{5}
\end{equation*}
$$

By means of an iterative Moser-type scheme, we can describe the set $S$ in the following way:
Proposition 2.1. There exists $c=c_{N}>0$ such that it holds $S=\Sigma_{c}$. In particular, $S$ is a finite set and, if $\mu=+\infty$, we have that $u_{n} \rightarrow 0$ in $C_{\mathrm{loc}}^{0}(\Omega \backslash S)$ (up to a subsequence).

Proof. First of all, we show the following implication:

$$
\int_{B_{2 r}(x)} u_{n}^{\frac{2 N}{N-2}} \leqslant\left(\frac{S_{N}}{q N(N-2)}\right)^{\frac{N}{2}}, q \geqslant 2 \Rightarrow\left(\int_{B_{r}(x)} u_{n}^{\frac{N}{N-2} q}\right)^{\frac{N-2}{N}} \leqslant \frac{8}{S_{N} r^{2}} \int_{B_{2 r}(x)} u_{n}^{q}
$$

for any $r<\frac{1}{2} \operatorname{dist}\left(x, \mathbb{R}^{N} \backslash \Omega\right)$. Let $\varphi \in C_{0}^{\infty}\left(B_{2 r}(x)\right)$ be so that $0 \leqslant \varphi \leqslant 1, \varphi \equiv 1$ in $B_{r}(x)$ and $\|\nabla \varphi\|_{\infty} \leqslant \frac{2}{r}$. Multiplying (4) by $\varphi^{2} u_{n}^{q-1}$ and integrating by parts, by (5) and Hölder's inequality we get that:

$$
\begin{aligned}
\int_{\Omega} \nabla u_{n} \nabla\left(\varphi^{2} u_{n}^{q-1}\right)+\mu_{n} \int_{\Omega} \varphi^{2} u_{n}^{q} & =N(N-2) \int_{\Omega} u_{n}^{\frac{4}{N-2}}\left(\varphi u_{n}^{\frac{q}{2}}\right)^{2} \\
& \leqslant \frac{N(N-2)}{S_{N}}\left(\int_{B_{2 r}(x)} u_{n}^{\frac{2 N}{N-2}}\right)^{\frac{2}{N}} \int_{\Omega}\left|\nabla\left(\varphi u_{n}^{\frac{q}{2}}\right)\right|^{2} .
\end{aligned}
$$

On the other hand, we can write:

$$
\begin{aligned}
\int_{\Omega} \nabla u_{n} \nabla\left(\varphi^{2} u_{n}^{q-1}\right) & =(q-1) \int_{\Omega} \varphi^{2} u_{n}^{q-2}\left|\nabla u_{n}\right|^{2}+2 \int_{\Omega} \varphi u_{n}^{q-1} \nabla \varphi \nabla u_{n} \\
& =\frac{2}{q} \int_{\Omega}\left|\nabla\left(\varphi u_{n}^{\frac{q}{2}}\right)\right|^{2}+\frac{q-2}{2} \int_{\Omega} \varphi^{2} u_{n}^{q-2}\left|\nabla u_{n}\right|^{2}-\frac{2}{q} \int_{\Omega}|\nabla \varphi|^{2} u_{n}^{q} \\
& \geqslant \frac{2}{q} \int_{\Omega}\left|\nabla\left(\varphi u_{n}^{\frac{q}{2}}\right)\right|^{2}-\frac{2}{q} \int_{\Omega}|\nabla \varphi|^{2} u_{n}^{q}
\end{aligned}
$$

Combining these two estimates, we get that

$$
\begin{aligned}
\frac{2}{q} \int_{\Omega}\left|\nabla\left(\varphi u_{n}^{\frac{q}{2}}\right)\right|^{2} & \leqslant \frac{2}{q} \int_{\Omega}|\nabla \varphi|^{2} u_{n}^{q}+\frac{N(N-2)}{S_{N}}\left(\int_{B_{2 r}(x)} u_{n}^{\frac{2 N}{N-2}}\right)^{\frac{2}{N}} \int_{\Omega}\left|\nabla\left(\varphi u_{n}^{\frac{q}{2}}\right)\right|^{2} \\
& \leqslant \frac{8}{q r^{2}} \int_{B_{2 r}(x)} u_{n}^{q}+\frac{1}{q} \int_{\Omega}\left|\nabla\left(\varphi u_{n}^{\frac{q}{2}}\right)\right|^{2}
\end{aligned}
$$

in view of $\int_{B_{2 r}(x)} u_{n}^{\frac{2 N}{N-2}} \leqslant\left(\frac{S_{N}}{q N(N-2)}\right)^{\frac{N}{2}}$. Therefore, by (5) we obtain that

$$
\left(\int_{B_{r}(x)} u_{n}^{\frac{N}{N-2} q}\right)^{\frac{N-2}{N}} \leqslant\left(\int_{\Omega}\left(\varphi u_{n}^{\frac{q}{2}}\right)^{\frac{2 N}{N-2}}\right)^{\frac{N-2}{N}} \leqslant \frac{1}{S_{N}} \int_{\Omega}\left|\nabla\left(\varphi u_{n}^{\frac{q}{2}}\right)\right|^{2} \leqslant \frac{8}{S_{N} r^{2}} \int_{B_{2 r}(x)} u_{n}^{q} .
$$

Since $\frac{N}{N-2} q>q$, we can iterate the procedure starting from $q=2$ up to get a-priori bounds in $L^{p}$-norms around $x$ for any $p>2$ provided the $L^{\frac{2 N}{N-2}}$ norm around $x$ is sufficiently small. Namely, we find $0<\delta<1, p>\frac{N+2}{N-2} \frac{N}{2}$ and $c=c_{N}>0$, depending only on $N$, such that, if $\int_{B_{2 r}(x)} u_{n}^{\frac{2 N}{N-2}} \leqslant c$, then

$$
\left(\int_{B_{\delta r}(x)} u_{n}^{p}\right)^{\frac{2}{p}} \leqslant C(N, r) \int_{B_{2 r}(x)} u_{n}^{2},
$$

for some constant $C(N, r)$ depending only on $N$ and $r$. Let $u_{n}^{(1)}$ be the solution of

$$
\begin{cases}-\Delta u_{n}^{(1)}=N(N-2) u_{n}^{\frac{N+2}{N-2}} & \text { in } B_{\delta r}(x) \\ u_{n}^{(1)}=0 & \text { on } \partial B_{\delta r}(x),\end{cases}
$$

and $u_{n}^{(2)}$ be an harmonic function such that $u_{n}^{(2)}=u_{n}$ on $\partial B_{\delta r}(x)$. Since

$$
\left\|N(N-2) u_{n}^{\frac{N+2}{N-2}}\right\|_{L^{s}\left(B_{\delta r}(x)\right)}=\mathrm{O}\left(\left(\int_{B_{2 r}(x)} u_{n}^{2}\right)^{\frac{1}{2}}\right)
$$

for some $s>\frac{N}{2}$, by elliptic regularity theory (cf. [14]) we get that

$$
\left\|u_{n}^{(1)}\right\|_{C^{0}\left(B_{\delta r}(x)\right)}=\mathrm{O}\left(\left(\int_{B_{2 r}(x)} u_{n}^{2}\right)^{\frac{1}{2}}\right)
$$

provided $\int_{B_{2 r}(x)} u_{n}^{\frac{2 N}{N-2}} \leqslant c$. By the representation formula for harmonic function, we get that

$$
\left\|u_{n}^{(2)}\right\|_{C^{0}\left(B_{\delta r / 2}(x)\right)}=\mathrm{O}\left(\int_{\partial B_{\delta r}(x)} u_{n}\right) .
$$

Since by the maximum principle $0<u_{n} \leqslant u_{n}^{(1)}+u_{n}^{(2)}$, we get that

$$
\begin{equation*}
\left\|u_{n}\right\|_{C^{0}\left(B_{\delta r / 2}(x)\right)} \leqslant C\left(\left(\int_{B_{2 r}(x)} u_{n}^{2}\right)^{\frac{1}{2}}+\int_{\partial B_{\delta r}(x)} u_{n}\right) \tag{6}
\end{equation*}
$$

for some $C>0$.
By the continuous embedding of $H^{1}(\Omega)$ into $L^{\frac{2 N}{N-2}}(\Omega)$ we get that $\sup _{n \in \mathbb{N}} \int_{\Omega} u_{n}^{\frac{2 N}{N-2}}<+\infty$ and therefore, $\Sigma_{c}$ is a finite set, where $c=c_{N}$ is as above. Moreover, up to a subsequence we can assume that $u_{n} \rightharpoonup u$ weakly in $H^{1}(\Omega)$ and $u_{n} \rightarrow u$ in $L^{2}(\Omega)$, in view of the compact embedding of $H^{1}(\Omega)$ into $L^{2}(\Omega)$. Integrating (4) against $\varphi u_{n}, \varphi \in C_{0}^{\infty}(\Omega)$, we get that $\mu_{n} u_{n}^{2}$ is uniformly bounded in $L_{\mathrm{loc}}^{1}(\Omega)$ and hence, $u=0$ if $\mu=+\infty$.

By the compactness of the embedding of $H^{1}\left(B_{2 r}(x)\right)$ into $L^{2}\left(B_{2 r}(x)\right)$ and of $H^{1}\left(B_{\delta r}(x)\right)$ into $L^{1}\left(\partial B_{\delta r}(x)\right)$ in the sense of traces, in view of (6) we get that $S=\Sigma_{c_{N}}$ is a finite set and, if $\mu=+\infty, u_{n} \rightarrow 0$ in $C_{\text {loc }}^{0}(\Omega \backslash S)$ (up to a subsequence).

Remark 2.2. Blowing up the sequence $u_{n}$ around a point $x \in S$, by means of the same techniques which we will exploit strongly in Appendix A, it is easy to show that:

$$
\limsup _{n \rightarrow+\infty} \int_{B_{r}(x)} u_{n}^{\frac{2 N}{N-2}} \geqslant\left(\frac{S_{N}}{N(N-2)}\right)^{\frac{N}{2}}
$$

for any $r>0$. Hence, the value $c=c_{N}$ in Proposition 2.1 can be taken as $c=\left(\frac{S_{N}}{N(N-2)}\right)^{\frac{N}{2}}$.
We are now in position to deduce Theorem 1.1 by Theorem 1.2.
Proof of Theorem 1.1. Multiplying (3) by $u_{n}$ and integrating by parts, we get that:

$$
\begin{equation*}
\int_{\Omega}\left(\left|\nabla u_{n}\right|^{2}+\lambda_{n} u_{n}^{2}\right) \leqslant \Lambda:=N(N-2) \sup _{n \in \mathbb{N}} \int_{\Omega} u_{n}^{\frac{2 N}{N-2}}<+\infty . \tag{7}
\end{equation*}
$$

We can define the blow-up set $S$ of the sequence $u_{n}$. By the validity of Theorem 1.2 , we deduce that $S$ has to be an empty set and therefore, $u_{n}$ is uniformly bounded in $C_{\text {loc }}^{0}(\Omega)$.

Otherwise, if $S \neq \emptyset$, up to a subsequence, we can assume that there exists $x_{0} \in S$ such that $\max _{x \in B_{r}\left(x_{0}\right)} u_{n}(x) \rightarrow$ $+\infty$ as $n \rightarrow+\infty$, for any $r>0$. By Proposition 2.1, we know that $S$ is a finite set. Let $0<r<\operatorname{dist}\left(x_{0}, S \backslash\left\{x_{0}\right\}\right)$ and $x_{n}$ be such that $u_{n}\left(x_{n}\right)=\max _{x \in B_{r}\left(x_{0}\right)} u_{n}(x) \rightarrow+\infty$ as $n \rightarrow+\infty$. Clearly, since $u_{n}$ is uniformly bounded in $C_{\mathrm{loc}}^{0}(\Omega \backslash S), x_{n} \rightarrow x_{0}$ as $n \rightarrow+\infty$.

Introduce $\varepsilon_{n}=u_{n}\left(x_{n}\right)^{-\frac{2}{N-2}} \rightarrow 0$ and define $U_{n}(y)=\varepsilon_{n}^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n} y+x_{n}\right)$ for $y \in B_{n}:=B_{\frac{r}{2 \varepsilon_{n}}}(0)$. We have that

$$
\left\{\begin{array}{l}
-\Delta U_{n}+\mu_{n} U_{n}=N(N-2) U_{n}^{\frac{N+2}{N-2}} \quad \text { in } B_{n} \\
0<U_{n}(y) \leqslant U_{n}(0)=1,
\end{array}\right.
$$

where $\mu_{n}=\lambda_{n} \varepsilon_{n}^{2}$. Assume that $\mu_{n}=\lambda_{n} \varepsilon_{n}^{2} \rightarrow \mu \in[0,+\infty]$. Since $U_{n}$ is uniformly bounded in $H_{\mathrm{loc}}^{1}\left(\mathbb{R}^{N}\right)$ and in $C_{\text {loc }}^{0}\left(\mathbb{R}^{N}\right)$, if $\mu=+\infty$, Proposition 2.1 implies that $U_{n} \rightarrow 0$ in $C_{\mathrm{loc}}^{0}\left(\mathbb{R}^{N}\right)$ (up to a subsequence) contradicting $U_{n}(0)=1$.

So, $\mu<+\infty$. By standard elliptic estimates (cf. [14]), we have that $U_{n} \rightarrow U$ in $C_{\text {loc }}^{2}\left(\mathbb{R}^{N}\right)$ where $U \in H^{1}\left(\mathbb{R}^{N}\right)$ is a solution of

$$
\left\{\begin{array}{l}
-\Delta U+\mu U=N(N-2) U^{\frac{N+2}{N-2}} \quad \text { in } \mathbb{R}^{N}  \tag{8}\\
0<U(y) \leqslant U(0)=1
\end{array}\right.
$$

(in view of (7)). By a Pohozaev identity on $\mathbb{R}^{N}$ (see [19]), we must have that

$$
\mu_{n}=\lambda_{n} \varepsilon_{n}^{2} \rightarrow \mu=0
$$

Now, we do the following rescaling. Let $v_{n}(x)=\lambda_{n}^{-\frac{N-2}{4}} u_{n}\left(x / \sqrt{\lambda_{n}}+x_{n}\right)$ be defined for $x \in B_{1}(0)$. The function $v_{n}$ satisfies:

$$
\begin{cases}-\Delta v_{n}+v_{n}=N(N-2) v_{n}^{\frac{N+2}{N-2}} & \text { in } B_{1}(0) \\ v_{n}>0 & \text { in } B_{1}(0) \\ \int_{B_{1}(0)}\left(\left|\nabla v_{n}\right|^{2}+v_{n}^{2}\right) \leqslant \Lambda, & \end{cases}
$$

since

$$
\int_{B_{1}(0)}\left(\left|\nabla v_{n}\right|^{2}+v_{n}^{2}\right)=\int_{B_{1 / \sqrt{\lambda_{n}}}\left(x_{n}\right)}\left(\left|\nabla u_{n}\right|^{2}+\lambda_{n} u_{n}^{2}\right) \leqslant \int_{\Omega}\left(\left|\nabla u_{n}\right|^{2}+\lambda_{n} u_{n}^{2}\right) \leqslant \Lambda
$$

By Theorem 1.2 we get that there exists $C>0$ such that

$$
\max _{x \in B_{1 / 2}(0)} v_{n}(x) \leqslant C
$$

So, we reach a contradiction since we have already shown that

$$
v_{n}(0)=\lambda_{n}^{-\frac{N-2}{4}} u_{n}\left(x_{n}\right)=\left(\frac{1}{\lambda_{n} \varepsilon_{n}^{2}}\right)^{\frac{N-2}{4}} \rightarrow+\infty
$$

as $n \rightarrow+\infty$. The proof is now complete.

## 3. Nonexistence of interior blow-up points

The proof of Theorem 1.2 is based on a contradiction argument. In view of Proposition 2.1, let us assume the existence of a solutions sequence $u_{n}$ of the following problem:

$$
\begin{cases}-\Delta u_{n}+u_{n}=N(N-2) u_{n}^{\frac{N+2}{N-2}} & \text { in } B_{1}(0) \\ u_{n}>0 & \text { in } B_{1}(0) \\ \sup _{n \in \mathbb{N}} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}}<+\infty\end{cases}
$$

which blows up in $B_{1}(0)$ only at $0: \max _{x \in B_{1}(0)} u_{n}(x) \rightarrow+\infty$ as $n \rightarrow+\infty$ and $u_{n}$ is uniformly bounded in $C_{\text {loc }}^{0}\left(B_{1}(0) \backslash\{0\}\right)$.

By means of Propositions A.1, A. 2 and by elliptic regularity theory (cf. [14]), up to a subsequence, we will assume throughout this section the existence of sequences $x_{n}^{1}, \ldots, x_{n}^{k} \rightarrow 0, \varepsilon_{n}^{1}, \ldots, \varepsilon_{n}^{k} \rightarrow 0$ and $x_{1}, \ldots, x_{k} \in \mathbb{R}^{N}$ such that for any $i=1, \ldots, k$ :

$$
\begin{align*}
& U_{n}^{i}(y)=\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n}^{i} y+x_{n}^{i}\right) \rightarrow \frac{1}{\left(1+\left|y-x_{i}\right|^{2}\right)^{\frac{N-2}{2}}} \quad \text { in } C_{\mathrm{loc}}^{2}\left(\mathbb{R}^{N} \backslash S_{i}\right) \text { as } n \rightarrow+\infty  \tag{9}\\
& u_{n} \rightarrow u_{0} \quad \text { in } C_{\mathrm{loc}}^{0}\left(B_{1}(0) \backslash\{0\}\right) \quad \text { as } n \rightarrow+\infty \tag{10}
\end{align*}
$$

$$
\begin{align*}
& d_{k}(x)^{\frac{N-2}{2}} u_{n}(x) \leqslant C \quad \text { for any } n \in \mathbb{N},|x|<1  \tag{11}\\
& \lim _{R \rightarrow+\infty} \limsup _{n \rightarrow+\infty} \max _{x \in B_{R}^{n}}\left(d_{k}(x)^{\frac{N-2}{2}}\left|u_{n}(x)-u_{0}(x)\right|\right)=0 \tag{12}
\end{align*}
$$

for some constant $C>0$ and for some smooth solution $u_{0} \geqslant 0$ of the equation:

$$
-\Delta u_{0}+u_{0}=N(N-2) u_{0}^{\frac{N+2}{N-2}} \quad \text { in } B_{1}(0),
$$

where $d_{k}(x)=\min \left\{\left|x-x_{n}^{i}\right|: i=1, \ldots, k\right\}, B_{R}^{n}=\left\{|x|<1:\left|x-x_{n}^{i}\right| \geqslant R \varepsilon_{n}^{i} \forall i=1, \ldots, k\right\}$ and

$$
S_{i}=\left\{y_{j}=\lim _{n \rightarrow+\infty} \frac{x_{n}^{j}-x_{n}^{i}}{\varepsilon_{n}^{i}}: j<i \text { s.t. } \frac{\left|x_{n}^{j}-x_{n}^{i}\right|}{\varepsilon_{n}^{i}}=\mathrm{O}(1)\right\} .
$$

Let now $x$ be so that $\left|x-x_{n}^{i}\right|=R \varepsilon_{n}^{i}$ for some $i=1, \ldots, k$. We have that $y=\frac{x-x_{n}^{i}}{\varepsilon_{n}^{i}}$ satisfies: $|y|=R$ and $\left|y-y_{j}\right| \geqslant$ $R-\left|y_{j}\right| \geqslant 1$, for $R$ large. Hence, by (9) we get that for any $R>0$ large and $C>1$ there exists $N_{0}$ such that for any $n \geqslant N_{0}$ and $\left|x-x_{n}^{i}\right|=R \varepsilon_{n}^{i}$

$$
u_{n}(x) \leqslant C U_{\varepsilon_{n}^{i}, x_{n}^{i}+\varepsilon_{n}^{i} x_{i}}(x),
$$

where

$$
U_{\varepsilon, y}(x)=\varepsilon^{-\frac{N-2}{2}} U\left(\frac{x-y}{\varepsilon}\right)=\frac{\varepsilon^{\frac{N-2}{2}}}{\left(\varepsilon^{2}+|x-y|^{2}\right)^{\frac{N-2}{2}}}
$$

Since $\left|x-\left(x_{n}^{i}+\varepsilon_{n}^{i} x_{i}\right)\right| \geqslant\left(1-\frac{\max _{i=1, \ldots, k}\left|x_{i}\right|}{R}\right)\left|x-x_{n}^{i}\right|$ for $\left|x-x_{n}^{i}\right|=R \varepsilon_{n}^{i}$, we obtain that for any $R>0$ large there exists $N_{0}$ such that

$$
\begin{equation*}
u_{n}(x) \leqslant 2 U_{\varepsilon_{n}^{i}, x_{n}^{i}}(x) \tag{13}
\end{equation*}
$$

for any $n \geqslant N_{0}$ and $\left|x-x_{n}^{i}\right|=R \varepsilon_{n}^{i}$.
The aim will be to estimate from above $u_{n}(x)$ in terms of the standard bubbles $U_{\varepsilon_{n}^{i}, x_{n}^{i}}(x), i=1, \ldots, k$, in $B_{\delta}(0) \backslash$ $\bigcup_{i=1}^{k} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right), 0<\delta<1$. By performing some simple asymptotic analysis we get the following result (see also the techniques developed by Schoen in [22] and exploited in [15,16]):

Lemma 3.1. Let $\alpha \in\left(0, \frac{N-2}{2}\right)$. There exist $R>0,0<\delta<1$ and $N_{0} \in \mathbb{N}$ such that

$$
u_{n}(x) \leqslant \sum_{i=1}^{k}\left(\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}-\alpha}\left|x-x_{n}^{i}\right|^{2-N+\alpha}+M_{n}\left|x-x_{n}^{i}\right|^{-\alpha}\right)
$$

for any $n \geqslant N_{0}$ and $|x| \leqslant \delta$ with $\left|x-x_{n}^{i}\right| \geqslant R \varepsilon_{n}^{i}, i=1, \ldots, k$, where $M_{n}=2 \delta^{\alpha} \sup _{|x|=\delta} u_{n}(x)$.
Proof. Let us introduce the operator $L_{n}=-\Delta+1-N(N-2) u_{n}^{\frac{4}{N-2}}$. Since $u_{n}$ is a positive solution of $L_{n} u_{n}=0$ in $B_{\delta}(0)$, we have that $L_{n}$ satisfies the minimum principle in $B_{\delta}(0)$ for any $0<\delta<1$ (see [14]). Since $u_{0}$ is a smooth function, by (12) we have that there exist $R>2^{\frac{1}{\alpha}}, 0<\delta<1$ and $N_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
d_{k}(x)^{\frac{N-2}{2}} u_{n}(x) \leqslant\left(\frac{\alpha(N-2-\alpha)}{k N(N-2)}\right)^{\frac{N-2}{4}} \tag{14}
\end{equation*}
$$

for any $n \geqslant N_{0}$ and $x \in B_{\delta}(0):\left|x-x_{n}^{i}\right| \geqslant R \varepsilon_{n}^{i}, i=1, \ldots, k$.
Define now a comparison function $\varphi_{n}$ in the form $\varphi_{n}=\sum_{i=1}^{k} \varphi_{n}^{i}$, where

$$
\varphi_{n}^{i}(x)=\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}-\alpha}\left|x-x_{n}^{i}\right|^{2-N+\alpha}+M_{n}\left|x-x_{n}^{i}\right|^{-\alpha},
$$

and compute $L_{n}$ on $\varphi_{n}-u_{n}$ :

$$
L_{n}\left(\varphi_{n}-u_{n}\right)=\sum_{i=1}^{k} L_{n} \varphi_{n}^{i}=\sum_{i=1}^{k}\left(\alpha(N-2-\alpha)\left|x-x_{n}^{i}\right|^{-2}+1-N(N-2) u_{n}^{\frac{4}{N-2}}\right) \varphi_{n}^{i}
$$

Let $x \in B_{\delta}(0)$ be such that $\left|x-x_{n}^{i}\right| \geqslant R \varepsilon_{n}^{i}, i=1, \ldots, k$. There exists $j \in\{1, \ldots, k\}$ so that $\left|x-x_{n}^{j}\right|=\min \left\{\mid x-x_{n}^{i}\right.$ : $i=\ldots, k\}$. Since $\left|x-x_{n}^{j}\right| \leqslant\left|x-x_{n}^{i}\right|$, we have that $\varphi_{n}^{j}(x) \geqslant \varphi_{n}^{i}(x)$ for any $i=1, \ldots, k$ and therefore,

$$
\begin{aligned}
L_{n}\left(\varphi_{n}-u_{n}\right)(x)= & \sum_{i=1}^{k}\left(\alpha(N-2-\alpha)\left|x-x_{n}^{i}\right|^{-2}+1-N(N-2) u_{n}(x)^{\frac{4}{N-2}}\right) \varphi_{n}^{i}(x) \\
& \geqslant \alpha(N-2-\alpha)\left|x-x_{n}^{j}\right|^{-2} \varphi_{n}^{j}(x)-N(N-2) u_{n}(x)^{\frac{4}{N-2}} \sum_{i=1}^{k} \varphi_{n}^{i}(x) \\
& \geqslant\left[\alpha(N-2-\alpha)-k N(N-2)\left(\left|x-x_{n}^{j}\right|^{\frac{N-2}{2}} u_{n}(x)\right)^{\frac{4}{N-2}}\right]\left|x-x_{n}^{j}\right|^{-2} \varphi_{n}^{j}(x) \geqslant 0
\end{aligned}
$$

in view of (14), for any $n \geqslant N_{0}$ and $x \in B_{\delta}(0):\left|x-x_{n}^{i}\right| \geqslant R \varepsilon_{n}^{i}, i=1, \ldots, k$. In view of the validity of (13) on $\partial B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right)$, we can always assume that $R$ and $N_{0}$ are such that

$$
u_{n}(x) \leqslant 2\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}}\left|x-x_{n}^{i}\right|^{2-N}
$$

for any $n \geqslant N_{0}$ and $\left|x-x_{n}^{i}\right|=R \varepsilon_{n}^{i}$. Therefore, we have that

$$
u_{n}(x) \leqslant 2 R^{-\alpha}\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}-\alpha}\left|x-x_{n}^{i}\right|^{2-N+\alpha} \leqslant\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}-\alpha}\left|x-x_{n}^{i}\right|^{2-N+\alpha} \leqslant \varphi_{n}^{i}(x) \leqslant \varphi_{n}(x)
$$

for $n \geqslant N_{0}$ and $\left|x-x_{n}^{i}\right|=R \varepsilon_{n}^{i}$ for some $i=1, \ldots, k$. Since

$$
u_{n}(x) \leqslant \frac{1}{2 \delta^{\alpha}} M_{n} \leqslant M_{n} \sum_{i=1}^{k}\left|x-x_{n}^{i}\right|^{-\alpha} \leqslant \varphi_{n}(x)
$$

for $|x|=\delta$ and $n$ large, by the minimum principle for $L_{n}$ we get the desired estimate in the region $x \in B_{\delta}(0)$ with $\left|x-x_{n}^{i}\right| \geqslant R \varepsilon_{n}^{i} \forall i=1, \ldots, k$.

We have to combine the estimate contained in Lemma 3.1 with the following Pohozaev-type inequality ("essentially" proved in [7], for the Pohozaev identity refer to [19]):

Lemma 3.2. There exists $C>0$, depending only on the dimension $N$, such that for any $|x|<1$ and $0<h<\frac{1-|x|}{4}$

$$
\begin{equation*}
\int_{B_{h}(x)} u_{n}^{2} \leqslant C \int_{B_{2 h}(x) \backslash B_{h}(x)}\left(\frac{1}{h^{2}} u_{n}^{2}+u_{n}^{\frac{2 N}{N-2}}\right) . \tag{15}
\end{equation*}
$$

Proof. Since Lemma 3.2 is written in a slightly different way with respect to [7], let us outline why some difference appears. By [7] we get that:

$$
\int_{B_{h}(x)} u_{n}^{2} \leqslant-\frac{2}{N} \int_{\Omega}\langle x, \nabla \varphi\rangle \varphi u_{n}^{\frac{2 N}{N-2}}+R_{n}
$$

where $\varphi \in C_{0}^{\infty}\left(B_{2 h}(x)\right)$ is such that $0 \leqslant \varphi \leqslant 1, \varphi=1$ on $B_{h}(x)$, and

$$
R_{n}=-\int_{\Omega} u_{n}^{2}\left[\langle\nabla \varphi, \nabla\langle x, \nabla \varphi\rangle\rangle+\frac{N}{2} \varphi \Delta \varphi+\frac{1}{2}\langle x, \nabla(\varphi \Delta \varphi)\rangle+\frac{N-2}{2}|\nabla \varphi|^{2}\right] .
$$

Assuming that $|\nabla \varphi| \leqslant \frac{2}{h}$, we have that $\langle x, \nabla \varphi\rangle \varphi$ vanishes outside $B_{2 h}(x) \backslash B_{h}(x)$ and

$$
|\langle x, \nabla \varphi\rangle \varphi| \leqslant 4
$$

in $B_{2 h}(x) \backslash B_{h}(x)$. Hence, we get that

$$
\left|\int_{\Omega}\langle x, \nabla \varphi\rangle \varphi u_{n}^{\frac{2 N}{N-2}}\right| \leqslant 4 \int_{B_{2 h}(x) \backslash B_{h}(x)} u_{n}^{\frac{2 N}{N-2}} .
$$

Similarly, assuming that $\left|\nabla^{2} \varphi\right| \leqslant \frac{2}{h^{2}}$ we show that

$$
\left|R_{n}\right| \leqslant \frac{C}{h^{2}} \int_{B_{2 h}(x) \backslash B_{h}(x)} u_{n}^{2}
$$

for some constant $C>0$. The proof is now complete.
Let $N>6$ and fix $0<\alpha<\frac{N-6}{3}$. Let us define the following sequence:

$$
r_{n}=\max \left\{\varepsilon_{n}^{i}: i=\ldots, k\right\}^{\frac{2+\alpha}{N-2-2 \alpha}} .
$$

Up to a subsequence and a re-labeling, let us assume that $\varepsilon_{n}^{k}=\max \left\{\varepsilon_{n}^{i}: i=\ldots, k\right\}$ and that, for some integer $s \in\{1, \ldots, k-1\}$, there hold:

$$
\begin{equation*}
\frac{\left|x_{n}^{i}-x_{n}^{k}\right|}{r_{n}} \rightarrow+\infty, \quad i=1, \ldots, s, \quad \frac{\left|x_{n}^{i}-x_{n}^{k}\right|}{r_{n}} \leqslant D-1, \quad i=s+1, \ldots, k \tag{16}
\end{equation*}
$$

as $n \rightarrow+\infty$, where $D>1$ is a constant. By (16), we obtain that for $x \in B_{2 D r_{n}}\left(x_{n}^{k}\right) \backslash B_{D r_{n}}\left(x_{n}^{k}\right)$ there hold:

$$
\begin{cases}\left|x-x_{n}^{i}\right| \geqslant\left|x_{n}^{i}-x_{n}^{k}\right|-\left|x-x_{n}^{k}\right| \geqslant\left|x_{n}^{i}-x_{n}^{k}\right|-2 D r_{n} \geqslant r_{n} & \text { if } i=1, \ldots, s  \tag{17}\\ \left|x-x_{n}^{i}\right| \geqslant\left|x-x_{n}^{k}\right|-\left|x_{n}^{k}-x_{n}^{i}\right| \geqslant D r_{n}-(D-1) r_{n}=r_{n} & \text { if } i=s+1, \ldots, k .\end{cases}
$$

We apply now (15) on $B_{D r_{n}}\left(x_{n}^{k}\right)$. Let $r=\frac{1}{2} \min \left\{\left|y_{j}\right|: y_{j} \in S_{k}, y_{j} \neq 0\right\}$ if $S_{k} \backslash\{0\} \neq \emptyset$ and $r=1$ otherwise. Since $r_{n} \gg \varepsilon_{n}^{k}$ for $n$ large in view of $\alpha<\frac{N-4}{3}$, we have that

$$
\int_{B_{D r_{n}}\left(x_{n}^{k}\right)} u_{n}^{2} \geqslant \int_{B_{r \varepsilon_{n}^{k}}\left(x_{n}^{k}\right)} u_{n}^{2}=\left(\varepsilon_{n}^{k}\right)^{2} \int_{B_{r}(0)}\left(U_{n}^{k}\right)^{2} \geqslant\left(\varepsilon_{n}^{k}\right)^{2} \int_{B_{r}(0) \backslash B_{r / 2}(0)}\left(U_{n}^{k}\right)^{2} .
$$

Since $S_{k} \cap\left\{\frac{r}{2} \leqslant|y| \leqslant r\right\}=\emptyset$, by (9) we get that

$$
\int_{B_{r}(0) \backslash B_{r / 2}(0)}\left(U_{n}^{k}\right)^{2} \rightarrow \int_{B_{r}(0) \backslash B_{r / 2}(0)} \frac{1}{\left(1+\left|y-x_{k}\right|^{2}\right)^{N-2}}>0
$$

and hence,

$$
\int_{B_{D r_{n}}\left(x_{n}^{k}\right)} u_{n}^{2} \geqslant \frac{1}{2}\left(\varepsilon_{n}^{k}\right)^{2} \int_{B_{r}(0) \backslash B_{r / 2}(0)} \frac{1}{\left(1+\left|y-x_{k}\right|^{2}\right)^{N-2}}
$$

for $n$ large. Let us remark that for $n$ large $\frac{1}{4 D} \delta>r_{n} \geqslant R \varepsilon_{n}^{k} \geqslant R \varepsilon_{n}^{i}$ for any $i=1, \ldots, k$. Therefore, we can use Lemma 3.1 and (17) to provide that

$$
\begin{equation*}
u_{n}(x) \leqslant \sum_{i=1}^{k}\left(\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}-\alpha}\left|x-x_{n}^{i}\right|^{2-N+\alpha}+M_{n}\left|x-x_{n}^{i}\right|^{-\alpha}\right) \leqslant k\left(\varepsilon_{n}^{k}\right)^{\frac{N-2}{2}-\alpha} r_{n}^{2-N+\alpha}+k M_{n} r_{n}^{-\alpha}, \tag{18}
\end{equation*}
$$

for any $n \geqslant N_{0}$ and $x \in B_{2 D r_{n}}\left(x_{n}^{k}\right) \backslash B_{D r_{n}}\left(x_{n}^{k}\right)$. Hence, by (18) we get that

$$
\begin{aligned}
& \quad \int_{B_{2 D r_{n}}\left(x_{n}^{k}\right) \backslash B_{D r_{n}}\left(x_{n}^{k}\right)}\left(\frac{1}{D^{2} r_{n}^{2}} u_{n}^{2}+u_{n}^{\frac{2 N}{N-2}}\right) \\
& \leqslant C\left(\left(\frac{\varepsilon_{n}^{k}}{r_{n}}\right)^{N-2-2 \alpha}+M_{n}^{2} r_{n}^{N-2-2 \alpha}+\left(\frac{\varepsilon_{n}^{k}}{r_{n}}\right)^{N-\alpha \frac{2 N}{N-2}}+M_{n}^{\frac{2 N}{N-2}} r_{n}^{N-\alpha \frac{2 N}{N-2}}\right) \\
& \leqslant C^{\prime}\left(\left(\frac{\varepsilon_{n}^{k}}{r_{n}}\right)^{N-2-2 \alpha}+M_{n}^{2} r_{n}^{N-2-2 \alpha}\right)=C^{\prime}\left(\left(\varepsilon_{n}^{k}\right)^{N-4-3 \alpha}+M_{n}^{2}\left(\varepsilon_{n}^{k}\right)^{2+\alpha}\right)
\end{aligned}
$$

since $\alpha<\frac{N-2}{2}$ and $M_{n}$ is bounded, in view of (10). Finally, by Lemma 3.2 we obtain that

$$
1 \leqslant C\left(\left(\varepsilon_{n}^{k}\right)^{N-6-3 \alpha}+\left(\varepsilon_{n}^{k}\right)^{\alpha}\right)
$$

which gives a contradiction for $n$ large, since $\varepsilon_{n}^{k} \rightarrow 0$ as $n \rightarrow+\infty$.

## Appendix A

In this appendix, we want to give a detailed local description of the blow-up phenomenon for a solutions sequence $u_{n}$ of the following problem:

$$
\begin{cases}-\Delta u_{n}+u_{n}=N(N-2) u_{n}^{\frac{N+2}{N-2}} & \text { in } B_{1}(0) \\ u_{n}>0 & \text { in } B_{1}(0), \\ \sup _{n \in \mathbb{N}} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}}<+\infty & \end{cases}
$$

We assume that 0 is the only blow-up point of $u_{n}$ in $B_{1}(0)$ :

$$
\begin{align*}
& \max _{x \in B_{1}(0)} u_{n}(x) \rightarrow+\infty \quad \text { as } n \rightarrow+\infty,  \tag{19}\\
& u_{n} \rightarrow u_{0} \quad \text { in } C_{\mathrm{loc}}^{0}\left(B_{1}(0) \backslash\{0\}\right) \tag{20}
\end{align*}
$$

(up to a subsequence), where $u_{0} \geqslant 0$ is a smooth solution of the limit equation:

$$
-\Delta u_{0}+u_{0}=N(N-2) u_{0}^{\frac{N+2}{N-2}} \text { in } B_{1}(0) .
$$

First of all, the following classical result holds (see for example [10,11] and [4,22]):
Proposition A.1. Up to a subsequence of $u_{n}$, there exist $s \in \mathbb{N}^{*}$ and sequences $x_{n}^{1}, \ldots, x_{n}^{s} \rightarrow 0$ such that for any $i, j=1, \ldots, s, i \neq j$, we have:

$$
\begin{align*}
& \varepsilon_{n}^{i}:=u_{n}\left(x_{n}^{i}\right)^{-\frac{2}{N-2}} \rightarrow 0, \quad \frac{\varepsilon_{n}^{i}+\varepsilon_{n}^{j}}{\left|x_{n}^{i}-x_{n}^{j}\right|} \rightarrow 0 \quad \text { as } n \rightarrow+\infty,  \tag{21}\\
& u_{n}\left(x_{n}^{i}\right)=\max _{x \in B_{\varepsilon_{n}}\left(x_{n}^{i}\right)} u_{n}(x) \text { forn large, }  \tag{22}\\
& U_{n}^{i}(y)=\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n}^{i} y+x_{n}^{i}\right) \rightarrow \frac{1}{\left(1+|y|^{2}\right)^{\frac{N-2}{2}}} \text { in } C_{\mathrm{loc}}^{2}\left(\mathbb{R}^{N}\right) \text { as } n \rightarrow+\infty,  \tag{23}\\
& d_{s}(x)^{\frac{N-2}{2}} u_{n}(x) \leqslant C \quad \text { for any } n \in \mathbb{N},|x|<1, \tag{24}
\end{align*}
$$

where $d_{j}(x)=\min \left\{\left|x-x_{n}^{i}\right|: i=1, \ldots, j\right\}, 1 \leqslant j \leqslant s$. In particular, there holds

$$
\begin{equation*}
\liminf _{n \rightarrow+\infty} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} \geqslant s\left(\frac{S_{N}}{N(N-2)}\right)^{\frac{N}{2}}+\int_{B_{1}(0)} u_{0}^{\frac{2 N}{N-2}} \tag{25}
\end{equation*}
$$

where $S_{N}$ is the Sobolev constant.

Proof. Let $x_{n}^{1}$ be the maximum point of $u_{n}$ in $B_{1}(0)$. By (19), (20) we deduce that $x_{n}^{1} \rightarrow 0$ and $\varepsilon_{n}^{1}=u_{n}\left(x_{n}^{1}\right)^{-\frac{2}{N-2}} \rightarrow 0$. Define $U_{n}^{1}(y)=\left(\varepsilon_{n}^{1}\right)^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n}^{1} y+x_{n}^{1}\right)$ for $y \in B_{n}:=B_{\left(\varepsilon_{n}^{1}\right)^{-1}}\left(-x_{n}^{1} / \varepsilon_{n}^{1}\right)$. We have that

$$
\left\{\begin{array}{l}
-\Delta U_{n}^{1}+\left(\varepsilon_{n}^{1}\right)^{2} U_{n}^{1}=N(N-2)\left(U_{n}^{1}\right)^{\frac{N+2}{N-2}} \text { in } B_{n}, \\
0<U_{n}^{1}(y) \leqslant U_{n}^{1}(0)=1
\end{array}\right.
$$

By standard elliptic estimates (cf. [14]) and a diagonal process, (up to a subsequence) we have that $U_{n}^{1} \rightarrow U$ in $C_{\mathrm{loc}}^{2}\left(\mathbb{R}^{N}\right)$ where $U$ is a solution of

$$
\left\{\begin{array}{l}
-\Delta U=N(N-2) U^{\frac{N+2}{N-2}} \quad \text { in } \mathbb{R}^{N}  \tag{26}\\
0<U(y) \leqslant U(0)=1
\end{array}\right.
$$

By the classification in [5], problem (26) admits only the solution

$$
U(y)=\frac{1}{\left(1+|y|^{2}\right)^{\frac{N-2}{2}}} .
$$

Hence, we get that

$$
U_{n}^{1}(y) \rightarrow \frac{1}{\left(1+|y|^{2}\right)^{\frac{N-2}{2}}} \quad \text { in } C_{\mathrm{loc}}^{2}\left(\mathbb{R}^{N}\right)
$$

Moreover, by taking the liminf as $n \rightarrow+\infty$ and the limit as $\delta \rightarrow 0, R \rightarrow+\infty$ in the following inequality chain:

$$
\int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} \geqslant \int_{B_{R \varepsilon_{n}^{1}}^{\left(\left(x_{n}^{1}\right)\right.}} u_{n}^{\frac{2 N}{N-2}}+\int_{B_{1}(0) \backslash B_{\delta}(0)} u_{n}^{\frac{2 N}{N-2}}=\int_{B_{R}(0)}\left(U_{n}^{1}\right)^{\frac{2 N}{N-2}}+\int_{B_{1}(0) \backslash B_{\delta}(0)} u_{n}^{\frac{2 N}{N-2}},
$$

which is true for $\delta, R>0$ and $n \geqslant N=N(\delta, R)$, we deduce that

$$
\liminf _{n \rightarrow+\infty} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} \geqslant \int_{\mathbb{R}^{N}} \frac{d y}{\left(1+|y|^{2}\right)^{N}}+\int_{B_{1}(0)} u_{0}^{\frac{2 N}{N-2}}=\left(\frac{S_{N}}{N(N-2)}\right)^{\frac{N}{2}}+\int_{B_{1}(0)} u_{0}^{\frac{2 N}{N-2}} .
$$

If (24) holds true for $x_{n}^{1}$, we take $s=1$ and the proof is complete since (21)-(23) are already verified. Otherwise, by induction assume that there exist $j$ sequences $x_{n}^{1}, \ldots, x_{n}^{j} \rightarrow 0$ as $n \rightarrow+\infty$ satisfying (21)-(23). If (24) holds true, we have done.

Otherwise, there exists a sequence $y_{n} \in B_{1}(0)$ such that

$$
\begin{equation*}
d_{j}\left(y_{n}\right)^{\frac{N-2}{2}} u_{n}\left(y_{n}\right)=\max _{x \in B_{1}(0)} d_{j}(x)^{\frac{N-2}{2}} u_{n}(x) \rightarrow+\infty \quad \text { as } n \rightarrow+\infty . \tag{27}
\end{equation*}
$$

Since $u_{n} \rightarrow u_{0}$ away from 0 , by (27) we have that $y_{n} \rightarrow 0$ and, for $\mu_{n}=u_{n}\left(y_{n}\right)^{-\frac{2}{N-2}}$, property (27) reads equivalently as:

$$
\begin{equation*}
\frac{\mu_{n}}{d_{j}\left(y_{n}\right)} \rightarrow 0 \quad \text { as } n \rightarrow+\infty \tag{28}
\end{equation*}
$$

Since (21)-(23) imply $d_{j}(x)^{\frac{N-2}{2}} u_{n}(x) \leqslant 1$ for $\left|x-x_{n}^{i}\right| \leqslant R \varepsilon_{n}^{i}, i=1, \ldots, j$, and $n \geqslant N=N(R)$, we deduce that $y_{n}$ must be outside this region, i.e.

$$
\begin{equation*}
\frac{\varepsilon_{n}^{i}}{\left|y_{n}-x_{n}^{i}\right|} \rightarrow 0 \quad \text { as } n \rightarrow+\infty, \forall i=1, \ldots, j \tag{29}
\end{equation*}
$$

Introduce the function $U_{n}(y)=\mu_{n}^{\frac{N-2}{2}} u_{n}\left(\mu_{n} y+y_{n}\right)$ for $y \in B_{n}$, where the balls $B_{n}=\left\{y \in \mathbb{R}^{N}:|y| \leqslant \frac{d_{j}\left(y_{n}\right)}{2 \mu_{n}}\right\}$ expand to $\mathbb{R}^{N}$ as $n \rightarrow+\infty$ in view of (28). Moreover, since

$$
\left|\mu_{n} y+y_{n}-x_{n}^{i}\right| \geqslant\left|y_{n}-x_{n}^{i}\right|-\mu_{n}|y| \geqslant \frac{1}{2}\left|y_{n}-x_{n}^{i}\right|
$$

for any $y \in B_{n}$ and $i=1, \ldots, j$, we have the following estimate:

$$
U_{n}(y)=\left(\frac{d_{j}\left(y_{n}\right)}{d_{j}\left(\mu_{n} y+y_{n}\right)}\right)^{\frac{N-2}{2}} \frac{d_{j}\left(\mu_{n} y+y_{n}\right)^{\frac{N-2}{2}} u_{n}\left(\mu_{n} y+y_{n}\right)}{d_{j}\left(y_{n}\right)^{\frac{N-2}{2}} u_{n}\left(y_{n}\right)} \leqslant 2^{\frac{N-2}{2}}
$$

for any $y \in B_{n}$, in view of the maximality property in the definition (27) of $y_{n}$. Hence, by standard elliptic estimates and a diagonal process, (up to a subsequence) we get that $U_{n} \rightarrow U$ in $C_{\mathrm{loc}}^{2}\left(\mathbb{R}^{N}\right)$, where $U$ is a solution of

$$
\left\{\begin{array}{l}
-\Delta U=N(N-2) U^{\frac{N+2}{N-2}} \quad \text { in } \mathbb{R}^{N} \\
0<U(y) \leqslant 2^{\frac{N-2}{2}}, \quad U(0)=1
\end{array}\right.
$$

By the classification in [5], the function $U$ satisfies:

$$
U(y)=\frac{\mu^{\frac{N-2}{2}}}{\left(\mu^{2}+\left|y-y_{0}\right|^{2}\right)^{\frac{N-2}{2}}},
$$

for some $\mu>0$ and $y_{0} \in \mathbb{R}^{N}$. Since $U(y)$ has a nondegenerate maximum point at $y=y_{0}$, for $n$ large $U_{n}(y)$ has a unique maximum point $z_{n}$ close to $y_{0}$. Hence, for the original sequence, the point $x_{n}^{j+1}:=\mu_{n} z_{n}+y_{n}$ is a maximum point of $u_{n}$ on $B_{2 \mu_{n}}\left(x_{n}^{j+1}\right)$ for $n$ large. Since $2^{\frac{N-2}{2}} \geqslant U_{n}\left(z_{n}\right)=\mu_{n}^{\frac{N-2}{2}} u_{n}\left(x_{n}^{j+1}\right) \geqslant U_{n}(0)=1$, we get that, for $\varepsilon_{n}^{j+1}=$ $u_{n}\left(x_{n}^{j+1}\right)^{-\frac{2}{N-2}}$, there holds:

$$
\frac{\varepsilon_{n}^{j+1}}{\mu_{n}} \rightarrow U\left(y_{0}\right)^{-\frac{2}{N-2}}=\mu \in\left[\frac{1}{2}, 1\right] .
$$

By (28), (29) we get that

$$
\left|x_{n}^{j+1}-x_{n}^{i}\right| \geqslant\left|y_{n}-x_{n}^{i}\right|-\left|y_{n}-x_{n}^{j+1}\right|=\left|y_{n}-x_{n}^{i}\right|-\mu_{n}\left|z_{n}\right| \gg\left\{\begin{array}{l}
2 \mu_{n} \geqslant \varepsilon_{n}^{j+1} \\
\varepsilon_{n}^{i}
\end{array}\right.
$$

for any $i=1, \ldots, j$ and hence (21), (22) hold. Moreover, there holds:

$$
\left(\varepsilon_{n}^{j+1}\right)^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n}^{j+1} y+x_{n}^{j+1}\right)=\left(\frac{\varepsilon_{n}^{j+1}}{\mu_{n}}\right)^{\frac{N-2}{2}} U_{n}\left(\frac{\varepsilon_{n}^{j+1}}{\mu_{n}} y+z_{n}\right) \rightarrow \mu^{\frac{N-2}{2}} U\left(\mu y+y_{0}\right)=\frac{1}{\left(1+|y|^{2}\right)^{\frac{N-2}{2}}}
$$

in $C_{\mathrm{loc}}^{2}\left(\mathbb{R}^{N}\right)$ and, so (23) holds. So, the inductive step holds for $j+1$. By (21) there holds

$$
\int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} \geqslant \sum_{i=1}^{j+1} \int_{B_{R i_{n}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}}+\int_{B_{1}(0) \backslash B_{\delta}(0)} u_{n}^{\frac{2 N}{N-2}}=\sum_{i=1}^{j+1} \int_{B_{R}(0)}\left(U_{n}^{i}\right)^{\frac{2 N}{N-2}}+\int_{B_{1}(0) \backslash B_{\delta}(0)} u_{n}^{\frac{2 N}{N-2}}
$$

for $\delta, R>0$ and $n \geqslant N=N(\delta, R)$, and hence, by (23) we deduce that

$$
\begin{align*}
\liminf _{n \rightarrow+\infty} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} & \geqslant(j+1) \int_{\mathbb{R}^{N}} \frac{d y}{\left(1+|y|^{2}\right)^{N}}+\int_{B_{1}(0)} u_{0}^{\frac{2 N}{N-2}} \\
& =(j+1)\left(\frac{S_{N}}{N(N-2)}\right)^{\frac{N}{2}}+\int_{B_{1}(0)} u_{0}^{\frac{2 N}{N-2}} \tag{30}
\end{align*}
$$

Since $\sup _{n \in \mathbb{N}} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}}<+\infty$, by (30) the induction process has to stop after $s$ steps giving that also (24) holds for the sequences $x_{n}^{1}, \ldots, x_{n}^{s}$. Further, there holds (25) by means of the validity of (30) for $s=j+1$.

Now, in the spirit of the blow-up techniques developed by Druet, Hebey and Robert in [10,11], we can perform a finer analysis. Proposition A. 2 below has already been showed in [10] for compact Riemannian manifolds. A careful
reader could observe that, in fact, their analysis is quite local and does not use any geometric features and/or compactness of the underlying space, and therefore it extends directly to our situation. For the sake of completeness, we re-write their proof in the simpler context of an Euclidean domain:

Proposition A.2. Up to a subsequence of $u_{n}$, there exist $k \in \mathbb{N}, k \geqslant s$, sequences $x_{n}^{s+1}, \ldots, x_{n}^{k} \rightarrow 0, \varepsilon_{n}^{s+1}, \ldots, \varepsilon_{n}^{k} \rightarrow 0$ and points $x_{s+1}, \ldots, x_{k} \in \mathbb{R}^{N}$ such that for any $i=s+1, \ldots, k, 1 \leqslant j<i$, we have:

$$
\begin{align*}
& \frac{\varepsilon_{n}^{j}}{\left|x_{n}^{i}-x_{n}^{j}\right|} \rightarrow 0, \quad \frac{\varepsilon_{n}^{j}}{\varepsilon_{n}^{i}} \rightarrow 0 \quad \text { as } n \rightarrow+\infty,  \tag{31}\\
& U_{n}^{i}(y)=\left(\varepsilon_{n}^{i}\right)^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n}^{i} y+x_{n}^{i}\right) \rightarrow \frac{1}{\left(1+\left|y-x_{i}\right|^{2}\right)^{\frac{N-2}{2}}} \quad \text { in } C_{\mathrm{loc}}^{2}\left(\mathbb{R}^{N} \backslash S_{i}\right) \text { as } n \rightarrow+\infty,  \tag{32}\\
& \lim _{R \rightarrow+\infty} \limsup _{n \rightarrow+\infty} \max _{x \in B_{R}^{n}}\left(d_{k}(x)^{\frac{N-2}{2}}\left|u_{n}(x)-u_{0}(x)\right|\right)=0, \tag{33}
\end{align*}
$$

where $x_{n}^{1}, \ldots, x_{n}^{s}$ are given in Proposition A.1, $d_{k}(x)=\min \left\{\left|x-x_{n}^{i}\right|: i=1, \ldots, k\right\}, B_{R}^{n}=\left\{|x|<1:\left|x-x_{n}^{i}\right| \geqslant\right.$ $\left.R \varepsilon_{n}^{i} \forall i=1, \ldots, k\right\}$ and

$$
S_{i}=\left\{y_{j}=\lim _{n \rightarrow+\infty} \frac{x_{n}^{j}-x_{n}^{i}}{\varepsilon_{n}^{i}}: j<i \text { s.t. } \frac{\left|x_{n}^{j}-x_{n}^{i}\right|}{\varepsilon_{n}^{i}}=\mathrm{O}(1)\right\}
$$

is a nonempty set. In particular, there holds

$$
\begin{equation*}
\liminf _{n \rightarrow+\infty} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} \geqslant k\left(\frac{S_{N}}{N(N-2)}\right)^{\frac{N}{2}}+\int_{B_{1}(0)} u_{0}^{\frac{2 N}{N-2}} . \tag{34}
\end{equation*}
$$

Proof. If (33) holds already true for $x_{n}^{1}, \ldots, x_{n}^{s}$, we take $k=s$ and the proof is complete. Moreover, since $\int_{\bigcup_{i=1}^{s} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}}=\sum_{i=1}^{s} \int_{B_{R}(0)}\left(U_{n}^{i}\right)^{\frac{2 N}{N-2}}$ for $R>0$ and $n \geqslant N=N(R)$, by (23) we deduce that

$$
\liminf _{n \rightarrow+\infty} \int_{\bigcup_{i=1}^{S} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}} \geqslant s \int_{B_{R}(0)} \frac{d y}{\left(1+|y|^{2}\right)^{N}}
$$

for any $R>0$. In particular, we get that

$$
\liminf _{n \rightarrow+\infty} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} \geqslant s \int_{B_{R}(0)} \frac{d y}{\left(1+|y|^{2}\right)^{N}}+\int_{B_{1}(0) \backslash B_{\delta}(0)} u_{0}^{\frac{2 N}{N-2}}
$$

for any $\delta, R>0$, and hence (34) holds.
Otherwise, by induction assume that there exist $j-s$ sequences $x_{n}^{s+1}, \ldots, x_{n}^{j} \rightarrow 0, \varepsilon_{n}^{s+1}, \ldots, \varepsilon_{n}^{k} \rightarrow 0$ as $n \rightarrow+\infty$ such that (31), (32) hold and

$$
\begin{equation*}
\operatorname{limin}_{n \rightarrow+\infty} \int_{\bigcup_{i=1}^{j} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}} \geqslant \sum_{i=1}^{j} \int_{B_{R}(0)} \frac{d y}{\left(1+\left|y-x_{i}\right|^{2}\right)^{N}}, \tag{35}
\end{equation*}
$$

where we take $x_{i}=0$ if $i=1, \ldots, s$. If (33) holds true, we take $k=j$ and, as before, (35) implies the validity of (34).
Otherwise, (up to a subsequence) there exist $R_{n} \rightarrow+\infty$ and $y_{n} \in B_{R_{n}}^{n}$ such that

$$
\begin{equation*}
d_{j}\left(y_{n}\right)^{\frac{N-2}{2}}\left|u_{n}\left(y_{n}\right)-u_{0}\left(y_{n}\right)\right|=\max _{x \in B_{R_{n}}^{n}}\left(d_{j}(x)^{\frac{N-2}{2}}\left|u_{n}(x)-u_{0}(x)\right|\right) \geqslant\left(4 \delta_{0}\right)^{\frac{N-2}{2}} \tag{36}
\end{equation*}
$$

for some $\delta_{0}>0$, where $B_{R_{n}}^{n}=\left\{|x|<1:\left|x-x_{n}^{i}\right| \geqslant R_{n} \varepsilon_{n}^{i} \forall i=1, \ldots, j\right\}$. Since $u_{n} \rightarrow u_{0}$ away from 0 , by (36) we have that $y_{n} \rightarrow 0$ and $d_{j}\left(y_{n}\right)^{\frac{N-2}{2}} u_{n}\left(y_{n}\right) \geqslant\left(2 \delta_{0}\right)^{\frac{N-2}{2}}$ for $n$ large. Hence, there exists a sequence $x_{n}^{j+1} \in B_{R_{n}}^{n}$ so that

$$
d_{j}\left(x_{n}^{j+1}\right)^{\frac{N-2}{2}} u_{n}\left(x_{n}^{j+1}\right)=\max _{x \in B_{R_{n}}^{n}}\left(d_{j}(x)^{\frac{N-2}{2}} u_{n}(x)\right) \geqslant\left(2 \delta_{0}\right)^{\frac{N-2}{2}} .
$$

For $\varepsilon_{n}^{j+1}=u_{n}\left(x_{n}^{j+1}\right)^{-\frac{2}{N-2}}$ there holds:

$$
\begin{equation*}
\frac{\left|x_{n}^{j+1}-x_{n}^{i}\right|}{\varepsilon_{n}^{j+1}} \geqslant 2 \delta_{0}, \quad \frac{\varepsilon_{n}^{i}}{\left|x_{n}^{j+1}-x_{n}^{i}\right|} \rightarrow 0 \quad \text { as } n \rightarrow+\infty \tag{37}
\end{equation*}
$$

for any $i=1, \ldots, j$. In particular, $\varepsilon_{n}^{j+1} \rightarrow 0$ as $n \rightarrow+\infty$. Moreover, observing that property (24) is still true if we add the points $x_{n}^{s+1}, \ldots, x_{n}^{j}$ :

$$
\begin{equation*}
d_{j}(x)^{\frac{N-2}{2}} u_{n}(x) \leqslant C, \quad|x|<1, n \in \mathbb{N}, \tag{38}
\end{equation*}
$$

we have that $\left|x_{n}^{j+1}-x_{n}^{i}\right| / \varepsilon_{n}^{j+1} \leqslant C^{\frac{2}{N-2}}$ for some $i=1, \ldots, j$ (up to a subsequence) and so, $S_{j+1} \neq \emptyset$ and (31) is satisfied. Up to a subsequence, assume that the limits $y_{i}=\lim _{n \rightarrow+\infty} x_{n}^{i}-x_{n}^{j+1} / \varepsilon_{n}^{j+1}$ exist for any $i=1, \ldots, j$ so that $\left|x_{n}^{i}-x_{n}^{j+1}\right| / \varepsilon_{n}^{j+1}=\mathrm{O}(1)$. Introduce the set $S_{j+1}=\left\{y_{i}: i \leqslant j\right\}$ and remark that by (37) we have that $S_{j+1} \cap$ $B_{\delta_{0}}(0)=\emptyset$.

Consider the function $U_{n}(y)=\left(\varepsilon_{n}^{j+1}\right)^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n}^{j+1} y+x_{n}^{j+1}\right)$ for $y \in B_{n}$, where the balls $B_{n}=\left\{y \in \mathbb{R}^{N}:|y| \leqslant \frac{1}{\varepsilon_{n}^{j+1}}\right\}$ expand to $\mathbb{R}^{N}$ as $n \rightarrow+\infty$. We want to show that $U_{n}$ is uniformly bounded in $C_{\text {loc }}^{0}$ ( $\mathbb{R}^{N} \backslash S_{j+1}$ ). Let $y \in B_{R}(0)$ be so that $\left|y-y_{i}\right| \geqslant \frac{1}{R}$ for any $y_{i} \in S_{j+1}$. Since for any $i=1, \ldots, j$ :

$$
\left|\varepsilon_{n}^{j+1} y+x_{n}^{j+1}-x_{n}^{i}\right|=\varepsilon_{n}^{j+1}\left|y-\frac{x_{n}^{i}-x_{n}^{j+1}}{\varepsilon_{n}^{j+1}}\right| \geqslant \begin{cases}\varepsilon_{n}^{j+1}\left(\left|y-y_{i}\right|-\left|y_{i}-\frac{x_{n}^{i}-x_{n}^{j+1}}{\varepsilon_{n}^{j+1}}\right|\right) \geqslant \frac{\varepsilon_{n}^{j+1}}{2 R} & \text { if } y_{i} \in S_{j+1}, \\ \varepsilon_{n}^{j+1}\left(\left|\frac{x_{n}^{i}-x_{n}^{j+1}}{\varepsilon_{n}^{j+1}}\right|-|y|\right) \geqslant \frac{\varepsilon_{n}^{j+1}}{2 R} & \text { otherwise }\end{cases}
$$

for $n \geqslant N=N(R)$, we get that $d_{j}\left(\varepsilon_{n}^{j+1} y+x_{n}^{j+1}\right) \geqslant \varepsilon_{n}^{j+1} /(2 R)$ and by (38) we get that

$$
U_{n}(y)=\left(\varepsilon_{n}^{j+1}\right)^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n}^{j+1} y+x_{n}^{j+1}\right) \leqslant(2 R)^{\frac{N-2}{2}} d_{j}\left(\varepsilon_{n}^{j+1} y+x_{n}^{j+1}\right)^{\frac{N-2}{2}} u_{n}\left(\varepsilon_{n}^{j+1} y+x_{n}^{j+1}\right) \leqslant C(2 R)^{\frac{N-2}{2}}
$$

for $n \geqslant N=N(R)$. By standard elliptic estimates and a diagonal process, (up to a subsequence) we get that $U_{n} \rightarrow U$ in $C_{\mathrm{loc}}^{2}\left(\mathbb{R}^{N} \backslash S_{j+1}\right)$, where $U$ is a nonnegative solution of

$$
\left\{\begin{array}{l}
-\Delta U=N(N-2) U^{\frac{N+2}{N-2}} \quad \text { in } \mathbb{R}^{N} \backslash S_{j+1}, \\
U(0)=1
\end{array}\right.
$$

since $B_{\delta_{0}}(0) \subset \mathbb{R}^{N} \backslash S_{j+1}$. By the result of Caffarelli, Gidas and Spruck [5], we have that:

$$
U(y)=\frac{\mu^{\frac{N-2}{2}}}{\left(\mu^{2}+\left|y-x_{i}\right|^{2}\right)^{\frac{N-2}{2}}},
$$

for some $\mu>0$ and $x_{i} \in \mathbb{R}^{N}$. Since

$$
\left(\mu \varepsilon_{n}^{j+1}\right)^{\frac{N-2}{2}} u_{n}\left(\mu \varepsilon_{n}^{j+1} y+x_{n}^{j+1}\right)=\mu^{\frac{N-2}{2}} U_{n}(\mu y) \rightarrow \mu^{\frac{N-2}{2}} U(\mu y)=\frac{1}{\left(1+\left|y-\mu^{-1} x_{i}\right|^{2}\right)^{\frac{N-2}{2}}}
$$

we can replace $\varepsilon_{n}^{j+1}$ with $\mu \varepsilon_{n}^{j+1}, x_{i}$ with $\mu^{-1} x_{i}$ and $S_{j+1}$ with respect to $\varepsilon_{n}^{j+1}$ with $S_{j+1}$ with respect to $\mu \varepsilon_{n}^{j+1}$ to obtain the validity of (32). Now, we want to show the validity of (35) with $j$ replaced by $j+1$. Let $I=$ $\left\{i=1, \ldots, j: y_{i} \in S_{j+1}\right\}$ and let $C>0$ be such that $\left|x_{n}^{i}-x_{n}^{j+1}\right| \leqslant C \varepsilon_{n}^{j+1}$ for any $i \in I$. Clearly, we have that

$$
\bigcup_{i \in I} B_{\delta \varepsilon_{n}^{j+1}}\left(x_{n}^{i}\right) \subset B_{(C+\delta) \varepsilon_{n}^{j+1}}\left(x_{n}^{j+1}\right)
$$

for any $\delta>0$, and by (31) we also deduce that

$$
\bigcup_{i \in I} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right) \subset B_{2 C \varepsilon_{n}^{j+1}}\left(x_{n}^{j+1}\right)
$$

for $n \geqslant N=N(R)$. Similarly, there holds

$$
\left(\bigcup_{i=1, \ldots, j, i \notin I} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right)\right) \cap B_{R \varepsilon_{n}^{j+1}}\left(x_{n}^{j+1}\right)=\emptyset
$$

for $n \geqslant N=N(R)$. Hence, we have that for $R \geqslant 2 C$ and $0<\delta<C$

$$
\begin{aligned}
\bigcup_{i=1}^{j+1} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right) & =B_{R \varepsilon_{n}^{j+1}}\left(x_{n}^{j+1}\right) \oplus \bigcup_{i=1, \ldots, j, i \notin I} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right) \\
& =\left(B_{R \varepsilon_{n}^{j+1}}\left(x_{n}^{j+1}\right) \backslash \bigcup_{i \in I} B_{\delta \varepsilon_{n}^{j+1}}\left(x_{n}^{i}\right)\right) \oplus \bigcup_{i \in I} B_{\delta \varepsilon_{n}^{j+1}}\left(x_{n}^{i}\right) \oplus \bigcup_{i=1, \ldots, j, i \notin I} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right) \\
& \supset\left(B_{R \varepsilon_{n}^{j+1}}\left(x_{n}^{j+1}\right) \backslash \bigcup_{i \in I} B_{\delta \varepsilon_{n}^{j+1}}\left(x_{n}^{i}\right)\right) \oplus \bigcup_{i=1}^{j} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right)
\end{aligned}
$$

for $n \geqslant N=N(\delta, R)$ in view of (31), where $\oplus$ stands for the union of disjoint sets. Hence, we can write

$$
\begin{aligned}
\int_{\bigcup_{i=1}^{j+1} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}} \geqslant & \int_{B_{R \varepsilon_{n}^{j+1}}\left(x_{n}^{j+1}\right) \backslash \bigcup_{i \in I} B_{\delta \delta_{n}^{j+1}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}}+\int_{\bigcup_{i=1}^{j} B_{R \varepsilon_{n}^{i}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}} \\
& =\int_{B_{R}(0) \backslash \bigcup_{i \in I} B_{\delta}\left(\frac{x_{n}^{i}-n_{n}^{j+1}}{\varepsilon_{n}^{j+1}}\right)}\left(U_{n}^{j+1}\right)^{\frac{2 N}{N-2}}+\int_{\bigcup_{i=1}^{j} B_{R E_{n}^{i}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}}
\end{aligned}
$$

for $n \geqslant N=N(\delta, R)$. Passing to the liminf as $n \rightarrow+\infty$, by the validity of (35), (32) for $j+1$ and $\left(x_{n}^{i}-x_{n}^{j+1}\right) / \varepsilon_{n}^{j+1} \rightarrow$ $y_{i} \in S_{j+1}$ as $n \rightarrow+\infty$ for any $i \in I$, we get that

$$
\operatorname{limin}_{n \rightarrow+\infty} \int_{\bigcup_{i=1}^{j+1} B_{R E_{n}^{i}}\left(x_{n}^{i}\right)} u_{n}^{\frac{2 N}{N-2}} \geqslant \int_{B_{R}(0) \backslash \bigcup_{i \in I} B_{\delta}\left(y_{i}\right)} \frac{\mathrm{d} y}{\left(1+\left|y-x_{j+1}\right|^{2}\right)^{N}}+\sum_{i=1}^{j} \int_{B_{R}(0)} \frac{\mathrm{d} y}{\left(1+\left|y-x_{i}\right|^{2}\right)^{N}}
$$

for any $R$ large and $\delta$ small. Letting $\delta \rightarrow 0^{+}$, we obtain that

$$
\begin{equation*}
\liminf _{n \rightarrow+\infty} \int_{\bigcup_{i=1}^{j+1}} u_{n}^{\frac{2 N}{N-2}} \geqslant \sum_{i=1}^{j+1} \int_{B_{R}(0)} \frac{\mathrm{d} y}{\left(1+\left|y-x_{n}^{i}\right|^{2}\right)^{N}} \tag{39}
\end{equation*}
$$

So, the inductive step holds for $j+1$. Since $\sup _{n \in \mathbb{N}} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}}<+\infty$, by (39) the induction process has to stop after $k-s$ steps giving that also (33) holds for the sequences $x_{n}^{1}, \ldots, x_{n}^{s}, x_{n}^{s+1}, \ldots, x_{n}^{k}$. Moreover, arguing as before, (39) for $k=j+1$ implies the validity of

$$
\liminf _{n \rightarrow+\infty} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} \geqslant \sum_{i=1}^{k} \int_{B_{R}(0)} \frac{\mathrm{d} y}{\left(1+\left|y-x_{i}\right|^{2}\right)^{N}}+\int_{B_{1}(0) \backslash B_{\delta}(0)} u_{0}^{\frac{2 N}{N-2}}
$$

for any $\delta, R>0$, and hence, taking $\delta \rightarrow 0$ and $R \rightarrow+\infty$,

$$
\liminf _{n \rightarrow+\infty} \int_{B_{1}(0)} u_{n}^{\frac{2 N}{N-2}} \geqslant k\left(\frac{S_{N}}{N(N-2)}\right)^{\frac{N}{2}}+\int_{B_{1}(0)} u_{0}^{\frac{2 N}{N-2}}
$$

Hence, (34) holds and the proof is complete.

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