The critical dimension for a 4th order problem with singular nonlinearity

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Abstract

We study the regularity of the extremal solution of the semilinear biharmonic equation $\Delta^2 u = \frac{\lambda}{(1-u)^2}$, which models a simple Micro-Electromechanical System (MEMS) device on a ball $B \subset \mathbb{R}^N$, under Dirichlet boundary conditions $u = \partial_\nu u = 0$ on ∂B. We complete here the results of F.H. Lin and Y.S. Yang [14] regarding the identification of a "pull-in voltage" $\lambda^* > 0$ such that a stable classical solution u_λ with $0 < u_\lambda < 1$ exists for $\lambda \in (0, \lambda^*)$, while there is none of any kind when $\lambda > \lambda^*$. Our main result asserts that the extremal solution u_{λ^*} is regular ($\sup_B u_{\lambda^*} < 1$) provided $N \leq 8$ while u_{λ^*} is singular ($\sup_B u_{\lambda^*} = 1$) for $N \geq 9$, in which case $1 - C_0|x|^{4/3} \leq u_{\lambda^*}(x) \leq 1 - |x|^{4/3}$ on the unit ball, where $C_0 := \left(\frac{\lambda^*}{\bar{\lambda}}\right)^{\frac{1}{3}}$ and $\bar{\lambda} := \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3})$.

1 Introduction

The following model has been proposed for the description of the steady-state of a simple Electrostatic MEMS device:

$$\begin{cases} \alpha \Delta^2 u = \left(\beta \int_\Omega |\nabla u|^2 dx + \gamma\right) \Delta u + \frac{\lambda f(x)}{(1-u)^2 \left(1 + \chi \int_\Omega \frac{dx}{(1-u)^2}\right)} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = \alpha \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\alpha, \beta, \gamma, \chi \geq 0$, $f \in C(\overline{\Omega}, [0,1])$ are fixed, Ω is a bounded domain in $\mathbb{R}^N$ and $\lambda \geq 0$ is a varying parameter (see for example Bernstein and Pelesko [19]). The function $u(x)$ denotes the height above a point $x \in \Omega \subset \mathbb{R}^N$ of a dielectric membrane clamped on $\partial\Omega$, once it deflects torwards a ground plate fixed at height $z = 1$, whenever a positive voltage – proportional to λ – is applied.

In studying this problem, one typically makes various simplifying assumptions on the parameters $\alpha, \beta, \gamma, \chi$, and the first approximation of (1) that has been studied extensively so far is the equation

$$\begin{cases} -\Delta u = \lambda \frac{f(x)}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{S}_{\lambda,f}$$

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where we have set $\alpha = \beta = \chi = 0$ and $\gamma = 1$ (see for example [6, 8, 9] and the monograph [7]) . This simple model, which lends itself to the vast literature on second order semilinear eigenvalue problems, is already a rich source of interesting mathematical problems. The case when the "permittivity profile" f is constant ($f = 1$) on a general domain was studied in [16], following the pioneering work of Joseph and Lundgren [13] who had considered the radially symmetric case. The case for a non constant permittivity profile f was advocated by Pelesko [18], taken up by [11], and studied in depth in [6, 8, 9]. The starting point of the analysis is the existence of a pull-in voltage $\lambda^*(\Omega, f)$, defined as

$$\lambda^*(\Omega, f) := \sup\left\{\lambda > 0 : \text{there exists a classical solution of } (S)_{\lambda, f}\right\}.$$

It is then shown that for every $0 < \lambda < \lambda^*$, there exists a smooth minimal (smallest) solution of $(S)_{\lambda, f}$, while for $\lambda > \lambda^*$ there is no solution even in a weak sense. Moreover, the branch $\lambda \mapsto u_\lambda(x)$ is increasing for each $x \in \Omega$, and therefore the function $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ can be considered as a generalized solution that corresponds to the pull-in voltage λ^*. Now the issue of the regularity of this extremal solution – which, by elliptic regularity theory, is equivalent to whether $\sup_\Omega u^* < 1$ – is an important question for many reasons, not the least of which being the fact that it decides whether the set of solutions stops there, or whether a new branch of solutions emanates from a bifurcation state (u^*, λ^*). This issue turned out to depend closely on the dimension and on the permittivity profile f. Indeed, it was shown in [9] that u^* is regular in dimensions $1 \leq N \leq 7$, while it is not necessarily the case for $N \geq 8$. In other words, the dimension $N = 7$ is critical for equation $(S)_\lambda$ (when $f = 1$, we simplify the notation $(S)_{\lambda, 1}$ into $(S)_\lambda$). On the other hand, it is shown in [8] that the regularity of u^* can be restored in any dimension, provided we allow for a power law profile $|x|^\eta$ with η large enough.

The case where $\beta = \gamma = \chi = 0$ (and $\alpha = 1$) in the above model, that is when we are dealing with the following fourth order analog of $(S)_\lambda$

$$\left\{\begin{array}{ll} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = \partial_\nu u = 0 & \text{on } \partial\Omega, \end{array}\right. \qquad (P)_\lambda$$

was also considered by [4, 14] but with limited success. One of the reasons is the lack of a "maximum principle" which plays such a crucial role in developing the theory for the Laplacian. Indeed, it is a well known fact that such a principle does not normally hold for general domains Ω (at least for the clamped boundary conditions $u = \partial_\nu u = 0$ on $\partial\Omega$) unless one restricts attention to the unit ball $\Omega = B$ in $\mathbb{R}^N$, where one can exploit a positivity preserving property of Δ^2 due to T. Boggio [3]. This is precisely what was done in the references mentioned above, where a theory of the minimal branch associated with $(P)_\lambda$ is developed along the same lines as for $(S)_\lambda$. The second obstacle is the well-known difficulty of extracting energy estimates for solutions of fourth order problems from their stability properties. This means that the methods used to analyze the regularity of the extremal solution for $(S)_\lambda$ could not carry to the corresponding problem for $(P)_\lambda$.

This is the question we address in this paper as we eventually show the following result.

Theorem 1.1. *The unique extremal solution u^* for $(P)_{\lambda^*}$ in B is regular in dimension $1 \leq N \leq 8$, while it is singular (i.e, $\sup_B u^* = 1$) for $N \geq 9$.*

In other words, the critical dimension for $(P)_\lambda$ in B is $N = 8$, as opposed to being equal to 7 in $(S)_\lambda$. We add that our methods are heavily inspired by the recent paper of Davila et al. [5] where it is shown that $N = 12$ is the critical dimension for the fourth order nonlinear eigenvalue problem

$$\left\{\begin{array}{ll} \Delta^2 u = \lambda e^u & \text{in } B \\ u = \partial_\nu u = 0 & \text{on } \partial B, \end{array}\right.$$

while the critical dimension for its second order counterpart (i.e., the Gelfand problem) is $N = 9$. There is, however, one major difference between our approach and the one used by Davila et al. [5]. It is related to the

most delicate dimensions – just above the critical one – where they use a computer assisted proof to establish the singularity of the extremal solution, while our method is more analytical and relies on improved and non standard Hardy-Rellich inequalities recently established by Ghoussoub-Moradifam [10] (See Appendix).

Throughout this paper, we will always consider problem $(P)_\lambda$ on the unit ball B. We start by recalling some of the results from [4] concerning $(P)_\lambda$, that will be needed in the sequel. We define

$$\lambda^* := \sup\left\{\lambda > 0 : \text{ there exists a classical solution of } (P)_\lambda\right\},$$

and note that we are not restricting our attention to radial solutions. We will deal also with weak solutions:

Definition 1.1. *We say that u is a weak solution of $(P)_\lambda$ if $0 \le u \le 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and*

$$\int_B u\Delta^2\phi = \lambda \int_B \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B).$$

We say that u is a weak super-solution (resp. weak sub-solution) of $(P)_\lambda$ if the equality is replaced with the inequality $\ge$ (resp. $\le$) for all $\phi \in C^4(\bar{B}) \cap H_0^2(B)$ with $\phi \ge 0$.

We also introduce notions of regularity and stability.

Definition 1.2. *Say that a weak solution u of $(P)_\lambda$ is regular (resp. singular) if $\|u\|_\infty < 1$ (resp. $=$) and stable (resp. semi-stable) if*

$$\mu_1(u) = \inf\left\{\int_B (\Delta\phi)^2 - 2\lambda \int_B \frac{\phi^2}{(1-u)^3} : \phi \in H_0^2(B), \|\phi\|_{L^2} = 1\right\}$$

is positive (resp. non-negative).

The following extension of Boggio's principle will be frequently used in the sequel (see [2, Lemma 16] and [5, Lemma 2.4]):

Lemma 1.1 (Boggio's Principle). *Let $u \in L^1(B)$. Then $u \ge 0$ a.e. in B, provided one of the following conditions hold:*

1. *$u \in C^4(\overline{B})$, $\Delta^2 u \ge 0$ on B, and $u = \frac{\partial u}{\partial n} = 0$ on ∂B.*

2. *$\int_B u\Delta^2\phi\, dx \ge 0$ for all $0 \le \phi \in C^4(\overline{B}) \cap H_0^2(B)$.*

3. *$u \in H^2(B)$, $u = 0$ and $\frac{\partial u}{\partial n} \le 0$ on ∂B, and $\int_B \Delta u\Delta\phi \ge 0$ for all $0 \le \phi \in H_0^2(B)$.*

Moreover, either $u \equiv 0$ or $u > 0$ a.e. in B.

The following theorem summarizes the main results in [4] that will be needed in the sequel:

Theorem 1.2. *The following assertions hold:*

1. *For each $0 < \lambda < \lambda^*$ there exists a classical minimal solution u_λ of $(P)_\lambda$. Moreover u_λ is radial and radially decreasing.*

2. *For $\lambda > \lambda^*$, there are no weak solutions of $(P)_\lambda$.*

3. *For each $x \in B$ the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$.*

4. *The pull-in voltage λ^* satisfies the following bounds:*

$$\max\left\{\frac{32(10N - N^2 - 12)}{27}, \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3})\right\} \le \lambda^* \le \frac{4\nu_1}{27}$$

where ν_1 denotes the first eigenvalue of Δ^2 in $H_0^2(B)$.

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5. *For each $0 < \lambda < \lambda^*$, u_λ is a stable solution (i.e., $\mu_1(u_\lambda) > 0$).*

Using the stability of u_λ, it can be shown that u_λ is uniformly bounded in $H_0^2(B)$ and that $\frac{1}{1-u_\lambda}$ is uniformly bounded in $L^3(B)$. Since now $\lambda \mapsto u_\lambda(x)$ is increasing, the function $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$ is well defined (in the pointwise sense), $u^* \in H_0^2(B)$, $\frac{1}{1-u^*} \in L^3(B)$ and u^* is a weak solution of $(P)_{\lambda^*}$. Moreover u^* is the unique weak solution of $(P)_{\lambda^*}$.

The second result we list from [4] is critical in identifying the extremal solution.

Theorem 1.3. *If $u \in H_0^2(B)$ is a singular weak solution of $(P)_\lambda$, then u is semi-stable if and only if $(u, \lambda) = (u^*, \lambda^*)$.*

2 The effect of boundary conditions on the pull-in voltage

As in [5], we are led to examine problem $(P)_\lambda$ with non-homogeneous boundary conditions such as

$$\begin{cases} \Delta^2 u = \frac{\lambda}{(1-u)^2} & \text{in } B \\ \alpha < u < 1 & \text{in } B \\ u = \alpha, \ \partial_\nu u = \beta & \text{on } \partial B, \end{cases} \qquad (P)_{\lambda, \alpha, \beta}$$

where α, β are given.

Notice first that some restrictions on α and β are necessary. Indeed, letting $\Phi(x) := (\alpha - \frac{\beta}{2}) + \frac{\beta}{2}|x|^2$ denote the unique solution of

$$\begin{cases} \Delta^2 \Phi = 0 & \text{in } B \\ \Phi = \alpha, \ \partial_\nu \Phi = \beta & \text{on } \partial B, \end{cases} \qquad (2)$$

we infer immediately from Lemma 1.1 that the function $u - \Phi$ is positive in B, which yields to

$$\sup_B \Phi < \sup_B u \leq 1.$$

To insure that Φ is a classical sub-solution of $(P)_{\lambda, \alpha, \beta}$, we impose $\alpha \neq 1$ and $\beta \leq 0$, and condition $\sup_B \Phi < 1$ rewrites as $\alpha - \frac{\beta}{2} < 1$. We will then say that the pair (α, β) is *admissible* if $\beta \leq 0$, and $\alpha - \frac{\beta}{2} < 1$.

This section will be devoted to obtaining results for $(P)_{\lambda, \alpha, \beta}$ when (α, β) is an admissible pair, which are analogous to those for $(P)_\lambda$. To cut down on notation, we shall sometimes drop α and β from our expressions whenever such an emphasis is not needed. For example in this section u_λ and u^* will denote the minimal and extremal solution of $(P)_{\lambda, \alpha, \beta}$.

We now introduce a notion of weak solution for $(P)_{\lambda, \alpha, \beta}$.

Definition 2.1. *We say that u is a weak solution of $(P)_{\lambda, \alpha, \beta}$ if $\alpha \leq u \leq 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and if*

$$\int_B (u - \Phi)\Delta^2 \phi = \lambda \int_B \frac{\phi}{(1-u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B),$$

where Φ is given in (2). We say that u is a weak super-solution (resp. weak sub-solution) of $(P)_{\lambda, \alpha, \beta}$ if the equality is replaced with the inequality $\geq$ (resp. $\leq$) for $\phi \geq 0$.

We now define as before

$$\lambda^* := \sup\{\lambda > 0 : (P)_{\lambda, \alpha, \beta} \text{ has a classical solution}\}$$

and

$$\lambda_* := \sup\{\lambda > 0 : (P)_{\lambda, \alpha, \beta} \text{ has a weak solution}\}.$$

Observe that by the Implicit Function Theorem, one can always solve $(P)_{\lambda, \alpha, \beta}$ for small λ's. Therefore, λ^* (and also λ_*) is well defined.

Let now U be a weak super-solution of $(P)_{\lambda, \alpha, \beta}$. Recall the following standard existence result.

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Theorem 2.1 ([2]). *For every $0 \leq f \in L^1(B)$, there exists a unique $0 \leq u \in L^1(B)$ which satisfies*

$$\int_B u \Delta^2 \phi = \int_B f \phi$$

for all $\phi \in C^4(\bar{B}) \cap H_0^2(B)$.

We can now introduce the following "weak iterative scheme": Start with $u_0 = U$ and (inductively) let u_n, $n \geq 1$, be the solution of

$$\int_B (u_n - \Phi) \Delta^2 \phi = \lambda \int_B \frac{\phi}{(1 - u_{n-1})^2} \qquad \forall \, \phi \in C^4(\bar{B}) \cap H_0^2(B)$$

given by Theorem 2.1. Since 0 is a sub-solution of $(P)_{\lambda,\alpha,\beta}$, one can easily show inductively by using Lemma 1.1 that $\alpha \leq u_{n+1} \leq u_n \leq U$ for every $n \geq 0$. Since

$$(1 - u_n)^{-2} \leq (1 - U)^{-2} \in L^1(B),$$

we get by Lebesgue Theorem, that the function $u = \lim_{n \to +\infty} u_n$ is a weak solution of $(P)_{\lambda,\alpha,\beta}$ such that $\alpha \leq u \leq U$. In other words, the following result holds.

Proposition 2.1. *Assume the existence of a weak super-solution U of $(P)_{\lambda,\alpha,\beta}$. Then there exists a weak solution u of $(P)_{\lambda,\alpha,\beta}$ so that $\alpha \leq u \leq U$ a.e. in B.*

In particular, we can find a weak solution of $(P)_{\lambda,\alpha,\beta}$ for every $\lambda \in (0, \lambda_*)$. Now we show that this is still true for regular weak solutions.

Proposition 2.2. *Let (α, β) be an admissible pair and let u be a weak solution of $(P)_{\lambda,\alpha,\beta}$. Then for every $0 < \mu < \lambda$, there is a regular solution for $(P)_{\mu,\alpha,\beta}$.*

Proof. Let $\varepsilon \in (0,1)$ be given and let $\bar{u} = (1 - \varepsilon)u + \varepsilon\Phi$, where Φ is given in (2). We have that

$$\sup_B \bar{u} \leq (1 - \varepsilon) + \varepsilon \sup_B \Phi < 1, \quad \inf_B \bar{u} \geq (1 - \varepsilon)\alpha + \varepsilon \inf_B \Phi = \alpha,$$

and for every $0 \leq \phi \in C^4(\bar{B}) \cap H_0^2(B)$ there holds:

$$\begin{aligned}
\int_B (\bar{u} - \Phi) \Delta^2 \phi &= (1 - \varepsilon) \int_B (u - \Phi) \Delta^2 \phi = (1 - \varepsilon)\lambda \int_B \frac{\phi}{(1 - u)^2} \\
&= (1 - \varepsilon)^3 \lambda \int_B \frac{\phi}{(1 - \bar{u} + \varepsilon(\Phi - 1))^2} \geq (1 - \varepsilon)^3 \lambda \int_B \frac{\phi}{(1 - \bar{u})^2}.
\end{aligned}$$

Note that $0 \leq (1 - \varepsilon)(1 - u) = 1 - \bar{u} + \varepsilon(\Phi - 1) < 1 - \bar{u}$. So $\bar{u}$ is a weak super-solution of $(P)_{(1-\varepsilon)^3\lambda,\alpha,\beta}$ satisfying $\sup_B \bar{u} < 1$. From Proposition 2.1 we get the existence of a weak solution w of $(P)_{(1-\varepsilon)^3\lambda,\alpha,\beta}$ so that $\alpha \leq w \leq \bar{u}$. In particular, $\sup_B w < 1$ and w is a regular weak solution. Since $\varepsilon \in (0,1)$ is arbitrarily chosen, the proof is complete. $\qquad\square$

Proposition 2.2 implies in particular the existence of a regular weak solution U_λ for every $\lambda \in (0, \lambda_*)$. Introduce now a "classical" iterative scheme: $u_0 = 0$ and (inductively) $u_n = v_n + \Phi$, $n \geq 1$, where $v_n \in H_0^2(B)$ is the (radial) solution of

$$\Delta^2 v_n = \Delta^2 (u_n - \Phi) = \frac{\lambda}{(1 - u_{n-1})^2} \qquad \text{in } B. \tag{3}$$

Since $v_n \in H_0^2(B)$, u_n is also a weak solution of (3), and by Lemma 1.1 we know that $\alpha \leq u_n \leq u_{n+1} \leq U_\lambda$ for every $n \geq 0$. Since $\sup_B u_n \leq \sup_B U_\lambda < 1$ for $n \geq 0$, we get that $(1 - u_{n-1})^{-2} \in L^2(B)$ and the

existence of v_n is guaranteed. Since v_n is easily seen to be uniformly bounded in $H_0^2(B)$, we have that $u_\lambda := \lim\limits_{n \to +\infty} u_n$ does hold pointwise and weakly in $H^2(B)$. By Lebesgue Theorem, we have that u_λ is a radial weak solution of $(P)_{\lambda,\alpha,\beta}$ so that $\sup\limits_B u_\lambda \leq \sup\limits_B U_\lambda < 1$. By elliptic regularity theory [1] $u_\lambda \in C^\infty(\bar{B})$ and $u_\lambda - \Phi = \partial_\nu(u_\lambda - \Phi) = 0$ on ∂B. So we can integrate by parts to get

$$\int_B \Delta^2 u_\lambda \phi = \int_B \Delta^2(u_\lambda - \Phi)\phi = \int_B (u_\lambda - \Phi)\Delta^2 \phi = \lambda \int_B \frac{\phi}{(1 - u_\lambda)^2}$$

for every $\phi \in C^4(\bar{B}) \cap H_0^2(B)$. Hence, u_λ is a radial classical solution of $(P)_{\lambda,\alpha,\beta}$ showing that $\lambda^* = \lambda_*$. Moreover, since Φ and $v_\lambda := u_\lambda - \Phi$ are radially decreasing in view of [20], we get that u_λ is radially decreasing too. Since the argument above shows that $u_\lambda < U$ for any other classical solution U of $(P)_{\mu,\alpha,\beta}$ with $\mu \geq \lambda$, we have that u_λ is exactly the minimal solution and u_λ is strictly increasing as $\lambda \uparrow \lambda^*$. In particular, we can define u^* in the usual way: $u^*(x) = \lim\limits_{\lambda \nearrow \lambda^*} u_\lambda(x)$.

Finally, we show the finiteness of the pull-in voltage.

Lemma 2.1. *If (α, β) is an admissible pair, then $\lambda^*(\alpha, \beta) < +\infty$.*

Proof. Let u be a classical solution of $(P)_{\lambda,\alpha,\beta}$ and let (ψ, ν_1) denote the first eigenpair of Δ^2 in $H_0^2(B)$ with $\psi > 0$. Now, let C be such that

$$\int_{\partial B} (\beta \Delta \psi - \alpha \partial_\nu \Delta \psi) = C \int_B \psi.$$

Multiplying $(P)_{\lambda,\alpha,\beta}$ by ψ and then integrating by parts one arrives at

$$\int_B \left(\frac{\lambda}{(1 - u)^2} - \nu_1 u - C \right) \psi = 0.$$

Since $\psi > 0$ there must exist a point $\bar{x} \in B$ where $\frac{\lambda}{(1 - u(\bar{x}))^2} - \nu_1 u(\bar{x}) - C \leq 0$. Since $\alpha < u(\bar{x}) < 1$, one can conclude that $\lambda \leq \sup_{\alpha < u < 1}(\nu_1 u + C)(1 - u)^2$, which shows that $\lambda^* < +\infty$. $\square$

The following summarizes what we have shown so far.

Theorem 2.2. *If (α, β) is an admissible pair, then $\lambda^* := \lambda^*(\alpha, \beta) \in (0, +\infty)$ and the following hold:*

1. *For each $0 < \lambda < \lambda^*$ there exists a classical, minimal solution u_λ of $(P)_{\lambda,\alpha,\beta}$. Moreover u_λ is radial and radially decreasing.*

2. *For each $x \in B$ the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing on $(0, \lambda^*)$.*

3. *For $\lambda > \lambda^*$ there are no weak solutions of $(P)_{\lambda,\alpha,\beta}$.*

2.1 Stability of the minimal branch of solutions

This section is devoted to the proof of the following stability result for minimal solutions. We shall need yet another notion of $H^2(B)-$weak solutions, which is an intermediate class between classical and weak solutions.

Definition 2.2. *We say that u is a $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$ if $u - \Phi \in H_0^2(B)$, $\alpha \leq u \leq 1$ a.e. in B, $\frac{1}{(1-u)^2} \in L^1(B)$ and if*

$$\int_B \Delta u \Delta \phi = \lambda \int_B \frac{\phi}{(1 - u)^2}, \qquad \forall \phi \in C^4(\bar{B}) \cap H_0^2(B),$$

where Φ is given in (2). We say that u is a $H^2(B)-$weak super-solution (resp. $H^2(B)-$weak sub-solution) of $(P)_{\lambda,\alpha,\beta}$ if for $\phi \geq 0$ the equality is replaced with $\geq$ (resp. $\leq$) and $u \geq \alpha$ (resp. $\leq$), $\partial_\nu u \leq \beta$ (resp. $\geq$) on ∂B.

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Theorem 2.3. *Suppose (α, β) is an admissible pair.*

1. *The minimal solution u_λ is then stable and is the unique semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$.*

2. *The function $u^* := \lim\limits_{\lambda \nearrow \lambda^*} u_\lambda$ is a well-defined semi-stable $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$.*

3. *When u^* is classical solution, then $\mu_1(u^*) = 0$ and u^* is the unique $H^2(B)-$weak solution of $(P)_{\lambda^*,\alpha,\beta}$.*

4. *If v is a singular, semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$, then $v = u^*$ and $\lambda = \lambda^*$*

The crucial tool is a comparison result which is valid exactly in this class of solutions.

Lemma 2.2. *Let (α, β) be an admissible pair and u be a semi-stable $H^2(B)-$weak solution of $(P)_{\lambda,\alpha,\beta}$. Assume U is a $H^2(B)-$weak super-solution of $(P)_{\lambda,\alpha,\beta}$ so that $U - \Phi \in H_0^2(B)$. Then*

1. *$u \leq U$ a.e. in B;*

2. *If u is a classical solution and $\mu_1(u) = 0$ then $U = u$.*

Proof. (i) Define $w := u - U$. Then by the Moreau decomposition [17] for the biharmonic operator, there exist $w_1, w_2 \in H_0^2(B)$, with $w = w_1 + w_2$, $w_1 \geq 0$ a.e., $\Delta^2 w_2 \leq 0$ in the $H^2(B)-$weak sense and $\int_B \Delta w_1 \Delta w_2 = 0$. By Lemma 1.1, we have that $w_2 \leq 0$ a.e. in B.
Given now $0 \leq \phi \in C_c^\infty(B)$, we have that

$$\int_B \Delta w \Delta \phi \leq \lambda \int_B (f(u) - f(U))\phi,$$

where $f(u) := (1 - u)^{-2}$. Since u is semi-stable, one has

$$\lambda \int_B f'(u)w_1^2 \leq \int_B (\Delta w_1)^2 = \int_B \Delta w \Delta w_1 \leq \lambda \int_B (f(u) - f(U))w_1.$$

Since $w_1 \geq w$ one also has

$$\int_B f'(u)w w_1 \leq \int_B (f(u) - f(U))w_1,$$

which once re-arranged gives

$$\int_B \tilde{f} w_1 \geq 0,$$

where $\tilde{f}(u) = f(u) - f(U) - f'(u)(u - U)$. The strict convexity of f gives $\tilde{f} \leq 0$ and $\tilde{f} < 0$ whenever $u \neq U$. Since $w_1 \geq 0$ a.e. in B one sees that $w \leq 0$ a.e. in B. The inequality $u \leq U$ a.e. in B is then established.

(ii) Since u is a classical solution, it is easy to see that the infimum in $\mu_1(u)$ is attained at some ϕ. The function ϕ is then the first eigenfunction of $\Delta^2 - \frac{2\lambda}{(1-u)^3}$ in $H_0^2(B)$. Now we show that ϕ is of fixed sign. Using the above decomposition, one has $\phi = \phi_1 + \phi_2$ where $\phi_i \in H_0^2(B)$ for $i = 1, 2$, $\phi_1 \geq 0$, $\int_B \Delta \phi_1 \Delta \phi_2 = 0$ and $\Delta^2 \phi_2 \leq 0$ in the $H_0^2(B)-$weak sense. If ϕ changes sign, then $\phi_1 \not\equiv 0$ and $\phi_2 < 0$ in B (recall that either $\phi_2 < 0$ or $\phi_2 = 0$ a.e. in B). We can write now:

$$0 = \mu_1(u) \leq \frac{\int_B (\Delta(\phi_1 - \phi_2))^2 - \lambda f'(u)(\phi_1 - \phi_2)^2}{\int_B (\phi_1 - \phi_2)^2} < \frac{\int_B (\Delta\phi)^2 - \lambda f'(u)\phi^2}{\int_B \phi^2} = \mu_1(u)$$

in view of $\phi_1\phi_2 < -\phi_1\phi_2$ in a set of positive measure, leading to a contradiction.
So we can assume $\phi \geq 0$, and by the Boggi's principle we have $\phi > 0$ in B. For $0 \leq t \leq 1$ define

$$g(t) = \int_B \Delta[tU + (1-t)u]\,\Delta\phi - \lambda \int_B f(tU + (1-t)u)\phi,$$

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where ϕ is the above first eigenfunction. Since f is convex one sees that

$$g(t) \geq \lambda \int_B [tf(U) + (1-t)f(u) - f(tU + (1-t)u)] \phi \geq 0$$

for every $t \geq 0$. Since $g(0) = 0$ and

$$g'(0) = \int_B \Delta(U-u)\Delta\phi - \lambda f'(u)(U-u)\phi = 0,$$

we get that

$$g''(0) = -\lambda \int_B f''(u)(U-u)^2 \phi \geq 0.$$

Since $f''(u)\phi > 0$ in B, we finally get that $U = u$ a.e. in B. $\qquad\square$

Based again on Lemma 1.1(3), we can show a more general version of the above Lemma 2.2.

Lemma 2.3. *Let (α, β) be an admissible pair and $\beta' \leq 0$. Let u be a semi-stable $H^2(B)-$weak sub-solution of $(P)_{\lambda,\alpha,\beta}$ with $u = \alpha$, $\partial_\nu u = \beta' \geq \beta$ on ∂B. Assume that U is a $H^2(B)-$weak super-solution of $(P)_{\lambda,\alpha,\beta}$ with $U = \alpha$, $\partial_\nu U = \beta$ on ∂B. Then $U \geq u$ a.e. in B.*

Proof. Let $\tilde{u} \in H_0^2(B)$ denote a weak solution to $\Delta^2 \tilde{u} = \Delta^2(u - U)$ in B. Since $\tilde{u} - u + U = 0$ and $\partial_\nu(\tilde{u} - u + U) \leq 0$ on ∂B, by Lemma 1.1 one has that $\tilde{u} \geq u - U$ a.e. in B. Again by the Moreau decomposition [17], we may write $\tilde{u}$ as $\tilde{u} = w + v$, where $w, v \in H_0^2(B)$, $w \geq 0$ a.e. in B, $\Delta^2 v \leq 0$ in a $H^2(B)-$weak sense and $\int_B \Delta w \Delta v = 0$. Then for $0 \leq \phi \in C^4(\bar{B}) \cap H_0^2(B)$ one has

$$\int_B \Delta\tilde{u}\Delta\phi = \int_B \Delta(u-U)\Delta\phi \leq \lambda \int_B (f(u) - f(U))\phi.$$

In particular, we have that

$$\int_B \Delta\tilde{u}\Delta w \leq \lambda \int_B (f(u) - f(U))w.$$

Since by semi-stability of u

$$\lambda \int_B f'(u)w^2 \leq \int_B (\Delta w)^2 = \int_B \Delta\tilde{u}\Delta w,$$

we get that

$$\int_B f'(u)w^2 \leq \int_B (f(u) - f(U))w.$$

By Lemma 1.1 we have $v \leq 0$ and then $w \geq \tilde{u} \geq u - U$ a.e. in B. So we see that

$$0 \leq \int_B (f(u) - f(U) - f'(u)(u - U))\, w.$$

The strict convexity of f implies as in Lemma 2.2 that $U \geq u$ a.e. in B. $\qquad\square$

We shall need the following a-priori estimates along the minimal branch u_λ.

Lemma 2.4. *Let (α, β) be an admissible pair. Then one has*

$$2 \int_B \frac{(u_\lambda - \Phi)^2}{(1 - u_\lambda)^3} \leq \int_B \frac{u_\lambda - \Phi}{(1 - u_\lambda)^2},$$

where Φ is given in (2). In particular, there is a constant $C > 0$ so that for every $\lambda \in (0, \lambda^*)$, we have

$$\int_B (\Delta u_\lambda)^2 + \int_B \frac{1}{(1 - u_\lambda)^3} \leq C. \tag{4}$$

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Proof. Testing $(P)_{\lambda,\alpha,\beta}$ on $u_\lambda - \Phi \in C^4(\bar{B}) \cap H_0^2(B)$, we see that

$$\lambda \int_B \frac{u_\lambda - \Phi}{(1 - u_\lambda)^2} = \int_B \Delta u_\lambda \Delta(u_\lambda - \Phi) = \int_B (\Delta(u_\lambda - \Phi))^2 \geq 2\lambda \int_B \frac{(u_\lambda - \Phi)^2}{(1 - u_\lambda)^3}$$

in view of $\Delta^2 \Phi = 0$. In particular, for $\delta > 0$ small we have that

$$
\begin{aligned}
\int_{\{|u_\lambda - \Phi| \geq \delta\}} \frac{1}{(1 - u_\lambda)^3} &\leq \frac{1}{\delta^2} \int_{\{|u_\lambda - \Phi| \geq \delta\}} \frac{(u_\lambda - \Phi)^2}{(1 - u_\lambda)^3} \leq \frac{1}{\delta^2} \int_B \frac{1}{(1 - u_\lambda)^2} \\
&\leq \delta \int_{\{|u_\lambda - \Phi| \geq \delta\}} \frac{1}{(1 - u_\lambda)^3} + C_\delta
\end{aligned}
$$

by means of Young's inequality. Since for δ small,

$$\int_{\{|u_\lambda - \Phi| \leq \delta\}} \frac{1}{(1 - u_\lambda)^3} \leq C'$$

for some $C' > 0$, we can deduce that for every $\lambda \in (0, \lambda^*)$,

$$\int_B \frac{1}{(1 - u_\lambda)^3} \leq C$$

for some $C > 0$. By Young's and Hölder's inequalities, we now have

$$\int_B (\Delta u_\lambda)^2 = \int_B \Delta u_\lambda \Delta \Phi + \lambda \int_B \frac{u_\lambda - \Phi}{(1 - u_\lambda)^2} \leq \delta \int_B (\Delta u_\lambda)^2 + C_\delta + C \left(\int_B \frac{1}{(1 - u_\lambda)^3} \right)^{\frac{2}{3}}$$

and estimate (4) is therefore established. $\qquad \square$

We are now ready to establish Theorem 2.3.

Proof (of Theorem 2.3): (1) Since $\|u_\lambda\|_\infty < 1$, the infimum defining $\mu_1(u_\lambda)$ is achieved at a first eigenfunction for every $\lambda \in (0, \lambda^*)$. Since $\lambda \mapsto u_\lambda(x)$ is increasing for every $x \in B$, it is easily seen that $\lambda \mapsto \mu_1(u_\lambda)$ is an increasing, continuous function on $(0, \lambda^*)$. Define

$$\lambda_{**} := \sup\{0 < \lambda < \lambda^* : \mu_1(u_\lambda) > 0\}.$$

We have that $\lambda_{**} = \lambda^*$. Indeed, otherwise we would have that $\mu_1(u_{\lambda_{**}}) = 0$, and for every $\mu \in (\lambda_{**}, \lambda^*)$ u_μ would be a classical super-solution of $(P)_{\lambda_{**},\alpha,\beta}$. A contradiction arises since Lemma 2.2 implies $u_\mu = u_{\lambda_{**}}$. Finally, Lemma 2.2 guarantees uniqueness in the class of semi-stable $H^2(B)$−weak solutions.
(2) By estimate (4) it follows that $u_\lambda \to u^*$ in a pointwise sense and weakly in $H^2(B)$, and $\frac{1}{1-u^*} \in L^3(B)$. In particular, u^* is a $H^2(B)$−weak solution of $(P)_{\lambda^*,\alpha,\beta}$ which is also semi-stable as limiting function of the semi-stable solutions $\{u_\lambda\}$.
(3) Whenever $\|u^*\|_\infty < 1$, the function u^* is a classical solution, and by the Implicit Function Theorem we have that $\mu_1(u^*) = 0$ to prevent the continuation of the minimal branch beyond λ^*. By Lemma 2.2 u^* is then the unique $H^2(B)$−weak solution of $(P)_{\lambda^*,\alpha,\beta}$. An alternative approach –which we do not pursue here– based on the very definition of the extremal solution u^* is available in [4] when $\alpha = \beta = 0$ (see also [15]) to show that u^* is the unique weak solution of $(P)_{\lambda^*}$, regardless of whether u^* is regular or not.
(4) If $\lambda < \lambda^*$, by uniqueness $v = u_\lambda$. So v is not singular and a contradiction arises.

By Theorem 2.2(3) we have that $\lambda = \lambda^*$. Since v is a semi-stable $H^2(B)$−weak solution of $(P)_{\lambda^*,\alpha,\beta}$ and u^* is a $H^2(B)$−weak super-solution of $(P)_{\lambda^*,\alpha,\beta}$, we can apply Lemma 2.2 to get $v \leq u^*$ a.e. in B. Since u^* is a semi-stable solution too, we can reverse the roles of v and u^* in Lemma 2.2 to see that $v \geq u^*$ a.e. in B. So equality $v = u^*$ holds and the proof is done.

3   Regularity of the extremal solution for $1 \leq N \leq 8$

We now return to the issue of the regularity of the extremal solution in problem $(P)_\lambda$. Unless stated otherwise, u_λ and u^* refer to the minimal and extremal solutions of $(P)_\lambda$. We shall show that the extremal solution u^* is regular provided $1 \leq N \leq 8$. We first begin by showing that it is indeed the case in small dimensions:

Theorem 3.1. u^* *is regular in dimensions* $1 \leq N \leq 4$.

Proof. As already observed, estimate (4) implies that $f(u^*) = (1 - u^*)^{-2} \in L^{\frac{3}{2}}(B)$. Since u^* is radial and radially decreasing, we need to show that $u^*(0) < 1$ to get the regularity of u^*. The integrability of $f(u^*)$ along with elliptic regularity theory shows that $u^* \in W^{4,\frac{3}{2}}(B)$. By the Sobolev imbedding Theorem we get that u^* is a Lipschitz function in B.
Now suppose $u^*(0) = 1$ and $1 \leq N \leq 3$. Since

$$\frac{1}{1 - u} \geq \frac{C}{|x|} \qquad \text{in } B$$

for some $C > 0$, one sees that

$$+\infty = C^3 \int_B \frac{1}{|x|^3} \leq \int_B \frac{1}{(1 - u^*)^3} < +\infty.$$

A contradiction arises and hence u^* is regular for $1 \leq N \leq 3$.
For $N = 4$ we need to be more careful and observe that $u^* \in C^{1,\frac{1}{3}}(\bar{B})$ by the Sobolev Imbedding Theorem. If $u^*(0) = 1$, then $\nabla u^*(0) = 0$ and

$$\frac{1}{1 - u^*} \geq \frac{C}{|x|^{\frac{4}{3}}} \qquad \text{in } B$$

for some $C > 0$. We now obtain a contradiction exactly as above.   □

We now tackle the regularity of u^* for $5 \leq N \leq 8$. We start with the following crucial result:

Theorem 3.2. *Let* $N \geq 5$ *and* (u^*, λ^*) *be the extremal pair of* $(P)_\lambda$. *When* u^* *is singular, then*

$$1 - u^*(x) \leq C_0 |x|^{\frac{4}{3}} \qquad \text{in } B,$$

where $C_0 := \left(\frac{\lambda^*}{\bar{\lambda}}\right)^{\frac{1}{3}}$ *and* $\bar{\lambda} = \bar{\lambda}_N := \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3})$.

Proof. First note that Theorem 1.2(4) gives the lower bound:

$$\lambda^* \geq \bar{\lambda} = \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}). \tag{5}$$

For $\delta > 0$, we define $u_\delta(x) := 1 - C_\delta |x|^{\frac{4}{3}}$ with $C_\delta := \left(\frac{\lambda^*}{\bar{\lambda}} + \delta\right)^{\frac{1}{3}} > 1$. Since $N \geq 5$, we have that $u_\delta \in H^2_{loc}(\mathbb{R}^N)$, $\frac{1}{1 - u_\delta} \in L^3_{loc}(\mathbb{R}^N)$ and u_δ is a H^2-weak solution of

$$\Delta^2 u_\delta = \frac{\lambda^* + \delta\bar{\lambda}}{(1 - u_\delta)^2} \qquad \text{in } \mathbb{R}^N.$$

We claim that $u_\delta \leq u^*$ in B, which will finish the proof by just letting $\delta \to 0$.

Assume by contradiction that the set $\Gamma := \{r \in (0, 1) : u_\delta(r) > u^*(r)\}$ is non-empty, and let $r_1 = \sup \Gamma$. Since

$$u_\delta(1) = 1 - C_\delta < 0 = u^*(1),$$

we have that $0 < r_1 < 1$ and one infers that

$$\alpha := u^*(r_1) = u_\delta(r_1)\,, \quad \beta := (u^*)'(r_1) \geq u_\delta'(r_1).$$

Setting $u_{\delta,r_1}(r) = r_1^{-\frac{4}{3}}\left(u_\delta(r_1 r) - 1\right)+1$, we easily see that u_{δ,r_1} is a $H^2(B)-$weak super-solution of $(P)_{\lambda^*+\delta\bar\lambda_N,\alpha',\beta'}$, where

$$\alpha' := r_1^{-\frac{4}{3}}(\alpha - 1) + 1\,, \quad \beta' := r_1^{-\frac{1}{3}}\beta.$$

Similarly, let us define $u_{r_1}^*(r) = r_1^{-\frac{4}{3}}\left(u^*(r_1 r) - 1\right) + 1$. The dilation map

$$w \to w_{r_1}(r) = r_1^{-\frac{4}{3}}\left(w(r_1 r) - 1\right) + 1 \tag{6}$$

is a correspondence between solutions of $(P)_\lambda$ on B and of $(P)_{\lambda,1-r_1^{-\frac{4}{3}},0}$ on $B_{r_1^{-1}}$ which preserves the H^2-integrability. In particular, $(u_{r_1}^*, \lambda^*)$ is the extremal pair of $(P)_{\lambda,1-r_1^{-\frac{4}{3}},0}$ on $B_{r_1^{-1}}$ (defined in the obvious way). Moreover, $u_{r_1}^*$ is a singular semi-stable $H^2(B)-$ weak solution of $(P)_{\lambda^*,\alpha',\beta'}$.

Since u^* is radially decreasing, we have that $\beta' \leq 0$. Define the function w as $w(x) := (\alpha' - \frac{\beta'}{2}) + \frac{\beta'}{2}|x|^2 + \gamma(x)$, where γ is a solution of $\Delta^2\gamma = \lambda^*$ in B with $\gamma = \partial_\nu\gamma = 0$ on ∂B. Then w is a classical solution of

$$\begin{cases} \Delta^2 w = \lambda^* & \text{in } B \\ w = \alpha'\,, \quad \partial_\nu w = \beta' & \text{on } \partial B. \end{cases}$$

Since $\frac{\lambda^*}{(1-u_{r_1}^*)^2} \geq \lambda^*$, by Lemma 1.1 we have $u_{r_1}^* \geq w$ a.e. in B. Since $w(0) = \alpha' - \frac{\beta'}{2} + \gamma(0)$ and $\gamma(0) > 0$, the bound $u_{r_1}^* \leq 1$ a.e. in B yields to $\alpha' - \frac{\beta'}{2} < 1$. Namely, (α', β') is an admissible pair and by Theorem 2.3(4) we get that $(u_{r_1}^*, \lambda^*)$ coincides with the extremal pair of $(P)_{\lambda,\alpha',\beta'}$ in B.

Since (α', β') is an admissible pair and u_{δ,r_1} is a $H^2(B)-$weak super-solution of $(P)_{\lambda^*+\delta\bar\lambda_N,\alpha',\beta'}$, by Proposition 2.1 we get the existence of a weak solution of $(P)_{\lambda^*+\delta\bar\lambda_N,\alpha',\beta'}$. Since $\lambda^* + \delta\bar\lambda_N > \lambda^*$, we contradict the fact that λ^* is the extremal parameter of $(P)_{\lambda,\alpha',\beta'}$. $\qquad\square$

Thanks to this lower estimate on u^*, we get the following result.

Theorem 3.3. *If $5 \leq N \leq 8$, then the extremal solution u^* of $(P)_\lambda$ is regular.*

Proof. Assume that u^* is singular. For $\varepsilon > 0$ set $\psi(x) := |x|^{\frac{4-N}{2}+\varepsilon}$ and note that

$$(\Delta\psi)^2 = (H_N + O(\varepsilon))|x|^{-N+2\varepsilon}$$

where

$$H_N := \frac{N^2(N-4)^2}{16}.$$

Given $\eta \in C_0^\infty(B)$, and since $N \geq 5$, we can use the test function $\eta\psi \in H_0^2(B)$ into the stability inequality to obtain

$$2\lambda \int_B \frac{\psi^2}{(1-u^*)^3} \leq \int_B (\Delta\psi)^2 + O(1),$$

where $O(1)$ is a bounded function as $\varepsilon \searrow 0$. By Theorem 3.2 we find that

$$2\bar\lambda_N \int_B \frac{\psi^2}{|x|^4} \leq \int_B (\Delta\psi)^2 + O(1),$$

and then

$$2\bar\lambda_N \int_B |x|^{-N+2\varepsilon} \leq (H_N + O(\varepsilon))\int_B |x|^{-N+2\varepsilon} + O(1).$$

Computing the integrals one arrives at

$$2\bar\lambda_N \leq H_N + O(\varepsilon).$$

As $\varepsilon \to 0$ finally we obtain $2\bar\lambda_N \leq H_N$. Graphing this relation one sees that $N \geq 9$. $\qquad\square$

We can now slightly improve the lower bound (5).

Corollary 3.1. *In any dimension $N \geq 1$, we have*

$$\lambda^* > \bar{\lambda}_N = \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}). \tag{7}$$

Proof. The function $\bar{u} := 1 - |x|^{\frac{4}{3}}$ is a $H^2(B)-$ weak solution of $(P)_{\bar{\lambda}_N, 0, -\frac{4}{3}}$. If by contradiction $\lambda^* = \bar{\lambda}_N$, then $\bar{u}$ is a $H^2(B)-$weak super-solution of $(P)_\lambda$ for every $\lambda \in (0, \lambda^*)$. By Lemma 2.2 we get that $u_\lambda \leq \bar{u}$ for all $\lambda < \lambda^*$, and then $u^* \leq \bar{u}$ a.e. in B.

If $1 \leq N \leq 8$, u^* is then regular by Theorems 3.1 and 3.3. By Theorem 2.3(3) there holds $\mu_1(u^*) = 0$. Lemma 2.2 then yields that $u^* = \bar{u}$, which is a contradiction since then u^* will not satisfy the boundary conditions.

If now $N \geq 9$ and $\bar{\lambda} = \lambda^*$, then $C_0 = 1$ in Theorem 3.2, and we then have $u^* \geq \bar{u}$. It means again that $u^* = \bar{u}$, a contradiction that completes the proof. □

4   The extremal solution is singular for $N \geq 9$

We prove in this section that the extremal solution is singular for $N \geq 9$. For that we have to distinguish between three different ranges for the dimension. For each range, we will need a suitable Hardy-Rellich type inequality that will be established in the appendix, by using the recent results of Ghoussoub-Moradifam [10]. As in the previous section (u^*, λ^*) denotes the extremal pair of $(P)_\lambda$.

• **Case $N \geq 17$:** To establish the singularity of u^* for these dimensions we shall need the following well known improved Hardy-Rellich inequality, which is valid for $N \geq 5$. There exists $C > 0$, such that for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2 \, dx \geq \frac{N^2(N-4)^2}{16} \int_B \frac{\phi^2}{|x|^4} \, dx + C \int_B \phi^2 \, dx. \tag{8}$$

• **Case $10 \leq N \leq 16$:** For this case, we shall need the following inequality valid for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2 \geq \frac{(N-2)^2(N-4)^2}{16} \int_B \frac{\phi^2}{(|x|^2 - |x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} \tag{9}$$
$$+ \frac{(N-1)(N-4)^2}{4} \int_B \frac{\phi^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}.$$

• **Case $N = 9$:** This case is the trickiest and will require the following inequality for all $\phi \in H_0^2(B)$

$$\int_B (\Delta\phi)^2 \geq \int_B Q(|x|)\left(P(|x|) + \frac{N-1}{|x|^2}\right)\phi^2, \tag{10}$$

where

$$P(r) = \frac{\Delta_N \varphi}{\varphi} \quad \text{and} \quad Q(r) = \frac{\Delta_{N-2}\psi}{\psi},$$

with φ and ψ being two appropriately chosen polynomials, namely

$$\varphi(r) := r^{-\frac{N}{2}+1} + r - 1.9 \quad \text{and} \quad \psi(r) := r^{-\frac{N}{2}+2} + 20r^{-1.69} + 10r^{-1} + 10r + 7r^2 - 48.$$

Recall that for a radial function φ, we set $\Delta_N\varphi(r) = \varphi''(r) + \frac{(N-1)}{r}\varphi'(r)$.

We shall first show the following upper bound on u^*.

Lemma 4.1. *If $N \geq 9$, then $u^* \leq 1 - |x|^{\frac{4}{3}}$ in B.*

Proof. Recall from Corollary 3.1 that $\bar{\lambda} := \frac{8}{9}(N - \frac{2}{3})(N - \frac{8}{3}) < \lambda^*$. We now claim that $u_\lambda \leq \bar{u}$ for all $\lambda \in (\bar{\lambda}, \lambda^*)$. Indeed, fix such a λ and assume by contradiction that

$$R_1 := \inf\{0 \leq R \leq 1 : u_\lambda < \bar{u} \text{ in } (R, 1)\} > 0.$$

From the boundary conditions, one has that $u_\lambda(r) < \bar{u}(r)$ as $r \to 1^-$. Hence, $0 < R_1 < 1$, $\alpha := u_\lambda(R_1) = \bar{u}(R_1)$ and $\beta := u'_\lambda(R_1) \leq \bar{u}'(R_1)$. Introduce, as in the proof of Theorem 3.2, the functions u_{λ,R_1} and $\bar{u}_{R_1}$. We have that u_{λ,R_1} is a classical super-solution of $(P)_{\bar{\lambda}_N,\alpha',\beta'}$, where

$$\alpha' := R_1^{-\frac{4}{3}}(\alpha - 1) + 1, \quad \beta' := R_1^{-\frac{1}{3}}\beta.$$

Note that $\bar{u}_{R_1}$ is a $H^2(B)$−weak sub-solution of $(P)_{\bar{\lambda}_N,\alpha',\beta'}$ which is also semi-stable in view of the Hardy-Rellich inequality (8) and the fact that

$$2\bar{\lambda}_N \leq H_N := \frac{N^2(N-4)^2}{16}.$$

By Lemma 2.3, we deduce that $u_{\lambda,R_1} \geq \bar{u}_{R_1}$ in B. Note that, arguing as in the proof of Theorem 3.2, (α', β') is an admissible pair. We have therefore shown that $u_\lambda \geq \bar{u}$ in B_{R_1} and a contradiction arises in view of the fact that $\lim_{x\to 0} \bar{u}(x) = 1$ and $\|u_\lambda\|_\infty < 1$. It follows that $u_\lambda \leq \bar{u}$ in B for every $\lambda \in (\bar{\lambda}_N, \lambda^*)$, and in particular $u^* \leq \bar{u}$ in B. $\qquad\square$

The following lemma is the key for the proof of the singularity of u^* in higher dimensions.

Lemma 4.2. *Let $N \geq 9$. Suppose there exist $\lambda' > 0$, $\beta > 0$ and a singular radial function $w \in H^2(B)$ with $\frac{1}{1-w} \in L^\infty_{loc}(\bar{B} \setminus \{0\})$ such that*

$$\begin{cases} \Delta^2 w \leq \frac{\lambda'}{(1-w)^2} & \text{for } 0 < r < 1, \\ w(1) = 0, \ w'(1) = 0, \end{cases} \tag{11}$$

and

$$2\beta \int_B \frac{\phi^2}{(1-w)^3} \leq \int_B (\Delta\phi)^2 \ \text{ for all } \ \phi \in H^2_0(B), \tag{12}$$

1. *If $\beta \geq \lambda'$, then $\lambda^* \leq \lambda'$.*

2. *If either $\beta > \lambda'$ or if $\beta = \lambda' = \frac{H_N}{2}$, then the extremal solution u^* is necessarily singular.*

Proof: 1) First, note that (12) and $\frac{1}{1-w} \in L^\infty_{loc}(\bar{B} \setminus \{0\})$ yield to $\frac{1}{(1-w)^2} \in L^1(B)$. By a density argument, (11) implies now that w is a $H^2(B)$−weak sub-solution of $(P)_{\lambda'}$ whenever $N \geq 4$. If now $\lambda' < \lambda^*$, then by Lemma 2.3 w would necessarily be below the minimal solution $u_{\lambda'}$, which is a contradiction since w is singular while $u_{\lambda'}$ is regular.
2) Suppose first that $\beta = \lambda' = \frac{H_N}{2}$ and that $N \geq 9$. Since by part 1) we have $\lambda^* \leq \frac{H_N}{2}$, we get from Lemma 4.1 and the improved Hardy-Rellich inequality (8) that there exists $C > 0$ so that for all $\phi \in H^2_0(B)$

$$\int_B (\Delta\phi)^2 - 2\lambda^* \int_B \frac{\phi^2}{(1-u^*)^3} \geq \int_B (\Delta\phi)^2 - H_N \int_B \frac{\phi^2}{|x|^4} \geq C \int_B \phi^2.$$

It follows that $\mu_1(u^*) > 0$ and u^* must therefore be singular since otherwise, one could use the Implicit Function Theorem to continue the minimal branch beyond λ^*.
Suppose now that $\beta > \lambda'$, and let $\frac{\lambda'}{\beta} < \gamma < 1$ in such a way that

$$\alpha := \left(\frac{\gamma\lambda^*}{\lambda'}\right)^{1/3} < 1. \tag{13}$$

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Setting $\bar{w} := 1 - \alpha(1 - w)$, we claim that

$$u^* \leq \bar{w} \quad \text{in} \quad B. \tag{14}$$

Note that by the choice of α we have $\alpha^3 \lambda' < \lambda^*$, and therefore to prove (14) it suffices to show that for $\alpha^3 \lambda' \leq \lambda < \lambda^*$, we have $u_\lambda \leq \bar{w}$ in B. Indeed, fix such λ and note that

$$\Delta^2 \bar{w} = \alpha \Delta^2 w \leq \frac{\alpha \lambda'}{(1-w)^2} = \frac{\alpha^3 \lambda'}{(1-\bar{w})^2} \leq \frac{\lambda}{(1-\bar{w})^2}.$$

Assume that $u_\lambda \leq \bar{w}$ does not hold in B, and consider

$$R_1 := \sup\{0 \leq R \leq 1 \mid u_\lambda(R) > \bar{w}(R)\} > 0.$$

Since $\bar{w}(1) = 1 - \alpha > 0 = u_\lambda(1)$, we then have $R_1 < 1$, $u_\lambda(R_1) = \bar{w}(R_1)$ and $(u_\lambda)'(R_1) \leq (\bar{w})'(R_1)$. Introduce, as in the proof of Theorem 3.2, the functions u_{λ,R_1} and $\bar{w}_{R_1}$. We have that u_{λ,R_1} is a classical solution of $(P)_{\lambda,\alpha',\beta'}$, where

$$\alpha' := R_1^{-\frac{4}{3}}(u_\lambda(R_1) - 1) + 1, \quad \beta' := R_1^{-\frac{1}{3}}(u_\lambda)'(R_1).$$

Since $\lambda < \lambda^*$ and then

$$\frac{2\lambda}{(1-\bar{w})^3} \leq \frac{2\lambda^*}{\alpha^3(1-w)^3} = \frac{2\lambda'}{\gamma(1-w)^3} < \frac{2\beta}{(1-w)^3},$$

by (12) $\bar{w}_{R_1}$ is a stable $H^2(B)$–weak sub-solution of $(P)_{\lambda,\alpha',\beta'}$. By Lemma 2.3, we deduce that $u_\lambda \geq \bar{w}$ in B_{R_1} which is impossible, since $\bar{w}$ is singular while u_λ is regular. Note that, arguing as in the proof of Theorem 3.2, (α', β') is an admissible pair. This establishes claim (14) which, combined with the above inequality, yields

$$\frac{2\lambda^*}{(1-u^*)^3} \leq \frac{2\lambda^*}{\alpha^3(1-w)^3} < \frac{2\beta}{(1-w)^3},$$

and therefore

$$\inf_{\phi \in H_0^2(B)} \frac{\int_B (\Delta\phi)^2 - \frac{2\lambda^* \phi^2}{(1-u^*)^3}}{\int_B \phi^2} > 0.$$

It follows that again $\mu_1(u^*) > 0$ and u^* must be singular, since otherwise, one could use the Implicit Function Theorem to continue the minimal branch beyond λ^*.

Consider for any $m > 0$ the following function:

$$w_m := 1 - \frac{3m}{3m-4} r^{4/3} + \frac{4}{3m-4} r^m, \tag{15}$$

which satisfies the right boundary conditions: $w_m(1) = w_m'(1) = 0$. We can now prove that the extremal solution is singular for $N \geq 9$.

Theorem 4.1. *Let $N \geq 9$. The following upper bounds on λ^* hold:*

1. *If $N \geq 31$, then Lemma 4.2 holds with $w := w_2$, $\lambda' = 27\bar{\lambda}_N$ and $\beta = \frac{H_N}{2}$, and therefore $\lambda^*(N) \leq 27\bar{\lambda}_N$.*

2. *If $17 \leq N \leq 30$, then Lemma 4.2 holds with $w := w_3$, $\lambda' = \beta = \frac{H_N}{2}$, and therefore $\lambda^*(N) \leq \frac{H_N}{2}$.*

3. *If $10 \leq N \leq 16$, then Lemma 4.2 holds with $w := w_3$, $\lambda_N' < \beta_N$ given in Table 1, and therefore $\lambda^*(N) \leq \lambda_N'$.*

4. *If $N = 9$, then Lemma 4.2 holds with $w := w_{2.8}$, $\lambda_9' := 366 < \beta_9 := 368.5$, and therefore $\lambda^*(9) \leq 366$.*

The extremal solution is therefore singular for dimension $N \geq 9$.

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Table 1: Summary

N	w	λ'_N	β_N
9	$w_{2.8}$	366	366.5
10	w_3	450	487
11	w_3	560	739
12	w_3	680	1071
13	w_3	802	1495
14	w_3	940	2026
15	w_3	1100	2678
16	w_3	1260	3469
$17 \le N \le 30$	w_3	$H_N/2$	$H_N/2$
$N \ge 31$	w_2	$27\bar{\lambda}_N$	$H_N/2$

Proof. 1) Assume first that $N \ge 31$, then $27\bar{\lambda} \le \frac{H_N}{2}$. We shall show that w_2 is a singular $H^2(B)-$weak sub-solution of $(P)_{27\bar{\lambda}}$ so that (12) holds with $\beta = \frac{H_N}{2}$. Indeed, write

$$w_2 := 1 - |x|^{\frac{4}{3}} - 2(|x|^{\frac{4}{3}} - |x|^2) = \bar{u} - \phi_0,$$

where $\phi_0 := 2(|x|^{\frac{4}{3}} - |x|^2)$, and note that $w_2 \in H_0^2(B)$, $\frac{1}{1-w_2} \in L^3(B)$, $0 \le w_2 \le 1$ in B, and

$$\Delta^2 w_2 = \frac{3\bar{\lambda}}{r^{\frac{8}{3}}} \le \frac{27\bar{\lambda}}{(1-w_2)^2} \qquad \text{in } B \setminus \{0\}.$$

So w_2 is $H^2(B)-$weak sub-solution of $(P)_{27\bar{\lambda}}$. Moreover, by $\phi_0 \ge 0$ and (8) we get that

$$H_N \int_B \frac{\phi^2}{(1-w_2)^3} = H_N \int_B \frac{\phi^2}{(|x|^{\frac{4}{3}} + \phi_0)^3} \le H_N \int_B \frac{\phi^2}{|x|^4} \le \int_B (\Delta\phi)^2$$

for all $\phi \in H_0^2(B)$. It follows from Lemma 4.2 that u^* is singular and that $\lambda^* \le 27\bar{\lambda} \le \frac{H_N}{2}$.
2) Assume $17 \le N \le 30$ and consider the function

$$w_3 := 1 - \frac{9}{5}r^{\frac{4}{3}} + \frac{4}{5}r^3.$$

We show that w_3 is a semi-stable singular $H^2(B)-$weak sub-solution of $(P)_{\frac{H_N}{2}}$. Indeed, we clearly have that $0 \le w_3 \le 1$ in B, $w_3 \in H_0^2(B)$ and $\frac{1}{1-w_3} \in L^3(B)$. To show the stability condition, we consider $\phi \in H_0^2(B)$ and write

$$H_N \int_B \frac{\phi^2}{(1-w_3)^3} = 125 H_N \int_B \frac{\phi^2}{(9r^{\frac{4}{3}} - 4r^3)^3} \le 125 H_N \sup_{0<r<1} \frac{1}{(9-4r^{\frac{5}{3}})^3} \int_B \frac{\phi^2}{r^4}$$

$$= H_N \int_B \frac{\phi^2}{r^4} \le \int_B (\Delta\phi)^2$$

by virtue of (8). An easy computation shows that

$$\frac{H_N}{2(1-w_3)^2} - \Delta^2 w_3 = \frac{25 H_N}{2(9r^{\frac{4}{3}} - 4r^3)^2} - \frac{9\bar{\lambda}}{5r^{\frac{8}{3}}} - \frac{12}{5}\frac{N^2-1}{r}$$

$$= \frac{25 N^2 (N-4)^2}{32(9r^{\frac{4}{3}} - 4r^3)^2} - \frac{8(N - \frac{2}{3})(N - \frac{8}{3})}{5r^{\frac{8}{3}}} - \frac{12}{5}\frac{N^2-1}{r}.$$

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By using Maple one can verify that this final quantity is nonnegative on $(0,1)$ whenever $17 \leq N \leq 30$, and hence w_3 is a $H^2(B)-$weak sub-solution of $(P)_{\frac{H_N}{2}}$. It follows from Lemma 4.2 that u^* is singular and that $\lambda^* \leq \frac{H_N}{2}$.

3) Assume $10 \leq N \leq 16$. We shall prove that again $w := w_3$ satisfies the assumptions of Lemma 4.2. Indeed, using Maple, we show that for each dimension $10 \leq N \leq 16$, inequality (11) holds with λ'_N given by Table 1. Then, by using Maple again, we show that for each dimension $10 \leq N \leq 16$, the following inequality holds

$$\frac{(N-2)^2(N-4)^2}{16} \frac{1}{(|x|^2 - |x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} + \frac{(N-1)(N-4)^2}{4} \frac{1}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}$$

$$\geq \frac{2\beta_N}{(1-w_3)^3}.$$

where β_N is again given by Table 1. The above inequality and the Hardy-Rellich inequality (9) guarantee that the stability condition (12) holds with $\beta := \beta_N$. Since $\beta_N > \lambda'_N$, we deduce from Lemma 4.2 that the extremal solution is singular for $10 \leq N \leq 16$.

4) Suppose now $N = 9$ and consider $w := w_{2.8}$. Using Maple on can see that

$$\Delta^2 w \leq \frac{366}{(1-w)^2} \quad \text{in} \quad B$$

and

$$\frac{723}{(1-w)^3} \leq Q(r) \left(P(r) + \frac{N-1}{r^2} \right) \quad \text{for all} \ \ r \in (0,1),$$

where P and Q are given in (10). Since $723 > 2 \times 366$, by Lemma 4.2 the extremal solution u^* is singular in dimension $N = 9$. $\qquad \square$

5 Appendix: Improved Hardy-Rellich Inequalities

We now prove the improved Hardy-Rellich inequalities used in section 4. They rely on the results of Ghoussoub-Moradifam in [10] which provide necessary and sufficient conditions for such inequalities to hold. At the heart of this characterization is the following notion of a Bessel pair of functions.

Definition 5.1. *Assume that B is a ball of radius R in $\mathbb{R}^N$, $V, W \in C^1(0,1)$, and $\int_0^R \frac{1}{r^{N-1}V(r)} dr = +\infty$. Say that the couple (V,W) is a Bessel pair on $(0,R)$ if the ordinary differential equation*

$$(B_{V,W}) \qquad\qquad y''(r) + (\tfrac{N-1}{r} + \tfrac{V_r(r)}{V(r)})y'(r) + \tfrac{W(r)}{V(r)}y(r) = 0$$

has a positive solution on the interval $(0,R)$.

The space of radial functions in $C_0^\infty(B)$ will be denoted by $C_{0,r}^\infty(B)$. The needed inequalities will follow from the following result.

Theorem 5.1. (Ghoussoub-Moradifam [10]) *Let V and W be positive radial C^1-functions on $B\backslash\{0\}$, where B is a ball centered at zero with radius R in $\mathbb{R}^N$ $(N \geq 1)$ such that $\int_0^R \frac{1}{r^{N-1}V(r)} dr = +\infty$ and $\int_0^R r^{N-1}V(r)dr < +\infty$. The following statements are then equivalent:*

1. *(V,W) is a Bessel pair on $(0,R)$.*

2. *$\int_B V(|x|)|\nabla\phi|^2 dx \geq \int_B W(|x|)\phi^2 dx$ for all $\phi \in C_0^\infty(B)$.*

3. *If $\lim_{r\to 0} r^\alpha V(r) = 0$ for some $\alpha < N - 2$, then the above are equivalent to*

$$\int_B V(|x|)(\Delta\phi)^2 dx \geq \int_B W(|x|)|\nabla\phi|^2 dx + (N-1)\int_B (\frac{V(|x|)}{|x|^2} - \frac{V_r(|x|)}{|x|})|\nabla\phi|^2 dx$$

   for all $\phi \in C_{0,r}^\infty(B)$.

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4. If in addition, $W(r) - \frac{2V(r)}{r^2} + \frac{2V_r(r)}{r} - V_{rr}(r) \geq 0$ on $(0, R)$, then the above are equivalent to

$$\int_B V(|x|)(\Delta\phi)^2 dx \geq \int_B W(|x|)|\nabla\phi|^2 dx + (N-1)\int_B \left(\frac{V(|x|)}{|x|^2} - \frac{V_r(|x|)}{|x|}\right)|\nabla\phi|^2 dx$$

for all $\phi \in C_0^\infty(B)$.

We shall now deduce the following corollary.

Corollary 5.1. Let $N \geq 5$ and B be the unit ball in $\mathbb{R}^N$. Then the following improved Hardy-Rellich inequality holds for all $\phi \in C_0^\infty(B)$:

$$
\begin{aligned}
\int_B (\Delta\phi)^2 \quad \geq \quad & \frac{(N-2)^2(N-4)^2}{16} \int_B \frac{\phi^2}{(|x|^2 - |x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})} \\
& + \frac{(N-1)(N-4)^2}{4} \int_B \frac{\phi^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}
\end{aligned}
\tag{16}
$$

Proof. Let $0 < \alpha < 1$ and define $y(r) := r^{-\frac{N}{2}+1} - \alpha$. Since

$$-\frac{y'' + \frac{(N-1)}{r}y'}{y} = \frac{(N-2)^2}{4}\frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}},$$

the couple $\left(1, \frac{(N-2)^2}{4}\frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}}\right)$ is a Bessel pair on $(0,1)$. By Theorem 5.1(4) the following inequality then holds:

$$\int_B (\Delta\phi)^2 dx \geq \frac{(N-2)^2}{4}\int_B \frac{|\nabla\phi|^2}{|x|^2 - \alpha|x|^{\frac{N}{2}+1}} + (N-1)\int_B \frac{|\nabla\phi|^2}{|x|^2} \tag{17}$$

for all $\phi \in C_0^\infty(B)$. Set $V(r) := \frac{1}{r^2 - \alpha r^{\frac{N}{2}+1}}$ and note that

$$\frac{V_r}{V} = -\frac{2}{r} + \frac{\alpha(N-2)}{2}\frac{r^{\frac{N}{2}-2}}{1 - \alpha r^{\frac{N}{2}-1}} \geq -\frac{2}{r}.$$

The function $y(r) = r^{-\frac{N}{2}+2} - 1$ is decreasing and is then a positive super-solution on $(0,1)$ for the ODE

$$y'' + \left(\frac{N-1}{r} + \frac{V_r}{V}\right)y'(r) + \frac{W_1(r)}{V(r)}y = 0,$$

where

$$W_1(r) = \frac{(N-4)^2}{4(r^2 - r^{\frac{N}{2}})(r^2 - \alpha r^{\frac{N}{2}+1})}.$$

Hence, by Theorem 5.1(2) we deduce

$$\int_B \frac{|\nabla\phi|^2}{|x|^2 - \alpha|x|^{\frac{N}{2}+1}} \geq \left(\frac{N-4}{2}\right)^2 \int_B \frac{\phi^2}{(|x|^2 - \alpha|x|^{\frac{N}{2}+1})(|x|^2 - |x|^{\frac{N}{2}})}$$

for all $\phi \in C_0^\infty(B)$. Similarly, for $V(r) = \frac{1}{r^2}$ we have that

$$\int_B \frac{|\nabla\phi|^2}{|x|^2} \geq \left(\frac{N-4}{2}\right)^2 \int_B \frac{\phi^2}{|x|^2(|x|^2 - |x|^{\frac{N}{2}})}$$

for all $\phi \in C_0^\infty(B)$. Combining the above two inequalities with (17) and letting $\alpha \to 1$ we get inequality (16). $\square$

Corollary 5.2. *Let $N = 9$ and B be the unit ball in $\mathbb{R}^N$. Define $\varphi(r) := r^{-\frac{N}{2}+1} + r - 1.9$ and $\psi(r) := r^{-\frac{N}{2}+2} + 20r^{-1.69} + 10r^{-1} + 10r + 7r^2 - 48$. Then the following improved Hardy-Rellich inequality holds for all $\phi \in C_0^\infty(B)$:*

$$\int_B (\Delta\phi)^2 \geq \int_B Q(|x|)\left(P(|x|) + \frac{N-1}{|x|^2}\right)\phi^2, \tag{18}$$

where

$$P(r) := -\frac{\varphi''(r) + \frac{N-1}{r}\varphi'(r)}{\varphi(r)} \quad and \quad Q(r) := -\frac{\psi''(r) + \frac{N-3}{r}\psi'(r)}{\psi(r)}.$$

Proof. By definition $(1, P(r))$ is a Bessel pair on $(0, 1)$. One can easily see that $P(r) \geq \frac{2}{r^2}$. Hence, by Theorem 5.1(4) the following inequality holds:

$$\int_B (\Delta\phi)^2 dx \geq \int_B P(|x|)|\nabla\phi|^2 + (N-1)\int_B \frac{|\nabla\phi|^2}{|x|^2} \tag{19}$$

for all $\phi \in C_0^\infty(B)$. Using Maple it is easy to see that

$$\frac{P_r}{P} \geq -\frac{2}{r} \quad \text{in} \quad (0, 1),$$

and therefore $\psi(r)$ is a positive super-solution for the ODE

$$y'' + \left(\frac{N-1}{r} + \frac{P_r(r)}{P(r)}\right)y'(r) + \frac{P(r)Q(r)}{P(r)}y = 0,$$

on $(0, 1)$. Hence, by Theorem 5.1(2) we have for all $\phi \in C_0^\infty(B)$

$$\int_B P(|x|)|\nabla\phi|^2 \geq \int_B P(|x|)Q(|x|)\phi^2,$$

and similarly

$$\int_B \frac{|\nabla\phi|^2}{|x|^2} \geq \int_B \frac{Q(|x|)}{|x|^2}\phi^2,$$

since $\psi(r)$ is a positive solution for the ODE

$$y'' + \frac{N-3}{r}y'(r) + Q(r)y = 0.$$

Combining the above two inequalities with (19) we get (18). $\qquad\square$

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