Equilibria of point-vortices on closed surfaces

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ABSTRACT. We discuss the existence of equilibrium configurations for the Hamiltonian point-vortex model on a closed surface Σ. The topological properties of Σ determine the occurrence of three distinct situations, corresponding to $\mathbb{S}^2$, to $\mathbb{RP}^2$ and to $\Sigma \neq \mathbb{S}^2, \mathbb{RP}^2$. As a by-product, we also obtain new existence results for the singular mean-field equation with exponential nonlinearity.

1. INTRODUCTION

Let Σ be a closed surface (i.e. compact and without boundary) endowed with a metric tensor g. We are concerned with equilibrium configurations of the Hamilton function

$$\mathcal{H}_0(\boldsymbol{\xi}) = \sum_i \Gamma_i^2 H(\xi_i, \xi_i) + \sum_{i \neq j} \Gamma_i \Gamma_j G(\xi_i, \xi_j)$$

for $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{N_0}) \in \Sigma^{N_0} \cap \{\xi_i \neq \xi_j \text{ for } i \neq j\}$, where $G(x, p)$ is the Green's function of $-\Delta_g$ over Σ with singularity at p and $H(x, p)$ is its regular part.

In an inviscid and incompressible fluid, the velocity field and the pressure obey the Euler equations. For a two-dimensional turbulent flow, the point-vortex ansatz $\omega = \sum_{i=1}^{N_0} \Gamma_i \delta_{\xi_i(t)}$ for the (scalar) vorticity function ω leads to the following Hamiltonian system:

$$\Gamma_i \, \partial_t \xi_i = J \, \nabla_{\xi_i} \mathcal{H}_0(\boldsymbol{\xi}) \qquad \forall \, i = 1, \dots, N_0, \tag{1.1}$$

where J denotes the symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The quantity $\Gamma_i \in \mathbb{R} \setminus \{0\}$ is the strength of the point-vortex ξ_i, whose sign determines the clockwise/counterclockwise rotation of the fluid near ξ_i. Based on ideas of Helmoltz ([33]), (1.1) has been derived by Kirchhoff ([37]) in $\mathbb{R}^2$. Extended by Routh ([47]) to a bounded domain in terms of the so-called hydrodynamic Green function (see also [38, 39]), the renormalized kinetic energy $\mathcal{H}_0$ is referred to as the Kirchhoff-Routh path function. The interested reader can look at [1, 45] for the case of a surface (like spheres, cylinders or tori), and refer to [32, 42, 44, 45, 48] for a modern treatment of the topic.

Apart from $\mathbb{R}^2$ and the case of special domains (like discs, half-discs, annuli, strips), very few is known concerning the existence of equilibrium configurations for $\mathcal{H}_0$. On a closed surface, notice that $\mathcal{H}_0$ has always a minimum point when the point-vortices have the same orientation (say, $\Gamma_i \geq 0$ for all $i = 1, \dots, N_0$). The presence of counter rotating vortices makes the problem very difficult. On a bounded domain, when $N_0 \leq 4$ point-vortices of alternating orientations have been considered in [9] with $\Gamma_i = (-1)^i$ and in [8] for the general case (see also [6] for $N_0 = 2$).

The assumption on N_0 prevents the collision of some ξ_i's with opposite orientations, the simplest case being given by three point-vortices with $\Gamma_i = 1$ collapsing onto one with $\Gamma_i = -1$ (see [31] in a PDE context).

In this paper we address the case where all the point-vortices with negative orientation are kept fixed. Denoting them by $p_1, \dots, p_\ell$ with strengths $-\frac{\alpha_1}{2}, \dots, -\frac{\alpha_\ell}{2}$, we are led to study

$$\mathcal{H}(\boldsymbol{\xi}) = \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \Gamma_j \Gamma_k G(\xi_j, \xi_k) - \sum_{i=1}^{\ell} \alpha_i \sum_{j=1}^{N} \Gamma_j G(\xi_j, p_i) + \sum_{j=1}^{N} h(\xi_j) \qquad (1.2)$$

for $\boldsymbol{\xi} = (\xi_1, \dots, \xi_N) \in \mathcal{M}$, where $N = N_0 - \ell$, $\alpha_i, \Gamma_i > 0$, $h \in C^1(\Sigma, \mathbb{R})$ and

$$\mathcal{M} = (\Sigma \setminus \{p_1, \dots, p_\ell\})^N \setminus \Delta, \quad \Delta = \{\boldsymbol{\xi} \in \Sigma^N : \ \xi_j = \xi_k \text{ for some } j \neq k\}.$$

Inspired by some arguments in [4, 10], the main aim of our paper is to investigate the interaction of the topology of Σ with the presence of singular sources $p_1, \dots, p_\ell$ toward the existence of equilibria for $\mathcal{H}$. As we will see below, the three cases $\Sigma = \mathbb{S}^2$, $\Sigma = \mathbb{RP}^2$ and $\Sigma \neq \mathbb{S}^2, \mathbb{RP}^2$ exhibit completely different phenomena.

The critical point of $\mathcal{H}$ will be found at the max-min energy level

$$\mathcal{H}^* = \sup_{\gamma \in \mathcal{F}} \min_{\boldsymbol{\xi} \in K} \mathcal{H}(\gamma(\boldsymbol{\xi})),$$

where $\mathcal{F}$ collects a suitable family of deformation maps from K into an open set $\mathcal{D} \subset\subset \mathcal{M}$ that keep fixed $K_0 \subset K$ (for some compact sets K, K_0). To prevent the collapsing for part of the ξ_j's onto some p_i, the following compactness condition is crucial:

$$\alpha_i \notin \left\{ \left(\sum_{\substack{j,k \in J \\ j \neq k}} \Gamma_j \Gamma_k \right) \left(\sum_{j \in J} \Gamma_j \right)^{-1} : \ J \subset \{1, \dots, N\} \right\} \qquad \forall\, i = 1, \dots, \ell. \qquad (1.3)$$

When $\Gamma_1 = \cdots = \Gamma_N = 1$, notice that (1.3) simply reduces to

$$\alpha_i \neq 1, \dots, N-1 \quad \forall\, i = 1, \dots, \ell. \qquad (1.4)$$

To produce the linking structure

$$\mathcal{H}^* < \min_{\boldsymbol{\xi} \in K_0} \mathcal{H}(\boldsymbol{\xi}),$$

we need that a crucial intersection property is accomplished: more precisely, by applying a topological degree argument, for all $\gamma \in \mathcal{F}$ we catch a point $\boldsymbol{\xi}_\gamma^* \in K$ with prescribed $\mathcal{P}_j(\gamma_j(\boldsymbol{\xi}_\gamma^*))$, $j = 1, \dots, N$, for suitable retraction maps $\mathcal{P}_j$. When $\Sigma \neq \mathbb{S}^2, \mathbb{RP}^2$, we take $\mathcal{P}_j = \mathcal{P}$ for all $j = 1, \dots, N$, $\mathcal{P}$ being a retraction of Σ onto a simple closed curve $\sigma \subset \Sigma \setminus \{p_1, \dots, p_\ell\}$. Since the fibers of $\mathcal{P}$ are well separated, the value $\mathcal{H}^*$ is uniformly (with respect to K_0) bounded from above, whereas $\min_{\boldsymbol{\xi} \in K_0} \Psi(\boldsymbol{\xi})$ can be made arbitrarily large by a suitable choice of K_0. Our first main result then reads as follows:

Theorem 1.1. *Let Σ be a closed surface topologically different from $\mathbb{S}^2$ and $\mathbb{RP}^2$. If (1.3) does hold, then $\mathcal{H}$ has a critical point.*

When $\Sigma = \mathbb{RP}^2$, every map $\mathcal{P}_j$, $j = 1, \ldots, N$, can be taken instead as a retraction $\mathcal{P}$ of $\mathbb{RP}^2 \setminus \{p_i\}$ onto a simple closed curve $\sigma \subset \Sigma \setminus \{p_1, \ldots, p_\ell\}$ for a fixed $i = 1, \ldots, \ell$. In this case the fibers of $\mathcal{P}$ are curves emanating from the singular source p_i and the assumption

$$\sum_{\substack{j,k=1 \\ j \neq k}}^N \Gamma_j \Gamma_k < \alpha_i \sum_{j=1}^N \Gamma_j \tag{1.5}$$

is required to assure that the mutual interactions between the components of $\boldsymbol{\xi}$ are dominated by the interplay between each component with p_i, which is essential to get a uniform control from above on $\mathcal{H}^*$. So, our second main result is the following:

Theorem 1.2. *Let Σ be a closed surface topologically equivalent to $\mathbb{RP}^2$. If (1.3) and*

$$\left(\sum_{\substack{j,k=1 \\ j \neq k}}^N \Gamma_j \Gamma_k \right) \left(\sum_{j=1}^N \Gamma_j \right)^{-1} < \max\{\alpha_1, \ldots, \alpha_\ell\} \tag{1.6}$$

do hold, then $\mathcal{H}$ has a critical point.

The Euclidean case ([22, 25]), which has been the starting point for our investigation, has a strong analogy with $\mathbb{RP}^2$. When $\Gamma_1 = \cdots = \Gamma_N = 1$, (1.6) becomes

$$N - 1 < \max\{\alpha_1, \ldots, \alpha_\ell\}.$$

The case $\Sigma = \mathbb{S}^2$ is more involved since $\mathbb{S}^2 \setminus \{p\}$ is contractible, therefore it is essential that we remove two points from $\mathbb{S}^2$ in order to find a suitable retraction. So, first of all we need to assume $\ell \geq 2$. Let us split

$$\{1, \ldots, N\} = \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_\ell$$

with disjoint union and set $N_i = \#\mathcal{N}_i \geq 0$. Then, each $i = 1, \ldots, \ell$ has to be coupled with $r(i) \neq i$ and we choose $\mathcal{P}_j = \mathcal{P}_i$ for all $j \in \mathcal{N}_i$, $\mathcal{P}_i$ being a retraction of $\mathbb{S}^2 \setminus \{p_i, p_{r(i)}\}$ onto a simple closed curve $\sigma_i \subset \mathbb{S}^2 \setminus \{p_1, \ldots, p_\ell\}$. The fibers of $\mathcal{P}_i$ are curves between p_i and $p_{r(i)}$, and then part of such N_i points could approach not only p_i but also $p_{r(i)}$. Now, by exchanging the role of i and r, for every $i = 1, \ldots, \ell$ we define the set $J_i \subset \{1, \ldots, \ell\} \setminus \{i\}$, as made up of those indices which are coupled in the above construction with i, and then $\{1, \ldots, \ell\}$ is the disjoint union of such J_i, $i = 1, \ldots, \ell$. To obtain an upper bound on $\mathcal{H}^*$ we need to require

$$\sum_{\substack{j,k \in \tilde{\mathcal{N}}_i \\ j \neq k}} \Gamma_j \Gamma_k < \alpha_i \sum_{j \in \tilde{\mathcal{N}}_i} \Gamma_j \quad \forall i = 1, \ldots, \ell, \tag{1.7}$$

where

$$\tilde{\mathcal{N}}_i = \mathcal{N}_i \cup \bigcup_{r \in J_i} \mathcal{N}_r.$$

When $\Gamma_1 = \cdots = \Gamma_N = 1$, notice that (1.7) turns into

$$N_i + \sum_{r \in J_i} N_r - 1 < \alpha_i \quad \forall i = 1, \ldots, \ell. \tag{1.8}$$

Our third main result reads as:

Theorem 1.3. *Let Σ be a closed surface topologically equivalent to $\mathbb{S}^2$ and $\ell \geq 2$. If (1.3) and (1.7) do hold, then $\mathcal{H}$ has a critical point.*

Hereafter, we restrict our attention to the case $\Gamma_1 = \cdots = \Gamma_N = 1$. The corresponding $\mathcal{H}$ can also be seen as the reduced energy ([16, 17, 29]) for the following singular mean-field problem

$$-\Delta_g u = \lambda \left(\frac{\kappa(x)e^u}{\int_\Sigma \kappa(x)e^u \, dV_g} - \frac{1}{|\Sigma|} \right) - 4\pi \sum_{i=1}^{\ell} \alpha_i \left(\delta_{p_i} - \frac{1}{|\Sigma|} \right) \qquad (1.9)$$

when looking for solutions blowing-up at distinct points $\xi_1, \ldots, \xi_N \in \Sigma \setminus \{p_1, \ldots, p_\ell\}$. Here, λ is a parameter close to $8\pi N$, $\kappa : \Sigma \to \mathbb{R}$ is a smooth positive function, and $p_1, \ldots, p_\ell \in \Sigma$ are singular sources with $\alpha_i > 0$. We denote by δ_p the Dirac measure supported at p, by dV_g the area element in (Σ, g) and by $|\Sigma| = \int_\Sigma dV_g$ the area of Σ.

Regular mean-field equations naturally arise in conformal geometry ([13, 14, 35]), in statistical mechanics ([11, 12, 15, 36]) and in the study of turbulent Euler flows ([50]). The singularities can model Euler flows interacting with sinks of opposite vorticities ([54]) or conical singularities on a surface ([4, 21, 53]), and naturally arise in connection with the Chern-Simons-Higgs model ([20, 26, 40, 46, 51, 55]) and the Electroweak theory ([3, 7, 49]). To attack existence issues, one can compute the topological degree ([16, 17, 18, 19, 20]), use a min-max variational approach ([4, 5, 7, 27, 43]) or perturbative arguments in the regime $\lambda \to 8\pi N$ ([16, 17, 29], see also [2, 23, 24, 25, 30, 41]). The topological degree is non-zero when $\Sigma \neq \mathbb{S}^2, \mathbb{RP}^2$, $\alpha_j \in \mathbb{N}$ and $\lambda \notin 8\pi\mathbb{N}$ ([19]). For $\mathbb{S}^2$ this is still true ([20]) when $\lambda \in (8\pi, 16\pi)$ and $\ell \geq 2$, but the topological degree vanishes in several cases like:

- $\ell = 1$ and $\lambda \in (8\pi, 8\pi(1 + \alpha_1)) \cup (8\pi(2 + \alpha_1), +\infty)$ (which is consistent with the necessary condition ([52]) for the existence: $\lambda \leq 8\pi$ or $\lambda \geq 8\pi(1 + \alpha_1)$);
- $\ell = 2$ and $\lambda \in (8\pi(1 + \alpha_1), 8\pi(1 + \alpha_2)) \cup (8\pi(2 + \alpha_1 + \alpha_2), +\infty)$ if $\alpha_1 \leq \alpha_2$ (in agreement with the necessary condition ([5]) for the existence: $\lambda < 8\pi(1 + \alpha_1)$ or $\lambda > 8\pi(1 + \alpha_2)$).

In a similar way, the topological degree can vanish when $\Sigma = \mathbb{RP}^2$. An alternative variational approach is also available, which is completely general for $\Sigma \neq \mathbb{S}^2, \mathbb{RP}^2$ ([4], see also [10]) and requires the following restrictions ([5, 43]) when $\Sigma = \mathbb{S}^2$:

a) $\alpha_1, \ldots, \alpha_\ell \leq 1$, $\lambda \in (8\pi, 16\pi) \setminus \{8\pi(1 + \alpha_1), \ldots, 8\pi(1 + \alpha_\ell)\}$ and $\#J_\lambda \geq 2$, where

$$J_\lambda = \{i = 1, \ldots, \ell : \lambda < 8\pi(1 + \alpha_i)\};$$

b) $\ell \geq 2$ and $\lambda \in (0, 8\pi \min_{i=1,\ldots,\ell} (1 + \alpha_i)) \setminus 8\pi\mathbb{N}$, which can be stated equivalently as $\#J_\lambda = \ell \geq 2$.

In the special regime $\lambda \to 8\pi N$ solutions of (1.9) may possibly exhibit concentration phenomena, a property of definite physical interest since the right hand side of (1.9) represents precisely the vorticity of the Euler flow. The concentration points, if different from $p_1, \ldots, p_\ell$, have to correspond to a critical point of a reduced energy having $\mathcal{H}$ (with $h(\xi) = H(\xi, \xi) + \frac{1}{4\pi} \log \kappa(\xi)$) as main order term. The existence of such concentrated solutions has been addressed, among other things, in [17] for non-degenerate critical points $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ of $\mathcal{H}$ with non-vanishing

$$A(\boldsymbol{\xi}) = \sum_{j=1}^{N} \kappa(\xi_j) e^{8\pi H(x,\xi_j) - 4\pi \sum_{i=1}^{\ell} \alpha_i G(\xi_j, p_i) + 8\pi \sum_{k \neq j} G(x,\xi_k)} \left[\Delta_g \log \kappa(\xi_j) + 4\pi \frac{2N - \Lambda}{|\Sigma|} - 2K(\xi_j) \right],$$

where $\Lambda = \sum_{i=1}^{\ell} \alpha_i$ and K is the Gaussian curvature of (Σ, g). The critical points provided by Theorems 1.1, 1.2, 1.3 may possibly be degenerate, but the critical value $\mathcal{H}^*$ is stable with respect to small C^1-perturbations of $\mathcal{H}$, since it has been found by a max-min scheme. Thanks

to the result in [29], stable critical values of $\mathcal{H}$, under the sign assumption $A < 0$ ($A > 0$, respectively), give rise to a family of solutions u_λ for (1.9) such that

$$\frac{\lambda \kappa(x) e^{u_\lambda}}{\int_\Sigma \kappa(x) e^{u_\lambda} dV_g} \to 8\pi \sum_{j=1}^N \delta_{\xi_j}$$

as $\lambda \to 8\pi N^-$ ($\lambda \to 8\pi N^+$, respectively) in the measure sense, for a critical point $\boldsymbol{\xi}$ of $\mathcal{H}$ with $\mathcal{H}(\boldsymbol{\xi}) = \mathcal{H}^*$. Consequently, as a byproduct of Theorems 1.1, 1.2 and 1.3, we provide solutions of multi-bubble type to equation (1.9) in the special regime $\lambda \to 8\pi N$. In many cases, we obtain the perturbative counterparts of global existence results already available in literature, obtained via degree theory or a global variational approach. However, compared to such previous results, in some situations one can still have existence when the degree of the equation vanishes even beyond the threshold on λ imposed by a) and b), as we will see by explicit examples (see Remark 1.5 and Example 1.6 below).

Setting

$$[\alpha]^- = \max\{n \mid n \in \mathbb{Z}, \ n < \alpha\} \qquad \forall \alpha \in \mathbb{R},$$

for $\Gamma_1 = \cdots = \Gamma_N = 1$ we summarize Theorems 1.1, 1.2 and 1.3 as follows:

Theorem 1.4. *Assume that (1.4) holds for N. Then*

$$\mathcal{H}(\boldsymbol{\xi}) = \sum_{\substack{j,k=1 \\ j \neq k}}^N G(\xi_j, \xi_k) - \sum_{i=1}^\ell \alpha_i \sum_{j=1}^N G(\xi_j, p_i) + \sum_{j=1}^N h(\xi_j)$$

has a C^1-stable critical value $\mathcal{H}^*$ if

- $\Sigma \neq \mathbb{S}^2, \mathbb{RP}^2$;
- $\Sigma = \mathbb{RP}^2$ *and*

$$N \leq \max\{1 + [\alpha_1]^-, \ldots, 1 + [\alpha_\ell]^-\};$$

- $\Sigma = \mathbb{S}^2$, $\ell \geq 2$ *and*

$$N_i + \sum_{r \in J_i} N_r \leq 1 + [\alpha_i]^- \qquad \forall\, i = 1, \ldots, \ell, \tag{1.10}$$

for $N_i \in \mathbb{N} \cup \{0\}$ and disjoint subsets $J_i \subset \{1, \ldots, \ell\} \setminus \{i\}$, $i = 1, \ldots, \ell$, so that

$$N = \sum_{i=1}^\ell N_i, \qquad \{1, \ldots, \ell\} = \bigcup_{i=1}^\ell J_i.$$

Remark 1.5. *For $\lambda \to 8\pi N$, on the standard sphere $(\mathbb{S}^2, g_0)$ let us compare (1.10) with a)$-$b). Assumption (1.10) turns out to be more general for $\ell \geq 3$ by allowing larger values of λ: notice that N may possibly overcome the value $\max_i(1 + \alpha_i)$. Moreover, the choice $N_i = 0$ for all $i \neq i_1, i_2$ and $J_{i_1} = \{i_2\}$, $J_{i_2} = \{i_1\}$ shows that $\#J_\lambda \geq 2$ implies the validity of (1.10), extending a) to general α_i and b). Since in this case $A < 0$ if $\kappa = 1$ in view of $K = 4\pi|\Sigma|^{-1}$, Theorem 1.4 provides new existence results for equation (1.9) on $(\mathbb{S}^2, g_0)$ with $\kappa = 1$ when $\lambda \to 8\pi N$ compared to [5, 43].*

Example 1.6. *Consider equation (1.9) on the standard sphere $(\mathbb{S}^2, g_0)$ with $\kappa = 1$ and let*

$$\ell = 3, \quad \alpha_1 = \alpha_2 = \alpha \in (1, 2), \quad \alpha_3 \geq 4.$$

According to the degree formula computed by Chen-Lin ([19]), it can be easily checked that the degree vanishes for $\lambda \in (8\pi(2+\alpha), 32\pi)$; moreover the existence results in [5, 43] do not work in such an interval since neither assumption a) nor assumption b) are satisfied. On the other hand, (1.10) is verified by taking $J_1 = \emptyset$, $J_2 = \{3\}$, $J_3 = \{1,2\}$, and $N_1 = N_2 = 2$, $N_3 = 0$, and it is immediate to check that $A < 0$ if $\kappa = 1$. Then, as a byproduct of Theorem 1.4, we deduce the existence of a solution to the Liouville equation (1.9) for λ in a small left neighborhood of 32π with $N = 4$ blow-up points. This example provides a new existence result in a perturbative regime for equation (1.9) which is not covered neither by the degree theory ([19]) nor by variational methods ([5, 43]).

Assumption (1.10) comes from (1.8) but is quite involved in such a general form. Finally, consider Theorem 1.4 restricted to the case $\#J_i = 1$, $i = 1, \ldots, \ell$, which, up to re-ordering, simply means that each p_i is coupled (in the construction of $\mathcal{P}_i$) with p_{i+1} (with the convention $p_{\ell+1} = p_1$, $\alpha_{\ell+1} = \alpha_1$ and $N_{\ell+1} = N_1$). Referred to as a consecutive coupling of the p_i's, assumption (1.10) reduces to

$$N_i + N_{i+1} \leq 1 + [\alpha_{i+1}]^- \qquad \forall\, i = 1, \ldots, \ell. \tag{1.11}$$

From now on we will use the following notation: the quantities $a_i = 1 + [\alpha_i]^-$, $i = 1, \ldots, \ell$, correspond to a general consecutive coupling, whereas $b_1, \ldots, b_\ell$ will denote the increasing ordering with $\alpha_1 \leq \cdots \leq \alpha_\ell$. Given $J \subset \{1, \ldots, k\}$, for any $2 \leq k \leq \frac{\ell}{2}$ let us define

$$s_k(J) = \sum_{j=2}^{k} \left[\chi_J(j) a_{2j} + (1 - \chi_J(j)) \left(a_{2j+1} + \chi_J(j-1) \min\{a_{2j-1}, a_{2j}\} \right) \right],$$

where χ_J denotes the characteristic function of J. Set $c_1 = a_1$, $g_1 = a_3$, $d_1 = f_1 = +\infty$, and for $k \geq 2$

$$c_k = \min\{a_2 + s_k(J) : J \text{ s.t. } 1, k \in J\}, \quad d_k = \min\{a_3 + s_k(J) : J \text{ s.t. } 1 \notin J, k \in J\}$$

and

$$f_k = \min\{a_2 + s_k(J) : J \text{ s.t. } 1 \in J, k \notin J\}, \quad g_k = \min\{a_3 + s_k(J) : J \text{ s.t. } 1, k \notin J\}.$$

In order to determine the maximal $N = \sum_{i=1}^{\ell} N_i$ so that (1.11) holds, in Appendix A simple but involved computations show the following:

Theorem 1.7. *For a general consecutive coupling, the maximal N is given by*

$$N = \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\}$$

for ℓ even and

$$N = \max\left\{ \min\{c_{\frac{\ell-1}{2}} + \hat{a}_1 - N_1, d_{\frac{\ell-1}{2}} + \hat{a}_1, f_{\frac{\ell-1}{2}}, g_{\frac{\ell-1}{2}} + N_1\} : \hat{a}_1 - \min\{a_1, a_\ell\} \leq N_1 \leq \min\{a_1, a_2\}\right\}$$

for ℓ odd, where $\hat{a}_1 = \min\{a_1, \min\{a_1, a_2\} + \min\{a_1, a_\ell\}\}$. For the increasing consecutive coupling, N takes the form

$$N = \begin{cases} \displaystyle\sum_{j=0}^{\frac{\ell-2}{2}} b_{2j+1} & \ell \text{ even}, \\[4ex] \displaystyle\min\left\{ b_1 + \sum_{j=1}^{\frac{\ell-1}{2}} b_{2j}, \frac{1}{2} \sum_{j=1}^{\ell} b_j \right\} & \ell \text{ odd}. \end{cases} \tag{1.12}$$

When $\ell = 2, 3$ consecutive (increasing or not) and non-consecutive couplings lead to the maximal N given by (1.12). However, for $\ell \geq 4$ non-consecutive couplings may possibly give rise to a larger maximal N than consecutive ones, which in turn may do better than the increasing consecutive coupling.

The paper is organized as follows. In Section 2 we set up the abstract max-min scheme to provide a stable critical level $\mathcal{H}^*$ of $\mathcal{H}$. Here we make use of a crucial compactness property which is established in Section 3. Finally, in Appendix A we derive the expression for the maximal N given in Theorem 1.7 along with a thorough discussion of the cases $\ell = 2, 3, 4$.

2. A MAX-MIN ARGUMENT AND THE ROLE OF THE TOPOLOGY OF Σ

Let us outline the variational argument we are going to set up. First, we need to construct compact sets K, K_0 (with K connected) and an open smooth set $\mathcal{D}$ so that

$$K_0 \subset K \subset \mathcal{D} \subset \overline{\mathcal{D}} \subset \mathcal{M},$$

where

$$\mathcal{M} = (\Sigma \setminus \{p_1, \ldots, p_\ell\})^N \setminus \Delta, \quad \Delta = \{\boldsymbol{\xi} \in \Sigma^N : \xi_j = \xi_k \text{ for some } j \neq k\}.$$

Let

$$\mathcal{F} = \left\{ \Gamma(1, \cdot) : \Gamma \in C([0,1] \times K, \mathcal{D}) \text{ s.t. } \Gamma(0, \cdot) = id_K, \ \Gamma(t, \cdot)\big|_{K_0} = id_{K_0} \ \forall t \in [0,1] \right\}$$

and

$$\mathcal{H}^* = \sup_{\gamma \in \mathcal{F}} \min_{\boldsymbol{\xi} \in K} \mathcal{H}(\gamma(\boldsymbol{\xi})).$$

Through a standard deformation argument, the existence of a critical point $\boldsymbol{\xi} \in \mathcal{D}$ of $\mathcal{H}$ with $\mathcal{H}(\boldsymbol{\xi}) = \mathcal{H}^*$ is driven by a change in the topology of superlevel sets for $\mathcal{H}$ in $\mathcal{D}$ at height $\mathcal{H}^*$, as expressed by

$$\mathcal{H}^* < \min_{\boldsymbol{\xi} \in K_0} \mathcal{H}(\boldsymbol{\xi}) \tag{2.1}$$

(with the convention $\min_{\boldsymbol{\xi} \in K_0} \mathcal{H}(\boldsymbol{\xi}) = +\infty$ if $K_0 = \emptyset$). To exclude the presence of constrained critical points of $\mathcal{H}\big|_{\partial \mathcal{D}}$ at level $\mathcal{H}^*$, we further require the following compactness condition:

$$\forall \boldsymbol{\xi} \in \{\mathcal{H} = \mathcal{H}^*\} \cap \partial \mathcal{D} \quad \exists \tau \in T_{\boldsymbol{\xi}}(\partial \mathcal{D}) \text{ s.t. } \langle \tau, \nabla \mathcal{H}(\boldsymbol{\xi}) \rangle \neq 0, \tag{2.2}$$

where $T_{\boldsymbol{\xi}}(\partial \mathcal{D})$ stands for the tangent space of $\partial \mathcal{D}$ at $\boldsymbol{\xi}$. Since properties (2.1)-(2.2) continue to hold also for functionals which are C^1-close to $\mathcal{H}$, notice that such critical points are stable under C^1-small perturbations of $\mathcal{H}$.

Let us set

$$\Phi(\boldsymbol{\xi}) = \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \Gamma_j \Gamma_k G(\xi_j, \xi_k) + \sum_{i=1}^{\ell} \alpha_i \sum_{j=1}^{N} \Gamma_j G(\xi_j, p_i) + \sum_{j=1}^{N} h(\xi_j), \tag{2.3}$$

and for $M > 0$ sufficiently large define $\mathcal{D}$ as

$$\mathcal{D} = \{\boldsymbol{\xi} \in \mathcal{M} : \Phi(\boldsymbol{\xi}) < M\}.$$

Since

$$\Phi(\boldsymbol{\xi}) \to +\infty \text{ as } \boldsymbol{\xi} \to \partial \mathcal{M}$$

in view of (2.5) below, it follows that $\mathcal{D}$ is an open set with $\overline{\mathcal{D}} \subset \mathcal{M}$. Letting $\sigma_1, \ldots, \sigma_N$ be (not necessarily distinct) simple, closed curves in $\Sigma \setminus \{p_1, \ldots, p_\ell\}$ and $\boldsymbol{\xi}^0 = (\xi_1^0, \ldots, \xi_N^0) \in \sigma_1 \times \cdots \times \sigma_N$ be a N-tuple of distinct points, introduce the sets K and K_0 as follows:

$$W \text{ connected component of } \left\{ \boldsymbol{\xi} \in \sigma_1 \times \cdots \times \sigma_N : \min_{j \neq k} d_g(\xi_j, \xi_k) > M^{-1} \right\} \text{ s.t. } \boldsymbol{\xi}^0 \in W$$

$$K = \overline{W}, \qquad K_0 = \left\{ \boldsymbol{\xi} \in K \,\Big|\, \min_{j \neq k} d_g(\xi_j, \xi_k) = M^{-1} \right\}$$

for $M > (\min_{j \neq k} d_g(\xi_j^0, \xi_k^0))^{-1}$. By construction K and K_0 are compact sets, K is connected and $K_0 \subset K$. Since σ_j, $j = 1, \ldots, N$, is a curve in $\Sigma \setminus \{p_1, \ldots, p_\ell\}$, we have that

$$\inf\{d_g(\xi_j, p_i) : \ \xi_j \in \sigma_j, \ i = 1, \ldots, \ell\} > 0. \tag{2.4}$$

Thanks to the decomposition

$$G(x, p) = -\frac{1}{2\pi} \log d_g(x, p) + H(x, p), \quad H \in C(\Sigma^2), \tag{2.5}$$

we can rewrite $\mathcal{H}$ in (1.2) and Φ in (2.3) as

$$\mathcal{H} = \Psi_+ + O(1), \qquad \Phi = \Psi_- + O(1)$$

$$\Psi_\pm(\boldsymbol{\xi}) = -\frac{1}{2\pi} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \Gamma_j \Gamma_k \log d_g(\xi_j, \xi_k) \pm \sum_{i=1}^{\ell} \frac{\alpha_i}{2\pi} \sum_{j=1}^{N} \Gamma_j \log d_g(\xi_j, p_i), \tag{2.6}$$

where

$$|(\mathcal{H} - \Psi_+)(\boldsymbol{\xi})| + |(\Phi - \Psi_-)(\boldsymbol{\xi})| \leq C_0 \qquad \forall \boldsymbol{\xi} \in \Sigma^N$$

for some $C_0 > 0$. By (2.4) and (2.6) we have that $\sup_K \Phi \leq C \log M$ for M large, with a universal $C > 0$, and then the inclusion $K \subset \mathcal{D}$ does hold for M sufficiently large, as required.

We are concerned now with the proof of (2.1), whereas (2.2) will be established in the next Section thanks to the validity of (1.3). We begin with the following lemma.

Lemma 2.1. *The following holds*

$$\min_{\boldsymbol{\xi} \in K_0} \mathcal{H}(\boldsymbol{\xi}) \to +\infty \ as \ M \to +\infty.$$

Proof. Assume by contradiction the existence of sequences $\boldsymbol{\xi}_n = (\xi_1^n, \ldots, \xi_N^n) \in \sigma_1 \times \cdots \times \sigma_N$ and M_n such that

$$\sup_n \mathcal{H}(\boldsymbol{\xi}_n) < +\infty, \quad \min_{j \neq k} d_g(\xi_j^n, \xi_k^n) = M_n^{-1} \to 0 \ as \ n \to +\infty.$$

Up to a subsequence, we can find $j_0 \neq k_0$ so that

$$d_g(\xi_{j_0}^n, \xi_{k_0}^n) = M_n^{-1} \to 0 \ as \ n \to +\infty.$$

By (2.4) and (2.6) we deduce that

$$\mathcal{H}(\boldsymbol{\xi}_n) \geq \frac{1}{2\pi} \Gamma_{j_0} \Gamma_{k_0} \log M_n + O(1) \to +\infty$$

as $n \to +\infty$, yielding a contradiction. $\qquad \square$

Thanks to Lemma 2.1, the validity of (2.1) will follow once we have obtained an upper bound on the max-min value $\mathcal{H}^*$ for M large.

To this aim, let $\mathcal{P}_j$, $j = 1, \ldots, N$, be a retraction of $\Sigma \setminus \{p_1, \ldots, p_\ell\}$ onto σ_j, i.e. $\mathcal{P}_j : \Sigma \setminus \{p_1, \ldots, p_\ell\} \to \sigma_j$ is a continuous map so that $\mathcal{P}_j\big|_{\sigma_j} = \mathrm{id}_{\sigma_j}$. A simple application of the topological degree yields the following crucial intersection property:

Theorem 2.2. *For all $\gamma \in \mathcal{F}$ there exists $\boldsymbol{\xi}_\gamma^* \in K$ so that $\mathcal{P}_j(\gamma_j(\boldsymbol{\xi}_\gamma^*)) = \xi_j^0$ for all $j = 1, \ldots, N$.*

Proof. Fix $\gamma \in \mathcal{F}$, and write it as $\gamma = \Gamma(1, \cdot)$, where $\Gamma \in C([0,1] \times K, \mathcal{D})$ satisfies $\Gamma(0, \cdot) = id_K$ and $\Gamma(t, \cdot)\big|_{K_0} = id_{K_0}$ for all $t \in [0, 1]$. Extend Γ from K to $\sigma_1 \times \cdots \times \sigma_N$ as $\tilde{\Gamma}$:

$$\tilde{\Gamma}(t, \boldsymbol{\xi}) = \Gamma(t, \boldsymbol{\xi}) \quad \text{if } \boldsymbol{\xi} \in K, \quad \tilde{\Gamma}(t, \boldsymbol{\xi}) = \boldsymbol{\xi} \quad \text{if } \boldsymbol{\xi} \in (\sigma_1 \times \cdots \times \sigma_N) \setminus K.$$

Notice that K_0 is the topological boundary of K relative to $\sigma_1 \times \cdots \times \sigma_N$, and then $\tilde{\Gamma} \in C([0,1] \times (\sigma_1 \times \cdots \times \sigma_N), \mathcal{D})$ in view of $\Gamma(t, \boldsymbol{\xi}) = \boldsymbol{\xi}$ for all $t \in [0, 1]$, $\boldsymbol{\xi} \in K_0$. Writing $\tilde{\Gamma}$ as $\tilde{\Gamma} = (\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_N)$, the map $H : [0,1] \times (\sigma_1 \times \cdots \times \sigma_N) \to (\sigma_1 \times \cdots \times \sigma_N)$ with components $H_j(t, \boldsymbol{\xi}) = (\mathcal{P}_j \circ \tilde{\Gamma}_j)(t, \boldsymbol{\xi})$, $j = 1, \ldots, N$, is a continuous map so that $H(0, \cdot) = id_{\sigma_1 \times \cdots \times \sigma_N}$.

To use a degree argument, we can identify each σ_j, $j = 1, \ldots, N$, with $\mathbb{S}^1$ through a suitable homeomorphism, and then regard H as a map $[0,1] \times (\mathbb{S}^1)^N \to (\mathbb{S}^1)^N$ with $H(0, \cdot) = id_{(\mathbb{S}^1)^N}$. Given the annulus $A = \left\{ \frac{1}{2} \le |x| \le 2 \right\}$, extend H from $(\mathbb{S}^1)^N$ to A^N as $\tilde{H} = (\tilde{H}_1, \ldots, \tilde{H}_N)$, with

$$\tilde{H}_j(t, x_1, \ldots, x_N) = |x_j| H_j\left(t, \frac{x_1}{|x_1|}, \ldots, \frac{x_N}{|x_N|}\right), \quad (x_1, \ldots, x_N) \in A^N.$$

By construction $\tilde{H}$ is a continuous map from $[0,1] \times A^N$ into A^N, owing to $|\tilde{H}_j(t, x_1, \ldots, x_N)| = |x_j|$ for all $t \in [0, 1]$, and $\tilde{H}(0, \cdot) = id_{A^N}$. Moreover, $\tilde{H}(t, \cdot)$ maps the boundary $\partial(A^N)$ into itself, and we are in the position to apply a degree argument: by homotopy invariance we have that

$$\deg(\tilde{H}(1, \cdot), A^N, \boldsymbol{\xi}^0) = \deg(\tilde{H}(0, \cdot), A^N, \boldsymbol{\xi}^0) = \deg(id, A^N, \boldsymbol{\xi}^0) = 1,$$

where $\boldsymbol{\xi}^0 \in (\mathbb{S}^1)^N$ corresponds to the original $\boldsymbol{\xi}^0 \in \sigma_1 \times \cdots \times \sigma_N$ through the identifications of σ_j, $j = 1, \ldots, N$, with $\mathbb{S}^1$. Then, there exists $x^* = (x_1^*, \ldots, x_N^*) \in A^N$ so that $\tilde{H}(1, x^*) = \boldsymbol{\xi}^0$, and consequently $x^* \in (\mathbb{S}^1)^N$ thanks to $|x_j^*| = |\tilde{H}_j(1, x_1^*, \ldots, x_N^*)| = |\xi_j^0| = 1$. Getting back, we have thus found $\boldsymbol{\xi}^* \in \sigma_1 \times \cdots \times \sigma_N$ so that $H(1, \boldsymbol{\xi}^*) = \boldsymbol{\xi}^0$. We claim that $\boldsymbol{\xi}^* \in K$: otherwise, if $\boldsymbol{\xi}^* \in (\sigma_1 \times \cdots \times \sigma_N) \setminus K$, then $H(1, \boldsymbol{\xi}^*) = \boldsymbol{\xi}^*$, which would lead to $\boldsymbol{\xi}^* = \boldsymbol{\xi}^0$, and this provides a contradiction with $\boldsymbol{\xi}^0 \in K$. So, $\boldsymbol{\xi}^* \in K$ and $H_j(1, \boldsymbol{\xi}^*) = \mathcal{P}_j(\gamma_j(\boldsymbol{\xi}^*)) = \xi_j^0$ for all $j = 1, \ldots, N$. $\square$

Since

$$\mathcal{H}^* = \sup_{\gamma \in \mathcal{F}} \min_{\boldsymbol{\xi} \in K} \mathcal{H}(\gamma(\boldsymbol{\xi})) \le \sup_{\gamma \in \mathcal{F}} \mathcal{H}(\gamma(\boldsymbol{\xi}_\gamma^*))$$

with $\boldsymbol{\xi}_\gamma^*$ given by Theorem 2.2, an upper bound on $\mathcal{H}^*$ is then reduced to show that

$$\sup_{\gamma \in \mathcal{F}} \Psi_+(\gamma(\boldsymbol{\xi}_\gamma^*)) \le C \tag{2.7}$$

does hold for M large, in view of (2.6). The topological properties of Σ play here a crucial role to find the retractions $\mathcal{P}_j$, $j = 1, \ldots, N$, and to investigate the structure of its fibers in order to prove (2.7). By the topological classification of closed surfaces, we have that Σ is homeomorphic to either the sphere $\mathbb{S}^2$ or the connected sum of tori T or a connected sum of real projective planes $\mathbb{RP}^2$. In the next Subsections, we will separately discuss the case $\Sigma \ne \mathbb{S}^2, \mathbb{RP}^2$ and the case $\Sigma = \mathbb{S}^2, \mathbb{RP}^2$ (up to homeomorphic equivalence), completing the proof of Theorems 1.1-1.3.

2.1. **The case $\Sigma \neq \mathbb{S}^2, \mathbb{RP}^2$.** By Dyck's theorem ([28]) Σ is homeomorphic either to the torus T or to the Klein bottle or to the connected sum $T \# \Sigma'$, for a closed surface Σ'. Recall that a torus and a Klein bottle can be represented by the fundamental square $ABA^{-1}B^{-1}$ and $ABA^{-1}B$, respectively. To fix the ideas, let A, B be a horizontal, vertical edge, respectively, and let us also assume that the singularities lie in the interior of the square. In this case, we can construct a retraction $\mathcal{P}$ of the surface onto A by simply projecting along vertical lines, where A represents a circle not passing through the singularities, and the fibers of $\mathcal{P}$ are well-separated:

$$\overline{\mathcal{P}^{-1}(\zeta_1)} \cap \overline{\mathcal{P}^{-1}(\zeta_2)} = \emptyset \tag{2.8}$$

for all $\zeta_1, \zeta_2 \in A$, $\zeta_1 \neq \zeta_2$. For $T \# \Sigma'$ the fundamental polygon looks like $ABA^{-1} \ldots$ and contains three edges of a square Q. Let v be one of the two vertices of Q which do not belong to B. The retraction $\mathcal{P}$ is the projection on A inside the square Q and takes constant value v outside Q, and we can still assume that A does not contain any singularities. The map $\mathcal{P}$ is continuous and its fibers satisfy (2.8).

Via the homeomorphism between Σ and one of the above models, in our hands we have a retraction map $\mathcal{P}$ from Σ onto a simple, closed curve σ in $\Sigma \setminus \{p_1, \ldots, p_\ell\}$ so that

$$\inf\{d_g(\xi_1, \xi_2) : \ \mathcal{P}(\xi_1) = \mu_1, \mathcal{P}(\xi_2) = \mu_2\} > 0 \tag{2.9}$$

for all $\mu_1, \mu_2 \in \sigma$, $\mu_1 \neq \mu_2$, in view of (2.8). We take $\sigma_1 = \cdots = \sigma_N = \sigma$, $\mathcal{P}_1 = \cdots = \mathcal{P}_N = \mathcal{P}$ and we fix N distinct points $\xi_1^0, \ldots, \xi_N^0 \in \sigma$. By Theorem 2.2 we have that $\gamma_j(\boldsymbol{\xi}_\gamma^*) \in \mathcal{P}^{-1}(\xi_j^0)$ for all $j = 1, \ldots, N$, and then

$$\inf\left\{ d_g(\gamma_j(\boldsymbol{\xi}_\gamma^*), \gamma_k(\boldsymbol{\xi}_\gamma^*)) : \ j \neq k, \gamma \in \mathcal{F}, M \geq 2(\min_{j \neq k} d_g(\xi_j^0, \xi_k^0))^{-1} \right\} > 0$$

does hold thanks to (2.9), yielding the validity of (2.7).

2.2. **The case $\Sigma = \mathbb{S}^2, \mathbb{RP}^2$.** Since Σ is a smooth surface, there exists (see for example [34]) a diffeomorphism ω from Σ to $\mathbb{S}^2$ or $\mathbb{RP}^2$. Let $\tilde{p}_1 = \omega(p_1), \ldots, \tilde{p}_\ell = \omega(p_\ell)$ be the corresponding singular sources, and $\tilde{g} = (\omega^{-1}) * g$ be the induced metric. In this way, (Σ, g) is isometrically equivalent to $\mathbb{S}^2$ or $\mathbb{RP}^2$ endowed with the metric $\tilde{g}$. In Theorem 1.3 let us first consider a consecutive coupling where each p_i is coupled with p_{i+1} ($p_{\ell+1} := p_1$), i.e. $J_1 = \{\ell\}$ and $J_i = \{i-1\}$ for all $i = 2, \ldots, \ell$. The argument is already involved and contains the main ideas of the general case, which will be discussed in a sketched way right after.

By using $\tilde{p}_\ell$ as north pole on $\mathbb{S}^2$, we can construct a diffeomorphism $\Pi : \Sigma \setminus \{p_\ell\} \to \mathbb{C}$ so that $\Pi(p_i) = q_i \in \mathbb{R}$ for all $i = 1, \ldots, \ell-1$, with $q_i = i$, and $\Pi \circ \omega^{-1}$ coincides with the stereographic projection through $\tilde{p}_\ell$ in a small neighborhood of $\tilde{p}_\ell$. Since

$$\frac{1}{C} g_0 \leq \tilde{g} \leq C g_0$$

does hold in every coordinate open set $\Omega \subset \mathbb{S}^2$ for some $C = C(\Omega) > 1$, where g_0 is the round metric on $\mathbb{S}^2$, by compactness of $\mathbb{S}^2$ we get

$$\frac{1}{C} \leq \frac{d_{g_0}(x, y)}{d_{\tilde{g}}(x, y)} \leq C \quad \forall x, y \in \mathbb{S}^2$$

for some $C > 1$. Since $(\mathbb{S}^2 \setminus \{\tilde{p}_\ell\}, g_0)$ is isometrically equivalent to $(\mathbb{C}, g_1)$, $g_1 = \frac{4}{(1+|z|^2)^2} dx dy$ ($z = x + iy$), via the stereographic projection, we have that there exists $C > 1$ so that

$$\frac{1}{C} \leq \frac{d_g(x, y)}{d_{g_1}(\Pi(x), \Pi(y))} \leq C \quad \forall x, y \in \Sigma \setminus \{p_\ell\}. \tag{2.10}$$

Indeed, (2.10) is true on compact subsets of $\Sigma \setminus \{p_\ell\}$, while near p_ℓ it follows by the property that $\Pi \circ \omega^{-1}$ coincides with the stereographic projection through $\tilde{p}_\ell$ near $\tilde{p}_\ell$.

Thanks to (2.10), we can work directly in $\mathbb{C}$. Let us now define a continuous map $\Upsilon_{i,r}$: $\mathbb{C} \setminus \{q_i, q_r\} \to \mathbb{S}^1$, $i \neq r$, as follows:

$$\Upsilon_{i,r}(z) = \begin{cases} e^{i\arg(z-q_i)} & \text{if } \operatorname{Re} z \leq \frac{i+r}{2} \\ e^{i(\pi-\arg(z-q_r))} & \text{if } \operatorname{Re} z \geq \frac{i+r}{2} \end{cases}$$

if $i < r \leq \ell - 1$, $\Upsilon_{i,\ell}(z) = e^{i\arg(z-q_i)}$ if $i < r = \ell$ and $\Upsilon_{i,r} = \frac{1}{\Upsilon_{r,i}}$ if $r < i$. For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ notice that the fibers $L_{i,r}(\theta) = \Upsilon_{i,r}^{-1}(e^{i\theta})$ represent

- the two vertical edges of the isosceles triangle with base $\overline{q_i q_r}$ and base angle θ

$$L_{i,r}(\theta) = \left\{ q_i + \rho e^{i\theta} \,\middle|\, 0 < \rho \leq \frac{r-i}{2\cos\theta} \right\} \cup \left\{ q_r - \rho e^{-i\theta} \,\middle|\, 0 < \rho \leq \frac{r-i}{2\cos\theta} \right\}$$

  for $i < r \leq \ell - 1$;
- the straight line starting from q_i with angle θ for $i < r = \ell$;
- the set $L_{r,i}(-\theta)$ for $r < i$.

Split $\{1, \ldots, N\}$ as the disjoint union of $\mathcal{N}_i$, $i = 1, \ldots, \ell$, and define continuous maps $\mathcal{P}_j$: $\Sigma \setminus \{p_1, \ldots, p_\ell\} \to \sigma_j$, $j = 1, \ldots, N$, as

$$\mathcal{P}_j = \Pi^{-1}\left(q_i + \frac{1}{4}\Upsilon_{i,i+1} \circ \Pi \right), \quad \sigma_j = \Pi^{-1}\left(q_i + \frac{1}{4}\mathbb{S}^1 \right)$$

when $j \in \mathcal{N}_i$, $i = 1, \ldots, \ell - 1$, and

$$\mathcal{P}_j = \Pi^{-1}\left(q_1 + \ell\Upsilon_{1,\ell} \circ \Pi \right), \quad \sigma_j = \Pi^{-1}\left(q_1 + \ell\mathbb{S}^1 \right)$$

when $j \in \mathcal{N}_\ell$. Notice that $\mathcal{P}_j\big|_{\sigma_j} = \operatorname{id}_{\sigma_j}$ for all $j = 1, \ldots, N$. Let us fix N distinct angles $\theta_1, \ldots, \theta_N \in (0, \frac{\pi}{2})$, and let $\boldsymbol{\xi}^0 = (\xi_1^0, \ldots, \xi_N^0) \in \sigma_1 \times \cdots \times \sigma_N$ be a N-tuple of distinct points, with

$$\xi_j^0 = \begin{cases} \Pi^{-1}(q_i + \frac{1}{4}e^{i\theta_j}) & \text{if } j \in \mathcal{N}_i, \ i = 1, \ldots, \ell - 1 \\ \Pi^{-1}(q_1 + \ell e^{-i\theta_j}) & \text{if } j \in \mathcal{N}_\ell. \end{cases}$$

Thanks to Theorem 2.2, for all $\gamma \in \mathcal{F}$ we can find $\boldsymbol{z}^\gamma = (z_1^\gamma, \ldots, z_N^\gamma) \in \mathbb{C}^N$, with $z_j^\gamma = \Pi[\gamma_j(\boldsymbol{\xi}_\gamma^*)]$ for $j = 1, \ldots, N$, so that

$$z_j^\gamma \in L_{i,i+1}(\theta_j), \quad \forall j \in \mathcal{N}_i, \ i = 1, \ldots, \ell$$

(with $\ell + 1 = 1$). In view of (2.10), (2.7) can be re-formulated as

$$\sup_{\gamma \in \mathcal{F}} \Psi(\boldsymbol{z}^\gamma) \leq C, \tag{2.11}$$

where

$$\Psi(\boldsymbol{z}) = -\frac{1}{2\pi} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \Gamma_j \Gamma_k \log d_{g_1}(z_j, z_k) + \sum_{i=1}^{\ell} \frac{\alpha_i}{2\pi} \sum_{j=1}^{N} \Gamma_j \log d_{g_1}(z_j, q_i) \tag{2.12}$$

for $\boldsymbol{z} = (z_1, \ldots, z_N) \in \mathbb{C}^N$, with $q_\ell = \infty$. Next lemma establishes the validity of (2.11).

Lemma 2.3. *If $\theta_1, \ldots, \theta_N \in (0, \frac{\pi}{2})$ are distinct angles, then*

$$\sup_{\boldsymbol{z} \in Z} \Psi(\boldsymbol{z}) < +\infty,$$

where Ψ is given by (2.12) and

$$Z = \left\{ \boldsymbol{z} = (z_1, \ldots, z_N) \in \mathbb{C}^N : \ z_j \in L_{i,i+1}(\theta_j) \ \forall j \in \mathcal{N}_i, \ i = 1, \ldots, \ell \right\}.$$

Proof. Observe that

$$\overline{L_{i,i+1}(\theta_j)} \cap \overline{L_{r,r+1}(\theta_k)} = \begin{cases} q_i, q_{i+1} & \text{if } r = i \\ q_{i+1} & \text{if } r = i+1 \\ \emptyset & \text{otherwise} \end{cases} \tag{2.13}$$

for all $j \in \mathcal{N}_i$ and $k \in \mathcal{N}_r$ with $j \neq k$, where $q_{\ell+1} = q_1$ and the closure is meant with respect to d_{g_1}. By (2.13) we can rewrite Ψ as

$$\Psi(\boldsymbol{z}) = -\frac{1}{2\pi} \sum_{i=1}^{\ell} \sum_{j \in \mathcal{N}_i} \sum_{\substack{k \in \mathcal{N}_{i-1} \cup \mathcal{N}_i \cup \mathcal{N}_{i+1} \\ k \neq j}} \Gamma_j \Gamma_k \log d_{g_1}(z_j, z_k) + \sum_{i=1}^{\ell} \frac{\alpha_i}{2\pi} \sum_{j \in \tilde{\mathcal{N}}_i} \Gamma_j \log d_{g_1}(z_j, q_i) + O(1),$$

where $\tilde{\mathcal{N}}_i = \mathcal{N}_i \cup \mathcal{N}_{i-1}$ for a consecutive coupling, $\mathcal{N}_0 = \mathcal{N}_\ell$, $\mathcal{N}_{\ell+1} = \mathcal{N}_1$ and $\alpha_{\ell+1} = \alpha_1$.

For $i = 1, \ldots, \ell$ and $j, k \in \mathcal{N}_i$, $j \neq k$, the two polygonals $L_{i,i+1}(\theta_j)$ and $L_{i,i+1}(\theta_k)$ approach the same end-points q_i and q_{i+1} with different angles $\theta_j \neq \theta_k$, yielding the inequality

$$d_{g_1}(z_j, z_k) \geq \delta \, d_{g_1}(z_j, q_i) \, d_{g_1}(z_j, q_{i+1}) \tag{2.14}$$

for all $z_j \in L_{i,i+1}(\theta_j)$ and $z_k \in L_{i,i+1}(\theta_k)$, where $\delta > 0$ depends only on $\theta_1, \ldots, \theta_N$. Indeed, for $i = 1, \ldots, \ell - 2$ we can write points z_j, z_k near q_i as $z_j = q_i + |z_j - q_i|e^{i\theta_j}$, $z_k = q_i + |z_k - q_i|e^{i\theta_k}$ to get

$$|z_j - z_k|^2 = |z_j - q_i|^2 - 2|z_j - q_i||z_k - q_i| \cos |\theta_k - \theta_j| + |z_k - q_i|^2 \geq |z_j - q_i|^2 \sin^2 |\theta_k - \theta_j|,$$

and (2.14) follows near q_i owing to the equivalence between d_{g_1} and the euclidean distance on compact subsets of $\mathbb{R}^2$. A similar argument works near q_{i+1}, and (2.14) is thus proved when $i = 1, \ldots, \ell - 2$. Inequality (2.14) is still valid near q_i for $i = \ell - 1$ and near q_{i+1} for $i = \ell$, and the difficult case is when approaching $q_\ell = \infty$. For $z_j = q_{\ell-1} + |z_j - q_{\ell-1}|e^{i\theta_j}$ and $z_k = q_{\ell-1} + |z_k - q_{\ell-1}|e^{i\theta_k}$ the following holds

$$\begin{aligned} \left| \frac{1}{z_j} - \frac{1}{z_k} \right|^2 &= \frac{|z_j - z_k|^2}{|z_j|^2 |z_k|^2} = \frac{|z_j - q_{\ell-1}|^2 - 2|z_j - q_{\ell-1}||z_k - q_{\ell-1}| \cos |\theta_k - \theta_j| + |z_k - q_{\ell-1}|^2}{|z_j|^2 |q_{\ell-1} + |z_k - q_{\ell-1}|e^{i\theta_k}|^2} \\ &\geq \frac{|z_k - q_{\ell-1}|^2 \sin^2 |\theta_k - \theta_j|}{2|z_j|^2 |z_k - q_{\ell-1}|^2} \geq \delta \left| \frac{1}{z_j} \right|^2, \end{aligned}$$

for $|z_j|, |z_k|$ large, providing the validity of (2.14) with $i = \ell - 1$ near $q_\ell = \infty$ in view of the invariance of g_1 under the map $z \to \frac{1}{z}$ and the equivalence between d_{g_1} and the euclidean distance near 0. A similar argument works for (2.14) with $i = \ell$ near $q_\ell = \infty$, and (2.14) is finally established for $i = 1, \ldots, \ell$.

For $i = 1, \ldots, \ell$ and $j \in \mathcal{N}_i$, $k \in \mathcal{N}_{i+1}$, the two polygonals $L_{i,i+1}(\theta_j)$ and $L_{i+1,i+2}(\theta_k)$ (with $L_{\ell+1,\ell+2}(\theta_k) = L_{1,2}(\theta_k)$) just have q_{i+1} as common end-point, and, arguing as above, we deduce

$$d_{g_1}(z_j, z_k) \geq \delta \max\{d_{g_1}(z_j, q_{i+1}) \, d_{g_1}(z_k, q_{i+1})\} \tag{2.15}$$

for all $z_j \in L_{i,i+1}(\theta_j)$ and $z_k \in L_{i+1,i+2}(\theta_k)$, where $\delta > 0$ just depends on $\theta_1, \ldots, \theta_N$.

For $i = 1, \ldots, \ell$ and $j \in \mathcal{N}_i$, by (2.14) and (2.15) we deduce that

$$
-\sum_{\substack{k \in \mathcal{N}_{i-1} \cup \mathcal{N}_i \cup \mathcal{N}_{i+1} \\ k \neq j}} \Gamma_k \log d_{g_1}(z_j, z_k)
$$

$$
= -\sum_{k \in \mathcal{N}_{i-1}} \Gamma_k \log d_{g_1}(z_j, z_k) - \sum_{\substack{k \in \mathcal{N}_i \\ k \neq j}} \Gamma_k \log d_{g_1}(z_j, z_k) - \sum_{k \in \mathcal{N}_{i+1}} \Gamma_k \log d_{g_1}(z_j, z_k)
$$

$$
\leq -\sum_{k \in \mathcal{N}_{i-1}} \Gamma_k \log d_{g_1}(z_j, q_i) - \sum_{\substack{k \in \mathcal{N}_i \\ k \neq j}} \Gamma_k \log d_{g_1}(z_j, q_i) - \sum_{\substack{k \in \mathcal{N}_i \\ k \neq j}} \Gamma_k \log d_{g_1}(z_j, q_{i+1})
$$

$$
- \sum_{k \in \mathcal{N}_{i+1}} \Gamma_k \log d_{g_1}(z_j, q_{i+1}) + O(1)
$$

$$
= -\sum_{\substack{k \in \tilde{\mathcal{N}}_i \\ k \neq j}} \Gamma_k \log d_{g_1}(z_j, q_i) - \sum_{\substack{k \in \tilde{\mathcal{N}}_{i+1} \\ k \neq j}} \Gamma_k \log d_{g_1}(z_j, q_{i+1}) + O(1).
$$

Therefore, we have shown that

$$
\begin{aligned}
\Psi(\boldsymbol{z}) &= -\frac{1}{2\pi} \sum_{i=1}^{\ell} \sum_{j \in \mathcal{N}_i} \sum_{\substack{k \in \tilde{\mathcal{N}}_i \\ k \neq j}} \Gamma_j \Gamma_k \log d_{g_1}(z_j, q_i) - \frac{1}{2\pi} \sum_{i=1}^{\ell} \sum_{j \in \mathcal{N}_i} \sum_{\substack{k \in \tilde{\mathcal{N}}_{i+1} \\ k \neq j}} \Gamma_j \Gamma_k \log d_{g_1}(z_j, q_{i+1}) \\
&\quad + \sum_{i=1}^{\ell} \frac{\alpha_i}{2\pi} \sum_{j \in \tilde{\mathcal{N}}_i} \Gamma_j \log d_{g_1}(z_j, q_i) + O(1) \\
&= -\frac{1}{2\pi} \sum_{i=1}^{\ell} \left[\sum_{\substack{j,k \in \tilde{\mathcal{N}}_i \\ k \neq j}} \Gamma_j \Gamma_k - \alpha_i \sum_{j \in \tilde{\mathcal{N}}_i} \Gamma_j \right] \log d_{g_1}(z_j, q_i) + O(1)
\end{aligned}
$$

and the above quantity is uniformly bounded from above with respect to $\boldsymbol{z} \in Z$ in view of (1.7), yielding $\sup_Z \Psi < +\infty$. $\qquad\square$

Letting $\sim$ be the equivalence relation between antipodal points, the surface $\mathbb{RP}^2$ can be represented as the quotient $\mathbb{S}^2/\sim$. We can find a retraction $\mathcal{P}_j : \mathbb{RP}^2 \setminus \{\tilde{p}_i\} \to C$, $j \in \mathcal{N}_i$, as the projection along great circles passing through $\tilde{p}_i$ onto a given equatorial circle C not passing through $\tilde{p}_1, \ldots, \tilde{p}_\ell$. The fibers of $\mathcal{P}_j$, $j \in \mathcal{N}_i$, intersect in $\tilde{p}_i$, a fact which can be controlled by an assumption like (1.5). However the fibers of $\mathcal{P}_j$ and $\mathcal{P}_k$ have intersection points outside $\tilde{p}_1, \ldots, \tilde{p}_\ell$, for all $j \in \mathcal{N}_{i_1}$ and $k \in \mathcal{N}_{i_2}$ with $i_1 \neq i_2$, and an upper bound on $\mathcal{H}^*$ is not generally available. In Theorem 1.2 we then restrict the attention to the special case $\mathcal{N}_i = \{1, \ldots, N\}$, with $\alpha_i = \max\{\alpha_1, \ldots, \alpha_\ell\}$ for some $i = 1, \ldots, \ell$. Taking $\tilde{p}_i$ as the north pole and C as the equator, we have that the upper hemisphere $\mathbb{S}^2_+$ (w.r.t. C) can be projected onto the equatorial plane and the unit disc D with identified antipodal boundary points is a model for $\mathbb{RP}^2$. Then we can find a diffeomorphism $\Pi : \Sigma \to D$ so that $\Pi(p_r) = q_r \in \mathbb{R}$, $r = 1, \ldots, \ell$, with $q_i = 0$ and $-1 \leq q_1 < \cdots < q_\ell \leq 1$. Moreover, by compactness of Σ the following holds

$$
\frac{1}{C} \leq \frac{d_g(x, y)}{|\Pi(x) - \Pi(y)|} \leq C \quad \forall \, x, y \in \Sigma \tag{2.16}
$$

for some $C > 1$. Letting $\pi : D \setminus \{0\} \to \partial D$ be the radial projection, we define $\mathcal{P}_j = \mathcal{P}$ for all $j = 1, \ldots, N$, where $\mathcal{P} = \Pi^{-1} \circ \pi \circ \Pi : \Sigma \setminus \{p_i\} \to \sigma$ and $\sigma = \Pi^{-1}(\partial D)$. Let us fix N distinct

points $\xi_1^0, \ldots, \xi_N^0 \in \sigma \setminus \{p_1, \ldots, p_\ell\}$. In view of (2.16), the validity of (2.7) will follow by

$$\sup_{\boldsymbol{z} \in Z} \Psi(\boldsymbol{z}) < +\infty, \quad \Psi(\boldsymbol{z}) = -\frac{1}{2\pi} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \Gamma_j \Gamma_k \log |z_j - z_k| + \sum_{i=1}^{\ell} \frac{\alpha_i}{2\pi} \sum_{j=1}^{N} \Gamma_j \log |z_j - q_i|, \quad (2.17)$$

where

$$Z = \left\{ \boldsymbol{z} = (z_1, \ldots, z_N) \in D^N : z_j \in \pi^{-1}(\zeta_j^0) \ \forall j = 1, \ldots, N \right\}, \quad \zeta_j^0 = \Pi(\xi_j^0).$$

Since $\overline{\pi^{-1}(\zeta_j^0)} \cap \overline{\pi^{-1}(\zeta_k^0)} = \{0\}$ for $j \neq k$, arguing as in Lemma 2.3 (2.17) can be deduced by (1.6).

For a general coupling in Theorem 1.3, we explain below the necessary changes. Denoting by $r(i)$ the unique index such that $i \in J_{r(i)}$, $i \in \{1, \ldots, \ell\}$, we split

$$\{1, \ldots, \ell\} = \bigcup_{r=1}^{m} X_r$$

in disjoint blocks X_r which satisfy

$$i \in X_r \Rightarrow J_i \subset X_r \ \& \ r(i) \in X_r \tag{2.18}$$

and are minimal (i.e. no proper subset of X_r satisfies (2.18)). Notice that such a partition $\{X_1, \ldots, X_m\}$ is unique, and (2.18) guarantees that there are no couplings between indices in different blocks. Thanks to the following result, we can provide each block with a nice order of all the indices but one (say, the last one):

Lemma 2.4. *Let $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ be a set of $n \geq 2$ elements. Let $J_1, \ldots, J_n$ be a partition of X so that $x_i \notin J_i$, $i = 1, \ldots, n$, and*

$$X' \subset X : J_i \subset X' \ \& \ x_{r(i)} \in X' \ \forall x_i \in X' \Longrightarrow X' = X, \tag{2.19}$$

where $r(i) \in \{1, \ldots, n\}$ is the unique index so that $x_i \in J_{r(i)}$. Then, there exists a permutation σ of $\{1, \ldots, n\}$ so that[1]

$$\sigma(i) < \sigma(j) \Rightarrow \ either \ J_{\sigma(i)}^* < x_{\sigma(i)} \leq J_{\sigma(j)}^* < x_{\sigma(j)} \ or \ J_{\sigma(j)}^* < J_{\sigma(i)}^* < x_{\sigma(i)} < x_{\sigma(j)}, \tag{2.20}$$

where $J_i^* = J_i \setminus \{x_{\sigma(n)}\}$, $i = 1, \ldots, n$.

Proof. We argue by induction on n. When $n = 2$ we have that $J_1 = \{x_2\}$, $J_2 = \{x_1\}$ and (2.20) is satisfied with $\sigma = \text{id}$. If Lemma 2.4 does hold for n, let us discuss its validity also for $n + 1$. If $J_i \neq \emptyset$ for any $i = 1, \ldots, n+1$, then $\#J_i = 1$ for all i, and by (2.19) we can find a permutation σ so that $J_{\sigma(i)} = \{x_{\sigma(i-1)}\}$ for $i \geq 2$ and $J_{\sigma(1)} = \{x_{\sigma(n)}\}$, and (2.20) easily follows. Otherwise, we can find a first permutation τ so that $J_{\tau(n+1)} = \emptyset$ and $\tau(n) = r(\tau(n+1))$. Letting $Y = \{x_{\tau(1)}, \ldots, x_{\tau(n)}\}$, we have that $\#Y = n$ and Y still satisfies (2.19) with the partition $J_{\tau(1)}, \ldots, J_{\tau(n-1)}, J_{\tau(n)} \setminus \{x_{\tau(n+1)}\}$. Since Lemma 2.4 is true for Y, we can find a permutation λ of $\{1, \ldots, n\}$ so that (2.20) does hold for Y with λ. The permutation σ of $\{1, \ldots, n+1\}$ constructed as $\sigma(1) = \tau \circ \lambda(1), \ldots, \sigma(i_0 - 1) = \tau \circ \lambda(i_0 - 1), \sigma(i_0) = \tau(n+1), \sigma(i_0 + 1) = \tau \circ \lambda(i_0), \ldots, \sigma(n+1) = \tau \circ \lambda(n)$ with i_0 defined by $\lambda(i_0) = \tau(n)$, satisfies (2.20), as it can be straightforwardly checked. $\qquad\square$

---

[1]Hereafter, the notation $A < B$ ($A \leq B$ resp.) stands for $\sup A < \inf B$ ($\sup A \leq \inf B$ resp.) if $A, B \neq \emptyset$. The inequality is always true if either $A = \emptyset$ or $B = \emptyset$, and points are identified with singletons.

We first make a permutation to have $X_1 < \cdots < X_m$ and then apply Lemma 2.4 to each X_r, $r = 1, \ldots, m$, to get the following:

Proposition 2.5. *Up to a permutation, there exist blocks $X_1 < \cdots < X_m$, $m \geq 1$, satisfying* (2.18) *and for all $r = 1, \ldots, m$:*

$$i, j \in X_r, \, i < j \Rightarrow \text{ either } J_i^* < i \leq J_j^* < j \text{ or } J_j^* < J_i^* < i < j, \tag{2.21}$$

where $J_i^* = J_i \setminus \{l_r\}$ and $l_r = \max X_r$.

Hereafter, let us assume that we have permuted the indices according to Proposition 2.5. Define continuous maps $\mathcal{P}_j : \Sigma \setminus \{p_1, \ldots, p_\ell\} \to \sigma_j$, $j = 1, \ldots, N$, as

$$\mathcal{P}_j = \Pi^{-1}\left(q_i + \frac{1}{4}\Upsilon_{i,r(i)} \circ \Pi\right), \quad \sigma_j = \Pi^{-1}\left(q_i + \frac{1}{4}\mathbb{S}^1\right)$$

when $j \in \mathcal{N}_i$, $i = 1, \ldots, \ell - 1$, and

$$\mathcal{P}_j = \Pi^{-1}\left(q_{r(\ell)} + \ell\Upsilon_{r(\ell),\ell} \circ \Pi\right), \quad \sigma_j = \Pi^{-1}\left(q_{r(\ell)} + \ell\mathbb{S}^1\right)$$

when $j \in \mathcal{N}_\ell$. Let us fix N angles

$$0 < \theta_N < \cdots < \theta_1 < \frac{\pi}{2}, \tag{2.22}$$

and let $\boldsymbol{\xi}^0 = (\xi_1^0, \ldots, \xi_N^0) \in \sigma_1 \times \cdots \times \sigma_N$ be a N−tuple of distinct points, with

$$\xi_j^0 = \begin{cases} \Pi^{-1}(q_i + \frac{1}{4}e^{i\theta_j}) & \text{if } j \in \mathcal{N}_i, \, i = 1, \ldots, \ell - 1, \\ \Pi^{-1}(q_{r(\ell)} + \ell e^{-i\theta_j}) & \text{if } j \in \mathcal{N}_\ell. \end{cases}$$

Thanks to Theorem 2.2, for all $\gamma \in \mathcal{F}$ we can find $\boldsymbol{z}^\gamma = (z_1^\gamma, \ldots, z_N^\gamma) \in Z$ (with $z_j^\gamma = \Pi[\gamma_j(\boldsymbol{\xi}_\gamma^*)]$ for $j = 1, \ldots, N$), where

$$Z = \{\boldsymbol{z} = (z_1, \ldots, z_N) \in \mathbb{C}^N : z_j \in L_{i,r(i)}(\theta_j) \; \forall j \in \mathcal{N}_i, \, i = 1, \ldots, \ell\}.$$

Arguing as in Lemma 2.3, the aim now is to discuss the set $\overline{L_{i,r(i)}(\theta_j)} \cap \overline{L_{s,r(s)}(\theta_k)}$ for $r(i) \leq r(s)$ and $j \in \mathcal{N}_i$, $k \in \mathcal{N}_s$ with $j \neq k$, where the closure is meant with respect to d_{g_1}. Since $i \in J_{r(i)}$, by (2.18) notice that $i \in X_r$ if and only if $r(i) \in X_r$, and by (2.21)-(2.22) the following distinct alternatives can arise:

- if $i = s$, then $\overline{L_{i,r(i)}(\theta_j)} \cap \overline{L_{s,r(s)}(\theta_k)} = \{q_i, q_{r(i)}\}$ (with $q_\ell = \infty$);
- if $i \neq s$ and $r(i) = r(s)$, then $\overline{L_{i,r(i)}(\theta_j)} \cap \overline{L_{s,r(s)}(\theta_k)} = \{q_{r(i)}\}$;
- if $i, s \in X_r$ with $r(i) < r(s)$ and $i = l_r$, then $s < r(s)$ and

$$\overline{L_{i,r(i)}(\theta_j)} \cap \overline{L_{s,r(s)}(\theta_k)} = \begin{cases} \{q_s, q_i\} & \text{if } r(i) = s, \, r(s) = l_r \\ \{q_i\} & \text{if } r(i) \neq s, \, r(s) = l_r \\ \{q_s\} & \text{if } r(i) = s, \, r(s) \neq l_r \\ \emptyset & \text{if } r(i) \neq s, \, r(s) \neq l_r; \end{cases}$$

- if $i, s \in X_r$ with $r(i) < r(s)$ and $s = l_r$, then $i < r(i) < r(s) < s$ and $\overline{L_{i,r(i)}(\theta_j)} \cap \overline{L_{s,r(s)}(\theta_k)} = \emptyset$;
- if $i, s \in X_r \setminus \{l_r\}$ with $r(i) < r(s)$, then either $i < r(i) \leq s < r(s)$ with

$$\overline{L_{i,r(i)}(\theta_j)} \cap \overline{L_{s,r(s)}(\theta_k)} = \begin{cases} \{q_s\} & \text{if } r(i) = s \\ \emptyset & \text{if } r(i) \neq s, \end{cases}$$

  or $s < i < r(i) < r(s)$ with $\overline{L_{i,r(i)}(\theta_j)} \cap \overline{L_{s,r(s)}(\theta_k)} = \emptyset$;
- if i and s belong to different blocks, then $\overline{L_{i,r(i)}(\theta_j)} \cap \overline{L_{s,r(s)}(\theta_k)} = \emptyset$.

Since the $L_{i,r(i)}(\theta_j)$'s can share at most endpoints among $p_1, \ldots, p_\ell$, we just need to analyze the behavior at each p_i. Every p_i is an endpoint of $L_{i,r(i)}(\theta_j)$, $j \in \mathcal{N}_i$, and of $L_{s,r(s)}(\theta_k)$, $s \in J_i$ and $k \in \mathcal{N}_s$. Letting

$$\tilde{\mathcal{N}}_i = \mathcal{N}_i \cup \bigcup_{r \in J_i} \mathcal{N}_r,$$

we can argue precisely as in Lemma 2.3 to show that (1.7) implies

$$\sup_{\boldsymbol{z} \in Z} \Psi(\boldsymbol{z}) < +\infty$$

with Ψ given in (2.12), which in turn is equivalent to (2.7).

3. A COMPACTNESS PROPERTY

We shall show that (2.2) holds provided that M is sufficiently large. Since the choice $\gamma = id_K$ in the definition of $\mathcal{H}^*$ leads to

$$\mathcal{H}^* \geq \min_{\boldsymbol{\xi} \in K} \Psi_+(\boldsymbol{\xi}) + O(1) \geq \frac{1}{2\pi}\left(\sum_{i=1}^\ell \alpha_i\right)\left(\sum_{j=1}^N \Gamma_j\right) \log d - \frac{1}{2\pi}\left(\sum_{\substack{j,k=1 \\ j \neq k}}^N \Gamma_j\Gamma_k\right) \log \operatorname{diam} \Sigma + O(1)$$

in view of (2.6), where $d = \inf\{d_g(\xi, p_i): \xi \in \sigma_j, i = 1, \ldots, \ell, j = 1, \ldots, N\} > 0$, by the previous Section we deduce that $\mathcal{H}^*$ is uniformly bounded in M. Therefore, it is enough to show that the tangential derivative of $\mathcal{H}$ on $\partial\mathcal{D}$ is non-zero for uniformly bounded values of $\mathcal{H}$ when M is large enough. By contradiction assume that there exist $\boldsymbol{\xi}_n = (\xi_1^n, \ldots, \xi_N^n) \in \mathcal{M}$ and $(\beta_1^n, \beta_2^n) \neq (0,0)$ such that

$$\Phi(\boldsymbol{\xi}_n) \to +\infty, \qquad -C \leq \mathcal{H}(\boldsymbol{\xi}_n) \leq C, \tag{3.1}$$
$$\beta_1^n \nabla\mathcal{H}(\boldsymbol{\xi}_n) + \beta_2^n \nabla\Phi(\boldsymbol{\xi}_n) = 0$$

for some $C > 0$, where the last expression accounts also for non-regular points of $\partial\mathcal{D}$ and can be re-written as

$$(\beta_2^n - \beta_1^n)\Gamma_j \sum_{i=1}^\ell \alpha_i \nabla_{\xi_j} G(\xi_j^n, p_i) + 2(\beta_1^n + \beta_2^n)\Gamma_j \sum_{\substack{k=1 \\ k \neq j}}^N \Gamma_k \nabla_{\xi_j} G(\xi_j^n, \xi_k^n) = O(1) \qquad \forall\, j. \tag{3.2}$$

To get a contradiction, our aim is to identify the leading term of the left hand side in (3.2). Without loss of generality we assume that

$$(\beta_1^n)^2 + (\beta_2^n)^2 = 1, \quad \beta_1^n - \beta_2^n \geq 0, \quad \min_{j \neq k} d_g(\xi_j^n, \xi_k^n) = o(1), \quad \min_{j,i} d_g(\xi_j^n, p_i) = o(1), \tag{3.3}$$

where we have used

$$2\sum_{\substack{j,k=1 \\ j \neq k}}^N \Gamma_j\Gamma_k G(\xi_j^n, \xi_k^n) = \mathcal{H}(\boldsymbol{\xi}_n) + \Phi(\boldsymbol{\xi}_n) + O(1) \to +\infty$$

and

$$2\sum_{i=1}^\ell \alpha_i \sum_{j=1}^N \Gamma_j G(\xi_j, p_i) = \Phi(\boldsymbol{\xi}_n) - \mathcal{H}(\boldsymbol{\xi}_n) + O(1) \to +\infty$$

thanks to (3.1). Given $r_0 > 0$ small enough (smaller than the injectivity radius of (Σ, g)), we introduce normal coordinates $y_\xi : y_\xi^{-1}(B_{r_0}(0)) \to B_{r_0}(0)$ which depend smoothly on $\xi \in \Sigma$. Since $y_\xi(\xi) = 0$ and $d_g(x, \xi) = |y_\xi(x)|$ for all $x \in y_\xi^{-1}(B_{r_0}(0))$, we have that

$$\nabla_{\xi_1} \log d_g(\xi_1, \xi_2) = \frac{y_{\xi_2}(\xi_1)}{|y_{\xi_2}(\xi_1)|^2} = \frac{y_\xi(\xi_1) - y_\xi(\xi_2)}{|y_\xi(\xi_1) - y_\xi(\xi_2)|^2} + o\left(\frac{1}{d_g(\xi_1, \xi_2)}\right) \tag{3.4}$$

as $\xi_1, \xi_2 \to \xi$, owing to

$$\begin{aligned}
y_\xi(\xi_1) - y_\xi(\xi_2) &= y_{\xi_2}(\xi_1) - y_{\xi_2}(\xi_2) + O(d_g(\xi_2, \xi))|\nabla_\xi y_{\tilde\xi}(\xi_1) - \nabla_\xi y_{\tilde\xi}(\xi_2)| \\
&= y_{\xi_2}(\xi_1) + o(|y_{\xi_2}(\xi_1)|)
\end{aligned}$$

as $\xi_1, \xi_2 \to \xi$ (where $\tilde\xi$ is "between" ξ and ξ_2).

Hereafter we might pass to subsequences without further notice. Let us split $\{1, \ldots, N\}$ as $Z_0 \cup \cdots \cup Z_\ell$, where

$$Z_0 = \{j : |\xi_j^n - p_i| \geq c \text{ for all } i\}, \quad Z_i = \{j : \xi_j^n \to p_i\} \quad i = 1, \ldots, \ell.$$

We begin with the following two Lemmas.

Lemma 3.1. *The following holds:*

a) *if $\#Z_i = 1$ for some $i = 1, \ldots, \ell$, then $\beta_1^n - \beta_2^n \to 0$;*
b) *if $d_g(\xi_j^n, \xi_k^n) = o(1)$ for some $j, k \in Z_0$, $j \neq k$, then $\beta_1^n + \beta_2^n \to 0$;*
c) *there exists $i \in \{1, \ldots, \ell\}$ such that $\#Z_i \geq 2$.*

Proof. If $Z_i = \{j_0\}$, the identity (3.2) with $j = j_0$ in the coordinate system y_{p_i} gives

$$(\beta_2^n - \beta_1^n)\alpha_i \Gamma_{j_0} \frac{y_{p_i}(\xi_{j_0}^n)}{|y_{p_i}(\xi_{j_0}^n)|^2} = O(1),$$

and then a) follows. Next, let $j_0, k_0 \in Z_0$, $j_0 \neq k_0$, be such that $d_g(\xi_{j_0}^n, \xi_{k_0}^n) = o(1)$. We may assume

$$d_g(\xi_{j_0}^n, \xi_{k_0}^n) = \min_{\substack{j,k \in Z_0 \\ j \neq k}} d_g(\xi_j^n, \xi_k^n) \qquad \forall\, n \in \mathbb{N}.$$

Letting

$$I = \left\{ j \in Z_0 : d_g(\xi_j^n, \xi_{j_0}^n) \sim d_g(\xi_{j_0}^n, \xi_{k_0}^n) \right\} \cup \{j_0\},$$

where $\sim$ denotes sequences of same order as $n \to +\infty$, observe that by construction

$$d_g(\xi_j^n, \xi_k^n) \sim d_g(\xi_{j_0}^n, \xi_{k_0}^n) \qquad \forall\, j, k \in I,\, j \neq k$$

and $d_g(\xi_{j_0}^n, \xi_{k_0}^n) = o(d_g(\xi_j^n, \xi_k^n))$ for all $j \in I$ and $k \in Z_0 \setminus I$, by which

$$\nabla_{\xi_j} G(\xi_j^n, \xi_k^n) = o\left(\frac{1}{d_g(\xi_{j_0}^n, \xi_{k_0}^n)}\right) \qquad \forall\, j \in I,\, k \in Z_0 \setminus I.$$

The identities (3.2) read in the coordinate system $y_{\xi_{j_0}^n}$ as

$$2(\beta_1^n + \beta_2^n)\Gamma_j \sum_{\substack{k \in I \\ k \neq j}} \Gamma_k \frac{y_{\xi_{j_0}^n}(\xi_j^n) - y_{\xi_{j_0}^n}(\xi_k^n)}{|y_{\xi_{j_0}^n}(\xi_j^n) - y_{\xi_{j_0}^n}(\xi_k^n)|^2} = o\left(\frac{1}{d_g(\xi_{j_0}^n, \xi_{k_0}^n)}\right) \qquad \forall\, j \in I \tag{3.5}$$

in view of (3.4). Since

$$2 \sum_{\substack{j,k \in I \\ j \neq k}} \Gamma_j \Gamma_k \frac{\langle z_j - z_k, z_j - z \rangle}{|z_j - z_k|^2} = 2 \sum_{\substack{j,k \in I \\ j < k}} \Gamma_j \Gamma_k = \sum_{\substack{j,k \in I \\ j \neq k}} \Gamma_j \Gamma_k \qquad (3.6)$$

for all $z_j, z \in \mathbb{R}^2$, by taking the inner product of (3.5) with $y_{\xi_{j_0}^n}(\xi_j^n)$ and summing up in $j \in I$ we deduce that

$$(\beta_1^n + \beta_2^n) \sum_{\substack{j,k \in I \\ j \neq k}} \Gamma_j \Gamma_k = o(1).$$

Since $j_0, k_0 \in I$, we get $\beta_1^n + \beta_2^n = o(1)$, and b) follows. Finally, if $\#Z_i \leq 1$ were true for all $i = 1, \ldots, \ell$, by (3.3) we would get that $\#Z_{i_0} = 1$ for some $i_0 = \{1, \ldots, \ell\}$. On the other hand, thanks to (3.3) we also have that $d_g(\xi_j^n, \xi_k^n) = o(1)$ for some $j, k \in Z_0$, $j \neq k$. Then, by a) and b) we would derive $\beta_1^n + \beta_2^n = o(1)$, $\beta_1^n - \beta_2^n = o(1)$, in contradiction with (3.3). $\square$

Lemma 3.2. *If $\#Z_i \geq 2$ for some $i = 1, \ldots, \ell$, then $d_g(\xi_j^n, p_i) = O(d_g(\xi_j^n, \xi_k^n))$ for all $j, k \in Z_i$, $j \neq k$.*

Proof. By contradiction, assume the existence of $j_0, k_0 \in Z_i$, $j_0 \neq k_0$, such that

$$\frac{d_g(\xi_{j_0}^n, \xi_{k_0}^n)}{d_g(\xi_{j_0}^n, p_i)} = \min_{\substack{j,k \in Z_i \\ j \neq k}} \frac{d_g(\xi_j^n, \xi_k^n)}{d_g(\xi_j^n, p_i)} \to 0. \qquad (3.7)$$

Letting

$$I = \left\{ j \in Z_i : \ d_g(\xi_j^n, \xi_{j_0}^n) \sim d_g(\xi_{j_0}^n, \xi_{k_0}^n) \right\} \cup \{j_0\},$$

observe that for $j \in I$ by construction

$$d_g(\xi_j^n, p_i) \sim d_g(\xi_{j_0}^n, p_i)$$

and

$$d_g(\xi_j^n, \xi_k^n) \sim d_g(\xi_{j_0}^n, \xi_{k_0}^n) \quad \forall \, k \in I, \, k \neq j, \qquad d_g(\xi_{j_0}^n, \xi_{k_0}^n) = o(d_g(\xi_j^n, \xi_k^n)) \quad \forall \, k \in Z_i \setminus I, \quad (3.8)$$

by which

$$\nabla_{\xi_j} G(\xi_j^n, \xi_k^n) = o\left(\frac{1}{d_g(\xi_{j_0}^n, \xi_{k_0}^n)} \right) \qquad \forall \, j \in I, \, k \in Z_i \setminus I.$$

The identities (3.2) in the coordinate system $y_{\xi_{j_0}^n}$ read as

$$(\beta_2^n - \beta_1^n)\alpha_i \Gamma_j \frac{y_{\xi_{j_0}^n}(\xi_j^n) - y_{\xi_{j_0}^n}(p_i)}{|y_{\xi_{j_0}^n}(\xi_j^n) - y_{\xi_{j_0}^n}(p_i)|^2} + 2(\beta_1^n + \beta_2^n)\Gamma_j \sum_{\substack{k \in I \\ k \neq j}} \Gamma_k \frac{y_{\xi_{j_0}^n}(\xi_j^n) - y_{\xi_{j_0}^n}(\xi_k^n)}{|y_{\xi_{j_0}^n}(\xi_j^n) - y_{\xi_{j_0}^n}(\xi_k^n)|^2}$$

$$= o\left(\frac{\beta_1^n + \beta_2^n}{d_g(\xi_{j_0}^n, \xi_{k_0}^n)} \right) + o\left(\frac{\beta_2^n - \beta_1^n}{d_g(\xi_{j_0}^n, p_i)} \right) + O(1) \qquad \forall \, j \in I \qquad (3.9)$$

in view of (3.4). By taking the inner product of (3.9) with $y_{\xi_{j_0}^n}(\xi_j^n)$ and summing up in $j \in I$ we deduce that

$$(\beta_1^n + \beta_2^n) \sum_{\substack{j,k \in I \\ j \neq k}} \Gamma_j \Gamma_k = o(\beta_1^n + \beta_2^n) + (\beta_2^n - \beta_1^n)O\left(\frac{d_g(\xi_{j_0}^n, \xi_{k_0}^n)}{d_g(\xi_{j_0}^n, p_i)} \right) + O(d_g(\xi_{j_0}^n, \xi_{k_0}^n))$$

thanks to (3.6), by which, using (3.7), we get

$$\beta_1^n + \beta_2^n = O\Big(\frac{d_g(\xi_{j_0}^n, \xi_{k_0}^n)}{d_g(\xi_{j_0}^n, p_i)}\Big) \to 0. \tag{3.10}$$

By taking the inner product of (3.9) with $y_{\xi_{j_0}^n}(\xi_j^n) - y_{\xi_{j_0}^n}(p_i)$ and summing up in $j \in I$ we obtain that

$$(\beta_1^n - \beta_2^n)\alpha_i \sum_{j \in I} \Gamma_j = (\beta_1^n + \beta_2^n) \sum_{\substack{j,k \in I \\ j \neq k}} \Gamma_j \Gamma_k + o(1) + (\beta_1^n + \beta_2^n)o\Big(\frac{d_g(\xi_{j_0}^n, p_i)}{d_g(\xi_{j_0}^n, \xi_{k_0}^n)}\Big)$$

thanks to (3.6). By (3.10) we arrive at $\beta_1^n \pm \beta_2^n \to 0$, in contradiction with (3.3). $\qquad\square$

If $\#Z_i \geq 2$, let us split Z_i as $Y_1 \cup \cdots \cup Y_l$, $l \geq 1$, in such a way that for all $j \in Y_r$

$$d_g(\xi_j^n, p_i) \sim d_g(\xi_k^n, p_i) \quad \forall\, k \in Y_r, \qquad d_g(\xi_j^n, p_i) = o(d_g(\xi_k^n, p_i)) \quad \forall\, k \in Y_{r+1} \cup \cdots \cup Y_l. \tag{3.11}$$

Notice that by construction $d_g(\xi_j^n, \xi_k^n) \sim d_g(\xi_k^n, p_i)$ for all $j \in Y_r$, $k \in Y_{r+1} \cup \cdots \cup Y_l$, and by Lemma 3.2 $d_g(\xi_j^n, \xi_k^n) \sim d_g(\xi_k^n, p_i)$ for all $j, k \in Y_r$ $(j \neq k)$, yielding

$$d_g(\xi_j^n, \xi_k^n) \sim d_g(\xi_k^n, p_i) \qquad \forall\, j \in Y_r,\ k \in Y_r \cup \cdots \cup Y_l,\ j \neq k. \tag{3.12}$$

Combining (3.11)-(3.12) we get

$$\nabla_{\xi_j} G(\xi_j^n, \xi_k^n) = o\Big(\frac{1}{d_g(\xi_j^n, p_i)}\Big) \qquad \forall\, j \in Y_r,\ k \in Y_{r+1} \cup \cdots \cup Y_l,$$

which inserted in (3.2) (written in the coordinate system y_{p_i}) gives that

$$2(\beta_1^n + \beta_2^n)\Gamma_j \sum_{\substack{k \in Y_1 \cup \cdots \cup Y_r \\ k \neq j}} \Gamma_k \frac{y_{p_i}(\xi_j^n) - y_{p_i}(\xi_k^n)}{|y_{p_i}(\xi_j^n) - y_{p_i}(\xi_k^n)|^2} = (\beta_1^n - \beta_2^n)\alpha_i \Gamma_j \frac{y_{p_i}(\xi_j^n)}{|y_{p_i}(\xi_j^n)|^2} + o\Big(\frac{1}{d_g(\xi_j^n, p_i)}\Big) \tag{3.13}$$

for all $j \in Y_r$ in view of (3.4). Since $|y_{p_i}(\xi_j^n)| = o(|y_{p_i}(\xi_j^n) - y_{p_i}(\xi_k^n)|)$ for all $j \in Y_1 \cup \cdots \cup Y_{r-1}$ and $k \in Y_r$ owing to (3.11)-(3.12), we can compute

$$\frac{\langle y_{p_i}(\xi_j^n) - y_{p_i}(\xi_k^n), y_{p_i}(\xi_j^n)\rangle}{|y_{p_i}(\xi_j^n) - y_{p_i}(\xi_k^n)|^2} = 1 + \frac{\langle y_{p_i}(\xi_j^n) - y_{p_i}(\xi_k^n), y_{p_i}(\xi_k^n)\rangle}{|y_{p_i}(\xi_j^n) - y_{p_i}(\xi_k^n)|^2} = 1 + o(1) \tag{3.14}$$

for all $j \in Y_r$ and $k \in Y_1 \cup \cdots \cup Y_{r-1}$. By taking the inner product of (3.13) with $y_{p_i}(\xi_j^n)$ and summing up in $j \in Y_r$ we get that

$$(\beta_1^n + \beta_2^n)\Big(\sum_{\substack{j,k \in Y_r \\ j \neq k}} \Gamma_j \Gamma_k + 2 \sum_{\substack{j \in Y_r \\ k \in Y_1 \cup \cdots \cup Y_{r-1}}} \Gamma_j \Gamma_k\Big) = (\beta_1^n - \beta_2^n)\alpha_i \sum_{j \in Y_r} \Gamma_j + o(1) \qquad \forall\, r \geq 1 \tag{3.15}$$

thanks to (3.6) and (3.14). Since $\#Z_i \geq 2$, notice that the coefficient in brackets on the left hand side of (3.15) is positive when $r = l$, and then $\beta_1^n - \beta_2^n$ and $\beta_1^n + \beta_2^n$ are positively proportional up to higher order terms. By (3.3) and (3.15) (with $r = l$) we deduce that

$$\beta_1^n - \beta_2^n,\ \beta_1^n + \beta_2^n, |\beta_1^n|,\ |\beta_2^n| \geq c > 0, \tag{3.16}$$

taking into account that $\beta_2^n = o(1)$ would imply $\beta_1^n = 1 + o(1)$ and consequently, by (3.15) (with $r = 1$),

$$\sum_{\substack{j,k \in Y_1 \\ j \neq k}} \Gamma_j \Gamma_k = \alpha_i \sum_{j \in Y_1} \Gamma_j,$$

contradicting (1.3). Setting

$$a = \lim_{n \to +\infty} \frac{\beta_2^n}{\beta_1^n} \neq 0, \tag{3.17}$$

let us evaluate the different pieces of the energy as follows:

$$\sum_{\substack{j,k \in Z_i \\ j \neq k}} \Gamma_j \Gamma_k G(\xi_j^n, \xi_k^n) - \alpha_i \sum_{j \in Z_i} \Gamma_j G(\xi_j^n, p_i) + O(1)$$

$$= \sum_{r=1}^{l} \sum_{\substack{j,k \in Y_r \\ j \neq k}} \Gamma_j \Gamma_k G(\xi_j^n, \xi_k^n) + 2 \sum_{r=1}^{l} \sum_{\substack{j \in Y_r \\ k \in Y_1 \cup \cdots \cup Y_{r-1}}} \Gamma_j \Gamma_k G(\xi_j^n, \xi_k^n) - \alpha_i \sum_{r=1}^{l} \sum_{j \in Y_r} \Gamma_j G(\xi_j^n, p_i) + O(1)$$

$$= -\frac{1}{2\pi} \sum_{r=1}^{l} \left[\sum_{\substack{j,k \in Y_r \\ j \neq k}} \Gamma_j \Gamma_k \log d_g(\xi_j^n, \xi_k^n) + 2 \sum_{\substack{j \in Y_r \\ k \in Y_1 \cup \cdots \cup Y_{r-1}}} \Gamma_j \Gamma_k \log d_g(\xi_j^n, \xi_k^n) - \alpha_i \sum_{j \in Y_r} \Gamma_j \log d_g(\xi_j^n, p_i) \right].$$

Since $d_g(\xi_j^n, \xi_k^n) \sim d_g(\xi_j^n, p_i)$ for all $j \in Y_r$ and $k \in Y_1 \cup \cdots \cup Y_r$ in view of (3.12), by (3.15) and (3.17) we have that

$$\sum_{\substack{j,k \in Y_r \\ j \neq k}} \Gamma_j \Gamma_k \log d_g(\xi_j^n, \xi_k^n) + 2 \sum_{\substack{j \in Y_r \\ k \in Y_1 \cup \cdots \cup Y_{r-1}}} \Gamma_j \Gamma_k \log d_g(\xi_j^n, \xi_k^n) - \alpha_i \sum_{j \in Y_r} \Gamma_j \log d_g(\xi_j^n, p_i)$$

$$= \left(\sum_{\substack{j,k \in Y_r \\ j \neq k}} \Gamma_j \Gamma_k + 2 \sum_{\substack{j \in Y_r \\ k \in Y_1 \cup \cdots \cup Y_{r-1}}} \Gamma_j \Gamma_k - \alpha_i \sum_{j \in Y_r} \Gamma_j \right) \log d_g(\xi_{j_r}^n, p_i) + O(1)$$

$$= -(a + o(1)) \left(\sum_{\substack{j,k \in Y_r \\ j \neq k}} \Gamma_j \Gamma_k + 2 \sum_{\substack{j \in Y_r \\ k \in Y_1 \cup \cdots \cup Y_{r-1}}} \Gamma_j \Gamma_k + \alpha_i \sum_{j \in Y_r} \Gamma_j \right) \log d_g(\xi_{j_r}^n, p_i) + O(1),$$

where $j_r \in Y_r$ is fixed. We have thus proved that

$$\frac{1}{a} \left(\sum_{\substack{j,k \in Z_i \\ j \neq k}} \Gamma_j \Gamma_k G(\xi_j^n, \xi_k^n) - \alpha_i \sum_{j \in Z_i} \Gamma_j G(\xi_j^n, p_i) \right) \to -\infty \tag{3.18}$$

for all Z_i with $\#Z_i \geq 2$. By (3.16) we deduce $\#Z_i \neq 1$ for all $i = 1, \dots, \ell$ and $G(\xi_j^n, \xi_k^n) = O(1)$ for all $(j,k) \notin \bigcup_{i=1}^{\ell} (Z_i \times Z_i)$ according to Lemma 3.1-a) and b), then we conclude that

$$\frac{1}{a} \mathcal{H}(\boldsymbol{\xi}_n) = \frac{1}{a} \left[\sum_{\substack{j,k=1 \\ j \neq k}}^{N} \Gamma_j \Gamma_k G(\xi_j^n, \xi_k^n) - \sum_{i=1}^{\ell} \alpha_i \sum_{j=1}^{N} \Gamma_j G(\xi_j^n, p_i) \right] + O(1)$$

$$= \frac{1}{a} \sum_{i=1}^{\ell} \left[\sum_{\substack{j,k \in Z_i \\ j \neq k}} \Gamma_j \Gamma_k G(\xi_j^n, \xi_k^n) - \alpha_i \sum_{j \in Z_i} \Gamma_j G(\xi_j^n, p_i) \right] + O(1) \to -\infty$$

by (3.18), in contradiction with (3.1).

APPENDIX A. PROOF OF THEOREM 1.7

Setting $a_i = 1 + [\alpha_i]^-$, the aim of this section is to compute the maximum

$$N := \max\{N_1 + \cdots + N_\ell : N_i \in \mathbb{N} \cup \{0\}, \ N_i + N_{i+1} \leq a_{i+1} \ \forall \ i = 1, \dots, \ell\}, \tag{A.1}$$

with the convention $a_{\ell+1} = a_1$ and $N_{\ell+1} = N_1$. The cases $\ell = 2, 3$ are easier to handle and will be treated later in details. From now on, let us assume $\ell \geq 4$. Notice that for any $i = 1, \dots, \ell$

$$0 \leq N_i \leq \min\{a_i, a_{i+1}\}$$

and for any $1 = 1, \dots, \ell - 1$

$$N_{i+1} \leq \min\{a_{i+1} - N_i, a_{i+2} - N_{i+2}\}.$$

Therefore, setting $J_i = [0, \min\{a_i, a_{i+1}\}] \cap (\mathbb{N} \cup \{0\})$, we have that (A.1) can be rewritten as

$$N = \max_{N_i \in J_i, \ i \ \text{odd}} \left(N_1 + \min\{a_2 - N_1, a_3 - N_3\} + N_3 + \cdots + \min\{a_\ell - N_{\ell-1}, a_1 - N_1\} \right)$$

when ℓ is even and

$$N = \max_{\substack{N_i \in J_i, \ i \ \text{odd} \\ N_1 + N_\ell \leq a_1}} \left(N_1 + \min\{a_2 - N_1, a_3 - N_3\} + N_3 + \cdots + \min\{a_{\ell-1} - N_{\ell-2}, a_\ell - N_\ell\} + N_\ell \right) \quad \text{(A.2)}$$

when ℓ is odd.

For the sake of clarity, we fix the following three Lemmas.

Lemma A.1. *Let* $f(t) = \min\{\alpha, \beta - t\} + t + \min\{\gamma - t, \delta\}$ *for* $0 \leq t \leq T$. *Then*

$$\max_{[0,T]} f = \min\{\alpha + \gamma, \beta + \gamma, \alpha + \delta + T, \beta + \delta\}.$$

Proof. For $t \in \mathbb{R}$ we can write

$$f(t) = \begin{cases} \alpha + \delta + t & \text{if } t \leq \min\{\beta - \alpha, \gamma - \delta\} \\ \min\{\beta + \delta, \alpha + \gamma\} & \text{if } \min\{\beta - \alpha, \gamma - \delta\} \leq t \leq \max\{\beta - \alpha, \gamma - \delta\} \\ \beta + \gamma - t & \text{if } t \geq \max\{\beta - \alpha, \gamma - \delta\}, \end{cases}$$

yielding

$$\max_{[0,T]} f = \begin{cases} \min\{\alpha + \delta + T, \beta + \delta, \alpha + \gamma\} & \text{if } \min\{\beta - \alpha, \gamma - \delta\} \geq 0 \\ \min\{\beta + \delta, \alpha + \gamma\} & \text{if } \min\{\beta - \alpha, \gamma - \delta\} \leq 0 \leq \max\{\beta - \alpha, \gamma - \delta\} \\ \beta + \gamma & \text{if } \max\{\beta - \alpha, \gamma - \delta\} \leq 0 \end{cases}$$
$$= \min\{\alpha + \gamma, \beta + \gamma, \alpha + \delta + T, \beta + \delta\}$$

as claimed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Let us fix $2 \leq k \leq \frac{\ell}{2}$ and consider the numbers c_k, d_k, f_k, g_k defined in the introduction. We get

$$c_2 = a_2 + a_4, \ d_2 = a_3 + a_4, \ f_2 = a_2 + \min\{a_3, a_4\} + a_5, \ g_2 = a_3 + a_5. \quad \text{(A.3)}$$

Lemma A.2. *The following identities hold:*

$$N = \max_{N_1 \in J_1} \left(\min\{c_{\frac{\ell}{2}}, d_{\frac{\ell}{2}} + N_1, f_{\frac{\ell}{2}} - N_1, g_{\frac{\ell}{2}}\} \right) \quad \text{(A.4)}$$

when ℓ is even and

$$N = \max_{\substack{N_1 \in J_1, N_\ell \in J_\ell \\ N_1 + N_\ell \leq a_1}} \left(\min\{c_{\frac{\ell-1}{2}} + N_\ell, d_{\frac{\ell-1}{2}} + N_1 + N_\ell, f_{\frac{\ell-1}{2}}, g_{\frac{\ell-1}{2}} + N_1\} \right) \quad \text{(A.5)}$$

when ℓ is odd.

Proof. We claim that for every $2 \le k \le \frac{\ell}{2}$ we have

$$\max_{N_i \in J_i \, i=3,\ldots,2k-1 \, \mathrm{odd}} \left(\min\{a_2 - N_1, a_3 - N_3\} + N_3 + \cdots + \min\{a_{2k} - N_{2k-1}, a_{2k+1} - N_{2k+1}\} \right)$$
$$= \min\{c_k - N_1, d_k, f_k - N_1 - N_{2k+1}, g_k - N_{2k+1}\}.$$
$$(A.6)$$

Indeed, (A.6) is valid for $k = 2$ owing to (A.3), and the validity of (A.6) with index k implies

$$\max_{N_i \in J_i \, i=3,\ldots,2k+1 \, \mathrm{odd}} \left(\min\{a_2 - N_1, a_3 - N_3\} + N_3 + \cdots + \min\{a_{2k+2} - N_{2k+1}, a_{2k+3} - N_{2k+3}\} \right)$$
$$= \max_{N_{2k+1} \in J_{2k+1}} \left(\min\{\alpha_k, \beta_k - N_{2k+1}\} + N_{2k+1} + \min\{a_{2k+2} - N_{2k+1}, a_{2k+3} - N_{2k+3}\} \right),$$

where $\alpha_k = \min\{c_k - N_1, d_k\}$ and $\beta_k = \min\{f_k - N_1, g_k\}$. By Lemma A.1 we have that

$$\max_{N_{2k+1} \in J_{2k+1}} \left(\min\{\alpha_k, \beta_k - N_{2k+1}\} + N_{2k+1} + \min\{a_{2k+2} - N_{2k+1}, a_{2k+3} - N_{2k+3}\} \right)$$
$$= \min\{\alpha_k + a_{2k+2}, \beta_k + a_{2k+2}, \alpha_k + \min\{a_{2k+1}, a_{2k+2}\} + a_{2k+3} - N_{2k+3}, \beta_k + a_{2k+3} - N_{2k+3}\}.$$

The validity of (A.6) with index $k + 1$ is now achieved through the identities

$$\begin{aligned}
c_{k+1} &= \min\{c_k + a_{2k+2}, f_k + a_{2k+2}\}, \qquad d_{k+1} = \min\{d_k + a_{2k+2}, g_k + a_{2k+2}\}, \\
f_{k+1} &= \min\{c_k + \min\{a_{2k+1}, a_{2k+2}\} + a_{2k+3}, f_k + a_{2k+3}\}, \\
g_{k+1} &= \min\{d_k + \min\{a_{2k+1}, a_{2k+2}\} + a_{2k+3}, g_k + a_{2k+3}\},
\end{aligned}$$

which follow by direct inspection of the definition of numbers c_k, d_k, f_k, g_k. Finally, by (A.6) we immediately get the thesis of the Lemma. $\qquad\square$

Lemma A.3. *The following inequalities hold:*
(a) $c_k + g_k \le d_k + f_k$ *for all* $2 \le k \le \frac{\ell}{2}$;
(b) $\min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} \le d_{\frac{\ell}{2}} + \min\{a_1, a_2\}$ *when* ℓ *is even.*

Proof. The inequality in (a) does hold for $k = 2$ thanks to (A.3) and its validity at step k implies that

$$c_{k+1} + g_{k+1} = \min\{c_k + a_{2k+2}, f_k + a_{2k+2}\} + \min\{d_k + \min\{a_{2k+1}, a_{2k+2}\} + a_{2k+3}, g_k + a_{2k+3}\}$$

$$\le \begin{cases}
c_k + a_{2k+2} + g_k + a_{2k+3} \le (d_k + a_{2k+2}) + (f_k + a_{2k+3}) \\
(d_k + a_{2k+2}) + (c_k + \min\{a_{2k+1}, a_{2k+2}\} + a_{2k+3}) \\
(g_k + a_{2k+2}) + (f_k + a_{2k+3}) \\
(g_k + a_{2k+2}) + (c_k + \min\{a_{2k+1}, a_{2k+2}\} + a_{2k+3})
\end{cases},$$

yielding

$$c_{k+1} + g_{k+1} \le \min\{d_k + a_{2k+2}, g_k + a_{2k+2}\} + \min\{c_k + \min\{a_{2k+1}, a_{2k+2}\} + a_{2k+3}, f_k + a_{2k+3}\}$$
$$= d_{k+1} + f_{k+1}.$$

By induction the inequality in (a) is true for all $k \ge 2$, which implies the validity of

$$\min\left\{c_{\frac{\ell}{2}} - d_{\frac{\ell}{2}}, g_{\frac{\ell}{2}} - d_{\frac{\ell}{2}}, \frac{1}{2}\left(f_{\frac{\ell}{2}} - d_{\frac{\ell}{2}}\right)\right\} = \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} - d_{\frac{\ell}{2}}$$

$$\max\left\{f_{\frac{\ell}{2}} - \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\}, \frac{1}{2}\left(f_{\frac{\ell}{2}} - d_{\frac{\ell}{2}}\right)\right\} = f_{\frac{\ell}{2}} - \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\}$$

for ℓ even, in view of $2\min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} \le f_{\frac{\ell}{2}} + d_{\frac{\ell}{2}}$. Concerning (b) notice that

$$s_k(J) = (1 - \chi_J(2))\chi_J(1)\min\{a_3, a_4\} + p_k(J^\#), \tag{A.7}$$

$$s_k(J) = a_{2k}\chi_J(k) + (1 - \chi_J(k))[a_{2k+1} + \chi_J(k-1)\min\{a_{2k-1}, a_{2k}\}] + q_k(J_\#) \tag{A.8}$$

for some functions p_k, q_k and $J^\# = J \cap \{2, \dots, k\}$, $J_\# = J \cap \{1, \dots, k-1\}$. Given $J \subset \{1, \dots, \frac{\ell}{2}\}$ so that $1 \notin J$, $\frac{\ell}{2} \in J$, for $\hat{J} = \{1\} \cup J$ by (A.7) we have that

$$
\begin{aligned}
c_{\frac{\ell}{2}} &\le a_2 + s_{\frac{\ell}{2}}(\hat{J}) = (a_2 - a_3 + (1 - \chi_{\hat{j}}(2))\min\{a_3, a_4\}) + \left(a_3 + p_{\frac{\ell}{2}}(\hat{J}^\#)\right) \\
&\le (a_2 - a_3 + \min\{a_3, a_4\}) + \left(a_3 + p_{\frac{\ell}{2}}(J^\#)\right) \le a_2 + (a_3 + s_{\frac{\ell}{2}}(J))
\end{aligned}
$$

yielding $c_{\frac{\ell}{2}} \le a_2 + d_{\frac{\ell}{2}}$. Similarly, for $\bar{J} = J \setminus \{\frac{\ell}{2}\}$ by (A.8) we get that

$$
\begin{aligned}
g_{\frac{\ell}{2}} &\le a_3 + s_{\frac{\ell}{2}}(\bar{J}) = \left(a_1 + \chi_{\bar{J}}\left(\frac{\ell}{2} - 1\right)\min\{a_{\ell-1}, a_\ell\}\right) + \left(a_3 + q_{\frac{\ell}{2}}(\bar{J}_\#)\right) \\
&\le \left(a_1 + \chi_J\left(\frac{\ell}{2} - 1\right)\min\{a_{\ell-1}, a_\ell\} - a_\ell\right) + \left(a_3 + s_{\frac{\ell}{2}}(J)\right) \le a_1 + \left(a_3 + s_{\frac{\ell}{2}}(J)\right)
\end{aligned}
$$

providing $g_{\frac{\ell}{2}} \le a_1 + d_{\frac{\ell}{2}}$. In conclusion, we have shown that

$$\min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} \le d_{\frac{\ell}{2}} + \min\{a_1, a_2\}$$

and (b) is thus established.

$\square$

Proof of Theorem 1.7. Thanks to inequalities (a)-(b) of Lemma A.3, for ℓ even we have that

$$
\min\{c_{\frac{\ell}{2}}, d_{\frac{\ell}{2}} + N_1, f_{\frac{\ell}{2}} - N_1, g_{\frac{\ell}{2}}\} =
\begin{cases}
d_{\frac{\ell}{2}} + N_1 & \text{if } N_1 \le \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} - d_{\frac{\ell}{2}} \\
\min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} & \text{if } \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} - d_{\frac{\ell}{2}} \le N_1 \le f_{\frac{\ell}{2}} - \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} \\
f_{\frac{\ell}{2}} - N_1 & \text{if } N_1 \ge f_{\frac{\ell}{2}} - \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\}
\end{cases}
$$

with $\min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} - d_{\frac{\ell}{2}} \le \min\{a_1, a_2\}$, yielding

$$N = \min\{c_{\frac{\ell}{2}}, g_{\frac{\ell}{2}}\} \tag{A.9}$$

when ℓ is even in view of (A.4). Unfortunately, when ℓ is odd the expression of N in (A.5) can be simply reduced to

$$N = \max_{\hat{a}_1 - \min\{a_1, a_\ell\} \le N_1 \le \min\{a_1, a_2\}} \left(\min\{c_{\frac{\ell-1}{2}} + \hat{a}_1 - N_1, d_{\frac{\ell-1}{2}} + \hat{a}_1, f_{\frac{\ell-1}{2}}, g_{\frac{\ell-1}{2}} + N_1\}\right) \tag{A.10}$$

because $N_1 + N_\ell \le \hat{a}_1 := \min\{a_1, \min\{a_1, a_2\} + \min\{a_1, a_\ell\}\}$ with the equality achieved for all the maximizers in (A.5).

An interesting situation corresponds to the case where the a_i's are ordered in an increasing way. To distinguish it from the general case, we will denote it by $b_1, \dots, b_\ell$. Given $J \subset \{1, \dots, k\}$ and $3 \le j \le k$, $s_k(J)$ depends on $j - 1$ only through the term

$$b_{2j-2}\chi_J(j-1) + (1 - \chi_J(j-1))[b_{2j-1} + \chi_J(j-2)b_{2j-3}] + (1 - \chi_J(j))\chi_J(j-1)b_{2j-1}$$

which is minimized by the choice $j - 1 \in J$ or $j - 1 \notin J$ depending on whether $j \in J$ or not, respectively. The same holds if $j = 2$. Therefore, the minimization in the definition of c_k and d_k is achieved by sets J with $\{2, \ldots, k\} \subset J$, yielding

$$c_k = \sum_{j=1}^{k} b_{2j}, \ d_k = b_3 + \sum_{j=1}^{k} b_{2j}, \tag{A.11}$$

whereas for f_k and g_k the minimizing sets J satisfy $J \cap \{2, \ldots, k\} = \emptyset$ and then

$$f_k = b_2 + \sum_{j=1}^{k} b_{2j+1}, \ g_k = \sum_{j=1}^{k} b_{2j+1}. \tag{A.12}$$

Since $g_{\frac{\ell}{2}} \leq c_{\frac{\ell}{2}}$ in view of (A.11)-(A.12), for ℓ even (A.9) becomes

$$N = g_{\frac{\ell}{2}} = \sum_{j=0}^{\frac{\ell-2}{2}} b_{2j+1}. \tag{A.13}$$

Since $c_{\frac{\ell-1}{2}} \leq d_{\frac{\ell-1}{2}}$ and $g_{\frac{\ell-1}{2}} + b_1 \leq f_{\frac{\ell-1}{2}}$ by (A.11)-(A.12), (A.10) gives that

$$\begin{aligned}
N &= \max\{\min\{c_{\frac{\ell-1}{2}} + b_1 - N_1, g_{\frac{\ell-1}{2}} + N_1\} : 0 \leq N_1 \leq b_1\} \\
&= \min\left\{c_{\frac{\ell-1}{2}} + b_1, \frac{1}{2}\left(c_{\frac{\ell-1}{2}} + b_1 + g_{\frac{\ell-1}{2}}\right)\right\} = \min\left\{b_1 + \sum_{j=1}^{\frac{\ell-1}{2}} b_{2j}, \frac{1}{2} \sum_{j=1}^{l} b_j\right\}
\end{aligned} \tag{A.14}$$

when ℓ is odd, in view of

$$\frac{1}{2}(c_{\frac{\ell-1}{2}} + b_1 - g_{\frac{\ell-1}{2}}) \leq \frac{b_1}{2}.$$

$\square$

Finally let us discuss the case $\ell = 2, 3, 4$.

$\boxed{\ell = 2}$ We clearly have that $N = \min\{a_1, a_2\}$.

$\boxed{\ell = 3}$ By (A.2) we deduce

$$\begin{aligned}
N &= \max\{\min\{a_2 + N_3, a_3 + N_1\} : \ N_1 + N_3 \leq a_1, 0 \leq N_1 \leq \min\{a_1, a_2\}, N_3 \leq \min\{a_1, a_3\}\} \\
&= \max\{\min\{a_2 + \hat{a}_1 - N_1, a_3 + N_1\} : \ \hat{a}_1 - \min\{a_1, a_3\} \leq N_1 \leq \min\{a_1, a_2\}\}
\end{aligned}$$

since $N_1 + N_3 \leq \hat{a}_1 := \min\{a_1, \min\{a_1, a_2\} + \min\{a_1, a_3\}\}$ with the equality achieved for all the maximizers. Then, we compute

$$N = \begin{cases} \frac{a_2 + a_3 + \hat{a}_1}{2} & \text{if} \ \hat{a}_1 - \min\{a_1, a_3\} \leq \frac{a_2 - a_3 + \hat{a}_1}{2} \leq \min\{a_1, a_2\} \\ a_3 + \min\{a_1, a_2\} & \text{if} \ \frac{a_2 - a_3 + \hat{a}_1}{2} \geq \min\{a_1, a_2\} \\ a_2 + \min\{a_1, a_3\} & \text{if} \ \frac{a_2 - a_3 + \hat{a}_1}{2} \leq \hat{a}_1 - \min\{a_1, a_3\}. \end{cases}$$

By discussing all the six possibilities for (a_1, a_2, a_3) $(a_1 < a_2 < a_3, a_2 < a_3 < a_1$ and so on), we immediately realize that $N = \min\{b_1 + b_2, \frac{b_1 + b_2 + b_3}{2}\}$, which actually corresponds to (A.14) for $\ell = 3$ for the increasing ordering b_1, b_2, b_3. We have thus proved that the maximal N in (A.1) is independent of the order of a_i's.

Instead of coupling the a_i's in a consecutive way (ordered or not), with regards to Theorem 1.4 let us now couple the pair a_1, a_2 with a_3 and a_3 with a_1: in order to satisfy (1.10), we decompose N as $N = N_1 + N_2 + N_3$ with

$$N_1 + N_3 \leq a_1, \quad N_1 + N_2 + N_3 \leq a_3, \quad N_2 \leq a_2.$$

We can easily see that the maximal N satisfies $N = \min\{a_1 + a_2, a_3\}$, giving $N \leq \min\{b_1 + b_2, \frac{b_1 + b_2 + b_3}{2}\}$. Since such a case represents the general situation for a non-consecutive coupling, we can summarize our discussion by saying that the consecutive increasing coupling b_1, b_2, b_3 gives rise to the best maximal $N = \min\{b_1 + b_2, \frac{b_1 + b_2 + b_3}{2}\}$ among all the possible couplings (consecutive or not, increasing or not). Such a property is peculiar for $\ell = 3$, as we will see below by discussing the case $\ell = 4$.

$\boxed{\ell = 4}$ By (A.9) we have that $N = \min\{a_1 + a_3, a_2 + a_4\}$, and the optimal choice is $(a_1, a_2, a_3, a_4) = (b_2, b_1, b_3, b_4)$ which gives rise to $N = \min\{b_2 + b_3, b_1 + b_4\}$. Since in general $\min\{b_2 + b_3, b_1 + b_4\} > b_1 + b_3$ (see (A.13)), we see that the increasing ordering is no longer the optimal among all the consecutive ones. Moreover, referring to non-consecutive couplings in Theorem 1.4, let us couple the pair b_1, b_2 with b_4, b_3 with b_2 and b_4 with b_3: in order to satisfy (1.10) we need to require that $N = N_1 + N_2 + N_3 + N_4$ with

$$N_1 \leq b_1, \quad N_2 + N_3 \leq b_2, \quad N_3 + N_4 \leq b_3, \quad N_1 + N_2 + N_4 \leq b_4.$$

The particular choice $N_1 = b_1$, $N_2 = \min\{b_2, b_4 - b_1\}$, $N_3 = 0$ and $N_4 = \min\{b_3, b_4 - b_1 - \min\{b_2, b_4 - b_1\}\}$ leads to $N \geq \min\{b_1 + b_2 + b_3, b_4\}$. Since $\min\{b_1 + b_2 + b_3, b_4\} > \min\{b_2 + b_3, b_1 + b_4\}$ when $b_1 + b_2 + b_3 \leq b_4$, we also see that consecutive couplings are not in general the optimal among all the possible ones.

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