Calc. Var.
DOI 10.1007/s00526-008-0184-3

Calculus of Variations

Degenerate elliptic equations with singular nonlinearities

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Received: 25 April 2007 / Accepted: 5 May 2008

Abstract The behavior of the "minimal branch" is investigated for quasilinear eigenvalue problems involving the p-Laplace operator, considered in a smooth bounded domain of $\mathbb{R}^N$, and compactness holds below a critical dimension $N^\#$. The nonlinearity $f(u)$ lies in a very general class and the results we present are new even for $p = 2$. Due to the degeneracy of p-Laplace operator, for $p \neq 2$ it is crucial to define a suitable notion of semi-stability: the functional space we introduce in the paper seems to be the natural one and yields to a spectral theory for the linearized operator. For the case $p = 2$, compactness is also established along unstable branches satisfying suitable spectral information. The analysis is based on a blow-up argument and stronger assumptions on the nonlinearity $f(u)$ are required.

Mathematics subject classification 35B35 · 35B45 · 35J70 · 35J60

1 Introduction and statement of main results

We deal with the analysis of solutions to the boundary value problem:

$$\begin{cases} -\Delta_p u = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda h(x) f(u) & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Authors are partially supported by MIUR, project "Variational methods and nonlinear differential equations".

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where $p > 1$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\lambda \geq 0$ is a parameter and $h(x) : \bar{\Omega} \to (0, +\infty)$ is an Hölder continuous function. Throughout the paper, the non-linearity $f(u)$ will be always assumed to be a non-decreasing, positive function defined on $[0, 1)$ with a singularity at $u = 1$:

$$\lim_{u \to 1^-} f(u) = +\infty. \tag{1.2}$$

Nonlinear eigenvalue problems as (1.1) with $p = 2$ and $f(u)$ a smooth nonlinearity unbounded at $+\infty$: $\lim_{u \to +\infty} f(u) = +\infty$, have been largely studied in last thirty years. Since the pioneering work of Crandall and Rabinowitz [8] for $f(u) = e^u$, there has been an intensive investigation to recover general smooth $f(u)$. Let us set up the problem in order to explain the contributions already available in literature. Let $f : [0, +\infty) \to (0, +\infty)$ be a smooth non-decreasing function so that $\liminf_{u \to +\infty} \frac{f(u)}{u} > 0$. By Implicit Function Theorem, there is a unique curve of positive solutions u_λ of (1.1) branching off $u = 0$, for λ small. It is possible to define the extremal parameter in the following way:

$$\lambda^* = \sup\{\lambda > 0 : (1.1) \text{ has a positive classical solution}\}, \tag{1.3}$$

and show that $\lambda^* < +\infty$. Since $f(0) > 0$, $u = 0$ is a subsolution of (1.1). By the method of sub/super solutions, the set of λ for which (1.1) is solvable coincides exactly with $[0, \lambda^*)$, and the associated iterative scheme provides a minimal solution u_λ (i.e. the smallest positive solution of (1.1) in a pointwise sense), for any $\lambda \in [0, \lambda^*)$. Moreover, the family $\{u_\lambda\}$ is non-decreasing in λ, and u_λ is a *semi-stable* solution of (1.1) in the sense:

$$\mu_1(u_\lambda) := \inf \left\{ \int_\Omega |\nabla \phi|^2 - \lambda h(x) \dot{f}(u_\lambda) \phi^2 : \phi \in H_0^1(\Omega), \int_\Omega \phi^2 = 1 \right\} \geq 0. \tag{1.4}$$

The main issues in such a topic are the following:
(1) compactness of the minimal branch u_λ

$$\sup_{\lambda \in [0, \lambda^*)} \|u_\lambda\|_\infty < +\infty \tag{1.5}$$

to guarantee that $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ -the so-called extremal solution- is a classical solution of (1.1) with $\lambda = \lambda^*$;
(2) study of u^* when compactness (1.5) along the minimal branch fails.
In general, u^* is a weak and still semi-stable solution: $\mu_1(u^*) \geq 0$ (defined as in (1.4)). In the non compact situation, u^* can be also computed explicitly in some special cases (see [3,4]). When compactness holds, let us stress that $\mu_1(u^*) = 0$ to prevent the continuation of the branch u_λ for $\lambda > \lambda^*$. In such a case (see [8]), by Implicit Function Theorem there is a second curve U_λ, different from u_λ, branching off $u = u^*$ for λ in a small left neighborhood of λ^*. The solutions U_λ turn out to be unstable, with Morse index one.

The validity of (1.5) depends on the dimension N and the nonlinearity $f(u)$: there is a critical "dimension" $N^\# \in \mathbb{R}$ so that compactness holds when $N < N^\#$ and fails when $N \geq N^\#$ (for some $h(x)$ and Ω). In [8,21], the critical dimension for the most typical examples $f(u) = e^u$ and $f(u) = (1 + u)^m$ are computed explicitly: $N^\# = 10$ when $f(u) = e^u$ and $N^\# \geq 11$ when $f(u) = (1 + u)^m$ (the expression of $N^\#$ in this case is rather involved). In [19], a thorough ODE analysis of solutions is achieved when Ω is a ball, $h(x) = 1$ and $f(u)$ as above.

For convex nonlinearities $f(u)$ so that $\lim_{u\to+\infty}\frac{f(u)}{u}=+\infty$, it is a long standing conjecture that the critical dimension should satisfy $N^\# > 9$ no matter $f(u)$ is. The first contribution is due to Crandall and Rabinowitz in [8] who prove, under the additional assumption:

$$0<\gamma=\liminf_{u\to+\infty}\frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)}\le\limsup_{u\to+\infty}\frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)}=\gamma_1<\infty,$$

that (1.5) holds for any $N<4+2\gamma+4\sqrt{\gamma}$, provided $\gamma_1<2+\sqrt{\gamma}+\gamma$. Recently, Ye and Zhou in [28] have improved Crandall-Rabinowitz statement: compactness holds for any $N<6+4\sqrt{\gamma}$, where

$$\gamma=\liminf_{u\to+\infty}\frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)}>0. \tag{1.6}$$

Let us remark that the critical dimension found by Ye and Zhou is 10 when $f(u)=e^u$. While, for $f(u)=(1+u)^m$ the dimension is not optimal but the optimal one can be easily recasted by a bootstrap argument. Without additional assumption, Nedev in [22] shows the validity of (1.5) for $N=2,3$ and Cabré in [5] has announced the result for $N=4$. When restricting the problem to radial solutions on the ball (with $h(x)=1$ for example), in [6] Cabré and Capella show compactness for any $N<10$ and possibly non-convex $f(u)$.

Problem (1.1) for a singular nonlinearity $f(u)=(1-u)^{-m}$, $m>0$, has been firstly considered by Joseph and Lundgreen in [19] in a radial setting. The analysis of the minimal branch u_λ has been pursued in [17,21] and the associated critical dimension has been computed. In [12,14] compactness of any unstable branch of solutions to (1.1) with uniformly bounded Morse indices is shown. The study in [12,14,17] is motivated by the theory of so-called MEMS devices and is focussed on $f(u)=(1-u)^{-2}$.

A MEMS device (Micro-Electro Mechanical System) is composed by a thin dielectric elastic membrane held fixed on $\partial\Omega$ (at level 0) placed below an upper plate (at level 1). An external voltage is applied, whose strength is measured by λ. The membrane deflects toward the upper plate, measured by $u(x)$ at any point $x\in\Omega$, and the deflection increases as λ increases. For an extremal λ^*, the membrane could touch the upper plate: $\max_\Omega u=1$, and the MEMS device would break down. The function $h(x)$ -referred to as permittivity profile- is directly related to the dielectricity of the membrane at the point $x\in\Omega$. Problem (1.1) with $p=2$ and $f(u)=(1-u)^{-2}$ arises as a model equation to describe MEMS devices theory. We investigate here problem (1.1) for general $p>1$ and nonlinearities $f(u)$, singular at $u=1$, with growth comparable to $(1-u)^{-m}$, $m>0$. We refer the interested reader to [23] for a complete account on this theory.

The difficulty is twofold. On one side, we allow more general functions $f(u)$ and the right assumption (in the spirit of [8,28]) has to be understood. On the other side, the p-Laplace operator is a nonlinear degenerate operator. Problem (1.1) for general $p>1$ has been considered in [15,16] for $f(u)=e^u$ and in [7] for $f(u)$ of polinomial-type growth. We will borrow some ideas and techniques from [7] to deal with singular nonlinearities, and some of their arguments will be refined here.

According to [11,20,26], Hölder continuity of first derivatives holds for any weak solution of (1.1) with $\|f(u)\|_\infty<+\infty$. Hence, we will say that u is a classical solution of (1.1) if u has Hölder continuous first derivatives, $0<u<1$ in Ω, $u=0$ on $\partial\Omega$ and u solves (1.1) in a weak sense:

$$\int_\Omega|\nabla u|^{p-2}\nabla u\nabla\phi=\lambda\int_\Omega h(x)f(u)\phi,\quad\forall\,\phi\in W_0^{1,p}(\Omega). \tag{1.7}$$

Let u be a classical solution of (1.1). The linearization of (1.7) is not possible along every direction in $W_0^{1,p}(\Omega)$, when $1 < p < 2$, while it is possible not only along directions in $W_0^{1,p}(\Omega)$, when $p > 2$. Then, the choice of a functional space $\mathcal{A}_u$, composed by admissible directions for which the linearization makes sense, is crucial. The stability of a minimal solution will be easier to establish as smaller the space $\mathcal{A}_u$ is. However, the class $\mathcal{A}_u$ should allow the choice of suitable test functions in order to prove a-priori energy estimates for semi-stable solutions.

In [7], the authors find a good candidate for $\mathcal{A}_u$, which however does not allow a spectral theory for the "linearized operator". We present here a different and more natural way to overcome the problem, by taking a space $\mathcal{A}_u$ larger than in [7].

Letting $\rho = |\nabla u|^{p-2}$, we introduce a weighted L^2-norm of the gradient: $|\phi| = \left(\int_\Omega \rho |\nabla \phi|^2\right)^{\frac{1}{2}}$. For $1 < p \le 2$, $\mathcal{A}_u$ is the following subspace of $H_0^1(\Omega)$:

$$\mathcal{A}_u = \{\phi \in H_0^1(\Omega) : |\phi| < +\infty\}.$$

Since $\int_\Omega |\nabla \phi|^2 \le \|\nabla u\|_\infty^{2-p} |\phi|^2$, $(\mathcal{A}_u, |\cdot|)$ is an Hilbert space.

For $p > 2$, the weight ρ is in $L^\infty(\Omega)$ and satisfies $\rho^{-1} \in L^1(\Omega)$, as shown in [9]. According to [27], the space

$$H_\rho^1(\Omega) = \{\phi \in L^2(\Omega) \text{ weakly differentiable} : |\phi| < +\infty\}$$

is an Hilbert space and is the completion of $C^\infty(\Omega)$ with respect to the $|\cdot|$-norm. For $p > 2$, the Hilbert space $\mathcal{A}_u$ is the closure of $C_0^\infty(\Omega)$ in $H_\rho^1(\Omega)$.

For convenience, we replace the $|\cdot|$-norm with the equivalent norm $\|\phi\| =< \phi, \phi >^{\frac{1}{2}}$, where

$$< \phi, \psi >= \int_\Omega |\nabla u|^{p-2} \nabla \phi \nabla \psi + (p-2)|\nabla u|^{p-2} \left(\frac{\nabla u}{|\nabla u|} \cdot \nabla \phi\right)\left(\frac{\nabla u}{|\nabla u|} \cdot \nabla \psi\right).$$

For any $p > 1$, the Hilbert space $\mathcal{A}_u$ is non-empty: $u \in \mathcal{A}_u$, and is compactly embedded in $L^2(\Omega)$, as we will derive in Appendix from the weighted Sobolev estimates of [10]. A first eigenfunction for the "linearized operator" then exists in $\mathcal{A}_u$:

Theorem 1.1 *Let u be a classical solution of* (1.1). *The infimum*

$$\mu_1(u) := \inf_{\phi \in \mathcal{A}_u \setminus \{0\}} \frac{\|\phi\|^2 - \lambda \int_\Omega h(x) \dot{f}(u) \phi^2}{\int_\Omega \phi^2}$$

is attained at some positive function ϕ_1, and any other minimizer is proportional to ϕ_1.

For our purposes, Theorem 1.1 is sufficient even if we guess a full spectral theory for the "linearized operator" to be in order. Once a right stability notion has been introduced, we show that the known results about the minimal branch are still available:

Theorem 1.2 *Let $p > 1$. Let $f(u)$ be a non-decreasing, positive function on* $[0, 1)$ *so that* (1.2) *holds. There exists $\lambda^* \in (0, +\infty)$ so that, for any $\lambda \in (0, \lambda^*)$,* (1.1) *has a unique minimal* (*classical*) *solution u_λ and, for any $\lambda > \lambda^*$ no classical solution of* (1.1) *exists. The following upper bound holds*:

$$\lambda^* \le f(0)^{-1} (\inf_\Omega h)^{-1} \lambda_1, \tag{1.8}$$

where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta_p$. Moreover, the family $\{u_\lambda\}$ is non-decreasing in λ and composed by semi-stable solutions: $\mu_1(u_\lambda) \ge 0$.

We are now concerned with compactness issues and, in the spirit of (1.6), we assume on $f(u)$:

$$\liminf_{u\to 1^-} \frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)} = \gamma > \frac{p-2}{p-1}, \qquad \liminf_{u\to 1^-} \frac{\ln f(u)}{\ln \frac{1}{1-u}} = m > 0. \tag{1.9}$$

Set

$$N^\# = \begin{cases} \frac{mp}{p+m-1}\left(\gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)} + \left(\gamma - 1 - \frac{1}{m}\right)_-\right) & \text{if } 1 < p \le 2 \\ \frac{mp}{p+m-1}\left(\gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}\right) & \text{if } p > 2, \end{cases} \tag{1.10}$$

where $u_- = \frac{|u|-u}{2}$ is the negative part of u. The result we have is:

Theorem 1.3 *Let $p > 1$ and $f(u)$ be a non-decreasing, positive function on $[0, 1)$ so that (1.9) holds. When $1 < p < 2$ assume the convexity of $f(u)$ near $u = 1$. Then*

$$\sup_{\lambda\in[0,\lambda^*)} \|u_\lambda\|_\infty < 1, \tag{1.11}$$

provided $N < N^\#$.

Remark 1.4 (1) Assumption $\gamma > \frac{p-2}{p-1}$ is necessary to obtain that $(q_-, q_+) \cap (-\min\{\gamma, 1\}, +\infty) \neq \emptyset$ in the basic integral estimate (3.30). In analogy with [22] it could be interesting to consider the case $\gamma = \frac{p-2}{p-1}$ when $p \ge 2$ (as in [25] for regular nonlinearities $f(u)$). However, when $p = 2$ Nedev result [22] applies for $N = 2, 3$ and can not be seen as the limiting case of Ye-Zhou result [28] because $\lim_{\gamma\to 0}(6 + 4\sqrt{\gamma}) = 6$. We are interested here in obtaining the maximal regularity we can (depending on γ) and we will consider only the case $\gamma > \frac{p-2}{p-1}$.

(2) Let us stress that the critical dimension $N^\#$ given in (1.10) has a jump discontinuity at $p = 2$ for $\gamma < 1 + \frac{1}{m}$: the method we will use (inspired by [28]) leads to stronger estimates when $1 < p \le 2$ and is based on the convexity of $f(u)$ near $u = 1$. The case $1 < p < 2$ and $\frac{p-2}{p-1} < \gamma \le 0$ could also be considered (as in [24] for regular nonlinearities $f(u)$) but this improved approach could not be used.

(3) Let us discuss assumption (1.9). Observe that, for singular polinomial nonlinearities $f(u) = (1-u)^{-m}$, there is a relation among m and γ: $\gamma = 1 + \frac{1}{m}$. In general, m and γ are not related, as the following convex nonlinearity shows:

$$f(u) = (1-u)^{-h(u)},\, h(u) = \frac{m_2 - m_1}{2} \sin\left\{\epsilon \ln\left[1 + \ln\left(1 + \ln\frac{1}{1-u}\right)\right]\right\} + \frac{m_1+m_2}{2},$$

where $\epsilon > 0$ small and $0 < m_1 < m_2 < \infty$. Observe that $h(u)$ oscillates taking all the values in $[m_1, m_2]$. Since $|\dot{h}(u) \ln \frac{1}{1-u}| \le \frac{\epsilon}{2}\frac{m_2-m_1}{1-u}$, for ϵ small there holds: $\dot{f}(u) = f(u)(\dot{h}(u) \ln \frac{1}{1-u} + \frac{h(u)}{1-u}) > 0$ for any $u \in [0, 1)$ (note that $f(u) = e^{h(u)\ln\frac{1}{1-u}}$). Since $|\ddot{h}(u) \ln \frac{1}{1-u}| + 2|\frac{\dot{h}(u)}{1-u}| \le 3\epsilon\frac{m_2-m_1}{(1-u)^2}$, for ϵ small we get:

$\ddot{f}(u) = \frac{\dot{f}^2(u)}{f(u)} + f(u)(\ddot{h}(u)\ln\frac{1}{1-u} + 2\frac{\dot{h}(u)}{1-u} + \frac{h(u)}{(1-u)^2}) > 0$ for any $u \in [0,1)$, and

$$\gamma = \liminf_{u\to 1^-} \frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)} = 1 + \liminf_{u\to 1^-} \frac{\ddot{h}(u)\ln\frac{1}{1-u} + 2\frac{\dot{h}(u)}{1-u} + \frac{h(u)}{(1-u)^2}}{\left(\dot{h}(u)\ln\frac{1}{1-u} + \frac{h(u)}{1-u}\right)^2}$$
$$= 1 + \liminf_{u\to 1^-} \frac{1}{h(u)} = 1 + \frac{1}{m_2}.$$

Since $\liminf_{u\to 1^-} \frac{\ln f(u)}{\ln\frac{1}{1-u}} = \liminf_{u\to 1^-} h(u) = m_1$, the values of m and γ in (1.9) are independent.

Moreover, this example features a general property (based on the validity of (1.2)):

$$\left(\liminf_{u\to 1^-} \frac{\ln f(u)}{\ln\frac{1}{1-u}}\right)^{-1} \le \limsup_{u\to 1^-} \frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)} - 1, \ \left(\limsup_{u\to 1^-} \frac{\ln f(u)}{\ln\frac{1}{1-u}}\right)^{-1}$$
$$\ge \liminf_{u\to 1^-} \frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)} - 1. \tag{1.12}$$

In strong analogy, let us remark that (1.6) on $[0,+\infty)$ implies:

$$\left(\liminf_{u\to +\infty} \frac{\ln f(u)}{\ln u}\right)^{-1} \le 1 - \liminf_{u\to +\infty} \frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)}.$$

Unfortunately, to establish energy estimates we need an assumption on $\liminf_{u\to 1^-} \frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)}$, which does not imply any control on $\liminf_{u\to 1^-} \frac{\ln f(u)}{\ln\frac{1}{1-u}}$. It explains somehow why we need to strengthen assumption (1.6) on $[0,\infty)$ when considering nonlinearities on $[0,1)$.

When (1.11) holds, the extremal function $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is so that: $\max_\Omega u^* < 1$. Since $\|f(u^*)\|_\infty < \infty$, by regularity theory u^* is a classical solution of (1.1) with $\lambda = \lambda^*$. Since u^* is the minimal solution, $\mu_1(u^*) \ge 0$. When $p = 2$, the Implicit Function Theorem provides $\mu_1(u^*) = 0$ and, following the classical argument of [8], there is $\delta > 0$ so that, for any $\lambda \in (\lambda^* - \delta, \lambda^*)$, a second solution U_λ of (1.1) exists so that $\lim_{\lambda\uparrow\lambda^*} U_\lambda = u^*$ in $C^1(\bar{\Omega})$.

For the analysis of the second branch U_λ, we will use a blowup approach developed in [12,14]. In order to identify a limiting equation on $\mathbb{R}^N$ and to have some useful information on such a limit problem, for $p = 2$ we will require:

$$\lim_{u\to 1^-} \frac{f(u)\ddot{f}(u)}{\dot{f}^2(u)} = \gamma > 1 \tag{1.13}$$

$$\sup_{0\le u<1}\sup_{0\le t\le u} \frac{(1-t)^{\frac{1}{\gamma-1}} f(t)}{(1-u)^{\frac{1}{\gamma-1}} f(u)} = c_0 < +\infty \tag{1.14}$$

$$\lim_{u\to 1^-} \sup_{1-M(1-u)\le t\le u} \left| \frac{(1-t)^{\frac{1}{\gamma-1}} f(t)}{(1-u)^{\frac{1}{\gamma-1}} f(u)} - 1 \right| = 0\,, \quad \forall\, M > 1. \tag{1.15}$$

By (1.12), the inequality $\gamma \geq 1$ in (1.13) always holds and $m = \lim_{u \to 1^-} \frac{\ln f(u)}{\ln \frac{1}{1-u}} = \frac{1}{\gamma - 1}$. Hence, $N^\#$ in (1.10) reduces to:

$$N^\# = \frac{2(\gamma + 2 + 2\sqrt{\gamma})}{\gamma}$$

(in case $p = 2$). The result we have is:

Theorem 1.5 *Let $p = 2$ and $f(u)$ be a convex, non-decreasing, positive function on $[0, 1)$ so that (1.13)–(1.15) hold. Let $\lambda_n \in (0, \lambda^*)$ be a sequence and u_n be associated solutions of (1.1). Assume that u_n has Morse index at most 1: $\mu_{2,n} \geq 0$ for any $n \in \mathbb{N}$, where $\mu_{2,n} = \mu_2(-\Delta - \lambda_n h(x)\dot{f}(u_n))$ is the 2nd eigenvalue of the linearized operator. Then*

$$\sup_{n \in \mathbb{N}} \|u_n\|_\infty < 1,$$

provided $N < \frac{2(\gamma+2+2\sqrt{\gamma})}{\gamma}$.

In [12], compactness of a solutions sequence u_n with uniformly bounded Morse indices is shown to hold for $f(u) = (1-u)^{-2}$, where $h(x)$ is allowed to vanish at p_i as $|x - p_i|^{\alpha_i}$, $\alpha_i > 0$, for $i = 1, \ldots, k$. This is still true in such a more general context but, for the sake of shortness and simplicity, we will consider in Theorem 1.5 only the case of Morse index one and $h > 0$ in $\bar{\Omega}$. Let us remark that assumptions (1.13)–(1.15) require that $f(u)$ behaves at main order like $(1-u)^{-\frac{1}{\gamma-1}}$ as $u \to 1^-$: an example is given by $f(u) = (1-u)^{-\frac{1}{\gamma-1}} \ln \frac{e}{1-u}$, $\gamma > 1$.

The paper is organized as follows. In Sect. 2 we illustrate how to adapt standard techniques to p-Laplace operator. In Sect. 3, we prove energy estimates and see how assumption (1.9) will allow us to prove Theorem 1.3. Let us stress that, by regularity theory, energy estimates on u_λ can provide useless L^∞-bounds on u_λ (in our context $\|u_\lambda\|_\infty \leq 1$). In particular, the second assumption in (1.9) will be crucial in our argument. In Sect. 4, we describe the blow up approach and, by an instability property of the limiting equation, we will derive Theorem 1.5. Existence of first eigenfunctions as in Theorem 1.1 and some technical Lemmata of Sect. 4 will be proved in the Appendix.

2 Minimal branch

In this section, we will establish Theorem 1.2 (we refer to [7] for related results). Since the Implicit Function Theorem does not produce solutions of (1.1) for $\lambda > 0$ small and $p \neq 2$, due to the degeneracy of p-Laplacian, we will use directly the sub/super solutions method.

Since $f(0) > 0$, $\underline{u} = 0$ is a sub-solution of (1.1). In order to produce a positive super-solution for λ small, let v be the solution of

$$\begin{cases} -\Delta_p v = h(x) f(0) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

Problem (2.16) has a unique positive solution $v \in C^1(\bar{\Omega})$. Let us fix $\beta > 0$ small so that $\overline{u} := \beta v$ satisfies $\|\overline{u}\|_\infty < 1$. By the monotonicity of $f(u)$, there holds:

$$-\Delta_p \overline{u} = \beta^{p-1} h(x) f(0) \geq \lambda h(x) f(\max_\Omega \overline{u}) \geq \lambda h(x) f(\overline{u})$$

for any $0 < \lambda < \frac{\beta^{p-1} f(0)}{f(\max_\Omega \overline{u})}$. Namely, for λ small $\overline{u}$ is a positive super-solution of (1.1).

Fix $\lambda > 0$ so that (1.1) has a super-solution $\overline{u}$: $0 < \overline{u} < 1$ in Ω. Let $u_1 \in C^1(\bar{\Omega})$ be the unique, positive solution of: $-\Delta_p u_1 = \lambda h(x) f(0)$ in Ω, $u_1 = 0$ on $\partial\Omega$. Since $-\Delta_p u_1 \leq \lambda h(x) f(\overline{u}) \leq -\Delta_p \overline{u}$, by the weak comparison principle $0 = \underline{u} \leq u_1 \leq \overline{u}$. Introduce now the following iteration scheme: let u_n, $n \geq 2$, be the unique, positive solution of

$$\begin{cases} -\Delta_p u_n = \lambda h(x) f(u_{n-1}) & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

We want to show that $0 = \underline{u} \leq u_n \leq \overline{u}$ for any $n \geq 1$. If such a property holds for some u_n, by the weak comparison principle applied to: $-\Delta_p u_{n+1} = \lambda h(x) f(u_n) \leq \lambda h(x) f(\overline{u}) \leq -\Delta_p \overline{u}$, we get $u_{n+1} \leq \overline{u}$. Since $u_1 \leq \overline{u}$, by induction $0 = \underline{u} \leq u_n \leq \overline{u}$ for any $n \geq 1$. In the same way, u_n is a non-decreasing sequence: $u_{n+1} \geq u_n$ for any $n \geq 1$. Set $u_\lambda(x) := \lim_{n \to +\infty} u_n(x)$.
Since $\|\overline{u}\|_\infty < 1$, for any $n \geq 1$ there holds:

$$\int_\Omega |\nabla u_n|^p = \lambda \int_\Omega h(x) f(u_{n-1}) u_n \leq \lambda \int_\Omega h(x) f(\overline{u}) \overline{u} < +\infty.$$

Up to a subsequence, we can assume that $u_n \rightharpoonup u_\lambda$ weakly in $W_0^{1,p}(\Omega)$ and by Lebesgue Theorem $f(u_n) \to f(u_\lambda)$ in $L^1(\Omega)$, as $n \to +\infty$. Since $\nabla u_n \to \nabla u_\lambda$ in $L^q(\Omega)$, $q < p$, as $n \to +\infty$ (see [1]), Eq. (2.17) passes to the limit yielding to a classical solution $0 = \underline{u} \leq u_\lambda \leq \overline{u}$ of (1.1) (classical in the sense specified in the Introduction). Since the scheme we defined is independent on $\overline{u}$, we get that $u_\lambda \leq \overline{u}$ for any super-solution of (1.1). In particular, u_λ defines the unique, positive minimal solution of (1.1).

Resuming what we did, for $\lambda > 0$ small a minimal solution u_λ exists. Then, $\lambda^* \in (0, +\infty]$, given as in (1.3), is a well defined number. To establish an upper bound on λ^*, let us compare (1.1) and

$$\begin{cases} -\Delta_p u = \beta u^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

Since $h(x) f(u) \geq h(x) f(0) \geq \delta u^{p-1}$ for any $0 \leq u < 1$, $\delta = f(0) \inf_\Omega h$, a solution u of (1.1) is a super-solution of (2.18) for $\beta = \delta\lambda$. Let λ_1 be the first eigenvalue of $-\Delta_p$ (λ_1 is the least value $\beta > 0$ so that (2.18) has a non trivial solution) and let φ_1 be an associated positive eigenfunction. For any $\beta \geq \lambda_1$, φ_1 is a sub-solution of (2.18). By Hopf Lemma and weak Harnack inequality, $\partial_\nu u < 0$ on $\partial\Omega$ and $u > 0$ in Ω, where $\nu(x)$ is the unit outer normal of $\partial\Omega$ at x. Hence, for $\epsilon > 0$ small, the function $\epsilon\varphi_1$ is still a first eigenfunction so that $\epsilon\varphi_1 < u$ in Ω.

If $\lambda^* > \frac{\lambda_1}{\delta}$, the sub/super solutions method explained above works as well yielding to a positive eigenfunction φ_β ($\varphi_\beta \geq \epsilon\varphi_1$) with associated eigenvalue β, for any $\delta\lambda^* > \beta > \lambda_1$. Since it is well known that the only positive eigenfunction of $-\Delta_p$ is the first one, we reach a contradiction. Hence (1.8) holds.

Since any classical solution u of (1.1), for some $\lambda = \bar{\lambda}$, is a super-solution of (1.1) for any $0 \leq \lambda \leq \bar{\lambda}$, a super-solution exists for any $\lambda \in [0, \lambda^*)$. The iterative scheme provides the existence of a (unique) classical, minimal solution u_λ for any $\lambda \in [0, \lambda^*)$ so that $u_\lambda \leq u_{\lambda'}$ for any $0 \leq \lambda \leq \lambda' < \lambda^*$. Next Lemma shows the semi-stability of u_λ and complete the proof of Theorem 1.2:

Lemma 2.1 *Let $\lambda \in [0, \lambda^*)$ and u_λ be the minimal solution of* (1.1). *Then, u_λ is semi-stable*:

$$\int_\Omega \left(|\nabla u_\lambda|^{p-2}|\nabla \phi|^2 + (p-2)|\nabla u_\lambda|^{p-2}\left(\frac{\nabla u_\lambda}{|\nabla u_\lambda|}\cdot \nabla\phi\right)^2 - \lambda h(x)\dot f(u_\lambda)\phi^2 \right) \geq 0 \quad \forall\, \phi \in \mathcal{A}_{u_\lambda}. \tag{2.19}$$

Proof Let $\mathcal{M} = \{u \in W_0^{1,p}(\Omega) \,:\, 0 \leq u \leq u_\lambda \text{ a.e.}\}$ and $F(u) = \int_0^u f(s)ds$. Since $\|u_\lambda\|_\infty < 1$, $F(u)$ is uniformly bounded for any $u \in \mathcal{M}$: $0 \leq F(u) \leq F(u_\lambda) \leq C < +\infty$ a.e. in Ω. Introduce the energy functional:

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \lambda \int_\Omega h(x)F(u), \quad u \in \mathcal{M}.$$

The functional E is well defined, bounded from below and weakly lower semi-continuous in $\mathcal{M}$. Then, it attains the minimum at some $u \in \mathcal{M}$: $E(u) = \inf_{v\in\mathcal{M}} E(v)$.

The key idea is to prove: $u_\lambda = u$. We need only to show that u is a classical positive solution of (1.1). Indeed, since $u \leq u_\lambda$ and u_λ is the minimal solution, necessarily $u = u_\lambda$.

Since u minimizes $E(v)$ on the convex set $\mathcal{M}$, the following inequality holds:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(\psi - u) - \lambda \int_\Omega h(x) f(u)(\psi - u) \geq 0 \qquad \forall\, \psi \in \mathcal{M}. \tag{2.20}$$

Let us introduce the notation

$$E^{'}(v)\varphi = \int_\Omega |\nabla v|^{p-2}\nabla v \nabla \varphi - \lambda \int_\Omega h(x) f(v)\varphi$$

for any $v \in \mathcal{M}$ and $\varphi \in W_0^{1,p}(\Omega)$.

Let now $\varphi \in C_0^\infty(\Omega)$. We use $\psi_\epsilon = u + \epsilon\varphi - (u+\epsilon\varphi - u_\lambda)^+ + (u+\epsilon\varphi)_- \in \mathcal{M}$, $\epsilon > 0$, as a test function in (2.20):

$$0 \leq E^{'}(u)(\psi_\epsilon - u) = \epsilon E^{'}(u)\varphi - E^{'}(u)(u+\epsilon\varphi - u_\lambda)^+ + E^{'}(u)(u+\epsilon\varphi)_-,$$

and then

$$E^{'}(u)\varphi \geq \frac{1}{\epsilon}E^{'}(u)(u+\epsilon\varphi - u_\lambda)^+ - \frac{1}{\epsilon}E^{'}(u)(u+\epsilon\varphi)_- \tag{2.21}$$

holds. Since u_λ solves (1.1) and $f(u)$ is non-decreasing, by Lebesgue Theorem we have:

$$\begin{aligned} \frac{1}{\epsilon}E^{'}(u)(u+\epsilon\varphi - u_\lambda)^+ &= \frac{1}{\epsilon}\left(E^{'}(u) - E^{'}(u_\lambda)\right)(u+\epsilon\varphi - u_\lambda)_+ \\ &\geq \int_{\{u_\lambda \leq u+\epsilon\varphi\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\nabla\varphi \\ &\to \int_{\{u_\lambda = u\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\nabla\varphi \end{aligned} \tag{2.22}$$

as $\epsilon \to 0^+$, in view of $u \leq u_\lambda$ and

$$(|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y) > 0 \qquad \forall\, x,\ y \in \mathbb{R}^N,\ x \neq y. \tag{2.23}$$

Recall now that, given $u,\ u' \in W_0^{1,p}(\Omega)$, by Stampacchia's Theorem it follows that $\nabla u = \nabla u'$ a.e. in the set $\{u = u'\}$. Therefore $\nabla u = \nabla u_\lambda$ a.e. in $\{u = u_\lambda\}$, and letting $\epsilon \to 0^+$ in (2.22) we get:

$$\liminf_{\epsilon\to 0^+} \frac{1}{\epsilon} E^{'}(u)(u+\epsilon\varphi - u_\lambda)^+ \geq \int\limits_{\{u_\lambda = u\}} \left(|\nabla u|^{p-2}\nabla u - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\nabla\varphi = 0$$

for any $\varphi \in C_0^\infty(\Omega)$. Since 0 is a sub-solution of (1.1), comparing $E^{'}(u)$ with $E^{'}(0)$, similarly we have:

$$\limsup_{\epsilon\to 0^+} \frac{1}{\epsilon} E^{'}(u)(u+\epsilon\varphi)_- \leq -\int\limits_{\{u=0\}} |\nabla u|^{p-2}\nabla u\nabla\varphi = 0.$$

Then, by (2.21) we get: $E^{'}(u)\varphi \geq 0$, for any $\varphi \in C_0^\infty(\Omega)$. By density we get that u is a non-negative weak bounded solution of (1.1). By regularity theory and weak Harnack inequality, u is then a positive classical solution for $0 < \lambda < \lambda^*$.

Once we have characterized u_λ as the minimum point of $E(u)$ in $\mathcal{M}$, we are in a good position to show semi-stability of u_λ. We should differentiate two times $E(u)$ at $u = u_\lambda$ but a lot of care is needed because $E(u)$ is not a C^2-functional. Let $0 \leq \varphi \in C_0^\infty(\Omega) \cap \mathcal{A}_{u_\lambda}$. Since u_λ is a positive continuous function, $u_\lambda \geq \delta > 0$ on Supp φ and $u_\lambda - t\varphi \in \mathcal{M}$ for $t > 0$ small. Compute the first derivative of $F(t) = E(u_\lambda - t\varphi)$, for $t > 0$ small:

$$\dot F(t) = -E^{'}(u_\lambda - t\varphi)\varphi = -\int\limits_\Omega \left(|\nabla u_\lambda - t\nabla\varphi|^{p-2}(\nabla u_\lambda - t\nabla\varphi)\nabla\varphi - \lambda h(x) f(u_\lambda - t\varphi)\varphi\right).$$

Since $F(t) \geq F(0)$, for $t > 0$ small, and $\dot F(0) = -E^{'}(u_\lambda)\varphi = 0$, we have that $\ddot F(0) \geq 0$, if $\ddot F(0)$ exists. Let $\Omega_t = \{x \in \Omega :\ 2t|\nabla\varphi|(x) < |\nabla u_\lambda|(x)\}$. Observe that:

$$\begin{aligned} I_2 := &\Big|\int\limits_{\Omega\setminus\Omega_t} \frac{|\nabla u_\lambda - t\nabla\varphi|^{p-2}(\nabla u_\lambda - t\nabla\varphi) - |\nabla u_\lambda|^{p-2}\nabla u_\lambda}{t}\nabla\varphi\Big| \\ &\leq C'\left(\int\limits_{\Omega\setminus\Omega_t} \frac{|\nabla u_\lambda|^{p-1}}{t}|\nabla\varphi| + \int\limits_{\Omega\setminus\Omega_t} t^{p-2}|\nabla\varphi|^p\right) \\ &\leq C\left(\int\limits_{\Omega\setminus\Omega_t} |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 + \int\limits_{\Omega\setminus\Omega_t} |\nabla\varphi|^p\right), \end{aligned}$$

because $\frac{1}{2}|\nabla u_\lambda| \leq t|\nabla\varphi|$ and $t^{p-2}|\nabla\varphi|^p = (t|\nabla\varphi|)^{p-2}|\nabla\varphi|^2 \leq 2^{2-p}|\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 + |\nabla\varphi|^p$ in $\Omega\setminus\Omega_t$. Also $\{|\nabla u_\lambda| = 0\}$ is a zero measure set (see [9]). Since $\Omega\setminus\Omega_t \to \{|\nabla u_\lambda| = 0\}$ in measure, as $t \to 0^+$, then we have:

$$\Omega\setminus\Omega_t \to \emptyset, \quad \Omega_t \to \Omega \qquad \text{in measure, as } t \to 0^+. \tag{2.24}$$

Since $|\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 + |\nabla\varphi|^p \in L^1(\Omega)$ for any $\varphi \in C_0^\infty(\Omega) \cap \mathcal{A}_{u_\lambda}$, then $I_2 \to 0$ as $t \to 0^+$. Compute now:

$$
\begin{aligned}
I_1 &:= \int_{\Omega_t} \frac{|\nabla u_\lambda - t\nabla\varphi|^{p-2}(\nabla u_\lambda - t\nabla\varphi) - |\nabla u_\lambda|^{p-2}\nabla u_\lambda}{t}\nabla\varphi \\
&= -\int_{\Omega_t} dx \int_0^1 ds\, \big(|\nabla u_\lambda - st\nabla\varphi|^{p-2}|\nabla\varphi|^2 \\
&\quad + (p-2)|\nabla u_\lambda - st\nabla\varphi|^{p-4}(\nabla u_\lambda\nabla\varphi - st|\nabla\varphi|^2)^2\big).
\end{aligned}
$$

Since $C \geq |\nabla u_\lambda - st\nabla\varphi| \geq |\nabla u_\lambda| - t|\nabla\varphi| > \frac{|\nabla u_\lambda|}{2}$ on Ω_t, observe that

$$
\begin{aligned}
&|\nabla u_\lambda - st\nabla\varphi|^{p-2}|\nabla\varphi|^2 + (p-2)|\nabla u_\lambda - st\nabla\varphi|^{p-4}(\nabla u_\lambda\nabla\varphi - st|\nabla\varphi|^2)^2 \\
&\quad \leq 2^{2-p}|\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 + (p-1)C^{p-2}|\nabla\varphi|^2 \in L^1(\Omega)
\end{aligned}
$$

for any $\varphi \in C_0^\infty(\Omega) \cap \mathcal{A}_{u_\lambda}$. By Lebesgue Theorem and (2.24) we get that

$$
I_1 \to -\int_\Omega \left(|\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 + (p-2)|\nabla u_\lambda|^{p-2}\left(\frac{\nabla u_\lambda}{|\nabla u_\lambda|}\cdot\nabla\varphi\right)^2 \right)
$$

as $t \to 0^+$.

We are now ready to conclude. Since the term $\lambda \int_\Omega h(x) f(u_\lambda - t\varphi)\varphi$ in $\dot{F}(t)$ is clearly a C^1-function at $t = 0$, the only difficulty to compute $\ddot{F}(0)$ is given by the term $\int_\Omega |\nabla u_\lambda - t\nabla\varphi|^{p-2}(\nabla u_\lambda - t\nabla\varphi)\nabla\varphi$. The limit of I_1, I_2 as $t \to 0^+$ provides the existence of $\ddot{F}(0)$ and:

$$
\ddot{F}(0) = \int_\Omega \left(|\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 + (p-2)|\nabla u_\lambda|^{p-2}\left(\frac{\nabla u_\lambda}{|\nabla u_\lambda|}\cdot\nabla\varphi\right)^2 - \lambda h(x)\dot{f}(u_\lambda)\varphi^2 \right) \geq 0, \tag{2.25}
$$

for any $0 \leq \varphi \in C_0^\infty(\Omega) \cap \mathcal{A}_{u_\lambda}$. For $p \geq 2$, by definition of $\mathcal{A}_{u_\lambda}$, $C_0^\infty(\Omega)$ is a dense subspace of $\mathcal{A}_{u_\lambda}$ in the $\|\cdot\|$-norm. Then, inequality (2.25) holds for any $0 \leq \varphi \in \mathcal{A}_{u_\lambda}$.
This is still true for $1 < p \leq 2$ but more care is needed in the density argument. For $1 < p \leq 2$, let us observe that (2.25) still holds for any $0 \leq \varphi \in L^\infty(\Omega) \cap \mathcal{A}_{u_\lambda}$ with supp $\varphi \subset \Omega$. The argument to derive (2.25) works as well because $\mathcal{A}_{u_\lambda} \subset H_0^1(\Omega) \subset W_0^{1,p}(\Omega)$ for $1 < p \leq 2$. Finally, let us show that any function $0 \leq \varphi \in \mathcal{A}_{u_\lambda}$ can be approximated in $\|\cdot\|$-norm by non-negative, essentially bounded functions φ_n with support in Ω. Indeed, let $\psi_n \in C_0^\infty(\Omega)$ be so that $\psi_n \to \varphi$ in $H_0^1(\Omega)$. By Hopf Lemma, there holds: $|\nabla u_\lambda| > 0$ in $\bar{\Omega}\setminus\Omega_{2\delta}$ for some $\delta > 0$ small, where $\Omega_\delta = \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > \delta\}$. Let χ be a cut-off function so that $\chi = 1$ in $\Omega_{2\delta}$ and $\chi = 0$ in $\Omega\setminus\Omega_\delta$. Define now $\varphi_n = \min\{\chi\varphi + (1-\chi)\psi_n, n\} \in L^\infty(\Omega)$. We have that

$$
\begin{aligned}
&\int_\Omega |\nabla u_\lambda|^{p-2}|\nabla(\varphi_n - \varphi)|^2 \\
&= \int_{\{\chi\varphi + (1-\chi)\psi_n > n\}} |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 + \int_{\{\chi\varphi + (1-\chi)\psi_n \leq n\}} |\nabla u_\lambda|^{p-2}|\nabla(1-\chi)(\psi_n - \varphi)|^2 \\
&\leq \int_{\{\chi\varphi + (1-\chi)\psi_n > n\}} |\nabla u_\lambda|^{p-2}|\nabla\varphi|^2 + C\int_{\Omega\setminus\Omega_{2\delta}} (\psi_n - \varphi)^2 + C\int_{\Omega\setminus\Omega_{2\delta}} (\nabla\psi_n - \nabla\varphi)^2 \to 0
\end{aligned}
$$

as $n \to +\infty$, because $\psi_n \to \varphi$ in $H_0^1(\Omega)$ and $L^2(\Omega)$, and

$$|\{\chi\varphi + (1-\chi)\psi_n > n\}| \le \frac{1}{n^2} \sup_{n\in\mathbb{N}} \int_\Omega (\chi\varphi + (1-\chi)\psi_n)^2 \to 0$$

as $n \to +\infty$. Hence, $0 \le \varphi_n \in L^\infty(\Omega) \cap \mathcal{A}_{u_\lambda}$ with $\operatorname{supp} \varphi_n \subset \operatorname{supp} \psi_n \cap \overline{\Omega_\delta} \subset \Omega$.

Once (2.25) is established for any $0 \le \varphi \in \mathcal{A}_{u_\lambda}$, since $\ddot{F}(0)$ is a quadratic form in φ and $\varphi^\pm \in \mathcal{A}_{u_\lambda}$ when $\varphi \in \mathcal{A}_{u_\lambda}$, we get that(2.25) holds for any $\varphi \in \mathcal{A}_{u_\lambda}$. The proof is done. □

3 Compactness of minimal branch

In this section, we will prove Theorem 1.3. Let us assume

$$f(u)\ddot{f}(u) \ge \gamma \dot{f}^2(u) \quad \forall\, t \le u < 1, \tag{3.26}$$

for some $t = t_\gamma \in (0,1)$, where $\gamma > \frac{p-2}{p-1}$ if $p \ge 2$ and $\gamma \ge 0$ if $1 < p < 2$. Observe that in particular $\gamma > \frac{p-2}{p-1}$.

For suitable test functions, semi-stability of u_λ and assumption (3.26) will provide integral bounds on the R.H.S. of (1.1).

Let $q > -\min\{\gamma, 1\}$. Introduce the following function:

$$g(u) = \begin{cases} 0 & \text{if } 0 \le u < t \\ \int_t^u \sqrt{q f^{q-1}(s)\dot{f}^2(s) + f^q(s)\ddot{f}(s)}ds & \text{if } t \le u < 1. \end{cases}$$

By (3.26), observe that $q f^{q-1}(s)\dot{f}^2(s) + f^q(s)\ddot{f}(s) \ge (q+\gamma) f^{q-1}(s)\dot{f}^2(s) \ge 0$ for any $t \le s < 1$. Then, $g(u)$ is well defined and, for any $t \le u < 1$:

$$g(u) \ge \sqrt{q+\gamma} \int_t^u f^{\frac{q-1}{2}}(s)\dot{f}(s)ds = \frac{2\sqrt{q+\gamma}}{q+1}\left(f^{\frac{q+1}{2}}(u) - f^{\frac{q+1}{2}}(t)\right). \tag{3.27}$$

Let us now test (1.1) against $\left(f^q(u_\lambda)\dot{f}(u_\lambda) - f^q(t)\dot{f}(t)\right)\chi_{\{u_\lambda \ge t\}} \in W_0^{1,p}(\Omega)$:

$$\begin{aligned} \lambda &\int_{\{u_\lambda \ge t\}} h(x) f^{q+1}(u_\lambda)\dot{f}(u_\lambda) \\ &\ge \int_{\{u_\lambda \ge t\}} |\nabla u_\lambda|^p \left(q f^{q-1}(u_\lambda)\dot{f}^2(u_\lambda) + f^q(u_\lambda)\ddot{f}(u_\lambda)\right) \\ &= \frac{1}{p-1}\int_\Omega \left(|\nabla u_\lambda|^{p-2}|\nabla g(u_\lambda)|^2 + (p-2)|\nabla u_\lambda|^{p-2}\left(\frac{\nabla u_\lambda}{|\nabla u_\lambda|}\cdot \nabla g(u_\lambda)\right)^2\right). \end{aligned} \tag{3.28}$$

Since $u_\lambda \in C^1(\bar{\Omega})$ and $\|u_\lambda\|_\infty < 1$, $g(u_\lambda) \in \mathcal{A}_{u_\lambda}$ for any $\lambda \in [0, \lambda^*)$. The semi-stability (2.19) of u_λ, inserted into (3.28), and estimate (3.27) yield to:

$$\begin{aligned} \int_{\{u_\lambda \ge t\}} h(x) f^{q+1}(u_\lambda)\dot{f}(u_\lambda) &\ge \frac{1}{p-1}\int_\Omega h(x)\dot{f}(u_\lambda)g^2(u_\lambda) \qquad (3.29) \\ &\ge \frac{4(q+\gamma)}{(p-1)(q+1)^2}\int_{\{u_\lambda \ge t\}} h(x)\dot{f}(u_\lambda)\left(f^{\frac{q+1}{2}}(u_\lambda) - f^{\frac{q+1}{2}}(t)\right)^2. \end{aligned}$$

Setting $q_\pm = \frac{2}{p-1} - 1 \pm \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}$, note that assumption $\gamma > \frac{p-2}{p-1}$ ensures $q_\pm \in \mathbb{R}$, $(q_-, q_+) \neq \emptyset$ and $q_+ > -\min\{\gamma, 1\}$. For any $q \in (q_-, q_+)$, there holds: $\frac{4(q+\gamma)}{(p-1)(q+1)^2} > 1$, and then:

$$\frac{4(q+\gamma)(1-\epsilon)}{(p-1)(q+1)^2} > 1,$$

for some $\epsilon > 0$ small. Since $\lim\limits_{u \to 1^-} f^{q+1}(u) = +\infty$, there exists $t_\epsilon \in (t, 1)$ so that

$$\dot{f}(u)(f^{\frac{q+1}{2}}(u) - f^{\frac{q+1}{2}}(t))^2 \geq (1-\epsilon)\dot{f}(u) f^{q+1}(u) \qquad \forall\, t_\epsilon \leq u < 1.$$

Combined with (3.29), finally we get that:

$$\int\limits_{\{u_\lambda \geq t\}} h(x) f^{q+1}(u_\lambda)\dot{f}(u_\lambda) \leq C \quad \forall\, q \in (q_-, q_+) \cap (-\min\{\gamma, 1\}, +\infty). \tag{3.30}$$

By integration, (3.26) gives that: $\dot{f}(u) \geq \frac{\dot{f}(t)}{f^\gamma(t)} f^\gamma(u)$ for any $t \leq u < 1$. Write $\dot{f}(u) = \dot{f}^\eta(u)\dot{f}^{1-\eta}(u)$ for $0 \leq \eta \leq 1$ and

$$\dot{f}^{1-\eta}(u) \geq \frac{\dot{f}^{1-\eta}(t)}{f^{\gamma(1-\eta)}(t)} f^{\gamma(1-\eta)}(u)$$

for any $t \leq u < 1$, we get the following result:

Theorem 3.1 *Assume* (3.26) *for* $\gamma > \frac{p-2}{p-1}$ *if* $p \geq 2$ *and* $\gamma \geq 0$ *if* $1 < p < 2$. *Given* $0 \leq \eta \leq 1$, *then*

$$\sup_{\lambda \in [0, \lambda^*)} \int\limits_\Omega h(x) f^q(u_\lambda)\dot{f}^\eta(u_\lambda) < +\infty \tag{3.31}$$

for any $1 \leq q < q_\eta = \gamma(1-\eta) + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}$.

For $\eta = 0$, Theorem 3.1 gives:

$$\sup_{\lambda \in [0, \lambda^*)} \int\limits_\Omega h(x) f^q(u_\lambda) < +\infty, \quad \forall\, 1 \leq q < q_0 = \gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}. \tag{3.32}$$

When $1 < p \leq 2$, estimate (3.32) can be improved with the following argument. Let us replace $f(u)$ with $\tilde{f}(u) = f(u) + u + Cu^2$, $C > 0$ large in order to have $\tilde{f}$ convex and strictly increasing on $[0, 1)$ (we use here the property of convexity of $f(u)$ near $u = 1$). Given $s \geq 1$, by (1.1) let us compute (in a weak sense):

$$\begin{cases} -\Delta_p\left(\tilde{f}^s(u_\lambda) - \tilde{f}^s(0)\right) \leq \lambda s^{p-1} h(x)\tilde{f}^{(p-1)(s-1)+1}(u_\lambda)\dot{\tilde{f}}^{p-1}(u_\lambda) & \text{in } \Omega \\ \tilde{f}^s(u_\lambda) - \tilde{f}^s(0) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.33}$$

Since $0 \leq u_\lambda < 1$, by (3.31) the R.H.S. in (3.33) is uniformly bounded in $L^{\frac{\eta}{p-1}}(\Omega)$, for any $p - 1 \leq \eta \leq 1$ and for any $1 \leq s < \frac{q_\eta}{\eta} - \frac{2-p}{p-1}$ (note that $\frac{q_\eta}{\eta} > \frac{2}{p-1}$ implies $\frac{q_\eta}{\eta} - \frac{2-p}{p-1} > 1$ for any $0 \leq \eta \leq 1$). Since it is possible to find $h_\lambda \in W_0^{1,p}(\Omega)$ so that

$$-\Delta_p h_\lambda = \lambda s^{p-1} h(x)\tilde{f}^{(p-1)(s-1)+1}(u_\lambda)\dot{\tilde{f}}^{p-1}(u_\lambda), \tag{3.34}$$

by weak comparison principle $0 \le \tilde f^s(u_\lambda) - \tilde f^s(0) \le h_\lambda$. Elliptic regularity theory for p-Laplace operator (see [18]) applies to (3.34): $L^{\frac{\eta}{p-1}}(\Omega)$-bounds on the R.H.S. of (3.34) gives estimates on h_λ and in turn, on $f^s(u_\lambda)$ for any $1 \le s < \frac{q_\eta}{\eta} - \frac{2-p}{p-1}$.

Let $1 < p \le 2$. If $N < \frac{p}{p-1}$, then $\frac{\eta}{p-1} > \frac{N}{p}$ for $\eta = 1$ and elliptic regularity theory provides:

$$\sup_{\lambda\in[0,\lambda^*)} \|f(u_\lambda)\|_\infty < \infty.$$

In particular, compactness (1.11) holds for $N < \frac{p}{p-1}$. When $N = \frac{p}{p-1}$, by elliptic regularity theory we get:

$$\sup_{\lambda\in[0,\lambda^*)} \|f(u_\lambda)\|_{L^q(\Omega)} < \infty,$$

for any $q \ge 1$. When $N > \frac{p}{p-1}$, for any $p-1 \le \eta \le 1$ we have that $\frac{\eta}{p-1} < \frac{N}{p}$ and $(\frac{\eta}{p-1})^{**} = \frac{N\eta(p-1)}{N(p-1)-\eta p}$ is well defined. Fix now $p-1 \le \eta \le 1$. Elliptic regularity theory gives that:

$$\sup_{\lambda\in[0,\lambda^*)} \|f(u_\lambda)\|_{L^q(\Omega)} < \infty, \quad \forall\, 1 \le q < \tilde q_\eta = \left(\frac{q_\eta}{\eta} - \frac{2-p}{p-1}\right)\left(\frac{\eta}{p-1}\right)^{**}. \tag{3.35}$$

We need now to maximize $\tilde q_\eta$ for $p-1 \le \eta \le 1$ in order to achieve the better integrability. It is a tedious but straightforward computation to see that:

$$\begin{aligned}\tilde q_\eta &= N\frac{p-1}{p}\left(\gamma + \frac{2-p}{p-1}\right) + \frac{N(p-1)}{N(p-1)-\eta p}\\ &\quad\times\left(-N\frac{p-1}{p}\left(\gamma + \frac{2-p}{p-1}\right) + \gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}\right).\end{aligned}$$

Then, the function $\tilde q_\eta$ is monotone in η. Define

$$N_p = \frac{p}{p-1}\,\frac{2+\gamma(p-1)+2\sqrt{2-p+\gamma(p-1)}}{2-p+\gamma(p-1)}.$$

Observe that $N_p > \frac{p}{p-1}$. If $\frac{p}{p-1} < N \le N_p$, the function $\tilde q_\eta$ is non-decreasing and achieves the maximum at $\eta = 1$. If $N > N_p$, $\tilde q_\eta$ decreases and achieves the maximum at $\eta = p-1$. We can compute now:

$$\begin{aligned}\tilde q &:= \sup_{p-1\le\eta\le1} \tilde q_\eta\\ &= \begin{cases} \frac{N}{N(p-1)-p}\left(p + 2\sqrt{2-p+\gamma(p-1)}\right) & \text{if } \frac{p}{p-1} < N \le N_p\\ \frac{N}{N-p}\left((2-p)(\gamma-1) + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}\right) & \text{if } N > N_p.\end{cases}\end{aligned}$$

When $1 < p \le 2$, observe that $\tilde q \ge q_0$ if and only if $N \le N_p$. Let us define

$$q_p = \begin{cases} +\infty & \text{if } N = \frac{p}{p-1},\ 1 < p \le 2\\ \frac{N}{N(p-1)-p}\left(p + 2\sqrt{2-p+\gamma(p-1)}\right) & \text{if } \frac{p}{p-1} < N \le N_p,\ 1 < p \le 2\\ \gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)} & \text{if either } N > N_p,\ 1 < p \le 2 \text{ or } p > 2.\end{cases}$$

Resuming (3.32), (3.35), the following result has been established:

Theorem 3.2 *Assume* (3.26) *for* $\gamma > \frac{p-2}{p-1}$ *if* $p \geq 2$ *and* $\gamma \geq 0$ *if* $1 < p < 2$. *When* $1 \leq N < \frac{p}{p-1}$ *and* $1 < p \leq 2$, *there holds*:

$$\sup_{\lambda \in [0,\lambda^*)} \|u_\lambda\|_\infty < 1.$$

When either $N \geq \frac{p}{p-1}$, $1 < p \leq 2$ *or* $p > 2$, *we have that*:

$$\sup_{\lambda \in [0,\lambda^*)} \|f(u_\lambda)\|_{L^q(\Omega)} < +\infty \tag{3.36}$$

for any $1 \leq q < q_p$.

Let $p > 1$, and assume $N \geq \frac{p}{p-1}$ if $1 < p \leq 2$. We want to understand when compactness (1.11) of u_λ holds. We will use now the following assumption:

$$f(u) \geq \frac{C_0}{(1-u)^m} \qquad \forall\, 0 \leq u < 1, \tag{3.37}$$

for some $m > 0$ and $C_0 > 0$. By (1.1) we have that (in a weak sense):

$$\begin{cases} -\Delta_p(\ln \frac{1}{1-u_\lambda}) \leq \lambda h(x) \frac{f(u_\lambda)}{(1-u_\lambda)^{p-1}} & \text{in } \Omega \\ \ln \frac{1}{1-u_\lambda} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.38}$$

By (3.37) we get that

$$0 \leq \lambda h(x) \frac{f(u_\lambda)}{(1-u_\lambda)^{p-1}} \leq \lambda C_0^{-\frac{p-1}{m}} f(u_\lambda)^{\frac{m+p-1}{m}} \quad \text{in } \Omega,$$

and then by (3.36)

$$\sup_{\lambda \in [0,\lambda^*)} \|\lambda h(x) \frac{f(u_\lambda)}{(1-u_\lambda)^{p-1}}\|_{L^q(\Omega)} < +\infty \qquad \forall\, 1 \leq q < \frac{m q_p}{m+p-1}.$$

If $\frac{mq_p}{m+p-1} > \frac{N}{p}$, arguing as before, by elliptic regularity theory [18] on (3.38) we get that

$$\sup_{\lambda \in [0,\lambda^*)} \|\ln \frac{1}{1-u_\lambda}\|_\infty < +\infty,$$

or equivalently (1.11) on u_λ holds.

We need to discuss the validity of

$$mpq_p > N(p-1+m). \tag{3.39}$$

Assume first $1 < p \leq 2$. If $\frac{p}{p-1} < N \leq N_p$, then $q_p = \frac{N}{N(p-1)-p}\left(p+2\sqrt{2-p+\gamma(p-1)}\right)$ and (3.39) is satisfied when

$$N < N_p^1 = \frac{p}{p-1}\left(1 + \frac{m}{m+p-1}(p+2\sqrt{2-p+\gamma(p-1)})\right).$$

Compute:

$$N_p - N_p^1 = -\frac{mp}{m+p-1}\frac{p+2\sqrt{2-p+\gamma(p-1)}}{2-p+\gamma(p-1)}\left(\gamma - 1 - \frac{1}{m}\right).$$

If $\gamma \leq 1 + \frac{1}{m}$, then $N_p^1 \leq N_p$ and (3.39) holds when

$$N < N_p - \frac{mp}{m+p-1}\frac{p+2\sqrt{2-p+\gamma(p-1)}}{2-p+\gamma(p-1)}\left(\gamma - 1 - \frac{1}{m}\right)_- .$$

Observing that

$$N_p = \frac{mp}{m+p-1}\left(\gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}\right)$$
$$-\frac{mp}{m+p-1}\left(1+\frac{p+2\sqrt{2-p+\gamma(p-1)}}{2-p+\gamma(p-1)}\right)\left(\gamma-1-\frac{1}{m}\right),$$

we get that, for $\gamma \le 1+\frac{1}{m}$, (3.39) holds when

$$N < N^{\#} = \frac{mp}{p+m-1}\left(\gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)} + \left(\gamma-1-\frac{1}{m}\right)-\right),$$

where $N^{\#}$ is defined in (1.10). When $\gamma > 1+\frac{1}{m}$, for $N \le N_p$ (3.39) is automatically satisfied holds for any $N \le N_p$. If $\gamma > 1+\frac{1}{m}$ and $N > N_p$, we have that $q_p = \gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}$ and (3.39) holds when

$$N_p < N < N^{\#} = \frac{mp}{p+m-1}\left(\gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}\right).$$

Hence (3.39) holds for any $N < N^{\#}$ also when $\gamma > 1+\frac{1}{m}$ and the case $1 < p \le 2$ has been completely discussed.
Assume now $p > 2$. Then, $q_p = \gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}$ and (3.39) holds when

$$N < N^{\#} = \frac{mp}{p+m-1}\left(\gamma + \frac{2}{p-1} + \frac{2}{p-1}\sqrt{2-p+\gamma(p-1)}\right).$$

Finally, we can conclude. Let $N^{\#}$ be defined by (1.10), where γ and m are given in (1.9), and let $N < N^{\#}$. For any $\epsilon > 0$, assumption (1.9) implies that (3.26) and (3.37) are valid for $\gamma - \epsilon > \frac{p-2}{p-1}$ and $m - \epsilon > 0$, respectively. When $1 < p < 2$ the convexity of $f(u)$ near $u = 1$ ensures that we can also assume $\gamma - \epsilon \ge 0$. For $\epsilon > 0$ small, N is less than the critical dimension $N^{\#}$ associated through (1.10) to $\gamma - \epsilon$, $m - \epsilon$. Hence, (1.11) holds and Theorem 1.3 is established. □

4 Compactness of the unstable branch

In this section we will give the proof of Theorem 1.5, namely the compactness of the first unstable branch (with Morse index one) for the problem (1.1) under the assumptions (1.13)–(1.15). The proof, adapted from the arguments in [12,14], will make use of two Lemmata which will be proved in the Appendix for the sake of simplicity.

Let $p = 2$. Let $\lambda_n \in (0, \lambda^*)$ be a sequence and let u_n be associated solutions of (1.1) of Morse index at most one, i.e. $\mu_{2,n} \ge 0$ for any $n \in \mathbb{N}$, where $\mu_{2,n} = \mu_2(-\Delta - \lambda_n h(x)\dot{f}(u_n))$ is the second eigenvalue of the linearized operator at u_n. We want to prove that any such sequence is compact in the sense that $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < 1$ for $N < N^{\#} = \frac{2(\gamma+2+2\sqrt{\gamma})}{\gamma}$.

Let us argue by contradiction and assume that this sequence is not compact, i.e. there exists $x_n \in \Omega$ such that $u_n(x_n) = \max_{\Omega} u_n(x) \underset{n\to\infty}{\longrightarrow} 1$. Suppose $x_n \to p \in \overline{\Omega}$ and set $\varepsilon_n = 1 - u_n(x_n) \underset{n\to\infty}{\longrightarrow} 0$.

Notice that

$$\lambda_n \frac{f(1-\varepsilon_n)}{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} \infty. \tag{4.40}$$

Indeed, if it were bounded we would have $\lambda_n \to 0$ since $\frac{f(1-\varepsilon_n)}{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} \infty$ by (1.2). Then, being f nondecreasing, we would have

$$0 \le \lambda_n h(x) f(u_n(x)) \le \lambda_n \|h\|_\infty f(1-\varepsilon_n) \le C\varepsilon_n$$

From elliptic regularity, up to a subsequence, we would have $u_n \to u$ in $C^1(\bar{\Omega})$, where u is a weak harmonic function such that $u = 0$ on $\partial\Omega$ and $\max_\Omega u = 1$, which is a contradiction. Let us introduce the following rescaled function:

$$U_n(y) \equiv \frac{1 - u_n\left(x_n + \left(\frac{\varepsilon_n}{\lambda_n f(1-\varepsilon_n)}\right)^{\frac{1}{2}} y\right)}{\varepsilon_n}, \quad y \in \Omega_n = \frac{\Omega - x_n}{\left(\frac{\varepsilon_n}{\lambda_n f(1-\varepsilon_n)}\right)^{\frac{1}{2}}}.$$

The following Lemma holds:

Lemma 4.1 *We have that $\Omega_n \to \mathbb{R}^N$ and there exists a subsequence $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$, where U is a solution of the problem*

$$\begin{cases} \Delta U = \frac{h(p)}{U^{\frac{1}{\gamma-1}}} & \text{in } \mathbb{R}^N \\ U(y) \ge U(0) = 1 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover, there exists $\phi_n \in C_0^\infty(\Omega)$ such that supp $\phi_n \subset B_{M\left(\frac{\varepsilon_n}{\lambda_n f(1-\varepsilon_n)}\right)^{\frac{1}{2}}}(x_n)$ for some $M > 0$ and

$$\int_\Omega |\nabla\phi_n|^2 - \lambda_n h(x)\dot{f}(u_n)\phi_n^2 < 0 \tag{4.41}$$

From this lemma (whose proof is in the Appendix) we get the existence of $\phi_n \in C_0^\infty(\Omega)$ which is identically zero outside a small ball $B_{r_n}(x_n)$, with $r_n \to 0$ as $n \to +\infty$, and the linearized operator is negative at ϕ_n. To conclude the proof we need the following estimate for Morse index one solutions, which says that the blow-up can essentially occur only along the maximum sequence x_n:

Lemma 4.2 *Given $0 < \delta < \gamma - 1$, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that*

$$f(u_n(x)) \le C\lambda_n^{-\frac{1}{\gamma-\delta}} |x - x_n|^{-\frac{2}{\gamma-\delta}} \tag{4.42}$$

for all $x \in \Omega$ and $n \ge n_0$.

From this lemma, thanks to estimate (4.42), we deduce that

$$0 \le \lambda_n h(x) f(u_n(x)) \le C\lambda_n^{\frac{\gamma-\delta-1}{\gamma-\delta}} \|h\|_\infty |x - x_n|^{-\frac{2}{\gamma-\delta}}$$

for any $0 < \delta < \gamma - 1$. Hence, $\lambda_n h(x) f(u_n(x))$ is uniformly bounded in $L^s(\Omega)$ for any $1 < s < \gamma\frac{N}{2}$. From standard elliptic regularity theory we have that u_n is uniformly bounded in $W^{2,s}(\Omega)$ for any $1 < s < \gamma\frac{N}{2}$. By the Sobolev embedding Theorem u_n is uniformly bounded also in $C^{0,\beta}(\bar{\Omega})$ for any $\beta \in (0, 2 - \frac{2}{\gamma})$. Then, up to a subsequence, we have that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_n \to u_0$ strongly in $C^{0,\beta}(\bar{\Omega})$ for any $\beta \in (0, 2 - \frac{2}{\gamma})$.

In case $\lambda_n \to 0$, u_0 is a $H_0^1(\Omega)$-weak harmonic function and, by the Maximum Principle, has to vanish in Ω. By uniform convergence, it holds that

$$u_0(p) = \max_\Omega u_0 = \lim_{n\to\infty} \max_\Omega u_n = 1, \quad p = \lim_{n\to\infty} x_n \tag{4.43}$$

and a contradiction arises.

Hence, $\lambda_n \to \lambda > 0$ and (4.43) implies $p \in \Omega$, since $u_0 = 0$ on $\partial\Omega$. By (4.42), it follows that $f(u_0) \le C|x-p|^{-\frac{2}{\gamma-\delta}}$ in $\Omega\backslash\{p\}$ and $f(u_n) \to f(u_0)$ in $C_{\text{loc}}(\bar\Omega\backslash\{p\})$. Since $H_0^1(\Omega\backslash\{p\}) = H_0^1(\Omega)$ and $f(u_0) \in L^{\frac{2N}{N+2}}(\Omega)$, then u_0 is an Hölderian $H_0^1(\Omega)$-weak solution of:

$$\begin{cases} -\Delta u_0 = \lambda h(x) f(u_0) & \text{in } \Omega \\ 0 \le u_0 \le 1 & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, consider the first eigenvalue of the linearized operator at u_0, namely

$$\mu_{1,\lambda}(u_0) \equiv \mu_1(-\Delta - \lambda h(x)\dot f(u_0)) = \inf_{\phi\in C_0^\infty(\Omega):\int\phi^2=1} \left(\int_\Omega |\nabla\phi|^2 - \lambda h(x)\dot f(u_0(x))\phi^2 \right).$$

For convex nonlinearities $f(u)$, uniqueness holds in the class of semi-stable $H_0^1(\Omega)$-weak solutions of (1.1) (see [14]). If $\mu_{1,\lambda}(u_0) \ge 0$, we deduce from Theorem 1.2 that $u_0 \equiv u_\lambda$, for some $\lambda \in [0, \lambda^*]$. But from Theorem 1.3 we know that $\max_\Omega u_\lambda < 1$ for any $\lambda \in [0, \lambda^*]$, and this contradicts $\max_\Omega u_0 = 1$.

So, we are left with the case $\mu_{1,\lambda}(u_0) < 0$, which means that there exists $\phi_0 \in C_0^\infty(\Omega)$ such that

$$\int_\Omega |\nabla\phi_0|^2 - \lambda h(x)\dot f(u_0)\phi_0^2 < 0$$

But from (4.41) we already had the existence of $\phi_n \in C_0^\infty(\Omega)$ such that supp $\phi_n \subset B_{r_n}(x_n)$ with $r_n \underset{n\to\infty}{\longrightarrow} 0$ and

$$\int_\Omega |\nabla\phi_n|^2 - \lambda_n h(x)\dot f(u_n)\phi_n^2 < 0.$$

We want to replace ϕ_0 with a truncated function ϕ_δ with $\delta > 0$ small enough so that

$$\int_\Omega |\nabla\phi_\delta|^2 - \lambda h(x)\dot f(u_0)\phi_\delta^2 < 0$$

and $\phi_\delta \equiv 0$ in $B_{\delta^2}(p) \cap \Omega$. So, for n large by Fatou's Lemma it holds

$$\int_\Omega |\nabla\phi_\delta|^2 - \lambda_n h(x)\dot f(u_n)\phi_\delta^2 < 0.$$

Since ϕ_n and ϕ_δ have disjoint compact supports, from the variational characterization of the second eigenvalue we would have a contradiction: $\mu_{2,n} < 0$. This ends the proof.

Take $\delta > 0$ and set $\phi_\delta = \chi_\delta\phi_0$ with

$$\chi_\delta(x) \equiv \begin{cases} 0 & \text{if } |x-p| \le \delta^2 \\ 2 - \frac{\log|x-p|}{\log\delta} & \text{if } \delta^2 \le |x-p| \le \delta \\ 1 & \text{if } |x-p| \ge \delta \end{cases}$$

By Fatou's Lemma we have

$$\liminf_{\delta\to 0}\int_\Omega \lambda h(x)\dot f(u_0)\phi_\delta^2 \geq \int_\Omega \lambda h(x)\dot f(u_0)\phi_0^2,$$

whereas for the gradient term we have

$$\int_\Omega |\nabla\phi_\delta|^2 = \int_\Omega \phi_0^2|\nabla\chi_\delta|^2 + \int_\Omega \chi_\delta^2|\nabla\phi_0|^2 + 2\int_\Omega \chi_\delta\phi_0\nabla\chi_\delta\nabla\phi_0.$$

We have the following estimates:

$$0 \leq \int_\Omega \phi_0^2|\nabla\chi_\delta|^2 \leq \|\phi_0\|_\infty^2 \int_{\delta^2\leq|x-p|\leq\delta} \frac{1}{|x-p|^2\log^2\delta} \leq \frac{C}{\log\frac{1}{\delta}}$$

and

$$\left|2\int_\Omega \chi_\delta\phi_0\nabla\chi_\delta\nabla\phi_0\right| \leq \frac{2\|\phi_0\|_\infty\|\nabla\phi_0\|_\infty}{\log\frac{1}{\delta}}\int_{B_1(0)}\frac{1}{|x|},$$

which give

$$\int_\Omega |\nabla\phi_\delta|^2 \underset{\delta\to 0}{\longrightarrow} \int_\Omega |\nabla\phi_0|^2.$$

In conclusion,

$$\limsup_{\delta\to 0}\int_\Omega |\nabla\phi_\delta|^2 - \lambda h(x)\dot f(u_0)\phi_\delta^2 \leq \int_\Omega |\nabla\phi_0|^2 - \lambda h(x)\dot f(u_0)\phi_0^2 < 0.$$

For $\delta > 0$ small enough, ϕ_δ is what we were searching for and this concludes the proof. □

5 Appendix

5.1 Embedding of the space $\mathcal{A}_u$

For $p \geq 2$, we will show below that the space $\mathcal{A}_u$ is compactly embedded in $L^2(\Omega)$, as it will follow by the weighted Sobolev estimates proved in [9]. The proof follows closely Theorem 9.16 in [2]. For the reader's convenience we give the details:

Lemma 5.1 *Let $p \geq 2$ and u be a solution of (1.1). For any $1 \leq q < \frac{2N(p-1)}{(N-2)(p-1)+2(p-2)}$ there exists $C > 0$ so that*

$$\|\phi\|_{L^q(\Omega)}^2 \leqslant C\int_\Omega |\nabla u|^{p-2}|\nabla\phi|^2, \quad \forall\, \phi \in \mathcal{A}_u. \tag{5.44}$$

Moreover, the embedding $\mathcal{A}_u \subset L^q(\Omega)$ is compact for any $1 \leq q < \frac{2N(p-1)}{(N-2)(p-1)+2(p-2)}$.

Proof Since $\bar q = \frac{2N(p-1)}{(N-2)(p-1)+2(p-2)} > 2$, for $2 < q < \bar q$ (5.44) follows by Theorem 2.2 in [10]. Then, by Hölder inequality (5.44) follows also for $1 \leq q \leq 2$.

Since $q < \bar{q}$, fix $\delta > 0$ so that $q + \delta < \bar{q}$, and set

$$\frac{1}{q} = \alpha + \frac{(1-\alpha)}{q+\delta}, \qquad \text{for } 0 < \alpha \leqslant 1.$$

Now, let $\omega \subset\subset \Omega$ and consider h such that

$$|h| < \text{dist}(\omega, \Omega^c).$$

By interpolation we have for $\phi \in L^2(\Omega)$

$$\|\phi(x+h) - \phi(x)\|_{L^q(\omega)} = \|\tau_h(\phi) - \phi\|_{L^q(\omega)} \leqslant \|\tau_h(\phi) - \phi\|^{\alpha}_{L^1(\omega)} \, \|\tau_h(\phi) - \phi\|^{1-\alpha}_{L^{q+\delta}(\omega)}.$$

Now we have that (see [9]) $\frac{1}{|\nabla u|^{p-2}} \in L^1(\Omega)$ and consequently

$$\mathcal{A}_u \subset W^{1,1},$$

with

$$\int_\Omega |\nabla \phi| \leqslant \left(\int_\Omega |\nabla u|^{p-2}|\nabla \phi|^2\right)^{\frac{1}{2}} \cdot \left(\int_\Omega \frac{1}{|\nabla u|^{p-2}}\right)^{\frac{1}{2}}.$$

Recall that the $|\cdot|$-norm and the $\|\cdot\|$-norm, as defined in the Introduction, are equivalent. Therefore, for every $\phi \in \mathcal{A}_u$ with $\|\phi\| \leq 1$ we have

$$\|\tau_h(\phi) - \phi\|_{L^1(\omega)} \leqslant |h| \|\nabla \phi\|_{L^1(\Omega)} \leqslant C_0 |h|$$

so that, exploiting (5.44) we get

$$\|\tau_h(\phi) - \phi\|_{L^q(\omega)} \leqslant C_0^{\alpha} \, |h|^{\alpha} \, \left(2\|\phi\|_{L^{q+\delta}(\Omega)}\right)^{1-\alpha} \leqslant C|h|^{\alpha}.$$

Since

$$\|\phi\|_{L^q(\Omega\backslash\omega)} \leqslant \|\phi\|_{L^{q+\delta}(\Omega\backslash\omega)} |\Omega\backslash\omega|^{\frac{1}{q} - \frac{1}{q+\delta}} \leqslant C \, |\Omega\backslash\omega|^{\frac{1}{q} - \frac{1}{q+\delta}} \leqslant \epsilon$$

for $\Omega\backslash\omega$ sufficiently small, by Corollary 4.27 in [2] we deduce that the unit ball of $\mathcal{A}_u$ is a compact set in $L^q(\Omega)$. Then, the embedding $\mathcal{A}_u \subset L^q(\Omega)$ is compact for any $1 \leq q < \bar{q}$. □

For $1 < p < 2$, as already remarked in the Introduction, $\mathcal{A}_u \subset H_0^1(\Omega)$. Since $H_0^1(\Omega) \subset L^q(\Omega)$ compactly for any $1 \leq q < \frac{2N}{N-2}$, by Lemma 5.1 we deduce

Lemma 5.2 *Let u be a solution of* (1.1). *The embedding $\mathcal{A}_u \subset L^2(\Omega)$ is compact.*

5.2 Proof of Theorem 1.1

- **Step 1. Existence of a first eigenfunction**
  Let us first note that $\dot{f}(u) \in L^\infty(\Omega)$, so that

$$\|\phi\|^2 - \lambda \int_\Omega h(x) \dot{f}(u) \phi^2$$

is bounded from below, for any $\phi \in \mathcal{A}_u$ with $\int_\Omega \phi^2 = 1$. Therefore, $\mu_1(u)$ is well defined:

$$\mu_1(u) = \inf_{\phi \in \mathcal{A}_u \backslash \{0\}} R_u(\phi) > -\infty, \quad R_u(\phi) = \frac{\|\phi\|^2 - \lambda \int_\Omega h(x) \dot{f}(u) \phi^2}{\int_\Omega \phi^2}.$$

Consider now a minimizing sequence $\phi_n \in \mathcal{A}_u$, $\int_\Omega \phi_n^2 = 1$, with

$$R_u(\phi_n) \to \mu_1(u) \quad \text{as } n \to +\infty.$$

Since $\dot{f}(u) \in L^\infty(\Omega)$, we have that

$$\sup_{n\in\mathbb{N}} \|\phi_n\| < +\infty.$$

Therefore, up to a subsequence, we get that

$$\phi_n \rightharpoonup \phi_1 \qquad \text{weakly in } \mathcal{A}_u$$

and by Lemma 5.2

$$\phi_n \to \phi_1 \qquad \text{strongly in } L^2(\Omega).$$

Now, the term $\lambda \int_\Omega h(x)\dot{f}(u)\phi^2$ is continuous in $L^2(\Omega)$ and $\|\cdot\|$ is weakly lower semicontinuous in $\mathcal{A}_u$. Therefore, $\phi_1 \in \mathcal{A}_u$ is so that $\int_\Omega \phi_1^2 = 1$ and $R_u(\phi_1) \le \mu_1(u)$. Hence, $\mu_1(u)$ is attained at ϕ_1.

- **Step 2. Every minimizer is positive (or negative) almost everywhere**
We show that $\phi_1 > 0$ (or $\phi_1 < 0$) in $\Omega\setminus Z_u$, where $Z_u = \{\nabla u = 0\}$ is a zero measure set (see [9]).
Assume

$$\mu_1(u) = \frac{\|\phi_1\|^2 - \lambda \int_\Omega h(x)\dot{f}(u)\phi_1^2}{\int_\Omega \phi_1^2}$$

so that for $\psi \in \mathcal{A}_u$ it follows

$$< \phi_1, \psi > -\lambda \int_\Omega h(x)\dot{f}(u)\phi_1\psi = \mu_1(u)\int_\Omega \phi_1\psi. \tag{5.45}$$

Taking $\phi_1^\pm$ as a test function in (5.45), we get that

$$\|\phi_1^\pm\|^2 - \lambda \int_\Omega h(x)\dot{f}(u)(\phi_1^\pm)^2 = \mu_1(u)\int_\Omega (\phi_1^\pm)^2,$$

showing that $\phi_1^\pm$ also minimizes $R_u(\phi)$. Then, there holds

$$< \phi_1^\pm, \psi > -\lambda \int_\Omega h(x)\dot{f}(u)\phi_1^\pm\psi = \mu_1(u)\int_\Omega \phi_1^\pm\psi, \;\; \forall\, \psi \in \mathcal{A}_u. \tag{5.46}$$

The differential operator in (5.46) is nondegenerate in $\Omega\setminus Z_u$. Moreover, by [9] $\Omega\setminus Z_u$ is connected and $|Z_u| = 0$, as already recalled. Therefore, the Strong Maximum Principle holds in $\Omega\setminus Z_u$, and $\phi_1^\pm$ is smooth in $\Omega\setminus Z_u$ by standard regularity results.
We have that either $\phi_1^\pm > 0$ in $\Omega\setminus Z_u$ or $\phi_1^\pm = 0$ in $\Omega\setminus Z_u$. Indeed, $C = \{x \in \Omega\setminus Z_u : \phi_1^\pm > 0\}$ is clearly an open set. By the classic Strong Maximum Principle exploited in $\Omega\setminus Z_u$, we have

$$\partial C \subset Z_u \cup \partial\Omega,$$

and then, $C = \bar{C} \cap (\Omega\setminus Z_u)$ is a closed set in the relative toplogy of $\Omega\setminus Z_u$. Since $\Omega\setminus Z_u$ is connected, the set C either is empty or coincides with $\Omega\setminus Z_u$.
Since $\phi_1 \neq 0$ a.e. in Ω, then either $\phi_1 > 0$ or $\phi_1 < 0$ in $\Omega\setminus Z_u$.

- **Step 3. The first eigenspace is one-dimensional**
  Let ϕ_1 be a first eigenfunction which can be assumed positive a.e. in Ω: $\phi_1 > 0$ in $\Omega \backslash Z_u$, by means of Step 2. Let now ϕ any other first eigenfunction. Since by Step 2 ϕ has constant sign, let us consider for example the case $\phi > 0$ in $\Omega \backslash Z_u$. Set

$$\bar{\beta} = \sup\{\beta \geq 0 \ : \ \phi - \beta\phi_1 \geq 0 \text{ a.e. in } \Omega\} < +\infty.$$

  Since by linearity $\phi - \bar{\beta}\phi_1$ is still a minimizer for R_u, by Step 2 we have that

$$\text{either } \phi - \bar{\beta}\phi_1 > 0 \quad \text{or} \quad \phi - \bar{\beta}\phi_1 = 0 \quad \text{a.e. in } \Omega.$$

  If $\phi - \bar{\beta}\phi_1 > 0$, we have that $\phi - (\bar{\beta} + \epsilon)\phi_1$ is still positive on a subset of positive measure, for small $\epsilon > 0$. As a minimizer, $\phi - (\bar{\beta} + \epsilon)\phi_1$ has constant sign and then, $\phi - (\bar{\beta} + \epsilon)\phi_1 > 0$ a.e. in Ω, against the definition of $\bar{\beta}$. Therefore, $\phi = \bar{\beta}\phi_1$ a.e. in Ω.

5.3 Proof of Lemma 4.1

The proof follows the arguments in [12,14], where similar results are proved for the nonlinearity $f(u) = \frac{1}{(1-u)^2}$. Suppose that $x_n \to p \in \bar{\Omega}$ and consider the rescaled function around x_n:

$$U_n(y) \equiv \frac{1 - u_n(x_n + \beta_n y)}{\varepsilon_n}, \quad y \in \Omega_n = \frac{\Omega - x_n}{\beta_n},$$

where $\beta_n = \left(\frac{\varepsilon_n}{\lambda_n f(1-\varepsilon_n)}\right)^{\frac{1}{2}} \underset{n\to\infty}{\longrightarrow} 0$ by means of (4.40). The function U_n verifies

$$\begin{cases} \Delta U_n = \frac{h(x_n+\beta_n y)}{f(1-\varepsilon_n)} f(1 - \varepsilon_n U_n(y)) & \text{in } \Omega_n \\ U(y) \geq U(0) = 1 & \text{in } \Omega_n. \end{cases} \tag{5.47}$$

First of all, we need to prove that $\Omega_n \underset{n\to\infty}{\longrightarrow} \mathbb{R}^N$. It will be sufficient to prove that

$$\beta_n d_n^{-1} \underset{n\to\infty}{\longrightarrow} 0,$$

where $d_n = \text{dist}\,(x_n, \partial\Omega)$. Indeed, arguing by contradiction and up to a subsequence, assume that $\beta_n^2 d_n^{-2} \to \delta > 0$. Obviously this implies that $d_n \to 0$ for $n \to \infty$.

Introduce the following rescaling

$$W_n(y) \equiv \frac{1 - u_n\,(x_n + d_n y)}{\varepsilon_n}, \quad y \in A_n = \frac{\Omega - x_n}{d_n}.$$

Since $d_n \to 0$ we have that $A_n \to T$, where T is a hyperspace containing 0 so that $d(0, \partial T) = 1$. The function W_n satisfies the following equation

$$\begin{aligned} \Delta W_n(y) &= \frac{\lambda_n d_n^2}{\varepsilon_n} h(x_n + d_n y) f(1 - \varepsilon_n W_n(y)) \\ &= \left(\frac{\lambda_n f(1-\varepsilon_n) d_n^2}{\varepsilon_n}\right) h(x_n + d_n y) \frac{f(1 - \varepsilon_n W_n(y))}{f(1-\varepsilon_n)} \\ &= \left(\frac{\lambda_n f(1-\varepsilon_n) d_n^2}{\varepsilon_n}\right) \left(\frac{f(1 - \varepsilon_n W_n(y)) \varepsilon_n^{\frac{1}{\gamma-1}} W_n^{\frac{1}{\gamma-1}}}{f(1-\varepsilon_n)\varepsilon_n^{\frac{1}{\gamma-1}}}\right) \frac{h(x_n + d_n y)}{W_n^{\frac{1}{\gamma-1}}}. \end{aligned}$$

From the hypothesis and condition (1.14), for any sufficiently large n we get

$$0 \leq \left(\frac{\lambda_n f(1-\varepsilon_n)d_n^2}{\varepsilon_n}\right)\left(\frac{f(1-\varepsilon_n W_n(y))\varepsilon_n^{\frac{1}{\gamma-1}} W_n^{\frac{1}{\gamma-1}}}{f(1-\varepsilon_n)\varepsilon_n^{\frac{1}{\gamma-1}}}\right) h(x_n+d_n y) \leq \frac{2}{\delta} c_0 \|h\|_\infty < \infty.$$

This means that the function W_n satisfies

$$\begin{cases} \Delta W_n = \frac{h_n}{W_n^{\frac{1}{\gamma-1}}} & \text{in } A_n \\ W_n(y) \geq C > 0 & \text{in } A_n \end{cases}$$

with $\sup\limits_{n\in\mathbb{N}} \|h_n\|_\infty < \infty$ and $C = W_n(0) = 1$.

Recall Lemma 4.1 in [14] (written there for the case $\gamma = \frac{3}{2}$):

Lemma 5.3 *Let h_n be a function on a smooth bounded domain A_n in $\mathbb{R}^N$. Let W_n be a solution of*:

$$\begin{cases} \Delta W_n = \dfrac{h_n(x)}{W_n^{\frac{1}{\gamma-1}}} & \text{in } A_n, \\ W_n(y) \geq C > 0 & \text{in } A_n, \\ W_n(0) = 1, \end{cases}$$

for some $C > 0$. Assume that $\sup\limits_{n\in\mathbb{N}} \| h_n \|_\infty < +\infty$ and $A_n \to T_\mu$ as $n \to +\infty$ for some $\mu \in (0, +\infty)$, where T_μ is an hyperspace so that $0 \in T_\mu$ and dist $(0, \partial T_\mu) = \mu$. Then, either $\inf\limits_{\partial A_n \cap B_{2\mu}(0)} W_n \leq C$ or $\inf\limits_{\partial A_n \cap B_{2\mu}(0)} \partial_\nu W_n \leq 0$, where ν is the unit outward normal of A_n.

Since $W_n|_{\partial A_n} \equiv \frac{1}{\varepsilon_n} \underset{n\to\infty}{\longrightarrow} \infty$ and Hopf Lemma provides $\partial_\nu W_n > 0$ on ∂A_n, a contradiction arises by means of Lemma 5.3. Hence, we have shown that $\beta_n d_n^{-1} \underset{n\to\infty}{\longrightarrow} 0$, i.e. $\Omega_n \to \mathbb{R}^N$.

We now want to prove that there exists a subsequence $\{U_n\}_{n\in\mathbb{N}}$ such that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^N)$, where U is a solution of the problem

$$\begin{cases} \Delta U = \frac{h(p)}{U^{\frac{1}{\gamma-1}}} & \text{in } \mathbb{R}^N \\ U(y) \geq U(0) = 1 & \text{in } \mathbb{R}^N. \end{cases} \tag{5.48}$$

Fix $R > 0$ and, for n large, decompose $U_n = U_n^1 + U_n^2$, where U_n^2 satisfies

$$\begin{cases} \Delta U_n^2 = \Delta U_n & \text{in } B_R(0) \\ U_n^2 = 0 & \text{in } \partial B_R(0). \end{cases}$$

By the Eq. (5.47) and the condition (1.14) we get that on $B_R(0)$:

$$0 \leq \Delta U_n(y) \leq \|h\|_\infty \left|\frac{f(1-\varepsilon_n U_n)}{f(1-\varepsilon_n)}\right| \leq c_0 \|h\|_\infty < \infty,$$

and then, standard elliptic regularity theory gives that U_n^2 is uniformly bounded in $C^{1,\beta}(B_R(0))$, $\beta \in (0,1)$. Up to a subsequence, we have that $U_n^2 \to U^2$ in $C^1(B_R(0))$.

Since $U_n^1 = U_n \geq 1$ on $\partial B_R(0)$, by harmonicity $U_n^1 \geq 1$ in $B_R(0)$. Through Harnack inequality, we have:

$$\sup_{B_{\frac{R}{2}}(0)} U_n^1 \leq C_R \inf_{B_{\frac{R}{2}}(0)} U_n^1 \leq C_R U_n^1(0) = C_R(1 - U_n^2(0)) \leq C_R \left(1 + \sup_n |U_n^2(0)|\right) < \infty.$$

Hence, U_n^1 is uniformly bounded in $C^{1,\beta}(B_{\frac{R}{4}}(0))$, $\beta \in (0,1)$. Up to a subsequence, we get that $U_n^1 \to U^1$ in $C^{1,\beta}(B_{\frac{R}{4}}(0))$ for any $R > 0$. Up to a diagonal process and a further subsequence, we can assume that $U_n \to U := U_1 + U_2$ in $C^1_{\text{loc}}(\mathbb{R}^N)$.
Notice that, by the condition (1.15) we have:

$$h(x_n + \beta_n y)\frac{f(1-\varepsilon_n U_n))}{f(1-\varepsilon_n)} = h(x_n+\beta_n y)\frac{f(1-\varepsilon_n U_n)U_n^{\frac{1}{\gamma-1}}}{f(1-\varepsilon_n)}\frac{1}{U_n^{\frac{1}{\gamma-1}}} \underset{n\to\infty}{\longrightarrow} \frac{h(p)}{U^{\frac{1}{\gamma-1}}}$$

in $C^0_{loc}(\mathbb{R}^N)$. This means that U is a solution of (5.48).

The following unstability property, a special case of a more general result in [13], will be crucial:

Theorem 5.4 *Let U be a solution of* (5.48). *Then, U is linearly unstable*:

$$\mu_1(U) = \inf\left\{\int |\nabla\phi|^2 - \frac{1}{\gamma-1}\int \frac{h(p)}{U^{\frac{\gamma}{\gamma-1}}}\phi^2 : \phi \in C_0^\infty(\mathbb{R}^N),\ \int \phi^2 = 1\right\} < 0,$$

provided $2 \le N < N^\# = \frac{2(\gamma+2+2\sqrt{\gamma})}{\gamma}$.

Theorem 5.4 provides the existence of $\phi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 - \frac{1}{\gamma-1}\frac{h(p)}{U^{\frac{\gamma}{\gamma-1}}}\phi^2 < 0.$$

Define

$$\phi_n(x) \equiv \beta_n^{-\frac{N-2}{2}}\phi\left(\frac{x-x_n}{\beta_n}\right) \in C_0^\infty(\Omega).$$

Condition (1.13) rewrites as:

$$\lim_{u\to1^-}\frac{f(u)\ddot{f}(u)}{(\dot{f}(u))^2} = \lim_{u\to1^-}\frac{(\ln\dot{f})'(u)}{(\ln f)'(u)} = \gamma,$$

which implies by (1.2) and L'Höpital rule:

$$\lim_{u\to1^-}\frac{\ln\dot{f}(u)}{\ln f(u)} = \gamma.$$

Hence, since $\gamma > 1$ we have that

$$\lim_{u\to1^-}\frac{\dot{f}(u)}{f(u)} = \lim_{u\to1^-} f(u)^{\gamma-1+o(1)} = +\infty,$$

where $o(1) \to 0$ as $u \to 1^-$. This means that by L'Höpital rule:

$$\lim_{u\to1^-}\frac{(1-u)\dot{f}(u)}{f(u)} = \lim_{u\to1^-}\frac{1-u}{\frac{f(u)}{\dot{f}(u)}} = \lim_{u\to1^-}\frac{1}{\frac{f(u)\ddot{f}(u)}{(\dot{f}(u))^2}-1} = \frac{1}{\gamma-1}. \tag{5.49}$$

Observe that by (1.15) and (5.49) it follows that

$$\frac{\varepsilon_n\dot{f}(1-\varepsilon_n U_n)}{f(1-\varepsilon_n)} = \frac{\varepsilon_n U_n\dot{f}(1-\varepsilon_n U_n)}{f(1-\varepsilon_n U_n)}\frac{f(1-\varepsilon_n U_n)\varepsilon_n^{\frac{1}{\gamma-1}}U_n^{\frac{1}{\gamma-1}}}{\varepsilon_n^{\frac{1}{\gamma-1}}f(1-\varepsilon_n)}\frac{1}{U_n^{\frac{\gamma}{\gamma-1}}} \to \frac{1}{\gamma-1}\frac{1}{U^{\frac{\gamma}{\gamma-1}}} \tag{5.50}$$

in $C_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$, in view of $U_n \to U$ locally uniformly.

Let us now evaluate the linearized operator at u_n on ϕ_n:

$$\int_{\Omega} |\nabla \phi_n|^2 - \lambda_n h(x) \dot{f}(u_n)\phi_n^2 = \int_{\Omega_n} |\nabla \phi|^2 - h(x_n + \beta_n y) \frac{\varepsilon_n \dot{f}(1-\varepsilon_n U_n)}{f(1-\varepsilon_n)} \phi^2$$
$$\to \int_{\mathbb{R}^N} |\nabla \phi|^2 - \frac{1}{\gamma - 1} \int_{\mathbb{R}^N} \frac{h(p)}{U^{\frac{\gamma}{\gamma-1}}} \phi^2 < 0 \quad \text{as } n \to +\infty$$

by means of $\phi \in C_0^\infty(\mathbb{R}^N)$ and (5.50). Hence, for any sufficiently large n we have:

$$\int_{\Omega} |\nabla \phi_n|^2 - \lambda_n h(x) \dot{f}(u_n)\phi_n^2 < 0$$

with supp $\phi_n \subset B_{M\beta_n}(x_n)$, for some $M > 0$. This concludes the proof. □

5.4 Proof of Lemma 4.2

Fix $0 < \delta < \gamma - 1$. Let us prove estimate (4.42): there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$f(u_n(x)) \leq C \lambda_n^{-\frac{1}{\gamma-\delta}} |x - x_n|^{-\frac{2}{\gamma-\delta}}$$

for any $x \in \Omega$ and for any $n \geq n_0$.

Let us argue by contradiction and assume that (4.42) does not hold. Up to a subsequence, we get the existence of a minimizing sequence $y_n \in \Omega$ such that:

$$\lambda_n^{-\frac{1}{\gamma-\delta}} f(u_n(y_n))^{-1} |x_n - y_n|^{-\frac{2}{\gamma-\delta}} = \lambda_n^{-\frac{1}{\gamma-\delta}} \min_{y \in \Omega} \left[f(u_n(x))^{-1} |x - x_n|^{-\frac{2}{\gamma-\delta}} \right] \underset{n\to\infty}{\longrightarrow} 0. \tag{5.51}$$

This means that $f(u_n(y_n)) \to +\infty$ as $n \to +\infty$, so that we have blow-up along the sequence y_n, i.e.

$$\mu_n \equiv 1 - u_n(y_n) \underset{n\to\infty}{\longrightarrow} 0.$$

By (5.49) and L'Höpital rule we get that

$$\lim_{u \to 1^-} \frac{\ln f(u)}{\ln \frac{1}{1-u}} = \frac{1}{\gamma - 1},$$

and then, we have:

$$f^{\gamma-\delta-1}(u) \leq \frac{C}{1-u} \qquad \text{in } [0, 1)$$

for some $C = C_\delta > 0$. Hence, this yields to:

$$\mu_n f(1-\mu_n)^{\gamma-1-\delta} \leq C. \tag{5.52}$$

From (5.51)–(5.52), we have that

$$\hat{\beta}_n := \lambda_n \frac{f(1-\mu_n)}{\mu_n} = (\lambda_n^{\frac{1}{\gamma-\delta}} f(1-\mu_n))^{\gamma-\delta} \frac{1}{\mu_n f(1-\mu_n)^{\gamma-1-\delta}} \to \infty \quad \text{as } n \to +\infty.$$

Assume that $y_n \to q \in \bar{\Omega}$ as $n \to +\infty$. Consider the following rescaled function

$$\hat{U}_n(y) \equiv \frac{1 - u_n(y_n + \hat{\beta}_n y)}{\mu_n}, \quad y \in \hat{\Omega}_n = \frac{\Omega - y_n}{\hat{\beta}_n}$$

We want to prove the following crucial convergence:

$$\lim_{n\to\infty}\frac{\hat\beta_n}{|x_n-y_n|}=\lim_{n\to\infty}\frac{\mu_n^{\frac12}}{\lambda_n^{\frac12}f(1-\mu_n)^{\frac12}|x_n-y_n|}=0. \tag{5.53}$$

From (5.51) and (5.52), we have that

$$\begin{aligned}\frac{\hat\beta_n^2}{|x_n-y_n|^2}&=\frac{\mu_n}{\lambda_n f(1-\mu_n)|x_n-y_n|^2}=\left(\mu_n f(1-\mu_n)^{\gamma-1-\delta}\right)\frac{1}{\lambda_n f(1-\mu_n)^{\gamma-\delta}|x_n-y_n|^2}\\&\le\frac{C}{\lambda_n f(1-\mu_n)^{\gamma-\delta}|x_n-y_n|^2}\underset{n\to\infty}{\longrightarrow}0.\end{aligned}$$

Namely, (5.53) holds. We enumerate now several properties of the crucial choice $R_n:=\hat\beta_n^{-\frac12}|x_n-y_n|^{\frac12}$:

(a) $R_n\underset{n\to\infty}{\longrightarrow}\infty$ from (5.53);

(b) $R_n\beta_n=\beta_n^{\frac12}|x_n-y_n|^{\frac12}\le\beta_n^{\frac12}(\operatorname{diam}\Omega)^{\frac12}\underset{n\to\infty}{\longrightarrow}0$;

(c) $\frac{R_n\beta_n}{|x_n-y_n|}=\beta_n^{\frac12}|x_n-y_n|^{-\frac12}=\left(\frac{|x_n-y_n|}{\beta_n}\right)^{-\frac12}\underset{n\to\infty}{\longrightarrow}0.$

Let us now focus our attention on the function $\hat U_n$. If $y\in\hat\Omega_n\cap B_{R_n}(0)$ we have that: either $u_n(y_n+\hat\beta_n y)\le u_n(y_n)$, which implies $\frac{1-u_n(y_n+\hat\beta_n y)}{1-u_n(y_n)}\ge1$; or $u_n(y_n)<u_n(y_n+\hat\beta_n y)$ and assumption (1.15) implies

$$\frac{(1-u_n(y_n+\hat\beta_n y))^{\frac{1}{\gamma-1}}f(u_n(y_n+\hat\beta_n y))}{(1-u_n(y_n))^{\frac{1}{\gamma-1}}f(u_n(y_n))}\ge\frac{1}{c_0},$$

or equivalently

$$\frac{1-u_n(y_n+\hat\beta_n y)}{1-u_n(y_n)}\ge\frac{1}{c_0^{\gamma-1}}\left(\frac{f(u_n(y_n)}{f(u_n(y_n+\hat\beta_n y))}\right)^{\gamma-1}.$$

In the latter situation, from the definition of y_n we have:

$$f(u_n(y_n))|x_n-y_n|^{\frac{2}{\gamma-\delta}}\ge f(u_n(y_n+\hat\beta_n y))|y_n+\hat\beta_n y-x_n|^{\frac{2}{\gamma-\delta}}$$

and, since (b) and (c) imply that for any $n\ge n_0$:

$$\frac{|y_n+\hat\beta_n y-x_n|}{|y_n-x_n|}\ge1-\frac{\hat\beta_n|y|}{|x_n-y_n|}\ge1-\frac{\hat\beta_n R_n}{|x_n-y_n|}\ge\frac12,\quad|y|\le R_n,$$

we get that

$$\begin{aligned}\frac{1-u_n(y_n+\hat\beta_n y)}{1-u_n(y_n)}&\ge\frac{1}{c_0^{\gamma-1}}\left(\frac{f(u_n(y_n))}{f(u_n(y_n+\hat\beta_n y))}\right)^{\gamma-1}\ge\frac{1}{c_0^{\gamma-1}}\left(\frac{|y_n+\hat\beta_n y-x_n|}{|y_n-x_n|}\right)^{2\frac{\gamma-1}{\gamma-\delta}}\\&\ge\frac{1}{c_0^{\gamma-1}}\left(\frac12\right)^{2\frac{\gamma-1}{\gamma-\delta}}.\end{aligned}$$

We finally get that for any $n\ge n_0$ and any $y\in\hat\Omega_n\cap B_{R_n}(0)$:

$$\hat U_n(y)=\frac{1-u_n(y_n+\hat\beta_n y)}{1-u_n(y_n)}\ge D_0=\min\left\{1,\frac{1}{c_0^{\gamma-1}}\left(\frac12\right)^{2\frac{\gamma-1}{\gamma-\delta}}\right\}$$

Setting $\hat{d}_n = \text{dist}\,(y_n, \partial\Omega)$, consider the rescaled function

$$\hat{W}_n(y) \equiv \frac{1 - u_n\left(y_n + \hat{d}_n y\right)}{\mu_n}, \quad y \in \hat{A}_n = \hat{\beta}_n \hat{d}_n^{-1}\left(\hat{\Omega}_n \cap B_{R_n}(0)\right).$$

Since $\hat{W}_n(y) \geq D_0$ in $\hat{A}_n$, we can apply again Lemma 5.3 to get: $\hat{\beta}_n \hat{d}_n^{-1} \underset{n\to\infty}{\longrightarrow} 0$, i.e. $\hat{\Omega}_n \cap B_{R_n}(0) \to \mathbb{R}^N$ as $n \to \infty$. Now, proceeding as in the proof of Lemma 4.1, we get that $\hat{U}_n \to \hat{U}$ in $C^1_{loc}(\mathbb{R}^N)$, where $\hat{U}$ solves

$$\begin{cases} \Delta \hat{U} = \frac{h(q)}{\hat{U}^{\frac{1}{\gamma-1}}} & \text{in } \mathbb{R}^N \\ \hat{U}(y) \geq D_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover, there exists $\psi_n \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_\Omega |\nabla \psi_n|^2 - \lambda_n h(x) f(u_n) \psi_n^2 < 0$$

with supp$\psi_n \subset B_{M\hat{\beta}_n}(y_n)$ for some $M > 0$. But from Lemma 4.1 we already had $\phi_n \in C_0^\infty(\mathbb{R}^N)$ with the same property and such that supp $\phi_n \subset B_{M'\beta_n}(x_n)$ for some $M' > 0$.

Since the nonlinearity $f(u)$ in non-decreasing and $1 - \varepsilon_n = u_n(x_n) \geq u_n(y_n) = 1 - \mu_n$, by (5.53) we get that:

$$\frac{\varepsilon_n^{\frac{1}{2}}}{\lambda_n^{\frac{1}{2}} f(1-\varepsilon_n)^{\frac{1}{2}} |x_n - y_n|} = \left(\frac{\varepsilon_n}{\mu_n}\right)^{\frac{1}{2}} \left(\frac{f(1-\mu_n)}{f(1-\varepsilon_n)}\right)^{\frac{1}{2}} \frac{\mu_n^{\frac{1}{2}}}{\lambda_n^{\frac{1}{2}} f(1-\mu_n)^{\frac{1}{2}} |x_n - y_n|}$$
$$\leq \frac{\mu_n^{\frac{1}{2}}}{\lambda_n^{\frac{1}{2}} f(1-\mu_n)^{\frac{1}{2}} |x_n - y_n|} \underset{n\to\infty}{\longrightarrow} 0.$$

This means that ϕ_n and ψ_n have disjoint compact support for n large, which contradicts the Morse index-one property of the solutions u_n and concludes ths proof. □

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