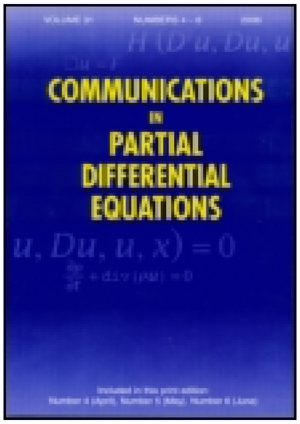


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# Pointwise Blow-Up Phenomena for a Dirichlet Problem

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*For the Dirichlet problem*

$$\begin{cases} -\Delta u + \lambda V(x) u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

*with* $\Omega \subset \mathbb{R}^N$, $N \geq 2$, *a bounded domain and* $p > 1$, *blow-up phenomena necessarily arise as* $\lambda \to +\infty$. *In the present paper, we address the asymptotic description for pointwise blow-up, as it occurs when either the "energy" or the Morse index is uniformly bounded.* A posteriori, *we obtain an equivalence between the two quantities in the form of a double-side bound with essentially optimal constants, a sort of improved Rozenblyum-Lieb-Cwikel inequality for the equation under exam. Moreover, we prove the nondegeneracy of any "low energy" or Morse index* 1 *solution under a suitable condition on the potential.*

**Keywords** Blow-up; Classification results; Morse index; Non-degeneracy; Rozenblyum-Lieb-Cwikel inequality.

**Mathematics Subject Classification** 35J60; 35B33; 35J25; 35J20; 35B40.

## 1. Introduction

We are concerned with the study of:

$$\begin{cases} -\Delta u + \lambda V(x) u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a large parameter, $p > 1$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$ and $V$ is a positive potential.

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Under the transformation $u(x) \to \lambda^{-\frac{1}{p-1}} u(x)$, $\lambda \to \varepsilon = \frac{1}{\sqrt{\lambda}}$, problem (1.1) reads equivalently as a singularly perturbed Dirichlet problem:

$$\begin{cases} -\varepsilon^2 \Delta u + Vu = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) and related ones have been widely considered in literature, as they arise as steady state equation in several biological and physical models, such as population dynamics, pattern formation theories and chemical reactor theory.

The asymptotic analysis for (1.2) in the sub-critical case ($p > 1$ if $N = 2$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$) and $V = 1$ has been considered for least energy solutions both with Dirichlet and Neumann boundary condition. By an asymptotic expansion of the corresponding critical value, in the Dirichlet case the sequence exhibits a single spike-layer with its unique peak situated near the most centered part of $\Omega$, where the distance function $d(\cdot, \partial\Omega)$ is maximal (see [37] and also [47]). More generally, when the corresponding "energy" is of order $\epsilon^N$, for the Dirichlet problem the energy density $\epsilon^{-N} u^{p+1}$ is expected to concentrate into a finite sum of Dirac masses as $\epsilon \to 0^+$, in the sense of measures. The centers of the Dirac masses (the blow-up points) are in $\Omega$ and their location depends on the distances from the boundary as well as on the mutual distances.

For domains with topology, it suggests the presence of multiple solutions of (1.1) for $\lambda$ large, as firstly shown [3] in terms of cat $\Omega$ (see also [4]). More recently, single-peak solutions have been obtained [16, 31, 34, 40, 44, 45, 47] for "good" c.p.s of $d(\cdot, \partial\Omega)$. The existence of k-peaks solutions has been addressed in [5, 6, 13–15, 17, 40], the main difficulty being related to the non-smoothness of distance functions.

Since solutions of (1.1) necessarily blow-up as $\lambda \to +\infty$, we aim to obtain an accurate description of the asymptotic behavior as $\lambda \to +\infty$ through an energy or a Morse index information. To be more precise, let $u_n$ be a solutions sequence of

$$\begin{cases} -\Delta u_n + \lambda_n V(x) u_n = u_n^p & \text{in } \Omega \\ u_n > 0 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda_n \to +\infty$ as $n \to +\infty$. First, observe that

$$\|u_n\|_\infty \to \infty \quad \text{as } n \to +\infty. \tag{1.4}$$

Indeed, if $\|u_n\|_\infty \leq C$ were valid along a sub-sequence, the integration of (1.3) against $u_n$ would provide

$$\int_\Omega |\nabla u_n|^2 + \lambda_n \int_\Omega V u_n^2 = \int_\Omega u_n^{p+1} \leq C^{p+1} |\Omega|.$$

Up to a further sub-sequence, the boundedness of $u_n$ in $H_0^1(\Omega)$ and the Sobolev embedding Theorem would imply that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and strongly in

$L^2(\Omega)$. Since

$$\int_\Omega u_n^2 \le \frac{C^{p+1}|\Omega|}{\lambda_n \inf_\Omega V} \to 0$$

as $n \to +\infty$, necessarily $u_n \to 0$ in $L^2(\Omega)$ and

$$\int_\Omega |\nabla u_n|^2 \le \int_\Omega u_n^{p+1} \le C^{p-1} \int_\Omega u_n^2 \to 0$$

as $n \to +\infty$. By the Hölder inequality and the Sobolev embedding Theorem we would have:

$$\int_\Omega |\nabla u_n|^2 \le \int_\Omega u_n^{p+1} \le C^{p+1-\frac{2N}{N-2}} S_N^{-\frac{N}{N-2}} \left( \int_\Omega |\nabla u_n|^2 \right)^{\frac{N}{N-2}}$$

for $p \ge \frac{N+2}{N-2}$ and

$$\int_\Omega |\nabla u_n|^2 \le \int_\Omega u_n^{p+1} \le \tilde{C} \Big( \int_\Omega u_n^{\frac{2N}{N-2}} \Big)^{\frac{N-2}{2N}(p+1)} \le \tilde{C}\, S_N^{-\frac{p+1}{2}} \Big( \int_\Omega |\nabla u_n|^2 \Big)^{\frac{p+1}{2}}$$

for $1 < p < \frac{N+2}{N-2}$, in contradiction with $u_n \to 0$ in $H_0^1(\Omega)$.

Once (1.4) is established, we can use the standard blow-up procedure to describe the asymptotic behavior. Let $P_n \in \Omega$ be a maximum point of $u_n$: $u_n(P_n) = \|u_n\|_\infty$, and set $\varepsilon_n = \lambda_n^{-\frac{1}{2}} V(P_n)^{-\frac{1}{2}}$, $\tilde{\varepsilon}_n = \|u_n\|_\infty^{-\frac{p-1}{2}}$. Let us define

$$U_n(y) = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n y + P_n), \quad y \in \Omega_n := \varepsilon_n^{-1}(\Omega - P_n).$$

Up to a sub-sequence, by the boundedness of $(\lambda_n \tilde{\varepsilon}_n^2)^{-1}$ the limiting profile $U = \lim_{n\to+\infty} U_n$ is a bounded solution of

$$\begin{cases} -\Delta U + U = U^p & \text{in } H \\ 0 < U \le U(0) & \text{in } H \\ U = 0 & \text{on } \partial H \end{cases} \tag{1.5}$$

with $H = \lim_{n\to+\infty} \Omega_n$ an half-space or $\mathbb{R}^N$. Towards a classification for (1.5) we require on $u_n$ one of the two following assumptions: either

$$\sup_{n\in\mathbb{N}} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} < +\infty \tag{1.6}$$

or

$$\sup_{n\in\mathbb{N}} m(u_n) < +\infty, \tag{1.7}$$

where $m(u_n)$ is the Morse index of $u_n$. Inspired by recent results [18, 23] (see also [11, 19, 22]), we have the following classification result:

**Theorem 1.1.** *Let $U$ be a nonnegative solution of $-\Delta U + U = U^p$ in $H$, where $H$ is an half-space or $\mathbb{R}^N$.*

- *If $H$ is a half-space and $U \in L^\infty(H)$ with $U = 0$ on $\partial H$, then $U \equiv 0$.*
- *If $H = \mathbb{R}^N$, assume $p$ sub-critical ($p > 1$ if $N = 2$ and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$) and either $\int_{\mathbb{R}^N} U^{p+1} < +\infty$ or $m(U) < +\infty$. If $U \not\equiv 0$, then $U$ coincides with the unique radial ground-state solution $U_0$ (see [33]). In particular, $U$ has Morse index one in $H^1(\mathbb{R}^N)$: the first negative eigenvalue $\mu_1 < 0$ is simple with eigenfunction $\varphi_1$, and the second eigenvalue $\mu_2 = 0$ has multiplicity $N$ with eigenspace given by*

$$span\{\partial_{x_1} U, \dots, \partial_{x_N} U\}.$$

Theorem 1.1 states that each sequence of blow-up points $P_n$ carries locally at least an energy $\int_{\mathbb{R}^N} U_0^{p+1}$ and one direction of negativity for the linearized operator. In this way, we can control the number of blow-up sequences thanks to (1.6) or (1.7), and the exponential decay of $U_0$ yields to strong pointwise estimates on $u_n$. A refined asymptotic analysis allows now the investigation of the link between the Morse index and the energy in case of pointwise blow-up:

**Theorem 1.2.** *Let $u_n$ be a solution of* (1.3) *and assume $p$ sub-critical. The assumptions* (1.6) *and* (1.7) *are equivalent, and there holds*

$$\frac{1}{N+1}(\inf_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}} \leq \liminf_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{\bar{m}(u_n) \int_{\mathbb{R}^N} U_0^{p+1}} \leq \limsup_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{m(u_n) \int_{\mathbb{R}^N} U_0^{p+1}}$$
$$\leq (\sup_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}},$$

*where $\bar{m}(u_n)$ denotes the large Morse index (i.e. the number of non-positive eigenvalues of $-\Delta + \lambda_n V - p u_n^{p-1}$).*

The constants in Theorem 1.2 are essentially optimal since $U_0$ has exactly energy $\int_{\mathbb{R}^N} U_0^{p+1}$ and $N+1$ non-positive eigenvalues for the linearized operator (counted with multiplicities). For general nonlinearities, one of the two implications (uniformly bounded Morse indices $\Rightarrow$ uniformly bounded energy) has been established for $N = 2$ [12] and $N = 3$ [10] for the problem in the form (1.2).

The double-side bound in Theorem 1.2 represents a sort of improved Rozenblyum-Lieb-Cwikel estimate [8, 35, 42]. Let us recall that this inequality is an estimate of the number of negative eigenvalues of a Schrödinger operator $-\Delta + V$ in terms of a suitable Lebesgue norm of the negative part $V_-$ of $V$ - a one side bound, where the universal constants are not explicit. Notice that the Morse index of $u_n$ coincides with the number of negative eigenvalues of $-\Delta + V_n$ in $H_0^1(\Omega_n)$, with $V_n = \frac{V(\varepsilon_n y + P_n)}{V(P_n)} - pU_n^{p-1}$, and

$$\int_{\Omega_n} [(V_n)_-]^{\frac{p+1}{p-1}} \leq p^{\frac{p+1}{p-1}} \int_{\Omega_n} U_n^{p+1} = p^{\frac{p+1}{p-1}} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} V(P_n)^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}.$$

Letting now

$$S := \left\{P \in \overline{\Omega} : \limsup_{n\to+\infty} \|u_n\|_{\infty, B_r(P)} = +\infty \ \forall\, r > 0\right\} \tag{1.8}$$

be the set of blow-up points, we have the following localization of $S$ in terms of $V$:

**Theorem 1.3.**

$$\nabla V(P) = 0 \quad \forall\, P \in S \cap \Omega, \qquad \partial_\nu V(P) \le 0 \quad \forall\, P \in S \cap \partial\Omega,$$

*where $\nu(Q)$ denotes the unit outward normal of $\partial\Omega$ at $Q$.*

Theorem 1.3 is reminiscent of what was already known for the Schrödinger equation in $\mathbb{R}^N$ in the semi-classical limit, see [43]. Thanks to the characterization of $S$, for suitable potentials $V$'s we can strengthen the previous analysis for either "low energy" or Morse index 1 solutions. To be more precise, assume that the potential $V$ is increasing at the boundary and is a Morse function:

$$\partial_\nu V > 0 \quad \text{on } \partial\Omega, \quad \nabla V(P) = 0 \Rightarrow \det D^2 V(P) \neq 0. \tag{1.9}$$

Inspired by the techniques in [30], we have the following:

**Theorem 1.4.** *Assume* (1.9). *Let $u_n$ be a solutions sequence of* (1.3) *so that either*

$$\limsup_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} < 2(\inf_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{\mathbb{R}^N} U_0^{p+1} \tag{1.10}$$

*or*

$$\limsup_{n\to+\infty} m(u_n) \le 1, \tag{1.11}$$

*where $U_0$ is given in Theorem* 1.1. *Then there exists $\delta_0 > 0$ so that*

$$\sigma(-\Delta + \lambda_n V - p u_n^{p-1}) \subset (-\infty, -\delta_0) \cup (\delta_0, +\infty) \tag{1.12}$$

*for $n$ large, where $\sigma$ denotes the spectrum. In particular $u_n$ is a non degenerate solution. Moreover, if $V$ has just one critical point we get that*

$$\lim_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{m(u_n)} = (\inf_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{\mathbb{R}^N} U_0^{p+1}. \tag{1.13}$$

In view of Theorem 3.1, the blow-up point $P$ is simple in the sense that $u_n$ admits just one blow-up sequence $P_n$ converging to $P$, given by maximum points of $u_n$ ($u_n(P_n) = \max_\Omega u_n$). By the exponential decay of $u_n$ away from the blow-up set, a localization argument should be in order to extend the result in case of multiple simple blow-up points, while the situation of non-simple blow-up points seems to be more delicate and still out of reach.

Finally, let us stress that the case of the critical nonlinearity $p = \frac{N+2}{N-2}$, $N \ge 3$, is quite different. Solutions of (1.1) with uniformly bounded energy do not exist [7]. In a forthcoming paper [39], the second named author extends the argument to solutions with uniformly bounded Morse indices. In the supercritical case a similar phenomenon is in order. We refer to [38] for a unified presentation of all these results.

The paper is organized as follows. In Section 2 we prove the classification result contained in Theorem 1.1. Section 3 will be devoted to give a global asymptotic description for a blowing-up sequence $u_n$ provided either (1.6) or (1.7) does hold. Theorem 1.2 is proved in Section 4 through an asymptotic analysis for the eigenfunctions of the corresponding linearized operator $L_n$. In Section 5 the characterization of $S$ given in Theorem 1.3 will follow from all the previous analysis, and an asymptotic analysis for the eigenvalues of $L_n$ will allow us to prove Theorem 1.4.

## 2. Classification Results

In order to state the results, let us introduce the notion of stability, stability outside a compact set, and Morse index $k$.

**Definition 2.1.** Let $\Omega$ be a domain in $\mathbb{R}^N$. We say that a solution $U$ of

$$-\Delta U + U = |U|^{p-1}U \quad \text{in } \Omega \tag{2.1}$$

- is stable if

$$Q_U(\varphi) := \int_\Omega |\nabla\varphi|^2 + \varphi^2 - p\,|U|^{p-1}\,\varphi^2 \,\geq 0 \quad \forall\,\varphi \in C_0^1(\Omega);$$

- is stable outside a compact set $K$ if $Q_U(\varphi) \geq 0$ for any $\varphi \in C_0^1(\Omega\backslash K)$;
- has Morse index $m(U)$ equal to $k$ if $k$ is the maximal dimension of a subspace $W \subset C_0^1(\Omega)$ so that $Q_u(\varphi) < 0$ for any $\varphi \in W\backslash\{0\}$.

**Remark 2.2.** Any finite Morse index solution $U$ is stable outside a compact set $K \subset \Omega$. Indeed, there exists a maximal subspace $W_k := \mathrm{span}\{\varphi_1, \ldots, \varphi_k\} \subset C_0^1(\Omega)$ of dimension $k = m(U)$ so that $Q_U(\varphi) < 0$ for any $\varphi \in W_k\backslash\{0\}$. So $Q_U(\varphi) \geq 0$ for every $\varphi \in C_0^1(\Omega\backslash K)$, where $K := \cup_{i=1}^k \mathrm{supp}\ \varphi_i$.

We have the following result:

**Theorem 2.3.** *Let $U$ be a solution of* (2.1) *on $\mathbb{R}^N$ for $p > 1$ when $N = 2$ and $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$. Assume that $U$ is stable outside a compact set $K$. Then*

$$U(x) \to 0 \quad \text{as } |x| \to +\infty.$$

*Proof.* First we show that

$$\int_{\mathbb{R}^N} U^2 + \int_{\mathbb{R}^N} |U|^{p+1} < +\infty. \tag{2.2}$$

To this aim, given $R_0 > 0$ so that $K \subset B_{R_0}(0)$ and $R > R_0 + 2$, introduce a radial function $\eta \in C_0^\infty(\mathbb{R}^N)$ so that

$$\begin{aligned} &0 \leq \eta \leq 1, \quad \eta \equiv 0 \quad \text{in } B_{R_0+1}(0) \cup B_{2R}^C(0), \quad \eta \equiv 1 \quad \text{in } B_R(0) \setminus B_{R_0+2}(0), \\ &R|\nabla\eta| \leq 2 \quad \text{in } B_{2R}(0) \setminus B_R(0). \end{aligned} \tag{2.3}$$

Multiply (2.1) by $\eta^{2m}U$, $m \geq \frac{p+1}{p-1}$, and integrate by parts to get

$$\int_{\mathbb{R}^N} \eta^{2m}|U|^{p+1} = \int_{\mathbb{R}^N} |\nabla(\eta^m U)|^2 + \int_{\mathbb{R}^N} (\eta^m U)^2 - \int_{\mathbb{R}^N} |\nabla \eta^m|^2 U^2.$$

Since $\eta^m U \in C_0^1(\mathbb{R}^N \backslash K)$, the stability condition gives

$$\int_{\mathbb{R}^N} |\nabla(\eta^m U)|^2 + \int_{\mathbb{R}^N} (\eta^m U)^2 \geq p \int_{\mathbb{R}^N} \eta^{2m}|U|^{p+1},$$

and (2.3) with Hölder inequality yield to

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla \eta^m|^2 U^2 &= m^2 \int_{\mathbb{R}^N} \eta^{2m-2-\frac{4m}{p+1}} |\nabla \eta|^2 \eta^{\frac{4m}{p+1}} U^2 \\ &\leq C_m \left[ \int_{B_{R_0+2}(0)\backslash B_{R_0+1}(0)} U^2 + R^{n\frac{p-1}{p+1}-2} \left( \int_{\mathbb{R}^N} \eta^{2m}|U|^{p+1} \right)^{\frac{2}{p+1}} \right].\end{aligned}$$

The sub-critical growth guarantees $n\frac{p-1}{p+1} - 2 < 0$, and then

$$\int_{\mathbb{R}^N} \eta^{2m}|U|^{p+1} \geq p \int_{\mathbb{R}^N} \eta^{2m}|U|^{p+1} - C_m \left( \int_{\mathbb{R}^N} \eta^{2m}|U|^{p+1} \right)^{\frac{2}{p+1}} - C_m \int_{B_{R_0+2}(0)\backslash B_{R_0+1}(0)} U^2$$

yields to $\int_{\mathbb{R}^N} |U|^{p+1} < +\infty$. By the estimate

$$\begin{aligned}\int_{\mathbb{R}^N} (\eta^m U)^2 &\leq \int_{\mathbb{R}^N} \eta^{2m}|U|^{p+1} + \int_{\mathbb{R}^N} |\nabla \eta^m|^2 U^2 \leq \int_{\mathbb{R}^N} |U|^{p+1} + C_m \left( \int_{\mathbb{R}^N} |U|^{p+1} \right)^{\frac{2}{p+1}} \\ &\quad + C_m \int_{B_{R_0+2}(0)\backslash B_{R_0+1}(0)} U^2\end{aligned}$$

we also get $\int_{\mathbb{R}^N} U^2 < +\infty$, and (2.2) is established.

Standard regularity theory now implies uniform continuity of $U$ in $\mathbb{R}^N$ and even global $C^2(\mathbb{R}^N)$-estimate. Together with the $L^{p+1}(\mathbb{R}^N)$-estimate, it implies that

$$U(x) \to 0 \quad \text{as } |x| \to +\infty$$

as claimed. □

**Remark 2.4.**

i) The proof of (2.2) works as well for solutions $U$ – stable outside a compact set- of (2.1) on an half-space $H$ with $U = 0$ on $\partial H$.
ii) Following the techniques used by Farina et al. [11, 23] and Esposito et al. [18, 19], we can get better estimates and cover all the exponents $p$, see [21], and get strong integrability properties: for any $q \in (0, 2p + 2\sqrt{p(p-1)}]$ and $\alpha \in \mathbb{R}$ we have

$$\int_H |U|^q (1 + |y|^2)^\alpha < \infty,$$

where $H$ is either an half-space or $\mathbb{R}^N$.

Once a decay property has been established in Theorem 2.3, one can use [26, 27] to show that positive solutions of (2.1) on $H = \mathbb{R}^N$ which are stable outside a compact set are necessarily radially symmetric and decreasing. The uniqueness of the positive, radially symmetric solution $U_0$ to (2.1) on $H = \mathbb{R}^N$ [33] leads to

*Proof (of Theorem* 1.1*).*

- Let $H$ be an half-space. Unless $U \equiv 0$, by the strong minimum principle we have that $U > 0$ in $H$. Since $U \in L^\infty(H)$, the moving plane method implies [9] that $\frac{\partial U}{\partial x_N} > 0$ in $H$. Since $\frac{\partial U}{\partial x_N}$ is a positive solution of the linearized equation, it is rather classical to see that $U$ is a stable solution of (2.1), see for example [1, 24, 36]. Then we can apply Remark 2.4 to have
$$U^{p+1},\ U^2 \in L^1(H). \tag{2.4}$$
It is standard to show that (2.4) implies $\nabla U \in L^2(H)$. By the non-existence result in [22] we have the desired conclusion $U \equiv 0$.
- As already discussed, if $U \not\equiv 0$ then $U$ coincides with $U_0$. Since $U_0$ can be obtained as a mountain-pass solution in $H^1(\mathbb{R}^N)$ for the corresponding energy functional and $U_0$ is unstable in view of
$$\int_{\mathbb{R}^N} |\nabla U_0|^2 + \int_{\mathbb{R}^N} U_0^2 - p\int_{\mathbb{R}^N} U_0^{p-1}\ U_0^2 = -(p-1)\int_{\mathbb{R}^N} U_0^{p+1} < 0,$$
we have that $U_0$ has exactly Morse index one in $H^1(\mathbb{R}^N)$ (see [25]). As far as the zero eigenvalue, it is known (see for example [31]) that
$$\text{kernel}\,(-\Delta + 1 - pU_0^{p-1}) = \text{span}\{\partial_{x_1} U_0, \ldots, \partial_{x_N} U_0\}$$
in $H^1(\mathbb{R}^N)$.

## 3. Asymptotic Analysis and Blow-up Profile

We focus now on the asymptotic behavior as $\lambda \to +\infty$ of solutions to (1.1). First we have a local description:

**Theorem 3.1.** *Let $u_n$ be a solutions sequence of* (1.3), *with $p > 1$ when $N = 2$ and $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$. Assume either*
$$\sup_n m(u_n) < +\infty$$
*or*
$$\sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} < +\infty.$$
*Let $P_n \in \Omega$ so that $u_n(P_n) = \max_{\Omega\cap B_{R_n\tilde{\varepsilon}_n}(P_n)} u_n$ for some $R_n \to +\infty$, where $\tilde{\varepsilon}_n = u_n(P_n)^{-\frac{p-1}{2}} \to 0$ as $n \to +\infty$. Setting $U_n(y) = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n y + P_n)$ for $y \in \Omega_n = \frac{\Omega - P_n}{\varepsilon_n}$, with $\varepsilon_n = \lambda_n^{-\frac{1}{2}}\, V(P_n)^{-\frac{1}{2}}$, then for a sub-sequence we have*:

- $\frac{\varepsilon_n}{d(P_n,\partial\Omega)} \to 0$ *as* $n \to +\infty$;
- $u_n(P_n) = \max_{\Omega\cap B_{R_n\varepsilon_n}(P_n)} u_n$ *for some* $R_n \to +\infty$;

- *$U_n \to U_0$ in $C^1_{\rm loc}(\mathbb{R}^N)$ as $n \to +\infty$;*
- *there exists $\phi_n \in C_0^\infty(\Omega)$ with supp $\phi_n \subset B_{R\varepsilon_n}(P_n)$, $R > 0$, so that for n large*

$$\int_\Omega |\nabla\phi_n|^2 + (\lambda_n V - p\, u_n^{p-1})\phi_n^2 dx < 0; \tag{3.1}$$

- *for all $R > 0$ there holds*

$$\lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{B_{R\varepsilon_n(P_n)}} u_n^{p+1} = \Big(\lim_{n\to+\infty} V(P_n)\Big)^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{B_R(0)} U_0^{p+1}, \tag{3.2}$$

*where $U_0$ is given in Theorem* 1.1.

*Proof.* Let us first introduce $\widetilde U_n(y) = \tilde\varepsilon_n^{\frac{2}{p-1}} u_n(\tilde\varepsilon_n y + P_n)$ for $y \in \tilde\Omega_n = \frac{\Omega - P_n}{\tilde\varepsilon_n}$, and let $d_n$ denote $d(P_n, \partial\Omega)$. Suppose that $\frac{\tilde\varepsilon_n}{d_n} \to L \in [0, +\infty]$, up to a sub-sequence. Then $\widetilde\Omega_n \to H$, with $H$ an half-space so that $0 \in \overline{H}$ and $d(0, \partial H) = \frac{1}{L}$. The function $\widetilde U_n$ satisfies

$$\begin{cases} -\Delta \widetilde U_n + \lambda_n \tilde\varepsilon_n^2 V(\tilde\varepsilon_n y + P_n)\widetilde U_n = \widetilde U_n^p, & \text{in } \widetilde\Omega_n \\ 0 < \widetilde U_n \le \widetilde U_n(0) = 1, & \text{in } \widetilde\Omega_n \cap B_{R_n}(0) \\ \widetilde U_n = 0 & \text{on } \partial\widetilde\Omega_n. \end{cases}$$

Since $P_n$ is a point of local maximum of $u_n$, we have

$$0 \le -\Delta \widetilde U_n(0) = 1 - \lambda_n \tilde\varepsilon_n^2 V(P_n) \Rightarrow \lambda_n \tilde\varepsilon_n^2 V(P_n) \le 1.$$

Setting $\omega(V) := [\sup_\Omega V][\inf_\Omega V]^{-1}$, it follows that

$$\lambda_n\, \tilde\varepsilon_n^2 V(x) \le \omega(V),$$

and, up to a sub-sequence,

$$\lambda_n \tilde\varepsilon_n^2 V(P_n) \to \tilde\lambda \quad \text{as } n \to +\infty,$$

for some $\tilde\lambda \in [0, 1]$. By elliptic regularity theory [29], up to a further sub-sequence $\widetilde U_n \to \widetilde U$ in $C^1_{\rm loc}(\overline{H})$ as $n \to +\infty$, where $\widetilde U$ is a solution of

$$\begin{cases} -\Delta \widetilde U + \tilde\lambda \widetilde U = \widetilde U^p & \text{in } H \\ 0 < \widetilde U \le \widetilde U(0) = 1 & \text{in } H \\ \widetilde U = 0 & \text{on } \partial H. \end{cases}$$

Since $\widetilde U$ is not trivial, by [28] we have necessarily $\tilde\lambda > 0$ in case $H$ is an half-space. If $H = \mathbb{R}^N$, observe that we have

$$m(\widetilde U) \le \sup_n m(u_n)\,, \quad \int_{\mathbb{R}^N} \widetilde U^{p+1} \le (\inf_\Omega V)^{\frac{N}{2}-\frac{p+1}{p-1}} \sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}. \tag{3.3}$$

Indeed, if $\phi_1, \ldots, \phi_k \in C_0^\infty(\mathbb{R}^N)$ are orthogonal in $L^2(\mathbb{R}^N)$ and satisfy

$$\int_{\mathbb{R}^N} |\nabla \phi_i|^2 + (\tilde{\lambda} - p\,\widetilde{U}^{p-1})\phi_i^2 < 0 \quad \forall\, i = 1, \ldots, k,$$

we have that $\phi_{i,n}(x) := \tilde{\varepsilon}_n^{-\frac{N-2}{2}} \phi_i(\frac{x-P_n}{\tilde{\varepsilon}_n})$ are orthogonal in $L^2(\Omega)$ with

$$\begin{aligned}\int_\Omega |\nabla \phi_{i,n}|^2 + (\lambda_n V - p\,u_n^{p-1})\phi_{i,n}^2 \mathrm{d}x &= \int_{\tilde{\Omega}_n} |\nabla \phi_i|^2 + (\lambda_n \tilde{\varepsilon}_n^2 V(\tilde{\varepsilon}_n y + P_n) - p\,\widetilde{U}_n^{p-1})\phi_i^2 \mathrm{d}x \\ &\to \int_{\mathbb{R}^N} |\nabla \phi_i|^2 + (\tilde{\lambda} - p\,\widetilde{U}^{p-1})\phi_i^2 \mathrm{d}x < 0\end{aligned}$$

as $n \to +\infty$ for all $i = 1, \ldots, k$. Hence, $m(\widetilde{U}) \le \sup_n m(u_n)$. Moreover, in view of $2\frac{p+1}{p-1} - N > 0$ and

$$\tilde{\varepsilon}_n^2 \le \lambda_n^{-1} V(P_n)^{-1} \le (\inf_\Omega V)^{-1} \lambda_n^{-1},$$

we let $R \to +\infty$ in

$$\begin{aligned}\int_{B_R(0)} \widetilde{U}^{p+1} &= \lim_{n\to+\infty} \int_{B_R(0)} \widetilde{U}_n^{p+1} = \lim_{n\to+\infty} \tilde{\varepsilon}_n^{2\frac{p+1}{p-1}-N} \int_{B_{R\tilde{\varepsilon}_n}(P_n)} u_n^{p+1} \\ &\le (\inf_\Omega V)^{\frac{N}{2}-\frac{p+1}{p-1}} \sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}\end{aligned} \tag{3.4}$$

to get the second estimate too. Our assumptions on $u_n$ then guarantee that either $m(\widetilde{U}) < +\infty$ or $\int_{\mathbb{R}^N} \widetilde{U}^{p+1} < +\infty$. If $m(\widetilde{U}) < +\infty$ and $\widetilde{U}$ is non trivial, by [2] we get $\tilde{\lambda} > 0$. If $\int_{\mathbb{R}^N} \widetilde{U}^{p+1} < +\infty$, we deduce also $\widetilde{U} \in L^2(\mathbb{R}^N)$ and then $\nabla\widetilde{U} \in L^2(\mathbb{R}^N)$. By the Pohozaev identity [41]

$$\left(\frac{N}{p+1} - \frac{N-2}{2}\right) \int_{\mathbb{R}^N} \widetilde{U}^{p+1} = \tilde{\lambda} \int_{\mathbb{R}^N} \widetilde{U}^2$$

for a non trivial solution $\widetilde{U}$ we still get $\tilde{\lambda} > 0$. Once we know that

$$\left(\frac{\tilde{\varepsilon}_n}{\varepsilon_n}\right)^2 = \lambda_n \tilde{\varepsilon}_n^2 V(P_n) \to \tilde{\lambda} \in (0, 1] \quad \text{as } n \to +\infty, \tag{3.5}$$

it is equivalent but more convenient to work with $U_n$. Since $U_n$ solves

$$\begin{cases} -\Delta U_n + \dfrac{V(\varepsilon_n y + P_n)}{V(P_n)} U_n = U_n^p & \text{in } \Omega_n \\ 0 < U_n \le U_n(0) = \left(\dfrac{\tilde{\varepsilon}_n}{\varepsilon_n}\right)^{-\frac{2}{p-1}} & \text{in } \Omega_n \cap B_{R_n \frac{\tilde{\varepsilon}_n}{\varepsilon_n}}(0) \\ U_n = 0 & \text{on } \partial\Omega_n, \end{cases}$$

in view of (3.5) and by elliptic regularity theory [29], up to a sub-sequence, we have that $U_n \to U$ in $C^1_{\mathrm{loc}}(\overline{H})$ as $n \to +\infty$, where $U$ is a non-trivial bounded solution of

(1.5). By Theorem 1.1 we have that $H = \mathbb{R}^N$: $\frac{\varepsilon_n}{d_n} \to 0$ as $n \to +\infty$. Arguing as for (3.3), observe that we have

$$m(U) \leq \sup_n m(u_n), \quad \int_{\mathbb{R}^N} U^{p+1} \leq \left(\inf_\Omega V\right)^{\frac{N}{2}-\frac{p+1}{p-1}} \sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}.$$

By Theorem 1.1 $U$ coincides with $U_0$ and is unstable: there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi \subset B_R(0)$, $R > 0$, and

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 + (1 - p\, U_0^{p-1})\phi^2 < 0.$$

As before, the function $\phi_n(x) := \varepsilon_n^{-\frac{N-2}{2}} \phi(\frac{x-P_n}{\varepsilon_n})$ is what we are looking for in (3.1). Arguing as for (3.4), we have that

$$\int_{B_R(0)} U_0^{p+1} = \lim_{n\to+\infty} \int_{B_R(0)} U_n^{p+1} = \left(\lim_{n\to+\infty} V(P_n)\right)^{\frac{N}{2}-\frac{p+1}{p-1}} \left(\lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{B_{R\varepsilon_n}(P_n)} u_n^{p+1}\right),$$

and (3.2) is also proved. □

Once the limiting problem has been identified and the local behavior around a blow up sequence $P_n$ has been described, we can prove global estimates. We will show in such a way that the sequence $u_n$ decays exponentially away from the blow-up points.

**Theorem 3.2.** *Let* $p > 1$ *when* $N = 2$ *and* $1 < p < \frac{N+2}{N-2}$ *when* $N = 3$. *Let* $u_n$ *be a solutions sequence to* (1.3) *so that either*

$$\overline{k} = \lim_{n\to+\infty} m(u_n) < +\infty$$

*or*

$$\overline{k} = \left(\inf_\Omega V\right)^{\frac{N}{2}-\frac{p+1}{p-1}} \left(\int_{\mathbb{R}^N} U_0^{p+1}\right)^{-1} \lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} < +\infty.$$

*Up to a sub-sequence, there exist* $P_n^1, \ldots, P_n^k$, $k \leq \overline{k}$ , *so that for all* $i, j = 1, \ldots, k$, $i \neq j$:

$$\lambda_n |P_n^i - P_n^j|^2 \to +\infty, \quad \lambda_n d(P_n^i, \partial\Omega)^2 \to +\infty \quad \text{as } n \to +\infty \tag{3.6}$$

*and*

$$u_n(P_n^i) = \max_{\Omega\cap B_{R_n\lambda_n^{-\frac{1}{2}}}(P_n^i)} u_n, \tag{3.7}$$

*for some* $R_n \to +\infty$ *as* $n \to +\infty$. *Moreover, there holds*

$$u_n(x) \leq C\lambda_n^{\frac{1}{p-1}} \sum_{i=1}^k e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^i|} \quad \forall\, x \in \Omega, \quad n \in \mathbb{N}, \tag{3.8}$$

*for some* $C, \gamma > 0$.

*Proof.* The proof is divided in two steps (see also [20]).

1[st] step *There exist* $k \le \bar{k}$ *sequences* $P_n^1, \ldots, P_n^k$ *satisfying* (3.6)–(3.7) *so that*

$$\lim_{R\to+\infty}\left(\limsup_{n\to+\infty}\left[\lambda_n^{-\frac{1}{p-1}} \max_{\left\{d_n(x)\ge R\lambda_n^{-\frac{1}{2}}\right\}} u_n(x)\right]\right) = 0, \tag{3.9}$$

*where* $d_n(x) = \min\{|x - P_n^i| : i = 1, \ldots, k\}$ *is the distance function from* $\{P_n^1, \ldots, P_n^k\}$.

Let $P_n^1$ be a point of global maximum of $u_n$: $u_n(P_n^1) = \max_\Omega u_n$. Since (3.7) holds, if (3.9) holds for $P_n^1$, then we take $k = 1$ and by Theorem 3.1 the claim is proved. Otherwise, set $\varepsilon_n^1 = \lambda_n^{-\frac{1}{2}} V(P_n^1)^{-\frac{1}{2}}$ and suppose by contradiction that

$$\limsup_{R\to+\infty}\left(\limsup_{n\to+\infty}\left[(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{|x-P_n^1|\ge R\varepsilon_n^1\}} u_n\right]\right) = 4\delta > 0.$$

By Theorem 3.1, up to a sub-sequence we have

$$(\varepsilon_n^1)^{\frac{2}{p-1}} u_n(\varepsilon_n^1 y + P_n^1) =: U_n^1(y) \to U_0(y) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N) \tag{3.10}$$

as $n \to +\infty$. Since $U_0 \to 0$ as $|x| \to +\infty$, we can find $R$ large so that

$$U_0(y) \le \delta \quad \forall\, |y| \ge R. \tag{3.11}$$

Up to take $R$ larger and up to a sub-sequence, we can assume that

$$(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{|x-P_n^1|\ge R\varepsilon_n^1\}} u_n \ge 2\,\delta. \tag{3.12}$$

Since $u_n = 0$ on $\partial\Omega$, we have that

$$\exists\, P_n^2 \in \Omega \setminus B_{R\,\varepsilon_n^1}(P_n^1) \quad \text{so that } u_n(P_n^2) = \max_{\Omega\setminus B_{R\,\varepsilon_n^1}(P_n^1)} u_n.$$

By (3.10)–(3.11) we have that $\frac{|P_n^2-P_n^1|}{\varepsilon_n^1} \to +\infty$. Indeed, if $\frac{|P_n^2-P_n^1|}{\varepsilon_n^1} \to R' \ge R$ were true, we would get

$$(\varepsilon_n^1)^{\frac{2}{p-1}} u_n(P_n^2) = U_n^1\left(\frac{P_n^2 - P_n^1}{\varepsilon_n^1}\right) \to U_0(R') \le \delta,$$

in contradiction with (3.12). Therefore, the first in (3.6) does hold for $\{P_n^1, P_n^2\}$. Set now $\tilde{\varepsilon}_n^2 = u_n(P_n^2)^{-\frac{p-1}{2}}$ and $R_n^2 = \frac{1}{2}\frac{|P_n^2-P_n^1|}{\tilde{\varepsilon}_n^2}$. By (3.12) we get $\tilde{\varepsilon}_n^2 \le (2\,\delta)^{-\frac{p-1}{2}}\varepsilon_n^1$, and then

$$R_n^2 \ge \frac{(2\delta)^{\frac{p-1}{2}}}{2}\frac{|P_n^2 - P_n^1|}{\varepsilon_n^1} \to +\infty \quad \text{as } n \to +\infty.$$

This implies

$$u_n(P_n^2) = \max_{\Omega\cap B_{R_n^2\tilde{\varepsilon}_n^2}(P_n^2)} u_n.$$

Indeed, since $\varepsilon_n^1 << |P_n^2 - P_n^1|$ for all $x \in B_{R_n^2 \tilde{\varepsilon}_n^2}(P_n^2)$ we have

$$|x - P_n^1| \geq |P_n^2 - P_n^1| - |x - P_n^2| \geq \frac{1}{2}|P_n^2 - P_n^1| \geq R\,\varepsilon_n^1,$$

and then $\Omega \cap B_{R_n^2\tilde{\varepsilon}_n^2}(P_n^2) \subset \Omega \backslash B_{R\,\varepsilon_n^1}(P_n^1)$. Since $R_n^2 \to +\infty$ as $n \to +\infty$, by Theorem 3.1 we also get that the second in (3.6) and (3.7) hold true for $\{P_n^1, P_n^2\}$. If (3.9) holds for $\{P_n^1, P_n^2\}$, we are done. Otherwise, we iterate the above argument: let $P_n^1, \ldots, P_n^s$ $s$ sequences so that (3.6)–(3.7) hold true, but (3.9) is not satisfied. As before, we can find $R > 0$ large and a sub-sequence so that

$$(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{d_n(x) \geq R\varepsilon_n^1\}} u_n(x) \geq 2\,\delta,$$

where $d_n(x) = \min\{|x - P_n^i| : i = 1, \ldots, s\}$. Up to a further sub-sequence, we can assume that

$$\frac{\varepsilon_n^1}{\varepsilon_n^i} \to \theta_i \in (0, +\infty) \quad \text{as } n \to +\infty,\ \forall\, i = 1, \ldots, s,$$

where $\varepsilon_n^i := \lambda_n^{-\frac{1}{2}} V(P_n^i)^{-\frac{1}{2}}$, and by Theorem 3.1 then deduce

$$(\varepsilon_n^1)^{\frac{2}{p-1}} u_n(\varepsilon_n^1 y + P_n^i) = \left(\frac{\varepsilon_n^1}{\varepsilon_n^i}\right)^{\frac{2}{p-1}} U_n^i\left(\frac{\varepsilon_n^1}{\varepsilon_n^i} y\right) \to \theta_i^{\frac{2}{p-1}} U_0(\theta_i\, y) \tag{3.13}$$

in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to +\infty$. Since $U_0 \to 0$ as $|x| \to +\infty$ we can find $R$ large so that $\theta_i^{\frac{2}{p-1}} U_0(\theta_i\, y) \leq \delta$ for $|y| \geq R$ and all $i = 1, \ldots, s$. We repeat the argument above, by replacing $|x - P_n^1|$ with $d_n(x)$. Let $P_n^{s+1}$ be so that

$$u_n(P_n^{s+1}) = \max_{\{d_n(x) \geq R\,\varepsilon_n^1\}} u_n \geq 2\,\delta\,(\varepsilon_n^1)^{-\frac{2}{p-1}}. \tag{3.14}$$

By (3.13) and $\theta_i^{\frac{2}{p-1}} U_0(\theta_i y) \leq \delta$ for $|y| \geq R$, we deduce as before that $\frac{|P_n^{s+1} - P_n^i|}{\varepsilon_n^1} \to +\infty$ as $n \to +\infty$, for all $i = 1, \ldots, s$, and the first in (3.6) does hold for $\{P_n^1, \ldots, P_n^{s+1}\}$. Setting $\tilde{\varepsilon}_n^{s+1} = u_n(P_n^{s+1})^{-\frac{p-1}{2}}$ and $R_n^{s+1} = \frac{1}{2}\frac{d_n(P_n^{s+1})}{\tilde{\varepsilon}_n^{s+1}}$, we still have by (3.14)

$$\tilde{\varepsilon}_n^{s+1} \leq (2\delta)^{-\frac{p-1}{2}} \varepsilon_n^1,$$

and then $R_n^{s+1} \to +\infty$ as $n \to +\infty$. Since as before

$$u_n(P_n^{s+1}) = \max_{\Omega \cap B_{R_n^{s+1}\tilde{\varepsilon}_n^{s+1}}(P_n^{s+1})} u_n,$$

by Theorem 3.1 we get that the second in (3.6) and (3.7) do hold for $\{P_n^1, \ldots, P_n^{s+1}\}$. For the sequence $P_n^i$, $i = 1, \ldots, s+1$, Theorem 3.1 also provides $\phi_n^i \in C_0^\infty(\Omega)$ with $\operatorname{supp} \phi_n^i \subset B_{R\,\varepsilon_n^i}(P_n^i)$, $R > 0$, which satisfy (3.1). By (3.6) $\phi_n^1, \ldots, \phi_n^{s+1}$ have disjoint compact supports for $n$ large yielding to $s + 1 \leq \lim_{n \to +\infty} m(u_n)$, and then the iterative procedure must stop after $k$ steps, $k \leq \lim_{n \to +\infty} m(u_n)$, providing sequences

$P_n^1, \dots, P_n^k$ so that (3.6)–(3.7) and (3.9) do hold. Alternatively, by (3.2) on each $P_n^1, \dots, P_n^{s+1}$ and (3.6) we get

$$\lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} \geq \lim_{n\to+\infty} \sum_{i=1}^{s+1} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_{B_{R\varepsilon_n^i}(P_n^i)} u_n^{p+1}$$
$$\geq (s+1)\left(\inf_\Omega V\right)^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{B_R(0)} U_0^{p+1}$$

for all $R > 0$, and then

$$s+1 \leq \left(\inf_\Omega V\right)^{\frac{N}{2}-\frac{p+1}{p-1}} \left(\int_{\mathbb{R}^N} U_0^{p+1}\right)^{-1} \lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}.$$

The conclusion follows also in this second case.

**2nd** step *Let $P_n^1, \dots, P_n^k$ be as in the* 1st *step. Then there are $\gamma, C > 0$ so that*

$$u_n(x) \leq C\, \lambda_n^{\frac{1}{p-1}} \sum_{i=1}^{k} e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^i|} \quad \forall x \in \Omega, \quad n \in \mathbb{N}.$$

By (3.9) for $R > 0$ large and $n \geq n(R)$ there holds

$$\lambda_n^{-\frac{1}{p-1}} \max_{\left\{d_n(x)\geq R\lambda_n^{-\frac{1}{2}}\right\}} u_n(x) \leq \left(\frac{1}{4p}\inf_\Omega V\right)^{\frac{1}{p-1}},$$

where $d_n(x) = \min\{|x - P_n^i| : i = 1, \dots, k\}$. Hence, in $\{d_n(x) \geq R\,\lambda_n^{-\frac{1}{2}}\}$ for $n \geq n(R)$ we have

$$\tilde{a}_n(x) := \frac{\lambda_n}{2} V(x) - p u_n^{p-1}(x) \geq \frac{\lambda_n}{4} \inf_\Omega V. \tag{3.15}$$

Compute the linear operator $-\Delta + \tilde{a}_n(x)$ on $\phi_n^i(x) = e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^i|}$ in $\{d_n(x) \geq R\,\lambda_n^{-\frac{1}{2}}\}$:

$$(-\Delta + \tilde{a}_n)(\phi_n^i) = \lambda_n \phi_n^i \left[ -\gamma^2 + (N-1)\frac{\gamma}{\lambda_n^{\frac{1}{2}}|x-P_n^i|} + \lambda_n^{-1}\tilde{a}_n(x)\right] \geq 0$$

for $n$ large, provided $0 < \gamma \leq \left(\frac{1}{4}\inf_\Omega V\right)^{\frac{1}{2}}$. Observe that for $R$ large

$$\left(e^{\gamma R}\phi_n^i(x) - \lambda_n^{-\frac{1}{p-1}} u_n(x)\right)\Big|_{\partial B_{R\lambda_n^{-\frac{1}{2}}}(P_n^i)} \to 1 - s_i^{\frac{2}{p-1}} U_0(s_i R) > 0$$

as $n \to +\infty$, where $s_i = \lim_{n\to+\infty} V(P_n^i)^{\frac{1}{2}}$.
Then, if we define $\phi_n := e^{\gamma R}\lambda_n^{\frac{1}{p-1}} \sum_{i=1}^k \phi_n^i$, for $L_n = -\Delta + \lambda_n V - u_n^{p-1}$ we have

$$L_n(\phi_n - u_n) \geq 0 \quad \text{in } \{d_n(x) > R\,\lambda_n^{-\frac{1}{2}}\}$$

and $\phi_n - u_n \geq 0$ on $\{d_n(x) = R\,\lambda_n^{-\frac{1}{2}}\} \cup \partial\Omega$. Note that by (3.6)–(3.7)

$$\left\{d_n(x) = R\,\lambda_n^{-\frac{1}{2}}\right\} = \cup_{i=1}^k \partial B_{R\,\lambda_n^{-\frac{1}{2}}}(P_n^i) \subset \Omega$$

for $n \geq n(R)$. Then, by the minimum principle

$$u_n \leq \phi_n = e^{\gamma R}\lambda_n^{\frac{1}{p-1}} \sum_{i=1}^k e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^i|}$$

in $\{d_n(x) \geq R\,\lambda_n^{-\frac{1}{2}}\}$, if $R$ is large and $n \geq n(R)$. Since by (3.5)

$$u_n(x) \leq \max_\Omega u_n = (\tilde{\varepsilon}_n^1)^{-\frac{2}{p-1}} \leq Ce^{\gamma R}\lambda_n^{\frac{1}{p-1}} \sum_{i=1}^k e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^i|}$$

for some $C > 0$ if $d_n(x) \leq R\,\lambda_n^{-\frac{1}{2}}$, we have that (3.8) holds true in $\Omega$ with a constant $Ce^{\gamma R}$ and $n \geq n(R)$. Up to take a larger constant $C$, we have the validity of (3.8) in $\Omega$ for every $n \in \mathbb{N}$. □



## 4. Morse Index Information and Energy Information

We address now the equivalence between Morse index and energy. The analysis of the previous Section provides us with the upper bound in Theorem 1.2.

**Theorem 4.1.** *Let $u_n$ be a solutions sequence of* (1.3) *so that* $\sup_n m(u_n) < +\infty$. *Then*

$$\limsup_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{m(u_n)} \leq \left(\sup_\Omega V\right)^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{\mathbb{R}^N} U_0^{p+1}. \tag{4.1}$$

*Proof.* By Theorem 3.2, up to a sub-sequence we can assume that $m(u_n) \to \bar{k}$ and there exist $\{P_n^1, \ldots, P_n^k\}$, $k \leq \bar{k}$, so that (3.6)–(3.8) do hold. Notice that $\bar{k} \geq 1$ since by Theorem 3.1 $\bar{k} = 0$ would imply $\sup_n \|u_n\|_\infty < +\infty$, in contradiction with (1.4). By (3.6) and (3.8) we can then write $\forall\, R > 0$

$$\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} = \sum_{i=1}^k (V(P_n^i))^{\frac{p+1}{p-1}-\frac{N}{2}} \int_{B_R(0)} (U_n^i)^{p+1} + O\left(\sum_{i=1}^k \int_{\mathbb{R}^N\setminus B_{R\lambda_n^{\frac{1}{2}}\varepsilon_n^i}(0)} e^{-\gamma(p+1)|y|}\mathrm{d}y\right),$$

and then get

$$\limsup_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} = \int_{B_R(0)} U_0^{p+1} \sum_{i=1}^k \left(\limsup_{n\to+\infty} V(P_n^i)\right)^{\frac{p+1}{p-1}-\frac{N}{2}}$$
$$+ O\left(\int_{\mathbb{R}^N\setminus B_{\delta R}(0)} e^{-\gamma(p+1)|y|}\mathrm{d}y\right)$$

for some $\delta > 0$. As $R \to +\infty$ we get

$$\limsup_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} = \int_{\mathbb{R}^N} U_0^{p+1} \sum_{i=1}^k \left( \limsup_{n\to+\infty} V(P_n^i) \right)^{\frac{p+1}{p-1}-\frac{N}{2}}$$
$$\leq k \int_{\mathbb{R}^N} U_0^{p+1} \left( \sup_\Omega V \right)^{\frac{p+1}{p-1}-\frac{N}{2}}$$
$$\leq \int_{\mathbb{R}^N} U_0^{p+1} \left( \sup_\Omega V \right)^{\frac{p+1}{p-1}-\frac{N}{2}} \lim_{n\to+\infty} m(u_n).$$

Since this is true for any sub-sequence so that $m(u_n)$ converges, we deduce the validity of (4.1). □

**Remark 4.2.** If $k = 1$, we have just one blow-up sequence $P_n := P_n^1$ and we can get that

$$\limsup_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{m(u_n)} \leq V^{\frac{p+1}{p-1}-\frac{N}{2}}(P) \int_{\mathbb{R}^N} U_0^{p+1},$$

where $P = \lim_{n\to+\infty} P_n$.

To complete the proof of Theorem 1.2, now we show the lower bound.

**Theorem 4.3.** *Let $u_n$ be a solutions sequence of* (1.3) *so that* $\sup_n \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1} < +\infty$. *Then*

$$\liminf_{n\to+\infty} \frac{\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}}{\bar{m}(u_n)} \geq \frac{1}{N+1} \int_{\mathbb{R}^N} U_0^{p+1} \ (\inf_\Omega V)^{\frac{p+1}{p-1}-\frac{N}{2}}. \tag{4.2}$$

*Proof.* Up to a sub-sequence, we can assume that $\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}$ converges as $n \to +\infty$. As before, Theorem 3.2 provides us with $\{P_n^1, \ldots, P_n^k\}$, $k \leq \bar{k}$, so that (3.6)–(3.8) do hold, and by Theorem 3.1 we have that $\bar{k} > 0$. Let $\varphi_n^m$ be the $m$th eigenfunction of $-\Delta + \lambda_n V - p u_n^{p-1}$ in $H_0^1(\Omega)$ corresponding to the eigenvalue $\mu_n^m$, with $\varphi_n^m$ normalized to have $\max_\Omega |\varphi_n^m| = \max_\Omega \varphi_n^m = 1$ (considered with multiplicities):

$$\begin{cases} -\Delta\varphi_n^m + \lambda_n V \varphi_n^m - p\, u_n^{p-1} \varphi_n^m = \mu_n^m \varphi_n^m & \text{in } \Omega \\ |\varphi_n^m| \leq \max_\Omega \varphi_n^m = 1 & \\ \varphi_n^m = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Fix now $m$ such that $\mu_n^m \leq 1$ (or bounded) for $n$ large. We have

First claim. *There exists $M_0 > 0$ so that $\mu_n^m \geq -M_0 \lambda_n$ and $Q_n^m \in \bigcup_{j=1}^k B_{M_0 \lambda_n^{-\frac{1}{2}}}(P_n^j)$, where $Q_n^m$ is so that $\varphi_n^m(Q_n^m) = 1$.*

Since $Q_n^m$ is a maximum point of $\varphi_n^m$, by (4.3) we have

$$\mu_n^m \geq \lambda_n V(Q_n^m) - p\, u_n^{p-1}(Q_n^m) \geq \lambda_n \inf_\Omega V - \frac{p}{(\tilde{\varepsilon}_n^1)^2} \geq -M_0 \lambda_n$$

in view of (3.5), for some $M_0 > 0$ large. Further, observe that by (3.8) we have

$$u_n^{p-1}(x) \le C\lambda_n \sum_{j=1}^{k} e^{-(p-1)\gamma\lambda_n^{\frac{1}{2}}|x-P_n^j|} \le Ck\lambda_n e^{-\gamma M_0(p-1)}$$

in $\Omega\setminus\bigcup_{j=1}^k B_{M_0\lambda_n^{-\frac{1}{2}}}(P_n^j)$, and then

$$\mu_n^m \ge \lambda_n V(Q_n^m) - p\,u_n^{p-1}(Q_n^m) \ge \lambda_n\Big[\inf_\Omega V - p\,C\,k\,e^{-(p-1)\gamma M_0}\Big] > 1$$

whenever $Q_n^m \in \Omega\setminus\bigcup_{j=1}^k B_{M_0\lambda_n^{-\frac{1}{2}}}(P_n^j)$, for some $M_0 > 0$ large. A contradiction to $\mu_n^m \le 1$. Hence, for some $M_0 > 0$ large

$$Q_n^m \in \bigcup_{j=1}^k B_{M_0\lambda_n^{-\frac{1}{2}}}(P_n^j).$$

Set $\Phi_n^{m,j}(y) = \varphi_n^m(\varepsilon_n^j y + P_n^j)$. The function $\Phi_n^{m,j}$ solves

$$\begin{cases} -\Delta\Phi_n^{m,j} + \dfrac{V(\varepsilon_n^j y + P_n^j)}{V(P_n^j)}\Phi_n^{m,j} - p(U_n^j)^{p-1}\Phi_n^{m,j} = (\varepsilon_n^j)^2\mu_n^m\,\Phi_n^{m,j} & \text{in } \frac{\Omega-P_n^j}{\varepsilon_n^j} \\ |\Phi_n^{m,j}| \le \Phi_n^{m,j}\Big(\dfrac{Q_n^m - P_n^j}{\varepsilon_n^j}\Big) = 1 & \text{in } \frac{\Omega-P_n^j}{\varepsilon_n^j} \\ \Phi_n^{m,j} = 0 & \text{on } \partial\Big(\dfrac{\Omega - P_n^j}{\varepsilon_n^j}\Big). \end{cases} \tag{4.4}$$

By the first claim we get

$$\lambda_n^{-1}(\inf_\Omega V)^{-1} \ge (\varepsilon_n^j)^2\mu_n^m \ge -M_0\lambda_n(\varepsilon_n^j)^2 = -M_0 V(P_n^j)^{-1} \ge -M_0(\inf_\Omega V)^{-1}.$$

Up to a sub-sequence, we can assume that

$$(\varepsilon_n^j)^2\mu_n^m \to \mu^{m,j} \le 0 \quad \text{as } n \to +\infty.$$

Multiply (4.4) by $\Phi_n^{m,j}$ and integrate on $\frac{\Omega-P_n^j}{\varepsilon_n^j}$ to get

$$\begin{aligned}
\int_{\frac{\Omega-P_n^j}{\varepsilon_n^j}} |\nabla\Phi_n^{m,j}|^2 + \Big[\frac{V(\varepsilon_n^j y + P_n^j)}{V(P_n^j)} - (\varepsilon_n^j)^2\mu_n^m\Big](\Phi_n^{m,j})^2 &\le p\int_{\frac{\Omega-P_n^j}{\varepsilon_n^j}} (U_n^j)^{p-1} \\
= p(\varepsilon_n^j)^{2-N}\int_\Omega u_n^{p-1} \le C(\varepsilon_n^j)^{2-N}\lambda_n \sum_{i=1}^k \int_\Omega e^{-(p-1)\gamma\lambda_n^{\frac{1}{2}}|x-P_n^i|} \\
= C(\varepsilon_n^j)^{2-N}\lambda_n^{1-\frac{N}{2}} \sum_{i=1}^k \int_{\lambda_n^{\frac{1}{2}}(\Omega-P_n^i)} e^{-(p-1)\gamma|y|} \le C\Big(\sup_\Omega V\Big)^{\frac{N}{2}-1} k\int_{\mathbb{R}^N} e^{-(p-1)\gamma|y|} < +\infty
\end{aligned}$$

in view of (3.8). In particular, $\|\Phi_n^{m,j}\|_{H^1(B_R(0))} \le C$ for all $R > 0$ and, up to a sub-sequence and a diagonal process, $\Phi_n^{m,j} \rightharpoonup \Phi^{m,j}$ in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and a.e. as $n \to +\infty$. Moreover $\Phi^{m,j} \in H^1(\mathbb{R}^N)$ solves

$$\begin{cases} -\Delta \Phi^{m,j} + \Phi^{m,j} - p\, U_0^{p-1}\, \Phi^{m,j} = \mu^{m,j}\, \Phi^{m,j} & \text{in } \mathbb{R}^N \\ |\Phi^{m,j}| \le 1. \end{cases} \tag{4.5}$$

We have that $\Phi^{m,j} \neq 0$ for some $j \in \{1, \dots, k\}$, as it follows by

Second claim. *Let $j \in \{1, \dots, k\}$ so that (up to a sub-sequence) $Q_n^m \in B_{M_0 \lambda_n^{-\frac12}}(P_n^j)$, for some $M_0 > 0$ large. Then $\Phi^{m,j} \neq 0$.*

Decompose $\Phi_n^{m,j}$ as $h_n + t_n$, where $h_n$ satisfies

$$\begin{cases} \Delta h_n = 0 & \text{in } B_{M_0+1}(0) \\ h_n = \Phi_n^{m,j} & \text{on } \partial B_{M_0+1}(0). \end{cases}$$

If $\Phi^{m,j} \equiv 0$, then $\Phi_n^{m,j} \rightharpoonup 0$ in $H^1(B_{M_0+1}(0))$, and by the trace Sobolev embedding Theorem $\Phi_n^{m,j} \to 0$ in $L^1(\partial B_{M_0+1}(0))$. By the mean value Theorem, then $h_n \to 0$ uniformly in $B_{M_0}(0)$. Since

$$\begin{cases} -\Delta t_n = -\Delta \Phi_n^{m,j} = O(1) & \text{in } B_{M_0+1}(0) \\ t_n = 0 & \text{on } \partial B_{M_0+1}(0), \end{cases}$$

by elliptic regularity theory [29] $t_n$ is uniformly bounded in $C^{0,\alpha}(B_{M_0+1}(0))$. In particular, by Ascoli-Arzelá Theorem $t_n \to t$ uniformly in $B_{M_0+1}(0)$. Hence, $\Phi_n^{m,j} = h_n + t_n \to 0$ uniformly in $B_{M_0}(0)$, and we reach the contradiction

$$\Phi_n^{m,j}\left(\frac{Q_n^m - P_n^j}{\varepsilon_n^j}\right) = 1 \to 0 \quad \text{as } n \to +\infty$$

in view of $Q_n^m \in B_{M_0 \lambda_n^{-\frac12} (\inf_\Omega V)^{-\frac12}}(P_n^j)$. Therefore, $\Phi^{m,j} \neq 0$.

By Theorem 1.1 recall that $-\Delta + 1 - p\, U_0^{p-1}$ has in $H^1(\mathbb{R}^N)$ a first negative eigenvalue $\mu_1 < 0$ (with corresponding eigenfunction $\varphi_1$), $\mu_2 = 0$ vanishes (with corresponding eigenfunctions $\varphi_2 = \partial_{x_1} U, \dots, \varphi_{N+1} = \partial_{x_N} U$), and all the other eigenvalues are positive. By (4.5) and $\varepsilon_n^i = \lambda_n^{-\frac12} V(P_n^i)^{-\frac12}$ we necessarily have that either $\mu^{m,j} < 0$ for all $j$ (with $\mu^{m,j} = \mu_1$ for all the $j$'s so that $\Phi^{m,j} \neq 0$) or $\mu^{m,j} = 0$ for all $j$. Assume that $\mu^{m,j} < 0$ for $m = 1, \dots, M$ and $\mu^{m,j} = 0$ for $m = M+1, \dots, M+S$. We want to estimate $M$ and $S$ thanks to

Third claim. *There exist $C, \gamma > 0$ so that*

$$|\varphi_n^m| \le C \sum_{j=1}^{k} e^{-\gamma \lambda_n^{\frac12} |x - P_n^j|} \quad \text{in } \Omega, \quad \forall n. \tag{4.6}$$

Let $\widetilde{L}_n = -\Delta + b_n(x)$, where $b_n = \lambda_n V - p\, u_n^{p-1} - \mu_n^m$. Notice that $b_n \ge \tilde{a}_n$ for $n$ large, where $\tilde{a}_n$ is given by (3.15). By the proof of 2nd step in Theorem 3.2, we have that

$$\widetilde{L}_n\bigg(\sum_{j=1}^k e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^j|}\bigg) \geq 0$$

in $\{d_n(x) \geq R\,\lambda_n^{-\frac{1}{2}}\} = \Omega\setminus \cup_{j=1}^k B_{R\,\lambda_n^{-\frac{1}{2}}}(P_n^j)$, provided $0 < \gamma \leq (\frac{1}{4}\inf_\Omega V)^{\frac{1}{2}}$. Since for $C \geq e^{\gamma R}$

$$\begin{aligned} |\varphi_n^m| &\leq 1 \leq C\sum_{j=1}^k e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^j|} \quad \text{on } \bigcup_{j=1}^k B_{R\,\lambda_n^{-\frac{1}{2}}}(P_n^j) \\ |\varphi_n^m| &= 0 \leq C\sum_{j=1}^k e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^j|} \quad \text{on } \partial\Omega, \end{aligned} \tag{4.7}$$

by the maximum principle

$$|\varphi_n^m| \leq C\sum_{j=1}^k e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^j|} \quad \text{in } \Omega\setminus\bigcup_{j=1}^k B_{R\,\lambda_n^{-\frac{1}{2}}}(P_n^j).$$

Hence, up to take $C$ larger, by (4.7) we get that

$$|\varphi_n^m| \leq C\sum_{j=1}^k e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n^j|} \quad \text{in } \Omega, \quad \forall\, n.$$

For $m, l \in \{1, \dots, M\}$, $m \neq l$, we want to take the limit of the orthogonality condition:

$$\begin{aligned} 0 = \int_\Omega \varphi_n^m\,\varphi_n^l &= \sum_{j=1}^k \int_{B_{R\,\varepsilon_n^j}(P_n^j)} \varphi_n^m\,\varphi_n^l + \int_{\Omega\setminus\bigcup_{j=1}^k B_{R\,\varepsilon_n^j}(P_n^j)} \varphi_n^m\,\varphi_n^l \\ &= \sum_{j=1}^k (\varepsilon_n^j)^N \int_{B_R(0)} \Phi_n^{m,j}\,\Phi_n^{l,j} + \int_{\Omega\setminus\bigcup_{j=1}^k B_{R\,\varepsilon_n^j}(P_n^j)} \varphi_n^m\,\varphi_n^l. \end{aligned}$$

By the 3rd claim we have that

$$\begin{aligned} \bigg|\int_{\Omega\setminus\bigcup_{j=1}^k B_{R\,\varepsilon_n^j}(P_n^j)} \varphi_n^m\,\varphi_n^l\bigg| &\leq C'\sum_{i=1}^k \int_{\Omega\setminus B_{R\,\varepsilon_n^i}(P_n^i)} e^{-2\,\gamma\lambda_n^{\frac{1}{2}}|x-P_n^i|} \\ &\leq C''k\lambda_n^{-\frac{N}{2}} \int_{\mathbb{R}^N\setminus B_{\delta R}(0)} e^{-2\,\gamma|y|}\mathrm{d}y \end{aligned}$$

for some $\delta > 0$ small. Since $\Phi_n^{m,j} \rightharpoonup \Phi^{m,j}$ and $\Phi_n^{l,j} \rightharpoonup \Phi^{l,j}$ in $H^1(B_R(0))$, we have that $\Phi_n^{m,j} \to \Phi^{m,j}$ and $\Phi_n^{l,j} \to \Phi^{l,j}$ in $L^2(B_R(0))$ for all $R$. Up to a subsequence, assume that $(\frac{V(P_n^1)}{V(P_n^j)})^{\frac{N}{4}} \to c_j > 0$ as $n \to +\infty$, for all $j = 1, \dots, k$. Finally, by

$$0 = \frac{1}{(\varepsilon_n^1)^N}\int_\Omega \varphi_n^m\varphi_n^l = \sum_{j=1}^k \bigg(\frac{\varepsilon_n^j}{\varepsilon_n^1}\bigg)^N \int_{B_R(0)} \Phi_n^{m,j}\,\Phi_n^{l,j} + O\bigg(\int_{\mathbb{R}^N\setminus B_R(0)} e^{-2\,\gamma|y|}\mathrm{d}y\bigg) \tag{4.8}$$

we get as $n \to +\infty$

$$0 = \sum_{j=1}^{k} c_j^2 \int_{B_R(0)} \Phi^{m,j}\, \Phi^{l,j} + O\Big( \int_{\mathbb{R}^N \setminus B_R(0)} e^{-2\gamma|y|} \mathrm{d}y \Big) \quad \forall R.$$

As $R \to +\infty$ we get

$$0 = \sum_{j=1}^{k} c_j^2 \int_{\mathbb{R}^N} \Phi^{m,j}\, \Phi^{l,j}.$$

For $m = 1, \ldots, M$, either $\Phi^{m,j} = 0$ or $\Phi^{m,j} \neq 0$ is an eigenfunction of $-\Delta + 1 - p\, U_0^{p-1}$ with eigenvalue $\mu^{m,j} = \mu_1 < 0$. In both cases, we can write

$$\Phi^{m,j} = \frac{\lambda_{m,j}}{c_j (\int_{\mathbb{R}^N} \varphi_1^2)^{\frac{1}{2}}} \varphi_1$$

for some $\lambda_{m,j}$, getting in this way

$$0 = \sum_{j=1}^{k} \lambda_{m,j}\, \lambda_{l,j}.$$

Set $\lambda_m = (\lambda_{m,1}, \ldots, \lambda_{m,k})$ $\forall\, m = 1, \ldots, M$. By the 2nd claim we have that $\lambda_m \neq 0$ $\forall\, m = 1, \ldots, M$ and $< \lambda_m, \lambda_l >= 0$ $\forall\, m \neq l,\ m, l = 1, \ldots, M$. Hence $M \leq k$. Next we want to show that $S \leq N\, k$. Indeed, by (4.8) we always have that

$$\sum_{j=1}^{k} c_j^2 \int_{\mathbb{R}^N} \Phi^{m,j}\, \Phi^{l,j} = 0 \quad \forall\, m \neq l, \quad m, l = M+1, \ldots, M+S.$$

Since $\Phi^{m,j}$ and $\Phi^{l,j}$ are eigenfunctions of $-\Delta + 1 - p\, U_0^{p-1}$ with eigenvalue $\mu^{m,j} = \mu^{l,j} = 0$, we can write

$$\Phi^{m,j} = \sum_{i=2}^{N+1} \frac{\beta_{m,j}^i\, \varphi_i}{c_j (\int_{\mathbb{R}^N} \varphi_i^2)^{\frac{1}{2}}}, \quad \Phi^{l,j} = \sum_{i=2}^{N+1} \frac{\beta_{l,j}^i\, \varphi_i}{c_j (\int_{\mathbb{R}^N} \varphi_i^2)^{\frac{1}{2}}}.$$

In this way, the orthogonality conditions rewrite as

$$\sum_{j=1}^{k} \sum_{i=2}^{N+1} \beta_{m,j}^i\, \beta_{l,j}^i = 0$$

in view of $\int_{\mathbb{R}^N} \varphi_i\, \varphi_j = 0$ for $i \neq j$. We consider $\beta_m = (\beta_{m,1}^2, \beta_{m,2}^2, \ldots, \beta_{m,k}^2, \ldots, \beta_{m,1}^{N+1}, \ldots, \beta_{m,k}^{N+1})$. We have that $\beta_m \neq 0$ $\forall\, m = M+1, \ldots, M+S$ and $< \beta_m, \beta_l >= 0$ $\forall\, m \neq l$, $m, l = M+1, \ldots, M+S$. Hence $S \leq Nk$. In conclusion, by Theorem 3.2 we have that

$$\begin{aligned} \limsup_{n \to +\infty} \bar{m}(u_n) &\leq M + S \leq (N+1)k \\ &\leq (N+1)(\inf_{\Omega} V)^{\frac{N}{2} - \frac{p+1}{p-1}} \Big( \int_{\mathbb{R}^N} U_0^{p+1} \Big)^{-1} \lim_{n \to +\infty} \lambda_n^{\frac{N}{2} - \frac{p+1}{p-1}} \int_{\Omega} u_n^{p+1}. \end{aligned}$$

Since this is true for any sub-sequence so that $\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}$ converges, we deduce the validity of (4.2). $\square$

## 5. Non-Degeneracy Issues

As far as the characterization of $S$, let $u_n$ be a solutions sequence of (1.3) so that either (1.6) or (1.7) does hold. By Theorem 3.2 we find a sub-sequence and $k$ points $P_n^1, \dots, P_n^k$ so that (3.6)–(3.8) do hold. Up to a further sub-sequence, assume that $P_n^i \to P^i \in \overline{\Omega}$ as $n \to +\infty$. Notice that by (3.8) the set $S$ of blow-up points given in (1.8) is so that

$$S = \{P^1, \dots, P^k\}.$$

Letting

$$J_i = \{j = 1, \dots, k \, : \, P_n^j \to P^i\}, \quad I_\delta^i := B_\delta(P^i) \cap \Omega,$$

where $\delta > 0$ is small so that $I_\delta^i \cap \{P^1, \dots, P^k\} = \{P^i\}$, we have the following.

*Proof (of Theorem* 1.3*).* We need the following integral expansion.

*Claim:* Let $g$ be a some smooth function in $\Omega$. For $q > 1$ and $i \in \{1, \dots, k\}$, then

$$\lambda_n^{\frac{N}{2}-\frac{q}{p-1}} \int_{I_\delta^i} g\, u_n^q \to g(P^i) V(P^i)^{\frac{q}{p-1}-\frac{N}{2}} \int_{\mathbb{R}^N} U_0^q \,\mathrm{card}(J_i) \tag{5.1}$$

as $n \to +\infty$, where $\mathrm{card}(J_i)$ is the number of elements in $J_i$.

Let $d_n(x) := \min\{|x - P_n^i| : i = 1, \dots, k\}$. Given $R > 0$, for $n > n(R)$

$$\{d_n(x) \le R\,\varepsilon_n^1\} \subset \Omega \quad \text{and} \quad I_\delta^i \cap \{d_n(x) \le R\,\varepsilon_n^1\} = \cup_{j \in J_i} \{|x - P_n^j| \le R\,\varepsilon_n^1\}$$

in view of (3.6). Since by (3.8)

$$u_n^q \le C \lambda_n^{\frac{q}{p-1}} \sum_{j=1}^k e^{-q\gamma \lambda_n^{\frac{1}{2}} |x - P_n^j|}, \tag{5.2}$$

we have

$$\begin{aligned}
\int_{I_\delta^i} g\, u_n^q &= \int_{I_\delta^i \cap \{d_n(x) \le R\,\varepsilon_n^1\}} g u_n^q + \int_{I_\delta^i \cap \{d_n(x) \ge R\,\varepsilon_n^1\}} g u_n^q \\
&= \sum_{j \in J_i} \int_{\{|x - P_n^j| \le R\,\varepsilon_n^1\}} g u_n^q + O\bigg( \lambda_n^{\frac{q}{p-1}} \sum_{j=1}^k \int_{\{|x - P_n^j| \ge R\,\varepsilon_n^1\}} e^{-q\gamma \lambda_n^{\frac{1}{2}} |x - P_n^j|} \bigg) \\
&= \sum_{j \in J_i} (\varepsilon_n^j)^{-\frac{2q}{p-1}+N} \int_{\{|y| \le \frac{R\varepsilon_n^1}{\varepsilon_n^j}\}} g(\varepsilon_n^j y + P_n^j)(U_n^j)^q + O\bigg( \lambda_n^{\frac{q}{p-1}-\frac{N}{2}} \sum_{j=1}^k \int_{\{|y| \ge R\lambda_n^{\frac{1}{2}}\,\varepsilon_n^1\}} e^{-q\gamma|y|} \bigg).
\end{aligned}$$

Notice that

$$\frac{\varepsilon_n^1}{\varepsilon_n^j} = \left( \frac{V(P_n^j)}{V(P_n^1)} \right)^{\frac{1}{2}} \to \theta_j := \left( \frac{V(P^j)}{V(P^1)} \right)^{\frac{1}{2}} \quad \text{as } n \to +\infty.$$

Since $U_n^j \to U_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ for any $j = 1, \ldots, k$, we find that

$$\lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{q}{p-1}} \int_{I_\delta^i} g\, u_n^q = g(P^i)V(P^i)^{\frac{q}{p-1}-\frac{N}{2}} \int_{\{|y|\le R\}} U_0^q \,\mathrm{card}(J_i) + O\bigg(\sum_{j=1}^k \int_{|y|\ge R\delta} e^{-q\gamma|y|}\bigg)$$

for some $\delta > 0$ and all $R > 0$. Letting $R \to +\infty$, we then get the validity of (5.1).

For $P^i \in \Omega$, we combine (5.1) with the following Pohozaev identity [41]. Multiply the equation $-\Delta u_n = u_n^p - \lambda_n V\, u_n$ by $\partial_h u_n$ on $I_\delta^i = B_\delta(P^i)$ and integrate by parts to get

$$\frac{\lambda_n}{2}\int_{I_\delta^i} \partial_h V u_n^2 = \int_{\partial I_\delta^i}\left(\frac{1}{2}|\nabla u_n|^2 \nu_h - \partial_\nu u_n\, \partial_h u_n\right) + \int_{\partial I_\delta^i}\left(\frac{\lambda_n}{2}Vu_n^2 - \frac{u_n^{p+1}}{p+1}\right)\nu_h.$$

By (3.8) and elliptic estimates [29] we get that $\lambda_n u_n^2, u_n^{p+1}, |\nabla u_n| \to 0$ uniformly on $\partial I_\delta^i$ as $n \to +\infty$ so to provide

$$\lambda_n \int_{I_\delta^i} \partial_h V\, u_n^2 \to 0 \quad \text{as } n \to +\infty$$

for all $h, i$. By (5.1) we get that

$$\lambda_n^{\frac{N-2}{2}-\frac{2}{p-1}}\left(\lambda_n \int_{I_\delta^i} \partial_h V\, u_n^2\right) \to \partial_h V(P^i)V(P^i)^{\frac{2}{p-1}-\frac{N}{2}} \int_{\mathbb{R}^N} U_0^2 \,\mathrm{card}(J_i)$$

for all $h, i$. Since $\frac{N-2}{2} - \frac{2}{p-1} < 0$, we get that

$$\partial_h V(P^i) = 0 \quad \forall\, h, i.$$

In conclusion, for all $P^i \in \Omega$ we have $\nabla V(P^i) = 0$. For $P^i \in \partial\Omega$, we use a different Pohozaev identity. Multiply the equation $-\Delta u_n = u_n^p - \lambda_n V\, u_n$ by $(x - P^i + \nu(P^i)) \cdot \nabla u_n$ on $I_\delta^i$ and integrate by parts to get

$$\begin{aligned}
&\frac{\lambda_n}{2}\int_{I_\delta^i} (x - P^i + \nu(P^i)) \cdot \nabla V u_n^2 \\
&\quad = \int_{\partial I_\delta^i}\left[\frac{1}{2}|\nabla u_n|^2 (x - P^i + \nu(P^i)) \cdot \nu - \partial_\nu u_n (x - P^i + \nu(P^i)) \cdot \nabla u_n - \frac{N-2}{2}\partial_\nu u_n u_n\right] \\
&\qquad + \int_{\partial I_\delta^i}\left(\frac{\lambda_n}{2}Vu_n^2 - \frac{u_n^{p+1}}{p+1}\right)(x - P^i + \nu(P^i)) \cdot \nu \\
&\qquad + \left(\frac{N}{p+1} - \frac{N-2}{2}\right)\int_{I_\delta^i} u_n^{p+1} - \lambda_n \int_{I_\delta^i} Vu_n^2.
\end{aligned}$$

Notice that the boundary contribution simply reduces to

$$-\frac{1}{2}\int_{B_\delta(P^i)\cap\partial\Omega} (\partial_\nu u_n)^2 (x - P^i + \nu(P^i)) \cdot \nu + o_n(1)$$

where $o_n(1) \to 0$ as $n \to +\infty$. For $\delta > 0$ small we also have that

$$(x - P^i + \nu(P^i)) \cdot \nu(x) \ge 0$$

for all $x \in B_\delta(P^i) \cap \partial\Omega$. The Pohozaev identity reduces to

$$\frac{\lambda_n}{2}\int_{I_\delta^i}(x - P^i + \nu(P^i)) \cdot \nabla V u_n^2 \le o_n(1) + \left(\frac{N}{p+1} - \frac{N-2}{2}\right)\int_{I_\delta^i} u_n^{p+1} - \lambda_n \int_{I_\delta^i} V u_n^2.$$

Multiplying by $\lambda_n^{\frac{N-2}{2}-\frac{2}{p-1}} \to 0$ and using (5.1) in the limit we get

$$\begin{aligned}&\frac{1}{2}\partial_\nu V(P^i)V(P^i)^{\frac{2}{p-1}-\frac{N}{2}}\int_{\mathbb{R}^N} U_0^2\,\mathrm{card}(J_i)\\ &\quad\le V(P^i)^{\frac{p+1}{p-1}-\frac{N}{2}}\,\mathrm{card}(J_i)\left[\left(\frac{N}{p+1} - \frac{N-2}{2}\right)\int_{\mathbb{R}^N} U_0^{p+1} - \int_{\mathbb{R}^N} U_0^2\right].\end{aligned}$$

By the exponential decay of $U_0$ and $|\nabla U_0|$ at infinity, the same Pohozaev identity does hold for $U_0$ on the whole $\mathbb{R}^N$ (no boundary terms):

$$\left(\frac{N}{p+1} - \frac{N-2}{2}\right)\int_{\mathbb{R}^N} U_0^{p+1} - \int_{\mathbb{R}^N} U_0^2 = 0.$$

In conclusion, for all $P^i \in \partial\Omega$ we have $\partial_\nu V(P^i) \le 0$. □

As an application of the characterization of the blow-up set $S$, we address now non degeneracy issues as stated in Theorem 1.4. Assumption (1.10) or (1.11) on $u_n$ ensures that $\bar{k} < 2$ in Theorem 3.2 (along any subsequence so that $\bar{k}$ exists). Letting $P_n$ be a maximum point of $u_n$: $u_n(P_n) = \max_\Omega u_n$, by (1.4) we have that Theorem 3.2 does hold with $k = 1$ and $P_n^1 = P_n$. Thanks to Theorem 1.4, assumption (1.9) on the potential $V$ guarantees that $S = \{P\}$, where $P = \lim_{n\to+\infty} P_n \in \Omega$ is a c.p. of $V$ with $\det D^2V(P) \neq 0$.

We adopt the same notations of Section 4. Let $\{\mu_n^m\}$ be the eigenvalues (counted with multiplicities) of $L_n = -\Delta + \lambda_n V - p u_n^{p-1}$ which are $\le 1$, and let $\varphi_n^m$ be the corresponding normalized eigenfunction. Up to a subsequence, from the analysis in Section 4 we know that $\lambda_n^{-1}V(P_n)^{-1}\mu_n^m \to \mu^m$, where $\mu^m = \mu_1 < 0$ or $\mu^m = 0$ according to Theorem 1.1. Assume that $\mu^m = \mu_1$ for $m = 1, \dots, M$ and $\mu^m = 0$ for $m = M+1, \dots, M+S$, and let $J \subset \{M+1, \dots, M+S\}$ be maximal so that $\mu_n^m \to 0$ as $n \to +\infty$ for all $m \in J$. The aim is now to show that $J = \emptyset$ so to provide that the eigenvalues $\mu_n^m$ can never approach zero, and in particular $u_n$ is a non-degenerate solution of (1.3).

We are left with proving $J = \emptyset$. We will use an integral representation (5.3) for $\mu_n^m$, which has revealed powerful [30] in dealing with non-degeneracy issues for problems with critical growth. The property $J = \emptyset$ will follow from the following claim and the assumption $\det D^2V(P) \neq 0$ in (1.9):

*Claim:* Let $\mu_n$ be an eigenvalue of $L_n$ so that $\lambda_n^{-1}\mu_n \to 0$ as $n \to +\infty$. Then the following expansion does hold:

$$\mu_n = N\frac{D^2V(P)[a,a]}{2V(P)|a|^2}\frac{\int_{\mathbb{R}^N} U_0^2}{\int_{\mathbb{R}^N}|\nabla U_0|^2} + o(\mu_n) + o(1),$$

for some $a \in \mathbb{R}^N$, $a \neq 0$.

If $V$ has just one critical point $P$, by (1.9) we get that $P$ is necessarily the global minimum point of $V$ in $\overline{\Omega}$ and $D^2V(P) > 0$. Letting $S' \le S$ be maximal so that $\mu_n^m \le 0$ for $m = M+1, \ldots, M+S'$, by the claim we get that $S' = 0$ and, by the proof of Theorem 4.3, $M \le k = 1$. Then by Theorem 3.2

$$\limsup_{n\to+\infty} m(u_n) = M \le (\inf_\Omega V)^{\frac{N}{2}-\frac{p+1}{p-1}} \left( \int_{\mathbb{R}^N} U_0^{p+1} \right)^{-1} \lim_{n\to+\infty} \lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}.$$

Since this is true for any sub-sequence so that $\lambda_n^{\frac{N}{2}-\frac{p+1}{p-1}} \int_\Omega u_n^{p+1}$ converges, and by the use of Remark 4.2, we deduce the validity of (1.13), yielding to a complete proof of Theorem 1.4.

*(Proof of the claim).* Let $\varphi_n$ be the corresponding eigenfunction. By differentiating (1.3) notice that the partial derivative $\partial_i u_n$ solves

$$L_n(\partial_i u_n) = -\lambda_n \partial_i V u_n.$$

Multiply it by $\varphi_n$ and integrate on $\Omega$ to get

$$\mu_n \int_\Omega \varphi_n \partial_i u_n = -\lambda_n \int_\Omega \partial_i V u_n \varphi_n - \int_{\partial\Omega} \partial_\nu \varphi_n \partial_i u_n. \tag{5.3}$$

By elliptic regularity theory, the boundedness of $\lambda_n^{-1}\mu_n$ and the estimates (3.8), (4.6), we get that $u_n$, $\phi_n$ and their derivatives up to order two tend uniformly to zero faster than any power of $\varepsilon_n := \lambda_n^{-\frac{1}{2}} V(P_n)^{-\frac{1}{2}}$ in a small neighborhood of $\partial\Omega$. In particular, we get that

$$\int_{\partial\Omega} \partial_\nu \varphi_n \partial_i u_n = o\left( (\varepsilon_n)^{N-\frac{p+1}{p-1}} \right)$$

as $n \to +\infty$. Moreover, arguing as in the proof of (3.8) by the two following estimates

$$|L_n(\partial_i u_n)| = |\lambda_n \partial_i V u_n| \le C\lambda_n^{\frac{p}{p-1}} e^{-\gamma\lambda_n^{\frac{1}{2}}|x-P_n|} \quad \text{in } \Omega$$
$$|\partial_i u_n| \le C\lambda_n^{\frac{p}{p-1}} e^{-\gamma'\lambda_n^{\frac{1}{2}}} \quad \text{on } \partial\Omega$$

one can easily deduce that

$$|\partial_i u_n| \le C'\lambda_n^{\frac{1}{p-1}+\frac{1}{2}} e^{-\gamma''\lambda_n^{\frac{1}{2}}|x-P_n|} \tag{5.4}$$

in $|x-P_n| \ge R\lambda_n^{-\frac{1}{2}}$, where $C, C'$ are positive constants, $\gamma'' > 0$ is small and $R$ large. Setting $U_n(y) = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n y + P_n)$ and $\Omega_n = \frac{\Omega - P_n}{\varepsilon_n}$, we have that (5.4) gives that

$$|\partial_i U_n| \le C' e^{-\gamma''|y|} \quad \text{in } |y| \ge R, \tag{5.5}$$

and a similar estimate does hold also for the second derivative of $U_n$. Re-write (5.3) in $\Omega_n$ to get:

$$\mu_n \int_{\Omega_n} \Phi_n \partial_i U_n = -\varepsilon_n^{-1} \int_{\Omega_n} \Phi_n \frac{\partial_i V(\varepsilon_n y + P_n)}{V(P_n)} U_n + o(1) \tag{5.6}$$

where $\Phi_n(y) := \varphi_n(\varepsilon_n y + P_n)$. Since we assume that $\varepsilon_n^2 \mu_n \to 0$ as $n \to +\infty$, by Section 4 recall that $\Phi_n \rightharpoonup \sum_{k=1}^N a_k \partial_k U_0$ in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ and a.e. as $n \to +\infty$, for some $a_1, \ldots, a_N$. Assuming that $B_{2\delta}(P) \subset \Omega$, let $\chi$ be a smooth cut-off function so that $0 \le \chi \le 1$, $\chi = 1$ in $B_\delta(P)$ and $\chi = 0$ in $\mathbb{R}^N \backslash B_{2\delta}(P)$. Introduce $\Delta_n = \Phi_n - \chi(\varepsilon_n y) \sum_{k=1}^N a_k^n \partial_k U_n$, where the coefficients $a_k^n$ are uniquely determined to have $\int_{\Omega_n} \Delta_n \partial_l U_n = 0$ for every $l = 1, \ldots, N$. Indeed, we can re-write these orthogonality conditions as

$$0 = \int_{\Omega_n} \Delta_n \partial_l U_n = \int_{\Omega_n} \Phi_n \partial_l U_n - \sum_{k=1}^N a_k^n \int_{\Omega_n} \chi(\varepsilon_n y) \partial_k U_n \partial_l U_n.$$

By Lebesgue Theorem and (5.5) we get that

$$\int_{\Omega_n} \chi(\varepsilon_n y) \partial_k U_n \partial_l U_n \to \int_{\mathbb{R}^N} \partial_k U_0 \partial_l U_0 = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla U_0|^2 \, \delta_{kl}$$

and

$$\int_{\Omega_n} \Phi_n \partial_l U_n \to \sum_{k=1}^N a_k \int_{\mathbb{R}^N} \partial_k U_0 \partial_l U_0 = \frac{a_l}{N} \int_{\mathbb{R}^N} |\nabla U_0|^2$$

as $n \to +\infty$, where $\delta_{kl}$ are the Kronecker's symbols. The coefficients $a_k^n$ have then to satisfy

$$\frac{a_l^n}{N} \int_{\mathbb{R}^N} |\nabla U_0|^2 + o\left( \sum_{k=1}^N |a_k^n| \right) = \int_{\Omega_n} \Phi_n \partial_l U_n,$$

which is a small perturbation of an uniquely solvable system. Then, the coefficients $a_k^n$ are uniquely determined and satisfy $a_k^n \to a_k$ as $n \to +\infty$. The function $\Delta_n$ solves the equation

$$\begin{cases} \widehat{L}_n \Delta_n = f_n & \text{in} \Omega_n \\ \Delta_n = 0 & \text{on } \partial\Omega_n, \end{cases}$$

where $\widehat{L}_n = -\Delta + \frac{V(\varepsilon_n y + P_n)}{V(P_n)} - pU_n^{p-1}$ and $f_n$ is defined as

$$f_n(y) = \varepsilon_n^2 \mu_n \Phi_n + \varepsilon_n^2 \Delta\chi(\varepsilon_n y) \sum_{k=1}^N a_k^n \partial_k U_n + 2\varepsilon_n \nabla\chi(\varepsilon_n y) \sum_{k=1}^N a_k^n \nabla(\partial_k U_n)$$
$$+ \varepsilon_n \chi(\varepsilon_n y) \sum_{k=1}^N a_k^n \frac{\partial_k V(\varepsilon_n y + P_n)}{V(P_n)} U_n.$$

Due to the orthogonality conditions $\int_{\Omega_n} \Delta_n \partial_l U_n = 0$ for all $l = 1, \ldots, N$, it is classical to show (see for example Proposition 6.1 in [45] and also [32, 46]) that

$$\|\Delta_n\|_{H^2(\Omega_n)} \le C \|f_n\|_{L^2(\Omega_n)}$$

for a suitable constant $C > 0$. By $\partial_k V(\varepsilon_n y + P_n) = \partial_k V(P_n) + O(\varepsilon_n |y|)$ and $\partial_k V(P_n) \to \partial_k V(P) = 0$ as $n \to +\infty$ we compute now

$$\|f_n\|_{L^2(\Omega_n)} = O(\mu_n \epsilon_n^2) + o(\varepsilon_n)$$

in view of (3.8), (4.6) and (5.5). Then we get that

$$\|\Delta_n\|_{H^2(\Omega_n)} = O(\mu_n \epsilon_n^2) + o(\varepsilon_n). \tag{5.7}$$

We use now (5.7) to get an expansion of (5.6). First, by Lebesgue Theorem and (3.8), (5.7) we have that

$$\begin{aligned}
&\int_{\Omega_n} \Phi_n \frac{\partial_i V(\varepsilon_n y + P_n)}{V(P_n)} U_n \\
&= \frac{1}{2} \sum_{k=1}^N a_k^n \int_{\Omega_n} \chi(\varepsilon_n y) \partial_k (U_n^2) \frac{\partial_i V(\varepsilon_n y + P_n)}{V(P_n)} + \int_{\Omega_n} \Delta_n \frac{\partial_i V(\varepsilon_n y + P_n)}{V(P_n)} U_n \\
&= -\frac{\varepsilon_n}{2} \sum_{k=1}^N a_k^n \int_{\Omega_n} \chi(\varepsilon_n y) U_n^2 \frac{\partial_{ik} V(\varepsilon_n y + P_n)}{V(P_n)} + O(\mu_n \epsilon_n^2) + o(\varepsilon_n) \\
&= -\frac{\varepsilon_n}{2V(P)} \sum_{k=1}^N a_k \partial_{ik} V(P) \int_{\mathbb{R}^N} U_0^2 + O(\mu_n \varepsilon_n^2) + o(\varepsilon_n).
\end{aligned}$$

Secondly, by Lebesgue Theorem and (4.6), (5.5) we get that

$$\int_{\Omega_n} \Phi_n \partial_i U_n = \sum_{k=1}^N a_k \int_{\mathbb{R}^N} \partial_k U_0 \partial_i U_0 + o(1) = \frac{a_i}{N} \int_{\mathbb{R}^N} |\nabla U_0|^2 + o(1).$$

Therefore, (5.6) re-writes as

$$\mu_n \frac{a_i}{N} \int_{\mathbb{R}^N} |\nabla U_0|^2 = \frac{1}{2V(P)} \sum_{k=1}^N a_k \partial_{ik} V(P) \int_{\mathbb{R}^N} U_0^2 + o(\mu_n) + o(1).$$

We multiply it by $a_i$ and sum over $i = 1, \dots, N$ to get

$$\mu_n \frac{|a|^2}{N} \int_{\mathbb{R}^N} |\nabla U_0|^2 = \frac{1}{2V(P)} D^2 V(P)[a, a] \int_{\mathbb{R}^N} U_0^2 + o(\mu_n) + o(1),$$

where $a = (a_1, \dots, a_n) \neq 0$. The Claim is established. $\square$

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