A nonlinear Lyapunov-Schmidt reduction and multiple solutions for some semilinear elliptic equation

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Abstract

We present a finite dimensional reduction for perturbed variational functionals and discuss some nonlinear elliptic PDE with Sobolev critical growth in bounded domains.

1 Introduction

We will present in this talk a general variational principle for perturbative problems in presence of a manifold of ”quasi critical points” for the unperturbed energy functional. A model problem is the following:

\[
(P) \begin{cases}
-\Delta u = u^p + f(\delta, x, u) & \text{in } \Omega \subset \mathbb{R}^N, \ N \geq 3 \\
u = 0 & \text{on } \partial \Omega \\
u > 0 & \text{in } \Omega
\end{cases}
\]

where \(p = \frac{N+2}{N-2}\) is the limiting Sobolev exponent for the immersion of \(H^1_0(\Omega)\) in \(L^q(\Omega)\), \(q \geq 1\). Here \(f(\delta, x, u)\) is a perturbation term, small if \(\delta\) is small, satisfying the growth condition

\[\exists \ c > o : |f(\delta, x, u)| \leq c(1 + |u|^p).\]

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For \( f(\delta, x, u) = \delta u \) and \( 0 < \delta < \lambda_1(\Omega) \), precise existence results for (P) were established in [9]; existence of multiple solutions and asymptotic behaviour for \( \delta \to 0^+ \) were discussed in [16], [17]. Problem (P) can be seen as the stationary equation for some variational functional \( E_\delta(u) := E(u) - G(\delta, u) \), where, for \( \delta \) small, \( G(\delta, u) \) is a small perturbation of the unperturbed energy \( E(u) \). Problem (P) fits into a general framework which we will describe in the next section (see [14] and [15] for some details).

2 A finite dimensional reduction

We consider a functional \( E_\delta(u) = E(u) - G(\delta, u) \), \( u \in H \) on some Hilbert space \( H \). We assume for \( E(u) \) the existence of a smooth manifold \( Z \subset H \) in the form

\[
Z = \{ z(\epsilon, y) : \epsilon > 0, y \in U \subset \mathbb{R}^N \},
\]

\( z \) a smooth parametrization, such that

\[(A1) \quad \| \nabla E(z(\epsilon, y)) \| \to_{\epsilon \to 0} 0 \text{ uniformly on compact subsets of } U.\]

We will refer to \( Z \) as "an almost critical manifold" for \( E(u) \). We require for \( Z \) some nondegeneracy property. Denoted by \( T_z \) the tangent space to \( Z \) at \( z \in Z \), let \( \pi_z : H \to T_z \) and \( \pi_z^\perp := Id - \pi_z \) be orthogonal projections, and set \( L_z := \pi_z^\perp E''(z) \mid_{T_z^\perp} \). We assume

\[(A2) \quad L_z \in Iso(T_z^\perp, T_z^\perp) \quad \text{and} \quad \sup_{z \in Z} \| L_z^{-1} \| < \infty.\]

We will also need a smallness assumption on \( G(\delta, u) \)

\[(A3) \quad G(\delta, u) \to_{\delta \to 0} 0 \text{ in } C^2_{\text{loc}}.\]

While looking for critical points of \( E_\delta \) close to \( Z \), one can perform a nonlinear Lyapunov-Schmidt reduction: given \( U_0 \subset \subset U \), there exist \( \bar{\epsilon} > 0 \), \( \bar{\delta} > 0 \) and a smooth map \( (\delta, z) \to w(\delta, z) \), \( z \in Z_0 := \{ z(\epsilon, y) : (\epsilon, y) \in (0, \bar{\epsilon}) \times U_0 \} \), \( \delta \leq \bar{\delta} \), such that

\[(i) \quad w(\delta, z) \in T_z^\perp \quad \forall \delta, z \quad \text{and} \quad \| w \| = O(\| \nabla E_\delta(z) \|)\]

\[(ii) \quad \pi_z^\perp \nabla E_\delta(z + w(\delta, z)) = 0, \quad \forall \delta, z.\]
It remains to solve the "bifurcation equation"

\[(b) \quad \pi_z \nabla E_\delta (z + w(\delta, z)) = 0, \quad z \in Z_0.\]

If \( Z \) is linear, equation \((b)\) is known to have a variational structure: it is equivalent to

\[(b)’ \quad \frac{\partial}{\partial z} E_\delta (z + w(\delta, z)) = 0, \quad z \in Z_0.\]

This is because \( \frac{\partial w}{\partial z}(\delta, h) \in Z^1 \quad \forall h \in Z \), in this case. Actually, to have equivalence between \((b)\) and \((b)’\) it is enough that \( \|\pi_z \frac{\partial w}{\partial z}\| = O(\|w\|). \) The argument goes as follows. Let \( z(t) \) be a smooth curve on \( Z_0 \), with \( z(0) = z_0 \) and \( \dot{z}(0) = \pi_{z_0} \nabla E_\delta (z_0 + w(\delta, z_0)) \). By assumption,

\[
0 = \frac{d}{dt} E_\delta (z(t) + w(\delta, z(t))) \bigg|_{t=0} = < \nabla E_\delta (z_0 + w(\delta, z_0)), \dot{z}(0) > + \frac{\partial w}{\partial z}(\delta, z_0) \dot{z}(0) > .
\]

Since \( \pi_{z_0} \nabla E_\delta (z_0 + w(\delta, z_0)) = 0 \), using the estimate for \( \|w\| \) we get

\[
|\dot{z}(0)|^2 \leq |\dot{z}(0)|^2 |\pi_{z_0} \frac{\partial w}{\partial z}(\delta, z_0)| \leq c \|w(\delta, z_0)\| |\dot{z}(0)|^2 \leq c |\dot{z}(0)|^2 \|\nabla E_\delta (z(0))\|
\]

and hence \( \dot{z}(0) = 0 \) because \( |\nabla E_\delta(z)| < 1 \) if \( \epsilon, \delta \) are small, by (A1)-(A3). In turn, the estimate for \( \|\pi_z \frac{\partial w}{\partial z}\| \) involves the variation of \( T_\delta \), and in fact it holds true assuming:

\[(A4) \quad \exists c > 0 : \quad \|\pi_z \frac{\partial w}{\partial z} (\pi_z^* v)\| \leq c \|\pi_z^* v\|, \quad \forall v \in H .\]

Let us derive from \((A4)\) the estimate for \( \|\pi_z \frac{\partial w}{\partial z}\| \). Let \( \bar{w} = w(\delta, \bar{z}) \) for some \( \bar{z} \in Z_0, \delta \) fixed.

From \( \pi_z w(\delta, z) \equiv 0 \) it follows \( \pi_z \frac{\partial w}{\partial z} = -\frac{\partial}{\partial z}(\pi_z \bar{w}) \) at \( z = \bar{z} \). Since \( -\frac{\partial}{\partial z}(\pi_z \bar{w}) = \frac{\partial}{\partial z}(\pi_z^* \bar{w}) \), we have, by \((A4)\),

\[
\|\pi_z \frac{\partial w}{\partial z}(\delta, \bar{z})\| \leq c \|\pi_z^* \bar{w}\|.
\]

This gives the desired estimate, because \( \pi_z^* \bar{w} = \bar{w} \). Relevant informations can be derived by the variational structure of \((b)\). After a Taylor expansion, \((b)’\) rewrites as

\[
\text{stat}_z \left[ E(z(\epsilon, y)) - G(\delta, z(\epsilon, y)) + O(\|\nabla E_\delta(z)\|^2) \right]
\]

and this leads to look for "stable" critical points of the "Melnikov function"

\[
\epsilon, y \rightarrow E(z(\epsilon, y)) - G(\delta, z(\epsilon, y)) .
\]

Of course, estimates for the "error term" \( O(\|\nabla E_\delta(z)\|^2) \), either in \( L^\infty \) or in \( C^1 \), are crucial.
Remark 2.1 1. Similar ( somehow different) procedure has been used by Rey, and then by many others for (P) in the perturbative case ( see, to quote a few, [1], [8].....). 2. Similar principle, but in case Z is a critical manifold ( i.e. $\nabla E(z) = 0$ for any $z \in Z$), goes back to Ambrosetti, Coti Zelati and Ekeland in [4] ( for compact Z) and to Ambrosetti and Badiale in [3] ( for non compact Z: problems with lack of compactness). Recent advances in much more complicated situations are due to Ambrosetti, Malchiodi and Ni in [5] ( solutions concentrating on codimension-1 manifolds for singularly perturbed problems). See also [6].

3 Applications to problem (P)

We consider the functional $E_\delta(u) = E(u) - G(\delta, u)$, $u \in H^1_0(\Omega)$, associated to problem (P), where

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega} |u|^{p+1}.$$ 

Here, the manifold Z is given by

$$Z = \{ PU_{\epsilon,y} : \epsilon > 0, y \in \Omega \},$$

where $P : H^1(\Omega) \to H^1_0(\Omega)$ is the orthogonal projection and

$$U_{\epsilon,y}(x) = \epsilon^{-\frac{N-2}{2}} U\left(\frac{x - y}{\epsilon}\right) \quad U(x) = \frac{c_N}{(1 + |x|^2)^{\frac{N-2}{2}}}$$

$c_N = [N(N-2)]^{\frac{N-2}{4}},$ are the extremal functions for the Sobolev inequality on $\mathbb{R}^N$.

Assumptions (A1)-(A2)-(A3) are satisfied ( see [17] and [15] for detailed proofs). We will consider two distinguished problems of type (P):

\[
\begin{align*}
\text{(NHD)} & \quad \begin{cases} 
-\Delta u = u^p & \text{in } \Omega \\
 u = \delta \varphi & \text{on } \partial \Omega \end{cases} \\
\varphi & \in C^1(\partial \Omega) \text{ positive somewhere}
\end{align*}
\]

\[
\begin{align*}
\text{(PSCE)} & \quad \begin{cases} 
-\Delta u = (1 + \delta a(x)) u^p & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega \\
 u > 0 & \text{in } \Omega.
\end{cases}
\end{align*}
\]

For problem (NHD), the existence of a small solution ( $u_\delta \to 0$ in $H^1(\Omega)$ as $\delta \to 0$) holds in much greater generality ( see [13]). The existence of a "large" solution for $\delta < \delta^*$, $\delta^*$ some positive number, is due, among other things, to Caffarelli and Spruck [10]. Some multiplicity results for problem (NHD) with $\delta$ small have been obtained by Rey [18].
We present here some improvements (a detailed proof is in [15]). Let $h(x)$ be the harmonic extension of $\varphi(x)$ and let $\psi(x, y)$ be the regular part of the Green function of $\Omega$. Set

$$K(x) = \frac{h^2(x)}{H(x)} \cdot H(x) = \psi(x, x).$$

Due to different estimates for the error term in dimension $N = 3$ and $N > 3$, we have the following

**Theorem 3.1** (i) Let $N \geq 3$. Let $C$ be a compact subset of $\Omega$ with $h(x) > 0$ on $C$ and $\max_{C'} K < \max_{C} K$. Then, for $\delta_j \to 0$ there exist $u_j$ solutions and $x \in C$ such that

$$|\nabla u_j|^2 \to S^2 \delta x$$

($S :=$ best Sobolev constant)

(ii) Let $N \geq 4$. Then the number of "large" solutions is greater (or equal) than the number of non degenerate critical points of $K$ with $h > 0$.

According to (i), highly oscillating boundary datas should give rise to a large number of solutions:

**Example 3.1** If $\varphi = \sum_{j=1}^m \frac{1}{|x - tx_j|^{N-2}}, x \in \Omega = B_1$ with $\{x_1, \ldots, x_m\} \subset \partial B_1$ and $1 < t < t(N, x_1, \ldots, x_m)$, problem (NHD) with boundary data $\delta \varphi$ admits for $\delta$ small at least $m$ solutions.

As for (PSCE), we recall that, if $\Omega = \mathbb{R}^N$, it is the prescribed scalar curvature problem for $S^N$. Known results, (see [11], [12]), recently improved by Ambrosetti, Azorero and Peral [2], are variations of the following basic one: assume that $a \in C^2$ has only a finite number of non degenerate critical points $x_j$ and

$$\Delta a(x_j) \neq 0 \forall j, \sum_{\Delta a(x_j) > 0} i(\nabla a, x_j) \neq 0$$

$$(*) \quad < \nabla a(x), x > < 0 \text{ for } |x| >> 1.$$  

Then (PSCE) is solvable. We recall that the "counting condition" (c.c.) was first introduced by Bahri-Coron in [7].

If $\Omega$ is a bounded domain in $\mathbb{R}^N$, different kind of results can be derived (see, e.g. [15]).

**Theorem 3.2** Let $N > 6$. Then, for $\delta$ small, the number of solutions for (PSCE) is greater (or equal) than the number of isolated critical points of $a$ with positive laplacian and non zero topological index.

**Remark 3.3** (i) If $< \nabla a(x), x > \leq 0$ for any $x \in \Omega$ and $\Omega$ is starshaped, no solution can exist.

(ii) There are no "ground state" solutions: all the solutions we find have Morse index greater or equal than 2.
The above results are obtained by an analysis of the corresponding "Melnikov functions". For the unperturbed energy along the manifold Z, we have that, for \( y \in K \subset \Omega \) compact:

\[
E(PU_{\epsilon,y}) = \frac{S_N^2}{N} + \alpha_N H(y)\epsilon^{N-2} + O(\epsilon^{N-1}), \quad \alpha_N = \frac{c_N^{p+1}}{2} \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{N/2}}.
\]

We first consider \((NHD)\), for which the functional \( G(\delta, u) \) turns out to be equal to

\[
G(\delta, u) = \frac{1}{p+1} \int_{\Omega} |u + \delta h|^{p+1} - |u|^{p+1}.\]

By Taylor formula, and for every \( N \geq 3 \), we have

\[
E_\delta(PU_{\epsilon,y} + w) = \frac{1}{N} S_N^2 + \alpha_N H(y)\epsilon^{N-2} - \frac{2\alpha_N}{c_N} h(y)\delta \epsilon^{N-2} + O(\epsilon^{N-1} + \delta\epsilon^{N} + \delta^{p+1}).
\]

After a change of variables \( \epsilon = (\theta\delta)^{\frac{1}{N-2}} \), we are lead to look for critical points of

\[
M_\delta(\theta, y) = c_N \theta^2 H(y) - 2\theta h(y) + \delta \frac{2}{N-2} O(1 + \theta^2(\frac{N-1}{N-2})).
\]

As for part i) in Theorem 3.1, we just remark that the above \( L^\infty \) estimate implies stability of strict local minima of \( M(\theta, y) := c_N \theta^2 H(y) - 2\theta h(y); \) on the other hand

\[
\max_{\partial C} \frac{h^2}{H} < \max_{C} \frac{h^2}{H} \Rightarrow \min_{[\theta^-, \theta^+]} M < \min_{\partial([\theta^-, \theta^+] \times C)} M
\]

where \( 0 < \theta^- < \min_{C} \frac{h_{\gamma \in \Omega}}{c_N H(y)}, \theta^+ > \max_{C} \frac{h_{\gamma \in \Omega}}{c_N H(y)}. \)

We now prove this claim. First, \( \theta(y) := \frac{h(y)}{c_N H(y)} \in (\theta^-, \theta^+) \) is the absolute minimizer of \( \theta \rightarrow M(\theta, y), y \in C. \) Also,

\[
M(\theta(y), y) = -\frac{1}{c_N} K(y) \quad \forall y \in \Omega.
\]

So, by assumption, \( \exists \gamma > 0: \)

\[
\min_{\partial C} M(\theta(y), y) \geq \gamma + \min_{C} M(\theta(y), y) \geq \gamma + \min_{[\theta^-, \theta^+]} M.
\]
Finally, set \( \min_C M(\theta, y) = M(\bar{\theta}, \bar{y}) \); since \( M(\theta, y) > M(\bar{\theta}, \bar{y}) \geq \min_{[\theta^- \theta^+]} M \), and similarly for \( \theta^+ \), the claim is proved.

Part ii) in Theorem 3.1 follows from

\[
\text{Let us prove this fact. As noticed above, } \frac{\partial M}{\partial y}(\theta, y) = \frac{\partial M}{\partial y}(\bar{\theta}, \bar{y}) = \frac{1}{c_n} \nabla K(y). \text{ Thus } (\theta(0), y, 0) \text{ is a critical point of } M \text{ because } \nabla K(y) = 0. \text{ We just have to check that it is nondegenerate. Using } \frac{\partial M}{\partial y}(y_0) = 2 \frac{H(y_0)}{\nabla h(y_0)} \frac{\partial h}{\partial y}(y_0) \text{ and the above relations, we find}
\]

\[
D^2 M(\theta(y_0), y_0) = \begin{pmatrix}
2c_n H(y_0) & 2\nabla h(y_0) \\
2\nabla h^t(y_0) & 2c_n H(y_0) \\
\end{pmatrix} \nabla h^t(y_0) \nabla h(y_0) - \frac{1}{c_n} D^2 K(y_0), \text{ and hence}
\]

\[
D^2 M(\theta(y_0), y_0) = \begin{pmatrix}
2c_n H(y_0) & 2\nabla h(y_0) \\
2\nabla h^t(y_0) & 2c_n H(y_0) \\
\end{pmatrix} \nabla h^t(y_0) \nabla h(y_0) - \frac{1}{c_n} D^2 K(y_0)
\]

Thus \( 0 = D^2 M(\theta_0, y_0)(\tau, \eta), \tau \in \mathbf{R}, \eta \in \mathbf{R}^p \) implies

\[
2c_n H(y_0) + 2 < \nabla h(y_0), \eta > = 0
\]

\[
2\nabla h(y_0) + \frac{2}{c_n H(y_0)} \nabla h(y_0) < \nabla h(y_0), \eta > - \frac{1}{c_n} D^2 K(y_0) \eta = 0.
\]

Inserting the first relation in the second one, we find

\[
H(y_0) D^2 K(y_0) \eta = \nabla h(y_0)[2c_n H(y_0) \tau + 2 < \nabla h(y_0), \eta >] = 0
\]
and hence \( \eta = 0 \) because \( D^2K(y_0) \) is invertible. This also implies \( \tau = 0 \), i.e. \( D^2M(\theta_0, y_0) \) is invertible.

Similarly, for problem (PSCE), we have that

\[
G(\delta, u) = \frac{\delta}{p+1} \int_\Omega |u|^{p+1} + \frac{S}{p+1} a(y) \delta + \frac{S}{4N} \Delta a(y) \delta^2 + o(h.o.t.)
\]

and the corresponding "Melnikov function" is

\[
(\epsilon, y) \rightarrow \frac{S}{N} + \alpha_N H(y) \epsilon^{N-2} - \frac{S}{p+1} a(y) \delta - \frac{S}{4N} \Delta a(y) \delta^2 + h.o.t..
\]

As above, we can define \( \theta := \frac{\epsilon}{\alpha_N H(y)} \) and, after a change of variable, we look for "stable" critical points of

\[
(\theta, y) \rightarrow -2(N-2)S^2 a(y) + \delta \frac{\alpha_N}{\alpha_N H(y)} H(y) y \theta^{N-2} - S^2 \Delta a(y) \theta^2
\]

( since the error term is of order \( O(\delta) \), the assumption \( N > 6 \) is needed to ensure that it is a fortiori of order \( o(\delta^{\frac{2}{3}}) \) ). Equivalently, we may look for zeros of the vector field

\[
\frac{\partial M_\delta}{\partial \theta} = -\frac{\delta}{\alpha_N} \left[ 4N(N-2)\alpha_N H(y) \theta^{N-3} - 2S^2 \Delta a(y) \theta + o(1) \right]
\]

\[
\frac{\partial M_\delta}{\partial y} = \frac{S}{p+1} \nabla a(y) - \frac{\delta}{\alpha_N} \left[ 4N \alpha_N \nabla H(y) \theta^{N-2} - S^2 \nabla \Delta a(y) \theta^2 + o(1) \right].
\]

We want to show that, if \( y_0 \) is an isolated critical point of \( a \) with positive laplacian and non zero topological index, then, for \( \delta \) small, \( \nabla M_\delta \) has a zero close to \( (\theta_0, y_0) \)

\[
\theta_0 := \theta(y_0) = \left( \frac{\delta \nabla a(y_0)}{N(N-2)DH(y_0)} \right)^{\frac{1}{N-2}}.
\]

To see this, we just consider the homothpy \( \Phi = (\Phi_1, \Phi_2) \), where

\[
\Phi_1(t; \theta, y) := -t \frac{\delta^2}{4N} \left[ 4N(N-2)\alpha_N H(y_0) \theta^{N-3} - 2S^2 \Delta a(y_0) \theta \right] + (1-t) \frac{\partial M_\delta}{\partial \theta}
\]

\[
\Phi_2(t; \theta, y) := \frac{S}{p+1} \nabla a(y) - (1-t) \frac{\delta}{\alpha_N} \left[ 4N \alpha_N \nabla H(y) \theta^{N-2} - S^2 \nabla \Delta a(y) \theta^2 + o(1) \right].
\]

Easily, \( \Phi \) does not vanish, for \( \delta \) small, on \( \theta(\theta, \bar{\theta}) \times U_{y_0} \) for suitable \( 0 < \theta < \theta_{y_0} < \bar{\theta}, U_{y_0} \) being a neighborhood of \( y_0 \) with \( \|\nabla a\| \geq c > 0 \) on \( \partial U_{y_0} \). Thus we conclude that

\[
\deg(\nabla M_\delta, (\theta_0, \bar{\theta}) \times U_{y_0}, 0) = \deg(\Phi(1; \theta, y), (\theta_0, \bar{\theta}) \times U_{y_0}, 0) = -\deg(\nabla a, U_{y_0}, 0).
\]
Remark 3.4 If $\Omega = B_R$, the finite dimensional reduction requires the bound $\delta << \frac{1}{R^N}$; hence, we cannot obtain solutions for (PSCE) on $\mathbb{R}^N$ sending $R$ to infinity: one needs uniform size of the perturbation as $R \to \infty$.

It is possible to prove that

Theorem 3.5 Let $N > 6$, $a \in C^3_b(\mathbb{R}^N)$ and assume (c.c.) and (*). Then (PSCE) has on $B_R$, for $R \geq \bar{R}$, $\delta \leq \bar{\delta}$, at least one solution.

The result follows from the fact that, for $\delta \leq \bar{\delta}$, $r << R$,

$$\deg(-\nabla E_\delta(\epsilon, y), \{\epsilon > \epsilon_\delta\} \times \{|y| < r\}, 0) = -\sum_{\nabla a(x) = 0, \Delta a(x) > 0} i(\nabla a, x) \neq 0$$

for some $\epsilon_\delta \to 0$ as $\delta \to 0$. The right hand side is the degree for $\delta$ small, whose computation relies on the local analysis of $\nabla M_\delta$ around its critical points. Assumption * provides the a priori bounds required to perform a continuation argument: property (c.c.) plays the role of an obstruction to the complete collapse of the bifurcating solutions given by Theorem 3.2

Remark 3.2 Assuming instead of (*) the opposite condition $< \nabla a(x), x > > 0$ for $|x| >> 1$ and

$$l := \sum_{\nabla a(x) = 0, \Delta a(x) < 0} i(\nabla a, x) \neq 0,$$

we get the same result, but with different argument: for $\delta \leq \bar{\delta}$ and $\delta >> \frac{1}{R^N}$, we get that

$$\deg(-\nabla E_\delta(\epsilon, y), \{\epsilon > \epsilon_\delta\} \times \{|y| < r\}, 0) = l \neq 0.$$

Notice that, for $\delta << \frac{1}{R^N}$, it still holds true that

$$\deg(-\nabla E_\delta(\epsilon, y), \{\epsilon > \epsilon_\delta\} \times \{|y| < r\}, 0) = -\sum_{\nabla a(x) = 0, \Delta a(x) > 0} i(\nabla a, x)$$

and this quantity is different from $l$; in particular, no a priori bounds are available in this case.

References


