A nonlinear Lyapunov-Schmidt reduction and multiple solutions for some semilinear elliptic equation

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Abstract

We present a finite dimensional reduction for perturbed variational functionals and discuss some nonlinear elliptic PDE with Sobolev critical growth in bounded domains.

1 Introduction

We will present in this talk a general variational principle for perturbative problems in presence of a manifold of "quasi critical points" for the unperturbed energy functional. A model problem is the following:

$$(P) \qquad \begin{cases} -\triangle u = u^p + f(\delta, x, u) & \text{in}\, \Omega \subset BbbR^N, \ N \geq 3 \\ \quad u = 0 & \text{on}\, \partial\Omega \\ \quad u > 0 & \text{in}\, \Omega \end{cases}$$

where $p = \frac{N+2}{N-2}$ is the limiting Sobolev exponent for the immersion of $H_0^1(\Omega)$ in $L^q(\Omega)$, $q \geq 1$. Here $f(\delta, x, u)$ is a perturbation term, small if δ is small, satisfying the growth condition

$$\exists \ c > o : \quad |f(\delta, x, u)| \leq c(1 + |u|^p).$$

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For $f(\delta, x, u) = \delta u$ and $0 < \delta < \lambda_1(\Omega)$, precise existence results for (P) were established in [9]; existence of multiple solutions and asymptotic behaviour for $\delta \to 0^+$ were discussed in [16], [17] . Problem (P) can be seen as the stationary equation for some variational functional $E_\delta(u) := E(u) - G(\delta, u)$, where, for δ small, $G(\delta, u)$ is a small perturbation of the unperturbed energy $E(u)$. Problem (P) fits into a general framework which we will describe in the next section (see [14] and [15] for some details).

2 A finite dimensional reduction

We consider a functional $E_\delta(u) = E(u) - G(\delta, u)$, $u \in H$ on some Hilbert space H. We assume for $E(u)$ the existence of a smooth manifold $Z \subset H$ in the form

$$Z = \{z(\epsilon, y) : \epsilon > 0, y \in U \subset \mathbf{R^N}\},$$

z a smooth parametrization, such that

$$(A1) \qquad \|\nabla E\,(z(\epsilon, y))\| \to_{\epsilon \to 0} 0 \text{ uniformly on compact subsets of } U.$$

We will refer to Z as "an almost critical manifold" for $E(u)$.
We require for Z some nondegeneracy property. Denoted by T_z the tangent space to Z at $z \in Z$, let $\pi_z : H \to T_z$ and $\pi_z^\perp := Id - \pi_z$ be orthogonal projections, and set $L_z := \pi_z^\perp\, E''(z)\,|_{T_z^\perp}$. We assume

$$(A2) \qquad L_z \in Iso(T_z^\perp, T_z^\perp) \qquad \text{and} \qquad \sup_{z \in Z} \|L_z^{-1}\| < \infty.$$

We will also need a smallness assumption on $G(\delta, u)$

$$(A3) \quad G(\delta, u) \to_{\delta \to 0} 0 \text{ in } C^2_{loc}.$$

While looking for critical points of E_δ close to Z, one can perform a nonlinear Lyapunov-Schmidt reduction: given $U_0 \subset\subset U$, there exist $\bar\epsilon > 0$, $\bar\delta > 0$ and a smooth map $(\delta, z) \to w(\delta, z)$, $z \in Z_0 := \{z(\epsilon, y) : (\epsilon, y) \in (0,\ \bar\epsilon) \times U_0\}$, $\delta \le \bar\delta$, such that

$$(i) \;\; w(\delta, z) \in T_z^\perp \;\; \forall\, \delta, z \quad \text{and} \quad \|w\| \;=\; O(\|\nabla E_\delta(z)\|)$$

$$(ii) \quad \pi_z^\perp\, \nabla E_\delta\,(z + w(\delta, z)) = 0, \qquad \forall\, \delta,\ z.$$

It remains to solve the "bifurcation equation"

$$(b) \qquad \pi_z \, \nabla E_\delta\left(z+w(\delta,z)\right)=0\,,\ z\in Z_0.$$

If Z is linear, equation (b) is known to have a variational structure: it is equivalent to

$$(b)' \qquad \frac{\partial}{\partial z}\ E_\delta\left(z+w(\delta,z)\right)=0\ \ z\in Z_0.$$

This is because $\frac{\partial w}{\partial z}(\delta,h)\in Z^\perp\ \forall h\in Z$, in this case. Actually, to have equivalence between (b) and $(b)'$ it is enough that $||\pi_z\frac{\partial w}{\partial z}|| = O(||w||)$. The argument goes as follows. Let $z(t)$ be a smooth curve on Z_0, with $z(0)=z_0$ and $\dot z(0)=\pi_{z_0}\nabla E_\delta(z_0+w(\delta,z_0))$. By assumption,

$$0=\frac{d}{dt}\,E_\delta\left(z(t)+w(\delta,z(t))\right)|_{t=0}=<\nabla E_\delta(z_0+w(\delta,z_0)),\dot z(0)+\frac{\partial w}{\partial z}(\delta,z_0)\dot z(0)>.$$

Since $\pi^\perp_{z_0}\nabla E_\delta(z_0+w(\delta,z_0))=0$, using the estimate for $||w||$ we get

$$\|\dot z(0)\|^2\le\|\dot z(0)\|^2\|\pi_{z_0}\frac{\partial w}{\partial z}(\delta,z_0)\|\le c\,\|w(\delta,z_0)\|\|\dot z(0)\|^2\le\tilde c\,\|\dot z(0)\|^2\|\nabla E_\delta(z(0))\|$$

and hence $\dot z(0)=0$ because $\|\nabla E_\delta(z)\|<<1$ if ϵ,δ are small, by (A1)-(A3).
In turn, the estimate for $||\pi_z\frac{\partial w}{\partial z}||$ involves the variation of T_z, and in fact it holds true assuming:

$$(A4) \qquad \exists\, c>0:\ \ \|\pi_z\,\frac{\partial}{\partial z}\left(\pi_z^\perp v\right)\|\ \le\ c\,\|\pi_z^\perp v\|\,,\ \ \forall\, v\in H \quad .$$

Let us derive from $(A4)$ the estimate for $||\pi_z\frac{\partial w}{\partial z}||$. Let $\bar w=w(\delta,\bar z)$ for some $\bar z\in Z_0$, δ fixed.
From $\pi_z\,w(\delta,z)\equiv 0$ it follows $\pi_z\frac{\partial w}{\partial z}=-\frac{\partial}{\partial z}(\pi_z\bar w)$ at $z=\bar z$. Since $-\frac{\partial}{\partial z}(\pi_z\bar w)=\frac{\partial}{\partial z}(\pi_z^\perp\bar w)$, we have, by (A4),

$$\|\pi_{\bar z}\frac{\partial w}{\partial z}(\delta,\bar z)\|\le c\,\|\pi^\perp_{\bar z}\bar w\|.$$

This gives the desired estimate, because $\pi^\perp_{\bar z}\bar w=\bar w$

Relevant informations can be derived by the variational structure of (b). After a Taylor expansion, (b)' rewrites as

$$stat_z\left[E\left(z(\epsilon,y)\right)-G\left(\delta,z(\epsilon,y)\right)+O(\|\nabla E_\delta(z)\|^2)\right]$$

and this leads to look for "stable" critical points of the "Melnikov function"

$$(\epsilon,y)\to E\left(z(\epsilon,y)\right)-G\left(\delta,z(\epsilon,y)\right).$$

Of course, estimates for the "error term" $O(\|\nabla E_\delta(z)\|^2)$, either in L^∞ or in C^1, are crucial.

Remark 2.1 *1. Similar (somehow different) procedure has been used by Rey, and then by many others for (P) in the perturbative case (see, to quote a few, [1], [8].....). 2. Similar principle, but in case Z is a critical manifold (i.e. $\nabla E(z) = 0$ for any $z \in Z$), goes back to Ambrosetti, Coti Zelati and Ekeland in [4] (for compact Z) and to Ambrosetti and Badiale in [3] (for non compact Z: problems with lack of compactness). Recent advances in much more complicated situations are due to Ambrosetti, Malchiodi and Ni in [5] (solutions concentrating on codimension-1 manifolds for singularly perturbed problems). See also [6].*

3 Applications to problem (P)

We consider the functional $E_\delta(u) = E(u) - G(\delta, u)$, $u \in H_0^1(\Omega)$, associated to problem (P), where

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1}.$$

Here, the manifold Z is given by

$$Z = \{PU_{\epsilon,y} : \epsilon > 0, y \in \Omega\},$$

where $P : H^1(\Omega) \to H_0^1(\Omega)$ is the orthogonal projection and

$$U_{\epsilon,y}(x) = \epsilon^{-\frac{N-2}{2}} U(\frac{x-y}{\epsilon}) \qquad U(x) = \frac{c_N}{(1+|x|^2)^{\frac{N-2}{2}}}$$

$c_N = [N(N-2)]^{\frac{N-2}{4}}$, are the extremal functions for the Sobolev inequality on $\mathbf{R^N}$.
Assumptions (A1)-(A2)-(A3) are satisfied (see [17] and [15] for detailed proofs). We will consider two distinguished problems of type (P):

$$(NHD) \qquad \begin{cases} -\triangle u = u^p & \text{in}\,\Omega \\ \quad u = \delta\varphi & \text{on}\,\partial\Omega \quad \varphi \in C^1(\partial\Omega) \text{ positive somewhere} \end{cases}$$

$$(PSCE) \qquad \begin{cases} -\triangle u = (1+\delta a(x))u^p & \text{in}\,\Omega \\ \quad u = 0 & \text{on}\,\partial\Omega \\ \quad u > 0 & \text{in}\,\Omega. \end{cases}$$

For problem (NHD), the existence of a small solution ($u_\delta \to 0$ in $H^1(\Omega)$ as $\delta \to 0$) holds in much greater generality (see [13]). The existence of a "large" solution for $\delta < \delta*$, $\delta*$ some positive number, is due, among other things, to Caffarelli and Spruck [10]. Some multiplicity results for problem (NHD) with δ small have been obtained by Rey [18].

We present here some improvements (a detailed proof is in [15]). Let $h(x)$ be the harmonic extension of $\varphi(x)$ and let $\psi(x, y)$ be the regular part of the Green function of Ω. Set

$$K(x) = \frac{h^2(x)}{H(x)} \ , \ H(x) = \psi(x, x).$$

Due to different estimates for the error term in dimension $N = 3$ and $N > 3$, we have the following

Theorem 3.1 *(i) Let $N \geq 3$. Let C be a compact subset of Ω with $h(x) > 0$ on C and $\max_{\partial C} K < \max_C K$. Then, for $\delta_j \to 0$ there exist u_j solutions and $x \in C$ such that*

$$|\nabla u_j|^2 \rightharpoonup S^{\frac{N}{2}} \delta_x \qquad\qquad (\ S := \textit{best Sobolev constant})$$

(ii) Let $N \geq 4$. Then the number of "large" solutions is greater (or equal) than the number of non degenerate critical points of K with $h > 0$.

According to (i), highly oscillating boundary datas should give rise to a large number of solutions:

Example 3.1 *If $\varphi = \sum_{j=1}^m \frac{1}{|x - tx_j|^{N-2}}, x \in \Omega = B_1$ with $\{x_1, .., x_m\} \subset \partial B_1$ and $1 < t < t(N, x_1, .., x_m)$, problem (NHD) with boundary data $\delta\varphi$ admits for δ small at least m solutions.*

As for (PSCE), we recall that, if $\Omega = \mathbf{R^N}$, it is the prescribed scalar curvature problem for S^N. Known results, (see [11], [12]), recently improved by Ambrosetti, Azorero and Peral [2], are variations of the following basic one: assume that $a \in C^2$ has only a finite number of non degenerate critical points x_j and

$$(c.c.) \qquad \Delta a(x_j) \neq 0 \ \forall\, j, \qquad \sum_{\Delta a(x_j) > 0} i(\nabla a, x_j) \neq 0$$

$$(*) \qquad < \nabla a(x), x > < 0 \text{ for } |x| >> 1.$$

Then (PSCE) is solvable. We recall that the "counting condition" (c.c.) was first introduced by Bahri-Coron in [7].
If Ω is a bounded domain in $\mathbf{R^N}$, different kind of results can be derived (see, e.g. [15]).

Theorem 3.2 *Let $N > 6$. Then, for δ small, the number of solutions for (PSCE) is greater (or equal) than the number of isolated critical points of a with positive laplacian and non zero topological index.*

Remark 3.3 *(i) If $< \nabla a(x), x > \leq 0$ for any $x \in \Omega$ and Ω is starshaped, no solution can exist.*
(ii) There are no "ground state" solutions: all the solutions we find have Morse index greater or equal than 2.

The above results are obtained by an analysis of the corresponding "Melnikov functions" . For the unperturbed energy along the manifold Z, we have that, for $y \in K \subset \Omega$ compact:

$$E(PU_{\epsilon,y}) = \frac{S^{\frac{N}{2}}}{N} + \alpha_N H(y)\epsilon^{N-2} + O(\epsilon^{N-1})\ ,\ \alpha_N = \frac{c_N^{p+1}}{2}\int_{\mathbf{R^N}} \frac{dx}{(1+|x|^2)^{\frac{N+2}{2}}}.$$

We first consider (NHD), for which the functional $G(\delta, u)$ turns out to be equal to

$$G(\delta,u) = \frac{1}{p+1}\int_\Omega |u+\delta h|^{p+1} - |u|^{p+1}.$$

By Taylor formula, and for every $N \geq 3$, we have

$$E_\delta(PU_{\epsilon,y}+w) = \tfrac{1}{N}S^{\frac{N}{2}} + \alpha_N H(y)\epsilon^{N-2} - \tfrac{2\alpha_N}{c_N}h(y)\delta\epsilon^{\frac{N-2}{2}} \quad + \\ +O(\epsilon^{N-1} + \delta\epsilon^{\frac{N}{2}} + \delta^2\epsilon + \delta^{p+1}).$$

After a change of variables $\epsilon = (\theta\delta)^{\frac{2}{N-2}}$, we are lead to look for critical points of

$$M_\delta(\theta,y) = c_N\theta^2 H(y) - 2\theta h(y) + \delta^{\frac{2}{N-2}} O(1+\theta^{\frac{2(N-1)}{N-2}})$$

As for part i) in Theorem 3.1, we just remark that the above L^∞ estimate implies stability of strict local minima of $M(\theta,y) := c_N\theta^2 H(y) - 2\theta h(y)$; on the other hand

$$\max_{\partial C}\frac{h^2}{H} \ < \ \max_{C}\frac{h^2}{H} \Rightarrow \min_{[\theta^-,\,\theta^+]\times C} M \ < \min_{\partial([\theta^-,\,\theta^+]\times C)} M$$

where $0 < \theta^- < \min_C \frac{h}{c_N H},\ \ \theta^+ > \max_C \frac{h}{c_N H}$.
We now prove this claim. First, $\theta(y) := \frac{h(y)}{c_n H(y)} \in (\theta^-,\ \theta^+)$ is the absolute minimizer of $\theta \to M(\theta,y)$, $y \in C$. Also,

$$M(\theta(y),y) \ = \ -\frac{1}{c_N}K(y) \ \ \forall\, y \in \Omega.$$

So, by assumption, $\exists \gamma > 0$:

$$\min_{\partial C} M(\theta(y),y) \geq \ \gamma + \min_{C} M(\theta(y),y) \geq \ \gamma + \min_{[\theta^-,\theta^+]\times C} M.$$

Hence, $\min_{[\theta^-,\theta^+]\times\partial C M(\theta,y)\geq\gamma+\min_{[\theta^-,\theta^+]\times C}} M\cdot$
Finally, set $\min_C M(\theta^-,y) = M(\theta^-,\bar{y})$; since $M(\theta^-,\bar{y}) > M(\theta(\bar{y}),\bar{y}) \geq \min_{[\theta-,\theta^+]\times C} M$, and similarly for θ^+, the claim is proved.

Part ii) in Theorem 3.1, deals with critical points of M_δ other than minima, so we also need C^1 estimates on the error terms. The following estimates hold true. Let $N \geq 4$. Then

$$\begin{aligned}\frac{\partial}{\partial y_i}E_\delta(PU_{\epsilon,y}+w) &= 2\alpha_N\frac{\partial H}{\partial y_i}(y)\epsilon^{N-2} - \frac{2\alpha_N}{c_N}\frac{\partial h}{\partial y_i}(y)\delta\epsilon^{\frac{N-2}{2}} + \\ &+O(\epsilon^{N-1}\log\frac{1}{\epsilon} + \delta\epsilon^{\frac{N}{2}}\log\frac{1}{\epsilon} + \delta^2\epsilon\log\frac{1}{\epsilon} + \frac{\delta^{2p}}{\epsilon}) \\ \frac{\partial}{\partial\epsilon}E_\delta(PU_{\epsilon,y}+w) &= (N-2)\alpha_N H(y)\epsilon^{N-3} - (N-2)\frac{\alpha_N}{c_N}h(y)\delta\epsilon^{\frac{N-4}{2}} + \\ &+O(\epsilon^{N-2} + \delta\epsilon^{\frac{N-2}{2}} + \delta^2\epsilon\log\frac{1}{\epsilon} + \delta^p\epsilon^{\frac{N-4}{2}} + \frac{\delta^{2p}}{\epsilon}).\end{aligned}$$

Part ii) in Theorem 3.1 follows from

$$\nabla K(y_0) = 0,\quad D^2K(y_0) \in Gl_n(\mathbf{R}),\ \mathbf{h(y_0) > 0}\ \Rightarrow$$

$$\nabla M(\theta(y_0),y_0) = 0\ \text{ and } D^2M(\theta(y_0),y_0) \in Gl_{n+1}(\mathbf{R}).$$

Let us prove prove this fact. As noticed above, $\frac{\partial M}{\partial\theta}(\theta(y),y) = 0\ \forall y \in C$, while $\frac{\partial M}{\partial y}(\theta(y),y) = -\frac{1}{c_n}\nabla K(y)$. Thus $(\theta(y_0),y_0)$ is a critical point of M because $\nabla K(y_0) = 0$. We just have to check that it is nondegenerate. Using $\frac{\partial H}{\partial y_j}(y_0) = 2\frac{H(y_0)}{h(y_0)}\frac{\partial h}{\partial y_j}(y_0)$ and the above relations, we find

$$D_y^2M(\theta(y_0),y_0) = \frac{2}{c_nH(y_0)}\nabla h^t(y_0)\nabla h(y_0) - \frac{1}{c_n}D^2K(y_0),\quad \text{and hence}$$

$$D^2M(\theta(y_0),y_0) = \begin{pmatrix} 2c_nH(y_0) & 2\nabla h(y_0) \\ 2\nabla h^t(y_0) & \frac{2}{c_nH(y_0)}\nabla h^t(y_0)\nabla h(y_0) - \frac{1}{c_n}D^2K(y_0)\end{pmatrix}.$$

Thus $0 = D^2M(\theta_0,y_0)(\tau,\eta),\ \tau \in \mathbf{R},\ \eta \in \mathbf{R^n}$ implies

$$2\tau c_nH(y_0) + 2 < \nabla h(y_0),\eta >= 0$$

$$2\tau\nabla h(y_0) + \frac{2}{c_nH(y_0)}\nabla h(y_0) < \nabla h(y_0),\eta > -\frac{1}{c_n}D^2K(y_0)\eta = 0.$$

Inserting the first relation in the second one, we find

$$H(y_0)D^2K(y_0)\eta = \nabla h(y_0)[2c_nH(y_0)\tau + 2 < \nabla h(y_0),\eta >] = 0$$

and hence $\eta = 0$ because $D^2K(y_0)$ is invertible. This also implies $\tau = 0$, i.e. $D^2M(\theta_0, y_0)$ is invertible.

Similarly, for problem (PSCE), we have that

$$G(\delta, u) = \frac{\delta}{p+1}\int_\Omega a(x)|u|^{p+1}$$

$$G(\delta, PU_{\epsilon,y}) = \frac{S^{\frac{N}{2}}}{p+1}a(y)\delta + \frac{S^{\frac{N}{2}}}{4N}\Delta a(y)\delta\epsilon^2 + h.o.t.$$

and the corresponding "Melnikov function" is

$$(\epsilon, y) \to \frac{S^{\frac{N}{2}}}{N} + \alpha_N H(y)\epsilon^{N-2} - \frac{S^{\frac{N}{2}}}{p+1}a(y)\delta - \frac{S^{\frac{N}{2}}}{4N}\Delta a(y)\delta\epsilon^2 + h.o.t..$$

As above, we can define $\theta := \frac{\epsilon}{\delta^{\frac{1}{N-4}}}$ and, after a change of variable, we look for "stable" critical points of

$$(\theta, y) \to -2(N-2)S^{\frac{N}{2}}a(y) + \delta^{\frac{2}{N-4}}[4N\alpha_N H(y)\theta^{N-2} - S^{\frac{N}{2}}\Delta a(y)\theta^2]$$

(since the error term is of order $O(\delta)$, the assumption $N > 6$ is needed to ensure that it is a fortiori of order $o(\delta^{\frac{2}{N-4}})$). Equivalently, we may look for zeros of the vector field

$$\frac{\partial M_\delta}{\partial\theta} = -\frac{\delta^{\frac{2}{N-4}}}{4N}\left[4N(N-2)\alpha_N H(y)\theta^{N-3} - 2S^{\frac{N}{2}}\Delta a(y)\theta + o(1)\right]$$

$$\frac{\partial M_\delta}{\partial y} = \frac{S^{\frac{N}{2}}}{p+1}\nabla a(y) - \frac{\delta^{\frac{2}{N-4}}}{4N}\left[4N\alpha_N\nabla H(y)\theta^{N-2} - S^{\frac{N}{2}}\nabla\Delta a(y)\theta^2 + o(1)\right].$$

We want to show that, if y_0 is an isolated critical point of a with positive laplacian and non zero topological index, then, for δ small, ∇M_δ has a zero close to (θ_0, y_0)

$\theta_0 := \theta(y_0) = \left(\frac{S^{\frac{N}{2}}\triangle a(y_0)}{N(N-2)DH(y_0)}\right)^{\frac{1}{N-4}}$. To see this, we just consider the homothopy $\Phi = (\Phi_1, \Phi_2)$, where

$$\Phi_1(t;\theta, y) := -t\frac{\delta^{\frac{2}{N-4}}}{4N}\left[4N(N-2)\alpha_N H(y_0)\theta^{N-3} - 2S^{\frac{N}{2}}\Delta a(y_0)\theta\right] + (1-t)\frac{\partial M_\delta}{\partial\theta}$$

$$\Phi_2(t;\theta, y) := \frac{S^{\frac{N}{2}}}{p+1}\nabla a(y) - (1-t)\frac{\delta^{\frac{2}{N-4}}}{4N}\left[4N\alpha_N\nabla H(y)\theta^{N-2} - S^{\frac{N}{2}}\nabla\Delta a(y)\theta^2 + o(1)\right].$$

Easily, Φ does not vanish, for δ small, on $\partial([\underline{\theta},\ \overline{\theta}] \times U_{y_0})$ for suitable $0 < \underline{\theta} < \theta_{y_0} < \overline{\theta}$, U_{y_0} being a neighborhood of y_0 with $\|\nabla a\| \geq c > 0$ on ∂U_{y_0}. Thus we conclude that

$$\deg(\nabla M_\delta, (\underline{\theta},\ \overline{\theta}) \times U_{y_0}, 0) = \deg(\Phi(1;\theta, y), (\underline{\theta},\ \overline{\theta}) \times U_{y_0}, 0) = -\deg(\nabla a, U_{y_0}, 0).$$

Remark 3.4 *If $\Omega = B_R$, the finite dimensional reduction requires the bound $\delta << \frac{1}{R^{N-2}}$; hence, we cannot obtain solutions for (PSCE) on $\mathbf{R^N}$ sending R to infinity: one needs uniform size of the perturbation as $R \to \infty$.*

It is possible to prove that

Theorem 3.5 *Let $N > 6$, $a \in C_b^3(\mathbf{R^N})$ and assume (c.c.) and (*). Then (PSCE) has on B_R, for $R \geq \bar{R}$, $\delta \leq \bar{\delta}$, at least one solution.*

The result follows from the fact that, for $\delta \leq \bar{\delta}$, $r << R$,

$$\deg(-\nabla E_\delta(\epsilon, y), \{\epsilon > \epsilon_\delta\} \times \{|y| < r\}, 0) = -\sum_{\nabla a(x)=0\ ,\ \Delta a(x)>0} i(\nabla a, x) \neq 0$$

for some $\epsilon_\delta \to 0$ as $\delta \to 0$. The right hand side is the degree for δ small, whose computation relies on the local analysis of ∇M_δ around its critical points. Assumption $*$ provides the a priori bounds required to perform a continuation argument: property (c.c.) plays the role of an obstruction to the complete collapse of the bifurcating solutions given by Theorem 3.2

Remark 3.2 *Assuming instead of (*) the opposite condition $< \nabla a(x), x > > 0$ for $|x| >> 1$ and*

$$l := \sum_{\nabla a(x)=0\ ,\ \Delta a(x)<0} i(\nabla a, x) \neq 0,$$

we get the same result, but with different argument: for $\delta \leq \bar{\delta}$ and $\delta >> \frac{1}{R^{N-2}}$, we get that

$$deg(-\nabla E_\delta(\epsilon, y), \{\epsilon > \epsilon_\delta\} \times \{|y| < r\}, 0) = l \neq 0.$$

Notice that, for $\delta << \frac{1}{R^{N-2}}$, it still holds true that

$$deg(-\nabla E_\delta(\epsilon, y), \{\epsilon > \epsilon_\delta\} \times \{|y| < r\}, 0) = -\sum_{\nabla a(x)=0\ \ \Delta a(x)>0} i(\nabla a, x)$$

and this quantity is different from l; in particular, no a priori bounds are available in this case.

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