Compactness of a nonlinear eigenvalue problem with a singular nonlinearity

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Abstract

We study the Dirichlet boundary value problem

\[-\Delta u = \frac{\lambda f(x)}{(1-u)^2} \text{ in } \Omega,\]

\[0 < u < 1 \text{ in } \Omega,\]

\[u = 0 \text{ on } \partial \Omega,\]

for \(2 \leq N \leq 7\) we characterize compactness for solutions sequence in terms of spectral informations. As a byproduct, we give an uniqueness result for \(\lambda\) close to 0 and \(\lambda^*\) in the class of all solutions with finite Morse index, \(\lambda^*\) being the extremal value associated to the nonlinear eigenvalue problem.

Keywords: Compactness, Morse index, Critical Parameter, Electrostatic MEMS.


1 Introduction

Let us consider the following problem:

\[
\begin{cases}
-\Delta u = \frac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\
0 < u < 1 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\lambda \geq 0, \Omega \subset \mathbb{R}^N\) is a bounded smooth domain and \(f \in C(\bar{\Omega})\) is a nonnegative function. We will say that \(u\) is a solution of (1.1) if \(u \in C^1(\bar{\Omega}) \cap W^{2,2}(\Omega)\) satisfies the equation a.e. in \(\Omega\) with \(u = 0\) on \(\partial \Omega\) and \(0 < u < 1\) in \(\Omega\).

The equation models the stationary regime of a simple electrostatic Micro-Electromechanical System (MEMS device), consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at +1 immersed in an external electric field, where \(u\) is the (normalized) deflection of the elastic membrane (see [7, 15] for a detailed discussion on MEMS devices). More generally, the model is described by a nonlinear parabolic problem which has been considered by Ghoussoub and Guo in [9].

Problems as in (1.1) with smooth nonlinearities (for example of the form \(e^u\) or \((1 + u)^p\) for \(p > 1\)) have been largely studied in the last thirty years and fine properties of the branch of minimal solutions have been established. We refer to the seminal works [4, 11, 12] and to [3] for a complete account on the topic.

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In [8] Ghoussoub and Guo extended this analysis to problem (1.1) (see [10, 14] for some interesting numerical results). Given $\lambda^* \in (0, +\infty)$ the so-called extremal value:

$$\lambda^* = \sup\{\lambda > 0 : \exists u \text{ solution of (1.1)}\},$$

for any $\lambda \in (0, \lambda^*)$ they proved the existence of the minimal solution $u_\lambda$, namely:

$$u_\lambda \leq u \quad \text{for any solution } u \text{ of (1.1).}$$

The map $\lambda \rightarrow u_\lambda$ is continuous and pointwise increasing for $\lambda \in (0, \lambda^*)$. Moreover, the minimal solution $u_\lambda$ is characterized as the unique semi-stable solution of (1.1) (in the sense that the linearized operator is a positive operator, see also [13]). Finally, they raised out the special role of dimension $N = 7$ for problem (1.1) by means of some energy estimates: the minimal branch satisfies

$$\sup_{\lambda \in (0, \lambda^*)} \|u_\lambda\|_\infty < 1$$

for $1 \leq N \leq 7$ and in general, this is not true anymore for $N \geq 8$. For $1 \leq N \leq 7$, as $\lambda \rightarrow \lambda^*$, the minimal branch $u_\lambda$ converges to $u^*$, the so-called extremal solution, the unique solution of (1.1) with $\lambda = \lambda^*$. By [4] $\lambda^*$ is a turning point and a second branch $U_\lambda$ of solutions for (1.1) comes out from $u^*$ for $\lambda$ close to $\lambda^*$ ($U_\lambda$ is a non degenerate solution with Morse index 1).

Unless semi-stable solutions are concerned, it is in general very difficult to show apriori bounds on the solutions energy and, for example, we were not able to establish energy estimates along $U_\lambda$. In [5], we exploited that the Morse index is 1 along the second branch, by developing a different approach to face non compactness phenomena based on this spectral information. Since in general it is relatively much easier to construct solutions satisfying good spectral informations (for example, by variational methods), assumption (1.3) below seems to be more natural than energy bounds in the study of (1.1).

The approach in [5], based on some fine asymptotic analysis, provided compactness along the second branch $U_\lambda$ in the same low dimensions (compactness of the minimal branch was also recovered).

The paper is a continuation and a strong improvement of the results in [5]. In [5], the Morse index 1 of a solutions sequence $u_n$ was used to ensure that blow up at 1 can occur “essentially” only along the maximum points $x_n$ of $u_n$. Boundedness on the Morse index allows the presence of multiple blow up points. By a careful asymptotic analysis, we are able to overcome the related technical difficulties and, by a non existence result for singular solutions of (1.1), to show:

**Theorem 1.1.** Assume $2 \leq N \leq 7$. Let $f \in C(\bar{\Omega})$ be such that:

$$f(x) = \left( \prod_{i=1}^{k} \|x - p_i\|^{\alpha_i} \right) g(x), \quad g(x) \geq C > 0 \text{ in } \Omega, \quad \text{(1.2)}$$

for some points $p_i \in \Omega$ and exponents $\alpha_i \geq 0$. Let $\{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence such that $\lambda_n \rightarrow \lambda \in [0, \lambda^*)$ and let $u_n$ be an associated solution such that:

$$\sup_{n \in \mathbb{N}} m(u_n, \lambda_n) < +\infty. \quad \text{(1.3)}$$

Then,

$$\sup_{n \in \mathbb{N}} \|u_n\|_\infty < 1. \quad \text{(1.4)}$$

Moreover, if in addition $\mu_{1,n} := \mu_{1,\lambda_n}(u_n) < 0$, then necessarily $\lambda > 0$.

Here and in the sequel, $\mu_k(\lambda)$ denotes the $k$-th eigenvalue of $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$ with the convention that eigenvalues are repeated according to their multiplicities, and the Morse index $m(u, \lambda)$ is the number of negative eigenvalues of $L_{u,\lambda}$. 

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Estimate (1.4) will be sometimes referred to as a “compactness property” of the solutions set of (1.1). Indeed, by elliptic regularity theory, for any $k \in \mathbb{N}$ the set $\{u : u \text{ is a solution of (1.1)}, m(u, \lambda) \leq k\}$ is a compact set in $C^m(\bar{\Omega})$-norm, where $m \geq 1$ depends on the regularity of $f(x)$.

Let us do some comments. For $2 \leq N \leq 7$ and $f(x) = 1$, Joseph and Lundgren in [11] showed that the bifurcation diagram of (1.1) on the ball has exactly the following form:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Plots of $\|u\|_{\infty}$ versus $\lambda$ in the case $f(x) = 1$ on the unit ball and $2 \leq N \leq 7$, where $\lambda_* = \frac{2(3N-4)}{9}$.}
\end{figure}

Namely, there are infinitely turning points oscillating around the value $\lambda_* = \frac{2(3N-4)}{9}$, the solutions number of (1.1) going to $+\infty$ as $\lambda$ approaches $\lambda_*$. For $2 \leq N \leq 7$ and $f(x) = |x|^\alpha$, $\alpha \geq 0$, numerically the diagram above still holds on the ball for $\lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ (see the thorough discussion in [8]).

Problem (1.1) presents in general a rich structure of the solutions set. The main goal now should be an existence theory for branches different from the first two, with Morse index higher than 1. Compactness properties are in general useful to establish existence results and Theorem 1.1 is a first step in the direction of an existence theory. An hopeful approach could be based on the analysis directly along the bifurcation diagram: any branch is characterized by a fixed Morse index and, when an eigenvalue of the linearized operator along the branch crosses zero, we have a “turning point” and the diagram turns into a new branch of higher Morse index (by [4] this the case for example of the first turning point $\lambda^*$).

In view of Theorem 1.1, we can show a posteriori the equivalence among energy bounds and Morse index bounds. Indeed, we provide the following characterization of blow up sequences $u_n$ (in the sense of blow up of $(1 - u_n)^{-1}$), to be compared with [1, 2] in the context of polynomial subcritical nonlinearities:

**Theorem 1.2.** Assume $2 \leq N \leq 7$. Let $f \in C(\bar{\Omega})$ be as in (1.2). Let $\{\lambda_n\}_{n \in \mathbb{N}}$ be a sequence such that $\lambda_n \to \lambda \in [0, \lambda^*]$ and let $u_n$ be an associated solution. Then,

1. $\max_{\Omega} u_n \to 1$ as $n \to +\infty$
2. $\int_{\Omega} \left( \frac{f(x)}{(1 - u_n)^{\alpha}} \right)^{\frac{N}{\alpha}} \to +\infty$ as $n \to +\infty$
3. $m(u_n, \lambda_n) \to +\infty$ as $n \to +\infty$
are equivalent.

As a direct consequence of Theorem 1.1, Theorems 1.3-1.4 below show that some features of the bifurcation
diagram on the ball hold for general domains. The following uniqueness result was strongly expected to be true:

**Theorem 1.3.** Assume \(2 \leq N \leq 7\). Let \(f \in C(\bar{\Omega})\) be as in (1.2). For any fixed \(k \in \mathbb{N}\) there exists \(\delta > 0\) small so that

1. for \(\lambda \in (0, \delta)\) the minimal solution \(u_\lambda\) is the unique solution \(u\) of (1.1) with \(m(u, \lambda) \leq k\);
2. for \(\lambda \in (\lambda^* - \delta, \lambda^*)\) \(u_\lambda\) and \(U_\lambda\) are the unique solutions \(u\) of (1.1) with \(m(u, \lambda) \leq k\).

As far as point (1) in Theorem 1.3 is concerned, in [6] the authors show that problem (1.1) on a two-
dimensional annulus with \(f(x) = 1\) has exactly two radial solutions for any \(\lambda \in (0, \lambda^*)\). The second solution
-the non minimal one- has Morse index unbounded in a neighborhood of \(\lambda = 0\).

Finally, based on a degree argument, we get the existence of a solutions sequence \(u_n\) whose Morse index
blows up (equivalently, by Theorem 1.2 the sequence blows up pointwise: \(\max_\Omega u_n \to 1\) as \(n \to +\infty\)):

**Theorem 1.4.** Assume \(2 \leq N \leq 7\). Let \(f \in C(\bar{\Omega})\) be as in (1.2). There exist a sequence \(\{\lambda_n\}_{n \in \mathbb{N}}\) and
associated solution \(u_n\) of (1.1) so that
\[
m(u_n, \lambda_n) \to +\infty \quad \text{as} \quad n \to +\infty.
\]

Let us point out that the equivalence among points (1) and (2) in Theorem 1.2 was already proved in [8] (even if it is not stated). Moreover, a weaker form of Theorem 1.3, point (1), was already shown in [8] as
uniqueness, for \(\lambda\) small, in the class of solutions of bounded energy:
\[
\int_\Omega \left(\frac{f(x)}{(1-u)^2}\right)^\frac{N}{2} \leq k \quad \text{for some} \quad k > 0.
\]

The paper is organized as follows. In [8], a regularity result for finite energy solutions of (1.1) was proved. In
Section 2, we extend it to discuss a non homogeneous Dirichlet version of (1.1), and to show a non existence
result for solutions of (1.1) with finite Morse index and finite singular set (where the solutions touch the value
1). Improving the approach of [5] for the second branch, in Section 3 we describe the asymptotic behavior
of a general blowing up sequence \(u_n\) (i.e. \(\max_\Omega u_n \to 1\) as \(n \to +\infty\)) to get a strong pointwise estimate on
the R.H.S. of (1.1). This provides the uniform convergence of \(u_n\) in \(\Omega\) to a limit singular solution \(u_0\) of (1.1)
having finite Morse index and finite singular set, which does not exist according to the regularity statements
of Section 2. In Section 4 we give proofs of Theorems 1.2-1.4. For reader’s convenience, in Appendix we
briefly sketch the proof of some results already proved in [5].

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2 Regularity properties

In this Section we establish some basic regularity results for the following boundary value problem:
\[
\begin{align*}
-\Delta u &= \frac{f(x)}{(1-u)^2} \quad \text{in} \quad \Omega, \\
u &= \bar{u} \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
where $f \in C(\Omega)$ satisfies (1.2) and $\bar{u} \in C^1(\bar{\Omega})$ is such that $0 \leq \bar{u} \leq \|\bar{u}\|_{\infty} < 1$. Solutions $u$ of (2.1) are to be considered in the following $H^1(\Omega)$-weak sense: $u - \bar{u} \in H^1_0(\Omega)$, $\frac{f(x)}{(1-u)^2} \in H^{-1}(\Omega)$ and $-\Delta u = \frac{f(x)}{(1-u)^2}$ in $H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the dual space of $H^1_0(\Omega)$.

The first regularity result we give is already contained in [8] for $\bar{u} = 0$. We extend it to cover non homogeneous boundary values and we slightly improve the original statement (for $N = 2$). The following holds:

**Proposition 2.1.** Let $N \geq 2$. Let $u$ be a $H^1(\Omega)$-weak solution of (2.1) so that

$$\frac{f}{(1-u)^3} \in L^{\frac{N}{2}}(\Omega).$$

Then, $u \in C^1(\Omega)$ and $0 < u \leq \|u\|_{\infty} < 1$.

**Proof:** First of all, by (2.2) the R.H.S. of (2.1) is in $L^{\frac{N}{2}}(\Omega)$. Standard regularity theory implies that $u \in C^{0,\frac{N}{2}}(\bar{\Omega})$. If $u(x_0) = 1$ at some $x_0 \in \Omega \setminus \{p_1, \ldots, p_k\}$, the Hölder continuity of $u$ implies $|1-u(x)| = |u(x_0) - u(x)| \leq C|x - x_0|^\frac{N}{2}$. Then, for $\delta$ small

$$\left(\inf_{B_\delta(x_0)} f\right)^{\frac{N}{2}} \int_{B_\delta(x_0)} \frac{f}{(1-u)^3} = \int_{B_\delta(x_0)} \frac{1}{1-u} \geq \int_{B_\delta(x_0)} \frac{1}{1-u} \frac{1}{2} \geq \frac{1}{\delta^N} \int_{B_{\delta}(x_0)} \frac{1}{1-u} = +\infty,$

in contradiction with (2.2). By continuity of $u$, $\|u\|_{\infty} \leq 1$ and $\{x \in \Omega : u(x) = 1\} \subset \{p_1, \ldots, p_k\}$.

We want to show now that (2.2) implies:

$$(1-u)^{-1} \in L^p(\Omega) \quad \forall \ p > 1.

(2.3)

Fix $p > 1$. Introduce $T_k u = \min\{u, 1-k\}$, the truncated function of $u$ at level $1-k$, $0 < k < 1$.

Let us first discuss the case $N = 2$. For $k$ small, take $(1-T_k u)^{-1} - (1-\bar{u})^{-1} \in H^1_0(\Omega)$ as a test function for (2.1):

$$\int_{\Omega} \frac{\nabla T_k u}{(1-T_k u)^2} = \int_{\Omega} \frac{\nabla u \nabla \bar{u}}{(1-u)^2} + \int_{\Omega} \frac{f(x)}{(1-u)^2} \left((1-T_k u)^{-1} - (1-\bar{u})^{-1}\right)$$

$$\leq \int_{\Omega} \frac{\nabla u \nabla \bar{u}}{(1-u)^2} + \int_{\Omega} \frac{f(x)}{1-u} < +\infty,$

because of $(1-T_k u)^{-1} \leq (1-u)^{-1}$ for $u \leq 1$ and (2.2). Classical consequence of the Moser-Trudinger inequality is the following: there exists $C > 0$ so that

$$\int_{\Omega} e^{p v} \leq C \exp \left(\frac{p^2}{16\pi} \|v\|^2_{H^1_0(\Omega)}\right) \quad \forall \ v \in H^1_0(\Omega), \ p > 1.

(2.5)

Since $\log \left(\frac{1-\bar{u}}{1-T_k u}\right) \in H^1_0(\Omega)$ for $k$ small, by (2.4)-(2.5) we get that for any $p > 1$:

$$\int_{\Omega} (1-T_k u)^{-p} \leq C \int_{\Omega} \left(\frac{1-\bar{u}}{1-T_k u}\right)^p \leq C \exp \left(\frac{p^2}{16\pi} \int_{\Omega} |\nabla \log \left(\frac{1-\bar{u}}{1-T_k u}\right)|^2\right) \leq C,$$

where $C$ denotes various positive constants depending only on $p$. Taking the limit as $k \to 0$, by $u \leq 1$ we get the validity of (2.3).
The case $N \geq 3$ is more involved. Since $\{u(x) = 1\}$ is a finite set and $u$ is continuous, we get that $|\{1 - u \leq \varepsilon\}| \to 0$ as $\varepsilon \to 0^+$, and by (2.2):

$$
\int_{\{1 - u \leq \varepsilon\}} \left( \frac{f(x)}{(1 - u)^{p+2}} \right)^\frac{2}{p} \leq \left( \frac{p + 1}{2p^2} S_N \right)^{\frac{2}{p}},
$$

(2.6)

for some $\varepsilon > 0$ small, where $|\cdot|$ stands for the Lebesgue measure and $S_N$ is the Sobolev constant for the embedding of $H^1_0(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$.

For $k$ small, take $(1 - T_k u)^{-p-1} - (1 - \bar{u})^{-p-1} \in H^1_0(\Omega)$ as a test function for (2.1), and by (2.2) we get:

$$(p + 1) \int_\Omega \frac{\vert \nabla T_k u \vert^2}{(1 - T_k u)^{p+2}} = (p + 1) \int_\Omega \nabla u \nabla \bar{u} + \int_\Omega \frac{f(x)}{(1 - u)^2} \left( (1 - T_k u)^{-p-1} - (1 - \bar{u})^{-p-1} \right) \leq \int_\Omega \frac{f(x)}{(1 - u)^3} (1 - T_k u)^{-p} + C.$$

(2.7)

In view of $(a + b)^2 = a^2 + b^2 + 2ab \leq (1 + \delta) a^2 + \frac{1 + \delta}{\delta} b^2$ for $a, b \in \mathbb{R}$ and $\delta > 0$, we deduce the following estimate:

$$(1 - T_k u)^{-p} \leq (1 + \delta) \left( (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right)^2 + \frac{1 + \delta}{\delta} (1 - \bar{u})^{-p}, \quad \delta > 0.
$$

(2.8)

Inserting (2.8) with $\delta = 1$ into (2.7), we get:

$$(p + 1) \int_\Omega \frac{\vert \nabla T_k u \vert^2}{(1 - T_k u)^{p+2}} \leq 2 \int_\Omega \frac{f(x)}{(1 - u)^3} \left( (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right)^2 + C.
$$

(2.9)

By (2.9) we get that:

$$
\int_\Omega \left\vert \nabla \left( (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right) \right\vert^2 \leq 2 \int_\Omega \left\vert \nabla \left( (1 - T_k u)^{-\frac{p}{2}} \right) \right\vert^2 + C = \frac{p^2}{2p} \int_\Omega \frac{\vert \nabla T_k u \vert^2}{(1 - T_k u)^{p+2}} + C
$$

$$
\leq \frac{p^2}{p + 1} \int_\Omega \frac{f(x)}{(1 - u)^3} \left( (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right)^2 + C
$$

$$
\leq \frac{p^2}{p + 1} \int_{\{1 - u \leq \varepsilon\}} \frac{f(x)}{(1 - u)^3} \left( (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right)^2 + C,
$$

(2.9)

where $C$ denotes various positive constants depending only on $\varepsilon$ and $p$. By Hölder inequality, (2.6) and the Sobolev embedding on $(1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \in H^1_0(\Omega)$, finally we get that

$$
\int_\Omega \left\vert \nabla \left( (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right) \right\vert^2
$$

$$
\leq \frac{p^2}{p + 1} \left( \int_{\{1 - u \leq \varepsilon\}} \left( \frac{f(x)}{(1 - u)^{3}} \right)^\frac{2}{p} \right)^{\frac{p}{2}} \left( \int_\Omega \left\vert (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right\vert^{\frac{2p}{p - 2}} \right)^{\frac{p - 2}{2}} + C
$$

$$
\leq \frac{1}{2} \int_\Omega \left\vert \nabla \left( (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right) \right\vert^2 + C.
$$

Hence, by Sobolev embedding:

$$
\left( \int_\Omega \left\vert (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right\vert^{\frac{2p}{p - 2}} \right)^{\frac{p - 2}{2}} \leq S_N^{-1} \int_\Omega \left\vert \nabla \left( (1 - T_k u)^{-\frac{p}{2}} - (1 - \bar{u})^{-\frac{p}{2}} \right) \right\vert^2 \leq C,
$$
and in turn
\[
\int_{\Omega} (1 - T_k u)^{-\frac{2}{pN}} \leq C,
\]
where \(C > 0\) does not depend on \(k\). Taking the limit as \(k \to 0\), as before we get the validity of (2.3).

Now property (2.3) implies \(\|u\|_{\infty} < 1\). Indeed, if \(u(x_0) = 1\) for some \(x_0 \in \Omega\), then \(|1-u(x)| = |u(x_0)-u(x)| \leq C|x-x_0|^\frac{2}{3}\), as already remarked. This is in contradiction with (2.3) for \(p\) large. Since \(\|u\|_{\infty} < 1\) implies that the R.H.S. of (2.1) is in \(L^p(\Omega)\) for any \(p > 1\), by elliptic regularity theory \(u \in C^1(\Omega)\) and then, we can conclude by maximum principle (in a weak form) that \(0 < u \leq \|u\|_{\infty} < 1\). \(\blacksquare\)

To cover non homogeneous boundary values, we adapt now the argument in [8] to show energy estimates for semi-stable solutions of (2.1). Proposition 2.1 applies to provide:

**Proposition 2.2.** Let \(2 \leq N \leq 7\). Let \(u\) be a \(H^1(\Omega)\)-weak solution of (2.1) so that \(\|u\|_{\infty} \leq 1\) and

\[
\int_{\Omega} \left( |\nabla \phi|^2 - \frac{2f(x)}{(1-u)^{4}} \phi^2 \right) \geq 0, \quad \forall \phi \in H^1_0(\Omega).
\] (2.10)

Then, \(u \in C^1(\Omega)\) and \(0 < u \leq \|u\|_{\infty} < 1\).

**Proof:** We will show that (2.10) gives energy estimates sufficiently good for \(2 \leq N \leq 7\). First of all, let us remark that (2.1) on \(u - \bar{u} \in H^1_0(\Omega)\) gives:

\[
\int_{\Omega} \frac{f(x)}{(1-u)^2} \leq C \int_{\Omega} \frac{f(x)}{(1-u)^2} (1-\bar{u}) = C \left( \int_{\Omega} \nabla u \nabla (u-\bar{u}) + \int_{\Omega} \frac{f(x)}{1-u} \right)
\]

\[
\leq C \left( \|u\|_{H^1} \|u-\bar{u}\|_{H^1} + \epsilon \int_{\Omega} \frac{f(x)}{(1-u)^2} + \frac{1}{4\epsilon} \int_{\Omega} f(x) \right)
\]

for any \(\epsilon > 0\), because of the inequality \(ab \leq \epsilon a^2 + \frac{1}{2\epsilon} b^2\). Hence, for \(\epsilon = \frac{1}{4C'}\) we get:

\[
\int_{\Omega} \frac{f(x)}{(1-u)^2} \leq \frac{1}{2} \int_{\Omega} \frac{f(x)}{(1-u)^2} + C'
\]

for some \(C' > 0\), and then, \(\int_{\Omega} \frac{f(x)}{(1-u)^2} < +\infty\). Now, (2.10) on \(u - \bar{u} \in H^1_0(\Omega)\) gives that

\[
\int_{\Omega} \frac{f(x)}{(1-u)^3} \leq 2C \int_{\Omega} \frac{f(x)}{(1-u)^3} (1-\bar{u}) \leq C \left( \int_{\Omega} |\nabla (u-\bar{u})|^2 + \int_{\Omega} \frac{2f(x)}{1-u} + \int_{\Omega} \frac{4f(x)}{(1-u)^2} (u-\bar{u}) \right)
\]

\[
= C \left( \int_{\Omega} |\nabla (u-\bar{u})|^2 + \int_{\Omega} \frac{2f(x)}{1-u} + 4 \int_{\Omega} \nabla u \nabla(u-\bar{u}) \right) \leq C.
\]

Fix \(0 < p < 4 + 2\sqrt{6}\) in order to have \(2 - \frac{2}{4(p+1)} > 0\). Introduce, as in the previous proof, \(T_k u = \min\{u, 1-k\}, 0 < k \leq 1\). For \(k\) small, taking \((1 - T_k u)^{-p-1} - (1 - \bar{u})^{-p-1} \in H^1_0(\Omega)\) as a test function in (2.1) we get:

\[
(p+1) \int_{\Omega} \frac{|\nabla T_k u|^2}{(1 - T_k u)^{p+2}} - \frac{\nabla u \nabla \bar{u}}{(1 - \bar{u})^{p+2}} = \int_{\Omega} \frac{f(x)}{(1-u)^2} \left( (1 - T_k u)^{-p-1} - (1 - \bar{u})^{-p-1} \right). \] (2.11)

Moreover, by (2.10) and the simple inequality \((a+b)^2 \leq (1+\delta)a^2 + \frac{1+\delta}{2} b^2\) we get:

\[
2 \int_{\Omega} \frac{f(x)}{(1-u)^3} \left( (1 - T_k u)^{-\frac{2}{3}} - (1 - \bar{u})^{-\frac{2}{3}} \right)^2 \leq \int_{\Omega} \left| \nabla \left( (1 - T_k u)^{-\frac{2}{3}} - (1 - \bar{u})^{-\frac{2}{3}} \right)^2 \right|^2 \geq \frac{p^2}{4} \int_{\Omega} \frac{|\nabla T_k u|^2}{(1 - T_k u)^{p+2}} + C \leq \frac{p^2}{4} \int_{\Omega} \frac{|\nabla T_k u|^2}{(1 - T_k u)^{p+2}} - \frac{\nabla u \nabla \bar{u}}{(1 - \bar{u})^{p+2}} + C,
\] (2.12)
for some $C > 0$ depending on $p$ and $\delta > 0$. Inserting (2.11) into (2.12) and using $(1 - T_k u)^{-1} \leq (1 - u)^{-1}$ for $u \leq 1$, we get that
\[
2 \int_\Omega \frac{f(x)}{(1-u)^3} \left( (1 - T_k u)^{-\frac{2}{p}} - (1 - \bar{u})^{-\frac{2}{p}} \right)^2 \leq \frac{p^2(1+\delta)^2}{4(p+1)} \int_\Omega \frac{f(x)}{(1-u)^3} \left( (1 - T_k u)^{-p-1} - (1 - \bar{u})^{-p-1} \right) + C
\]
\[
\leq \frac{p^2(1+\delta)^2}{4(p+1)} \int_\Omega \frac{f(x)}{(1-u)^3} (1 - T_k u)^{-p} + C
\]
\[
\leq \frac{p^2(1+\delta)^2}{4(p+1)} \int_\Omega \frac{f(x)}{(1-u)^3} \left( (1 - T_k u)^{-\frac{2}{p}} - (1 - \bar{u})^{-\frac{2}{p}} \right)^2 + C
\]
in view of (2.8), where $C > 0$ does not depend on $k$. Since $p < 4 + 2\sqrt{6}$, $2 - \frac{p^2(1+\delta)^2}{4(p+1)} > 0$ for $\delta$ small and then,
\[
\int_\Omega \frac{f(x)}{(1-u)^3} (1 - T_k u)^{-p} \leq 2 \int_\Omega \frac{f(x)}{(1-u)^3} \left( (1 - T_k u)^{-\frac{2}{p}} - (1 - \bar{u})^{-\frac{2}{p}} \right)^2 + C \leq C
\]
for some $C > 0$ not depending on $k$. Taking the limit as $k \to 0$, we get that
\[
\int_\Omega \frac{f(x)}{(1-u)^{3+p}} \leq C,
\]
and then, $\frac{f(x)}{(1-u)^{3+p}} \in L^\frac{4}{3+p}(\Omega)$ for any $0 < p < 4 + 2\sqrt{6}$. Since $\frac{2}{3} < \frac{1}{3} (3 + (4 + 2\sqrt{6}))$ for $2 \leq N \leq 7$, we get the validity of (2.2) and hence, applying Proposition 2.1 the proof is complete. \hfill \blacksquare

We conclude the Section providing a non existence result for solutions of (2.1) with finite Morse index and finite singular set. We have that:

**Theorem 2.3.** Let $2 \leq N \leq 7$. Let $u \in C(\bar{\Omega})$ be a $H^1(\Omega)$-weak solution of (2.1) so that $\|u\|_\infty \leq 1$ and the singular set $S = \{x \in \Omega : u(x) = 1\}$ is a non empty set. Assume that $u$ has finite Morse index: there exists a finite dimensional subspace $T \subset H^1_0(\Omega)$ so that
\[
\int_\Omega \left( |\nabla \phi|^2 - \frac{2f(x)}{(1-u)^3} \phi^2 \right) \geq 0, \quad \text{for any } \phi \in T^\perp = \{\phi \in H^1_0(\Omega) : \int_\Omega \nabla \phi \cdot \nabla \psi = 0 \forall \psi \in T\}. \tag{2.13}
\]
Then, the singular set $S$ has no isolated points.

**Proof:** Assume by contradiction that $x_0 \in S$ is an isolated point of $S$. Let $\delta_0$ be such that $B_{2\delta_0}(x_0) \cap S = \{x_0\}$. We want to show that:
\[
\int_{B_\delta} \left( |\nabla \phi|^2 - \frac{2f(x)}{(1-u)^3} \phi^2 \right) \geq 0, \quad \text{for any } \phi \in H^1_0(B_\delta), \tag{2.14}
\]
for some $0 < \delta \leq \delta_0$ small, where $B_\delta := B_\delta(x_0)$.

By contradiction, assume that (2.14) is false for any $0 < \delta \leq \delta_0$. Then, there exists $\phi_0 \in C^\infty_{0^+}(B_{2\delta_0})$ such that
\[
\int_{B_{2\delta_0}} \left( |\nabla \phi_0|^2 - \frac{2f(x)}{(1-u)^3} \phi_0^2 \right) < 0. \tag{2.15}
\]
We can assume that \( \phi_0 = 0 \) in \( B_\delta \) for some \( 0 < \delta < \delta_0 \) small. Indeed, let us replace \( \phi_0 \) with a truncated function \( \phi_\delta \), \( \delta > 0 \) small, so that (2.15) is still true while \( \phi_\delta = 0 \) in \( B_\delta \). Set \( \phi_\delta = \chi_\delta \phi_0 \), where \( \chi_\delta \) is a cut-off function defined as:

\[
\chi_\delta(x) = \begin{cases} 
0 & \quad |x - x_0| \leq \delta, \\
2 \left(1 - \frac{\log |x - x_0|}{\log \delta}\right) & \quad \delta \leq |x - x_0| \leq \sqrt{\delta}, \\
1 & \quad |x - x_0| \geq \sqrt{\delta}.
\end{cases}
\]

By Fatou’s Lemma, we have:

\[
\int_{B_{\delta_0}} \frac{2f(x)}{(1-u)^3} \phi_0^2 \leq \liminf_{\delta \to 0} \int_{B_\delta} \frac{2f(x)}{(1-u)^3} \phi_\delta^2. 
\tag{2.16}
\]

For the gradient term, we have the expansion:

\[
\int_{B_{\delta_0}} |\nabla \phi_\delta|^2 = \int_{B_{\delta_0}} \phi_0^2 |\nabla \chi_\delta|^2 + \int_{B_{\delta_0}} \chi_\delta^2 |\nabla \phi_0|^2 + 2 \int_{B_{\delta_0}} \chi_\delta \phi_0 \nabla \chi_\delta \nabla \phi_0.
\]

The following estimates hold:

\[
0 \leq \int_{B_{\delta_0}} \phi_0^2 |\nabla \chi_\delta|^2 \leq 4\|\phi_0\|_\infty^2 \int_{\delta \leq |x - x_0| \leq \sqrt{\delta}} \frac{1}{|x - x_0|^2 \log^2 \delta} \leq \frac{C}{\log \frac{\delta}{\delta_0}}
\]

and

\[
|2 \int_{B_{\delta_0}} \chi_\delta \phi_0 \nabla \chi_\delta \nabla \phi_0| \leq \frac{4\|\phi_0\|_\infty \|\nabla \phi_0\|_\infty}{\log \frac{\delta}{\delta_0}} \int_{B_1(0)} \frac{1}{|x|},
\]

and provide by Lebesgue’s Theorem:

\[
\int_{B_{\delta_0}} |\nabla \phi_\delta|^2 \to \int_{B_{\delta_0}} |\nabla \phi_0|^2 \quad \text{as} \quad \delta \to 0. \tag{2.17}
\]

Combining (2.16)-(2.17), we get that:

\[
\int_{B_{\delta_0}} \left( |\nabla \phi_\delta|^2 - \frac{2f(x)}{(1-u)^3} \phi_\delta^2 \right) < 0
\]

for \( \delta > 0 \) sufficiently small.

In this way, we find \( 0 < \delta_1 < \delta_0 \) small and \( \phi_0 \in C_0(B_{\delta_1} \setminus B_{\delta_0}) \cap H_0^1(\Omega) \) such that (2.15) holds. Since by contradiction we are assuming that (2.14) is false for any \( \delta > 0 \), we can iterate now the argument to find a strictly decreasing sequence \( \delta_n \) and \( \phi_n \in C_0(B_{\delta_n} \setminus B_{\delta_{n+1}}) \cap H_0^1(\Omega) \) such that:

\[
\int_{B_{\delta_n}} \left( |\nabla \phi_n|^2 - \frac{2f(x)}{(1-u)^3} \phi_n^2 \right) < 0.
\]

Since \( \{\phi_n\}_{n \in \mathbb{N}} \) are mutually orthogonal having disjoint supports, we have found an infinite dimensional set \( M = \text{Span} \{\phi_n : n \in \mathbb{N}\} \subset H_0^1(\Omega) \) so that

\[
\int_{\Omega} \left( |\nabla \phi|^2 - \frac{2f(x)}{(1-u)^3} \phi^2 \right) < 0 \quad \forall \phi \in M.
\]
Since $M$ is an infinite dimensional subspace of $H^1_0(\Omega)$, we have that $M \cap T^\perp \neq \emptyset$, in contradiction with (2.13). Hence, (2.14) holds for some $\delta = \delta(x_0) \leq \delta_0$.

By elliptic regularity theory, we get that $u \in C^1_{\text{loc}}(B_{2\delta_0} \setminus \{x_0\})$. Since $u \in C^1(\partial B_\delta)$ and $\max u < 1$ in view of $0 < \delta \leq \delta_0$, we extend it on $B_\delta$ as a function $\bar{u} \in C^1(B_\delta)$ satisfying $0 \leq \bar{u} \leq \|\bar{u}\|_{C^1(\partial B_\delta)} < 1$. Since (2.14) holds on $B_\delta$, we can apply Proposition 2.2 to get that $\|u\|_{C^1(\partial B_\delta)} < 1$, contradicting $u(x_0) = 1$. Hence, $S$ has no isolated points.

3 Compactness issues

In this Section we turn to the compactness result stated in Theorem 1.1. We follow the approach developed in [5] to prove compactness of the second branch of solutions. To deal with higher branches, we improve the argument to discuss multiple blow up (for the second branch the blow up occurs only at the maximum point).

Let $2 \leq N \leq 7$. Assume that $f \in C(\bar{\Omega})$ is in the form (1.2), and let $(u_n)_n$ be a solutions sequence of (1.1) associated to $\lambda_n \to \lambda \in [0, \lambda^*]$. Since we want to show that $\sup_{n \in \mathbb{N}} \|u_n\|_{C^2} < 1$, by contradiction and up to a subsequence, we will assume all along the Section that $u_n(x_n) = \max_{\Omega} u_n \to 1^-$ as $n \to +\infty$, $x_n$ being a maximum point of $u_n$.

3.1 A blow-up approach

Let $y_n \in \Omega$ be a sequence of points so that $u_n(y_n) \to 1^-$ as $n \to +\infty$. Set $\mu_n = 1 - u_n(y_n)$. As we will see later, for our purposes it is not restrictive to assume that $\mu_n^3 \lambda_n^{-1} \to 0$ and $y_n \to p \in \Omega$ as $n \to +\infty$.

Depending on the location of $p$ and the rate of $|y_n - p|$, the length scale to see around $y_n$ some non trivial limit profile is the following:

$$r_n = \begin{cases} \mu_n^3 \lambda_n^{-\frac{1}{2}} & \text{if } p \notin Z \\ \mu_n^3 \lambda_n^{-\frac{3}{2}} |y_n - p_i|^{-\frac{1}{2\alpha_i}} & \text{if } p = p_i, \mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha_i + 2} \to +\infty \text{ as } n \to +\infty \\ \mu_n^{-3} \lambda_n^{-\frac{3}{2\alpha_i}} & \text{if } \limsup_{n \to +\infty} \mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha_i + 2} < +\infty, \end{cases} \quad (3.1)$$

where $Z = \{p_1, \ldots, p_k\}$ is the zero set of the potential $f(x)$ and $\alpha_1, \ldots, \alpha_k$ are the related multiplicities given by (1.2). Let us remark that $\mu_n^3 \lambda_n \to 0$ implies $r_n \to 0$ as $n \to +\infty$.

Only to give an idea, let us establish the following rough correspondence: the first situation in the definition of $r_n$ corresponds to a blow up at some point outside $Z$, the second one to a “slow” blow up at some $p_i \in Z$, while the third one is a “fast” blow at some $p_i \in Z$. Let us now introduce the following rescaled function around $y_n$:

$$U_n(y) = \frac{1 - u_n(r_n y + y_n)}{\mu_n}, \quad y \in \Omega_n = \frac{\Omega - y_n}{r_n}.$$ 

Since $U_n(0) = 1$ by construction, in order to get a limit profile equation we should add a condition avoiding vanishing on compact sets of $\Omega_n$. Let us remark that, for $x_n$ the maximum point of $u_n$ and $\varepsilon_n = 1 - u_n(x_n)$, the associated rescaled function $U_n$ satisfies: $U_n \geq U_n(0) = 1$ in $\Omega_n$.

**Proposition 3.1.** Assume that

$$\mu_n^3 \lambda_n^{-1} (\text{dist } (y_n, \partial \Omega))^2 \to 0 \text{ as } n \to +\infty \quad (3.2)$$

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and
\[ U_n \geq C > 0 \quad \text{in} \quad \Omega_n \cap B_{R_n}(0), \quad (3.3) \]
for some \( R_n \to +\infty \) as \( n \to +\infty \). Then, up to a subsequence, \( U_n \to U \) in \( C^1_{\text{loc}}(\mathbb{R}^N) \), where \( U \) is a solution of the equation:
\[
\begin{cases}
\Delta U = s\frac{|y + y_0|^\gamma}{U^2} & \text{in} \quad R^N, \\
U(y) \geq C > 0 & \text{in} \quad R^N,
\end{cases}
\]
for some \( s > 0, \gamma \in \{0, \alpha_1, \ldots, \alpha_k\} \) and \( y_0 \in \mathbb{R}^N \) (depending on the type of blow up). Moreover, there exists a function \( \phi_n \in C_0^\infty(\Omega) \) such that:
\[ \int_{\Omega} \left( |\nabla \phi_n|^2 - \frac{2\lambda_n f(x)}{(1 - u_n)^3} \phi_n^2 \right) < 0 \]
and \( \text{Supp} \ \phi_n \subset B_{Mr_n}(y_n) \) for some \( M > 0 \).

To establish property \((3.5)\), it will be crucial the knowledge of the linear instability for solutions of \((3.4)\) in low dimensions:

**Theorem 3.2.** ([5]) Assume either \( 1 \leq N \leq 7 \) or \( N \geq 8, \gamma > \frac{3N - 14 - 4\sqrt{6}}{4 + 2\sqrt{6}} \). Let \( U \) be a solution of
\[
\begin{cases}
\Delta U = \frac{|y|^\gamma}{U^2} & \text{in} \quad R^N, \\
U(y) \geq C > 0 & \text{in} \quad R^N.
\end{cases}
\]

Then,
\[ \mu_1(U) = \inf \left\{ \int_{\mathbb{R}^N} (|\nabla \phi|^2 - \frac{2|y|^\gamma}{U^3} \phi^2) ; \phi \in C_0^\infty(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \phi^2 = 1 \right\} < 0. \]

Moreover, if \( N \geq 8 \) and \( 0 \leq \gamma \leq \frac{3N - 14 - 4\sqrt{6}}{4 + 2\sqrt{6}} \), then there exists at least a solution \( U \) of \((3.6)\) such that \( \mu_1(U) \geq 0 \).

For the sake of completeness, we will sketch the proof of Theorem 3.2 in the Appendix.

**Proof (of Proposition 3.1):** First of all, let us remark that \((3.2)\) implies \( \mu_2^3 \lambda_n^{-1} \to 0 \), and then \( r_n \to 0 \) as \( n \to +\infty \). If \( p \in \partial \Omega \), we have that \( r_n = \frac{2}{\mu_2^3 \lambda_n} \) and, by \((3.2)\)
\[ \text{dist} \ (0, \partial \Omega_n) = \frac{\text{dist} \ (y_n, \partial \Omega)}{r_n} = \left( \mu_2^3 \lambda_n^{-1} \text{dist} \ (y_n, \partial \Omega)^{-2} \right)^{-\frac{1}{2}} \to +\infty \]
as \( n \to +\infty \). Arguing in a simpler way if \( p \in \Omega \), we get that \( \Omega_n \to \mathbb{R}^N \) as \( n \to +\infty \). Introduce the following notation
\[ f_i(x) = \left( \prod_{j=1, j \neq i}^k |x - p_j|^{\alpha_j} \right) g(x). \]
(3.8)
The function \( U_n \) satisfies \( \Delta U_n = \frac{f_n(y)}{U^2} \) in \( \Omega_n \), where \( f_n(y) \) is given by:
\[
f_n = \begin{cases}
f(r_n y + y_n) & \text{if } p \notin Z \\
\left| y + \frac{y - p_n}{r_n} \right| + \alpha f_i(r_n y + y_n) & \text{if } p = p_i, \ \mu_3^3 \lambda_n |y_n - p_n|^{\alpha_i + 2} \to +\infty \text{ as } n \to +\infty \\
\limsup_n \mu_3^3 \lambda_n |y_n - p_n|^{\alpha_i + 2} < +\infty, \end{cases}
\]
(3.9)
and $p = \lim_{n \to +\infty} y_n$. Only in the latter situation $\limsup_{n \to +\infty} \mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha_i + 2} < +\infty$, up to a subsequence assume that

$$\frac{y_n - p_i}{r_n} \to y_0 \ \text{as} \ n \to +\infty. \quad (3.10)$$

Let $R > 0$. For $n$ large, decompose $U_n = U_{n,1} + U_{n,2}$, where $U_{n,2}$ satisfies:

$$\begin{cases}
\Delta U_{n,2} = \Delta U_n & \text{in } B_R(0), \\
U_{n,2} = 0 & \text{on } \partial B_R(0).
\end{cases}$$

Since (3.3) implies $0 \leq \Delta U_n \leq C_R$ on $B_R(0)$, by elliptic regularity theory we get that $U_{n,2}$ is uniformly bounded in $C^{1,\beta}(B_R(0))$, $\beta \in (0,1)$. Since $U_{n,1} = U_n \geq C$ on $\partial B_R(0)$, by harmonicity $U_{n,1} \geq C$ in $B_R(0)$. Since $U_n(0) = 1$, by Harnack inequality we get:

$$\sup_{B_R(0)} U_{n,1} \leq C_R \inf_{B_{R/2}(0)} U_{n,1} \leq C_R U_{n,1}(0) = C_R \left(1 - U_{n,2}(0)\right) \leq C_R \left(1 + \sup_{n \in \mathbb{N}} |U_{n,2}(0)|\right) < \infty.$$  

Hence, $U_{n,1}$ is uniformly bounded in $C^{1,\beta}(B_{R/4}(0))$, $\beta \in (0,1)$. Since $U_n = U_{n,1} + U_{n,2}$ is uniformly bounded in $C^{1,\beta}(B_{R/4}(0))$ for any $R > 0$, by a diagonal process and up to a subsequence, we get that $U_n \to U$ in $C^{1}_{\text{loc}}(\mathbb{R}^N)$. According to the three situations described in the definition (3.9) of $f_n$, the function $U \geq C > 0$ is a solution of (3.4) with: $s = f(p)$, $\gamma = 0$ in the first case; $s = f_i(p)$, $\gamma = 0$ in the second case; $s = f_i(p)$, $\gamma = \alpha_i$ and $y_0$ as in (3.10) in the third case. Set $f_\infty(y) := \lim_{n \to +\infty} f_n(y) = s|y + y_0|^\gamma$.

Since $2 \leq N \leq 7$ and $s > 0$, by Theorem 3.2 we get that $\mu_1(U) < 0$ and then, we find $\phi \in C^\infty_0(\mathbb{R}^N)$ so that:

$$\int \left(\frac{|\nabla \phi|^2}{U^3} - \frac{2f_\infty(y)}{U^3} \phi^2\right) < 0.$$

Define now $\phi_n(x) = r_n^{-N/2} \phi(\frac{1}{r_n} x)$. We have that:

$$\int_{\Omega} \left(\frac{|\nabla \phi_n|^2}{1 - u_n} - \frac{2\lambda_n f(x)}{(1 - u_n)^3} \phi_n^2\right) = \int_{\Omega} \left(\frac{|\nabla \phi|^2}{U^3} - \frac{2f_\infty(y)}{U^3} \phi^2\right) < 0$$

as $n \to +\infty$, since $\phi$ has compact support and $U_n \to U$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R})$. \hfill \blacksquare

Remark 3.1. In case of fast blow up at $p_i$: $\limsup_{n \to +\infty} \mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha_i + 2} < +\infty$, Proposition 3.1 is still true if, instead of condition (3.3), we assume:

$$U_n \geq C|y + \frac{y_n - p_i}{r_n}|^{\frac{\alpha_i + 2}{\alpha_i}} \quad \text{in } \Omega_n \cap B_{R_n}(0), \quad (3.11)$$

for some $R_n \to +\infty$ as $n \to +\infty$ and $C > 0$. Recall that in this situation $r_n = \mu_n^{\frac{\alpha_i + 2}{\alpha_i}} \lambda_n^{-\frac{1}{\alpha_i}}$. By (3.11), we get easily that on $\Omega_n \cap B_{R_n}(0)$:

$$0 \leq \Delta U_n \leq C|y + \frac{y_n - p_i}{r_n}|^{\frac{\alpha_i + 2}{\alpha_i}}.$$  

Given $R > 0$, then $0 \leq \Delta U_n \leq C_R$ on $B_R(0)$ for $n$ large. Arguing as in the proof of Proposition 3.1, up to a subsequence, we get that $U_n \to U$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$, where $U \in C^{1}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{-y_0\})$ is a solution of the equation

$$\begin{cases}
\Delta U = |y + y_0|^{\alpha_i} \frac{f_i(p_i)}{U^{\frac{\alpha_i + 2}{\alpha_i}}} & \text{in } \mathbb{R}^N \setminus \{-y_0\}, \\
U(y) \geq C|y + y_0|^{\frac{\alpha_i + 2}{\alpha_i}} & \text{in } \mathbb{R}^N.
\end{cases}$$
By Hopf Lemma, we have that \( U(-y_0) > 0 \). Indeed, let \( B \) some ball so that \(-y_0 \in \partial B\) and assume by contradiction that \( U(-y_0) = 0 \). Since
\[
-\Delta U + c(y) U = 0 \quad \text{in } B, \quad U \in C^2(B) \cap C(\bar{B}), \quad U(y) > U(-y_0) \quad \text{in } B,
\]
and \( c(y) = f(p_i) \frac{|y+y_0|^n}{|y|^n} \geq 0 \) is a bounded function, by Hopf Lemma we get that \( \partial_n U(-y_0) < 0 \), where \( \nu \) is the unit outward normal of \( B \). Hence, along the outward normal direction of \( B \) at \(-y_0 \) \( U \) becomes negative in contradiction with the positivity of \( U \). Hence, \( U(-y_0) > 0 \) and \( U(y) \geq C := \inf \frac{U(y)}{\nu^{N}} \) in \( \mathbb{R}^N \). The argument now goes as in the proof of Proposition 3.1.

### 3.2 A pointwise estimate

Let us assume now the validity of (1.3), namely \( m(u_n, \lambda_n) \leq k \) for any \( n \in \mathbb{N} \) and some \( k \in \mathbb{N} \). This information, combined with Proposition 3.1, will permit us to control the blow up behavior of \( u_n \). Indeed, the following pointwise estimate on \( u_n \) is available:

**Theorem 3.3.** Assume \( 2 \leq N \leq 7 \). Let \( f \in C(\bar{\Omega}) \) be as in (1.2). Let \( u_n \) be a solution of (1.1) associated to \( \lambda_n \in [0, \lambda^*) \). Assume that \( \lambda_n \to \lambda \) and \( u_n(x_n) = \max u_n \to 1 \) as \( n \to +\infty \). Then, up to a subsequence, there exist constants \( C > 0 \), \( N_0 \in \mathbb{N} \) and \( m \)-sequences \( x_1, \ldots, x_m \), \( m \leq k \), such that
\[
1 - u_n(x) \geq C\lambda_n^{-\frac{1}{2}} (d(x)^{\alpha}) \frac{d_n(x)\lambda_n^{-\frac{1}{2}}}{\tilde{\nu}^{N}} - \frac{\lambda_n}{\nu} \quad \forall x \in \Omega, \quad \forall n \geq N_0,
\]
where \( d(x)^{\alpha} := \min\{|x-p_i|^{\alpha_i} : i = 1, \ldots, k\} \) is a “distance function” from \( Z \) and \( d_n(x) = \min\{|x-x_n^i| : i = 1, \ldots, m\} \) is the distance function from \( \{x_1, \ldots, x_m\} \).

More precisely, letting \( r_n^i \) be associated to \( x_n^i \) by means of (3.1), for any \( i, j = 1, \ldots, m, i \neq j \), there holds:
\[
(\varepsilon_n^i)^3 \lambda_n^{-1} \to 0, \quad U_n^i(y) = \frac{1 - u_n(r_n^i y + x_n^i)}{\varepsilon_n^i} \to U^i(y) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N), \quad r_n^i + r_n^j \to 0
\]
as \( n \to +\infty \), where \( \varepsilon_n^i := 1 - u_n(x_n^i) \) and \( U^i \) satisfies an equation of type (3.4). In addition, there exist \( m \)-sequences of test functions \( \phi_1^i, \ldots, \phi_m^i \in C^\infty(\bar{\Omega}) \) so that
\[
\int_{\Omega} \left( |\nabla \phi_n^i|^2 - \frac{2\lambda_n f(x)}{(1 - u_n)^3} (\phi_n^i)^2 \right) < 0, \quad \text{Supp } \phi_n^i \subset B_{Mr_n^i}(x_n^i), \quad \forall i = 1, \ldots, m,
\]
for some \( M > 0 \) large.

**Proof:** Let \( \varepsilon_n = 1 - u_n(x_n) \), where \( x_n \) is a maximum point of \( u_n \). By the inequality
\[
0 \leq \frac{\lambda_n f(x)}{(1 - u_n)^2} \leq \frac{\lambda_n}{\nu} \| f \|_{\infty},
\]
we get that:
\[
(\varepsilon_n^i)^3 \lambda_n^{-1} \to 0 \quad \text{as } n \to +\infty. \tag{3.15}
\]
By contradiction, if \( (\varepsilon_n^i)^3 \lambda_n^{-1} \geq \delta > 0 \) along a subsequence, the right hand side of (1.1) would converge uniformly to zero as \( n \to +\infty \) and, by elliptic regularity theory, \( u_n \to u \) in \( C^1(\bar{\Omega}) \) (up to a further subsequence), where \( u \) is an harmonic function so that \( u = 0 \) on \( \partial \Omega \), \( \max u = 1 \). A contradiction. Hence, (3.15) must hold.

As needed in Proposition 3.1, (3.15) now implies:
\[
(\varepsilon_n^i)^3 (\text{dist}(x_n, \partial \Omega))^{-2} \to 0 \quad \text{as } n \to +\infty. \tag{3.16}
\]
In order to prove it, we will use the following Lemma and we refer to Appendix for the proof:
Lemma 3.4. Let $A_n$ be a bounded domain in $\mathbb{R}^N$ so that $A_n \to T$ as $n \to +\infty$, where $T$ is an hyperspace so that $0 \in T$ and dist $(0, \partial T) = 1$. Let $h_n$ be a function on $A_n$ and $W_n$ be a solution of:

$$\begin{cases}
\Delta W_n = \frac{h_n(y)}{W_n^2} & \text{in } A_n, \\
W_n(y) \geq C > 0 & \text{in } A_n, \\
W_n(0) = 1,
\end{cases}$$

for some $C > 0$. Assume that $\sup_{n \in \mathbb{N}} \| h_n \|_{\infty} < +\infty$ and $\partial A_n \cap B_2(0)$ is smooth. Then, either

$$\min_{\partial A_n \cap B_2(0)} W_n \leq C$$

or

$$\min_{\partial A_n \cap B_2(0)} \partial_{\nu} W_n \leq 0,$$

where $\nu$ is the unit outward normal of $A_n$.

Assume by contradiction that (3.16) is false, namely, up to a subsequence, $\varepsilon_n^3 \lambda_n^{-1} d_n^{-2} \to \delta > 0$ as $n \to +\infty$, where $d_n = \text{dist} (x_n, \partial \Omega)$. In view of (3.15) we get $d_n \to 0$ as $n \to +\infty$. Introduce a rescaled function $W_n$:

$$W_n(y) = \frac{1 - u_n(d_n y + x_n)}{\varepsilon_n}, \quad y \in A_n = \frac{\Omega - x_n}{d_n}.$$  

Since $d_n \to 0$ and

$$\text{dist} (0, \partial \Omega_n) = \frac{\text{dist} (x_n, \partial \Omega)}{d_n} = 1,$$

we get that $A_n \to T$ as $n \to +\infty$, where $T$ is an hyperspace so that $0 \in T$ and dist $(0, \partial T) = 1$. The function $W_n$ solves problem (3.17) with $h_n(y) = \frac{\lambda_n d_n^2}{\varepsilon_n} f(d_n y + x_n)$ and $C = W_n(0) = 1$. We have that for $n$ large:

$$\| h_n \|_{\infty} \leq \frac{\lambda_n d_n^2}{\varepsilon_n} \| f \|_{\infty} \leq 2 \| f \|_{\infty}.$$

Since $W_n = \frac{1}{\varepsilon_n} \to +\infty$ on $\partial A_n$, by Lemma 3.4 we get that (3.19) must hold. A contradiction to Hopf Lemma applied to $u_n$. Hence, the validity of (3.16).

Let $r_n$ be associated to $x_n$ according to (3.1). Up to a subsequence, Proposition 3.1 gives:

$$\frac{1 - u_n(r_n y + x_n)}{\varepsilon_n} \to U(y)$$

in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$,

where $U$ satisfies an equation of type (3.4), and provides the existence of $\phi_n \in C^\infty_0(\Omega)$ such that (3.5) holds with $\text{Supp} \phi_n \subset B_{Mr_n}(x_n)$, $M > 0$.

Let now $x^1_n = x_n$, $\varepsilon^1_n = \varepsilon_n$, $r^1_n = r_n$, $U^1 = U$ and $\phi^1_n = \phi_n$. If (3.12) is true for some subsequence of $u_n$ with $x^1_n$, take $m = 1$ and the proof is done. Otherwise, we proceed by an inductive method. Indeed, assume that, up to a subsequence, we have already found $l$-sequences $x^1_n, \ldots, x^l_n$ associated $r^1_n, \ldots, r^l_n$ (defined by (3.1)) and test functions $\phi^1_n, \ldots, \phi^l_n \in C^\infty_0(\Omega)$ so that (3.13)-(3.14) hold at $l$-th step. If (3.12) holds for some subsequence of $u_n$ with $x^1_n, \ldots, x^l_n$, we take $m = l$ and the proof is done. Otherwise, up to a subsequence, we will show the existence of $x^{l+1}_n, r^{l+1}_n$ and $\phi^{l+1}_n$ so that (3.13)-(3.14) are still true at $(l+1)$-th step. Since (3.13)-(3.14) at $l$-th step imply that $\phi^1_n, \ldots, \phi^l_n$ have mutually disjoint compact supports, we get that
m(u_n, \lambda_n) \geq l$. Then, by (1.3) the inductive process must stop after a finite number of steps, say $m$ steps, with $m \leq k$, and (3.12) holds with $x_1^n, \ldots, x_m^n$.

In order to complete the proof, we need to show how the induction process works. Assume that (3.13)-(3.14) hold at $l$-th step and (3.12) is not true for any subsequence of $u_n$ with $x_1^n, \ldots, x_l^n$. Let $x_{l+1}^n \in \Omega$ be such that

$$\lambda_n^{-\frac{1}{2}} (d(x_{l+1}^n)\alpha)^{-\frac{1}{2}} d_n(x_{l+1}^n)^{-\frac{1}{2}} \left(1 - u_n(x_{l+1}^n)\right) = \lambda_n^{-\frac{1}{2}} \min_{x \in \Omega} \left((d(x)\alpha)^{-\frac{1}{2}} d_n(x)^{-\frac{1}{2}} \left(1 - u_n(x)\right)\right) \rightarrow 0$$

as $n \to +\infty$, where $d_n(x)$ is the distance function from $(x_1^n, \ldots, x_l^n)$. Let $\varepsilon_{l+1} := 1 - u_n(x_{l+1}^n)$.

Formula (3.20) gives a lot of informations about the blow up around $x_{l+1}^n$. First of all, it can be rewritten in the more convenient form:

$$\frac{(\varepsilon_{l+1}^n)^{\frac{1}{2}} \lambda_n^{-\frac{1}{2}}}{|x_{l+1}^n - x_i^n| x_{l+1}^n - p_j|^\frac{1}{2}} \rightarrow 0 \quad \text{as } n \to +\infty, \forall i = 1, \ldots, l, j = 1, \ldots, k.$$  \hspace{1cm} (3.21)

The inductive assumption gives $\frac{r_i + r_j}{|x_i^n - x_j^n|} \to 0$ as $n \to +\infty$ for any $i, j = 1, \ldots, l, i \neq j$. Then, by definition of $r^n_k$, we get for $|y| \leq R$ and $n \geq n_R$:

$$\lambda_n^{-\frac{1}{2}} \left(d(r^n_j y + x_i^n)\alpha\right)^{-\frac{1}{2}} d_n(r^n_j y + x_i^n)^{-\frac{1}{2}} \left(1 - u_n(r^n_j y + x_i^n)\right)$$

$$= \left\{ \begin{array}{ll}
\left|\frac{d(r^n_j y + x_i^n)\alpha}{y + x_i^n - p}\right|^{-\frac{1}{2}} U_{l}^j(y) & \text{if } x_i^n \to p \notin Z \\
\left|\frac{d(r^n_j y + x_i^n)\alpha}{y + x_i^n - p}\right|^{-\frac{1}{2}} U_{l}^j(y) & \text{if } x_i^n \to p \in Z, (\varepsilon_i^n)^{-3} \lambda_n|x_i^n - p_i|^{\alpha + 2} \to +\infty \\
\left|\frac{d(r^n_j y + x_i^n)\alpha}{y + x_i^n - p}\right|^{-\frac{1}{2}} U_{l}^j(y) & \text{if } (\varepsilon_i^n)^{-3} \lambda_n|x_i^n - p_i|^{\alpha + 2} \leq C
\end{array} \right.$$  \hspace{1cm} (3.22)

for any $j = 1, \ldots, l$. By inductive assumption, we have that $U_{l}^j(y) = \frac{1 - u_n(r^n_j y + x_i^n)}{\varepsilon_i^n} \to U^j(y)$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ for any $j = 1, \ldots, l$. Associating (eventually) to $x_i^n$ the limit point $y_0$ as in (3.10), we get that:

$$\lambda_n^{-\frac{1}{2}} \left(d(r^n_j y + x_i^n)\alpha\right)^{-\frac{1}{2}} d_n(r^n_j y + x_i^n)^{-\frac{1}{2}} \left(1 - u_n(r^n_j y + x_i^n)\right)$$

$$\rightarrow \left\{ \begin{array}{ll}
\left|\frac{(d(y)\alpha)}{y - U^j(y)}\right|^{-\frac{1}{2}} U^j(y) & \text{if } x_i^n \to p \notin Z \\
\left|\frac{d(y)\alpha}{y - U^j(y)}\right|^{-\frac{1}{2}} U^j(y) & \text{if } x_i^n \to p \in Z, (\varepsilon_i^n)^{-3} \lambda_n|x_i^n - p_i|^{\alpha + 2} \to +\infty \\
\left|\frac{d(y)\alpha}{y - U^j(y)}\right|^{-\frac{1}{2}} U^j(y) & \text{if } (\varepsilon_i^n)^{-3} \lambda_n|x_i^n - p_i|^{\alpha + 2} \leq C
\end{array} \right.$$  \hspace{1cm} (3.22)

uniformly for $|y| \leq R$ as $n \to +\infty$. Since $U^j$ is bounded away from zero, then (3.20) gives also that $x_{l+1}^n$ cannot asymptotically lie in balls centered at $x_i^n$ of radius $\approx r_i^n$, $i = 1, \ldots, l$, namely:

$$\frac{r_i^n}{|x_{l+1}^n - x_i^n|} \rightarrow 0 \quad \text{as } n \to +\infty, \forall i = 1, \ldots, l.$$  \hspace{1cm} (3.22)

Finally, the choice of $x_{l+1}^n$ as a minimum point in (3.20) gives that:

$$\frac{1 - u_n(\beta_n y + x_{l+1}^n)}{\varepsilon_{l+1}^n} \geq \left(\frac{(d(\beta_n y + x_{l+1}^n)\alpha)}{d(x_{l+1}^n)\alpha}\right)^{\frac{1}{2}} \left(\frac{d_n(\beta_n y + x_{l+1}^n)}{d_n(x_{l+1}^n)\alpha}\right)^{\frac{1}{2}},$$  \hspace{1cm} (3.23)

for any sequence $\beta_n$. Indeed, by the following chain of estimates:

$$\varepsilon_{l+1}^n \leq \left(\frac{(d(x_{l+1}^n)\alpha)^{\frac{1}{2}} d_n(x_{l+1}^n)\alpha}{d(x_{l+1}^n)\alpha}\right) \min_{x \in \Omega} \left((d(x)\alpha)^{-\frac{1}{2}} d_n(x)^{-\frac{1}{2}} \left(1 - u_n(x)\right)\right)$$

$$\leq \left(\frac{(d(x_{l+1}^n)\alpha)^{\frac{1}{2}} d_n(x_{l+1}^n)\alpha}{d(x_{l+1}^n)\alpha}\right) \left(\frac{d_n(\beta_n y + x_{l+1}^n)\alpha}{d_n(x_{l+1}^n)\alpha}\right)^{\frac{1}{2}} \left(1 - u_n(\beta_n y + x_{l+1}^n)\right),$$
the validity of (3.23) follows. Here and in the sequel of the proof, the crucial point to establish the validity of (3.3) (or (3.11)) for suitable rescaled functions around \( x_n^{l+1} \) is exactly given by the validity of (3.23). By (3.21), we get that in particular \((\varepsilon_n^{l+1})^3 \lambda_n^{-1} \to 0\) as \(n \to +\infty\). We need now to discuss all the possible types of blow up at \( x_n^{l+1} \).

1st Case Assume that \( x_n^{l+1} \to q \notin Z \). Then, \(|x_n^{l+1} - p_j| \geq C > 0\) for any \( j = 1, \ldots, k \) which reduces (3.21) to:

\[
\frac{(\varepsilon_n^{l+1})^2 \lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^j|} \to 0 \quad \text{as } n \to +\infty, \quad \forall \ i = 1, \ldots, l.
\]

(3.24)

In order to apply Proposition 3.1 to \( x_n^{l+1} \), first of all we need to show that (3.2) holds for \( x_n^{l+1} \):

\[
(\varepsilon_n^{l+1})^3 \lambda_n^{-1} \left( \text{dist} \ (x_n^{l+1}, \partial \Omega) \right)^{-2} \to 0 \quad \text{as } n \to +\infty.
\]

We proceed exactly as in the proof of (3.16). By contradiction, up to a subsequence, assume that \((\varepsilon_n^{l+1})^3 \lambda_n^{-1} \to \delta > 0\) as \(n \to +\infty\), where \(d_n = \text{dist} \ (x_n^{l+1}, \partial \Omega)\) (do not confuse \(d_n\) with \(d_n(x)\)), and then by (3.24):

\[
\frac{d_n}{|x_n^{l+1} - x_n|} = \frac{d_n}{(\varepsilon_n^{l+1})^2 \lambda_n^{-\frac{1}{2}} |x_n^{l+1} - x_n^j|} \leq \frac{2}{\sqrt{\delta}} \frac{(\varepsilon_n^{l+1})^2 \lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^j|} \to 0 \quad \text{as } n \to +\infty, \quad \forall \ i = 1, \ldots, l.
\]

Let \(M_n = \left( \frac{d_n(x_n^{l+1})}{d_n} \right)^{\frac{1}{2}} \to +\infty\) as \(n \to +\infty\). We introduce the rescaling \(W_n\) of \(u_n\) in the form:

\[
W_n(y) = \frac{1 - u_n(d_n y + x_n^{l+1})}{\varepsilon_n^{l+1}} \quad \text{for } y \in A_n = \frac{\Omega - x_n^{l+1}}{d_n} \cap B_{M_n}(0).
\]

Since \(d_n = \text{dist} \ (x_n^{l+1}, \partial \Omega) \to 0\) and \(M_n \to +\infty\), we get that \(A_n \to T\) as \(n \to +\infty\), where \(T\) is an hyperspace so that \(0 \in T\) and \(\text{dist} \ (0, \partial T) = 1\). Since \(\{d_n y + x_n^{l+1} : y \in A_n\}\) is uniformly far away from \(Z = \{p_1, \ldots, p_k\}\), by (3.23) we get for \(W_n\) (here, \(\beta_n\) is exactly \(d_n\)):

\[
W_n(y) \geq C_0 \left( 1 - \frac{d_n M_n}{d_n(x_n^{l+1})^\frac{2}{3}} \right) \geq C_0 \cdot \frac{2}{3}
\]

for any \(n\) large and \(y \in A_n\). We have used here the following estimate:

\[
\frac{d_n(\beta_n y + x_n^{l+1})}{d_n(x_n^{l+1})} = \min \left\{ \frac{|x_n^{l+1} - x_n^i|}{d_n(x_n^{l+1})}, \frac{\beta_n}{d_n(x_n^{l+1})} \right\} \geq 1 - \frac{\beta_n}{d_n(x_n^{l+1})} \cdot |y|.
\]

(3.25)

Hence, the function \(W_n\) solves problem (3.17) with \(h_n(y) = \frac{\lambda_n d_n^2}{(\varepsilon_n^{l+1})^3} f(d_n y + x_n^{l+1})\) and \(C = \frac{C_0}{2}\). Since

\[
\| h_n \|_\infty \leq \frac{\lambda_n d_n^2}{(\varepsilon_n^{l+1})^3} \| f \|_\infty \leq \frac{2}{\delta} \| f \|_\infty
\]

and \(W_n = \frac{1}{\varepsilon_n^{l+1}} \to +\infty\) on \(\partial A_n \cap B_2(0)\), Lemma 3.4 provides that (3.19) must hold, contradicting Hopf Lemma for \(u_n\). Hence, (3.2) holds for \(x_n^{l+1}\).

Associated to \(x_n^{l+1}\), let \(r_n^{l+1} = (\varepsilon_n^{l+1})^2 \lambda_n^{-\frac{1}{2}}\) be defined according to (3.1). By (3.24) we get that

\[
\frac{r_n^{l+1}}{|x_n^{l+1} - x_n^i|} \to 0 \quad \text{as } n \to +\infty, \quad \forall \ i = 1, \ldots, l.
\]

(3.26)
and then, \( R_n = \left( \frac{d(x_{n+1}^l)}{r_n^l} \right)^\frac{1}{2} \to +\infty \) as \( n \to +\infty \). Since \( \{x_{n+1}^l + x_{n+1}^l : |y| \leq R_n\} \) is uniformly far away from \( Z \), by (3.23) and (3.25) we get that:

\[
U_{n+1}^l(y) := \frac{1 - u_n(r_{n+1}^l y + x_{n+1}^l)}{e_{n+1}^l} \geq C_0 \left( 1 - \frac{r_{n+1}^l R_n}{d_n(x_{n+1}^l)} \right)^\frac{\xi}{2} \geq \frac{C_0}{2}
\]

for \( n \) large and \( y \in \frac{\Omega - x_{n+1}^l}{r_n^l} \cap B_{R_n}(0) \). Up to a subsequence, Proposition 3.1 provides \( U_{n+1}^l \to U_{l+1}^l \) in \( C^1_{loc}(\mathbb{R}^N) \) as \( n \to +\infty \), \( U_{l+1}^l \) being a solution of an equation of type (3.4), and some \( \phi_{n+1}^l \in C_0^\infty(\Omega) \) such that (3.5) holds with \( \text{Supp } \phi_{n+1}^l \subset B_{M r_{n+1}^l}(x_{n+1}^l), M > 0 \). By (3.26), combined with (3.22), we get that (3.13)-(3.14) are still true at (\( l + 1 \))-th step, as needed.

2nd Case Assume that \( x_{n+1}^l \to p_j \) with the following rate:

\( (\epsilon_{n+1}^l)^{-3} \lambda_n |x_{n+1}^l - p_j|^{\alpha_j+2} \to +\infty \) as \( n \to +\infty \).

Let \( r_{n+1}^l = (\epsilon_{n+1}^l)^\frac{1}{2} \lambda_n^{\frac{1}{2}} |x_{n+1}^l - p_j|^{-\frac{\alpha_j}{2}} \) according to (3.1). By (3.21) we get that (3.26) still holds and then, \( R_n = \min\{r_{n+1}^l, d_n(x_{n+1}^l)\} \to +\infty \) as \( n \to +\infty \). Since \( \{x_{n+1}^l + x_{n+1}^l : |y| \leq R_n\} \) is uniformly close to \( p_j \in Z \), estimates (3.23) and (3.25) imply:

\[
U_{n+1}^l(y) := \frac{1 - u_n(r_{n+1}^l y + x_{n+1}^l)}{e_{n+1}^l} \geq \left( \frac{|r_{n+1}^l y + x_{n+1}^l - p_j|}{|x_{n+1}^l - p_j|} \right)^\frac{\alpha_j}{2} \left( \frac{d_n(r_{n+1}^l y + x_{n+1}^l)}{d_n(x_{n+1}^l)} \right)^\frac{\xi}{2} \geq \frac{1}{2^n}
\]

for \( n \) large and \( |y| \leq R_n \). Up to a subsequence, Proposition 3.1 provides \( U_{l+1}^l \to U_{l+1}^l \) in \( C^1_{loc}(\mathbb{R}^N) \) as \( n \to +\infty \), where \( U_{l+1}^l \) solves an equation of type (3.4), and the existence of \( \phi_{n+1}^l \in C_0^\infty(\Omega) \) such that (3.5) holds, \( \text{Supp } \phi_{n+1}^l \subset B_{M r_{n+1}^l}(x_{n+1}^l) \) for some \( M > 0 \). Finally, (3.22) with (3.26) gives that (3.13)-(3.14) are still true at (\( l + 1 \))-th step, also in this second case.

3rd Case Assume that \( x_{n+1}^l \to p_j \) and

\( (\epsilon_{n+1}^l)^{-3} \lambda_n |x_{n+1}^l - p_j|^{\alpha_j+2} \leq C. \)

By (3.21) \( x_{n+1}^l \neq p_j \) and for any \( i = 1, \ldots, l \) there holds:

\[
\frac{|x_{n+1}^l - p_j|}{|x_{n+1}^l - x_{n+1}^l|} = \frac{(\epsilon_{n+1}^l)^\frac{1}{2} \lambda_n^{\frac{1}{2}}}{(|x_{n+1}^l - x_{n+1}^l| |x_{n+1}^l - p_j|^{\alpha_j} \geq (\epsilon_{n+1}^l)^{-3} \lambda_n |x_{n+1}^l - p_j|^{\alpha_j+2})^\frac{1}{2}} \to 0 \) as \( n \to +\infty. \)

(3.27)

Let \( r_{n+1}^l = (\epsilon_{n+1}^l)^\frac{1}{2} \lambda_n^{\frac{1}{2}} \) according to (3.1). By (3.21) and (3.27) we get that for any \( i = 1, \ldots, l \):

\[
\frac{r_{n+1}^l}{|x_{n+1}^l - x_{n+1}^l|} = \left( \frac{(\epsilon_{n+1}^l)^\frac{1}{2} \lambda_n^{\frac{1}{2}}}{(|x_{n+1}^l - x_{n+1}^l| |x_{n+1}^l - p_j|^{\alpha_j} \geq (\epsilon_{n+1}^l)^{-3} \lambda_n |x_{n+1}^l - p_j|^{\alpha_j+2})^\frac{1}{2}} \to 0 \) as \( n \to +\infty. \)

(3.28)
providing the validity of (3.26). Let $R_n = \left(\frac{d_n(x_n^{l+1})}{r_n^{l+1}}\right)^{\frac{1}{2}} \to +\infty$ as $n \to +\infty$. Since \( \{r_n^{l+1}y + x_n^{l+1} : |y| \leq R_n\} \) is uniformly close to $p_j \in Z$, by (3.23) and (3.25) we get:

\[
U_n^{l+1}(y) := \frac{1 - u_n(r_n^{l+1}y + x_n^{l+1})}{r_n^{l+1}} \geq \frac{\left(\frac{|r_n^{l+1}y + x_n^{l+1} - p_j|}{|x_n^{l+1} - p_j|}\right)^{\alpha_j}}{(1 - \frac{r_n^{l+1}R_n}{d_n(x_n^{l+1})})^{\frac{3}{2}}} \geq \frac{1}{2} \left(\frac{|r_n^{l+1}y - p_j|}{r_n^{l+1}}\right)^{-\alpha_j} |y + \frac{r_n^{l+1}y - p_j}{r_n^{l+1}}|^{\alpha_j} \geq C |y + \frac{x_n^{l+1} - p_j}{r_n^{l+1}}|^{\alpha_j}
\]

for $n$ large and $|y| \leq R_n$, where $C > 0$ is a constant. We have used that $\frac{|r_n^{l+1}y - p_j|}{r_n^{l+1}} \leq C$, which is true for assumption in this case. We use now Proposition 3.1 in combination with Remark 3.1 to get that, up to a subsequence, $U_n^{l+1} \to U^{l+1}$ in $C^{0,\beta}_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ and $U^{l+1}$ is a solution of an equation of type (3.4). Moreover, we find $\phi_n^{l+1} \in C_0^\infty(\Omega)$ such that (3.5) holds and $\text{Supp} \phi_n^{l+1} \subset B_M(x_n^{l+1})$, $M > 0$. Since (3.22) together with (3.28) gives the validity of (3.13)-(3.14) at $(l + 1)$-th step, the induction scheme also works in this last case and the proof of Theorem 3.3 is complete. \[\blacksquare\]

### 3.3 Compactness of unstable branches

We are now in position to give the proof of Theorem 1.1. The essential ingredient will be the pointwise estimate of Theorem 3.3. The contradiction will come out from the non existence result of Theorem 2.3.

**Proof of Theorem 1.1:** By contradiction, up to a subsequence, let us assume that $\max u_n \to 1$ as $n \to +\infty$. Up to a further subsequence, Theorem 3.3 gives the existence of $m$-sequences $x_n^1, \ldots, x_n^m$ so that $x_n^i \to x^i \in \Omega$ as $n \to +\infty$ and the following pointwise estimate holds:

\[
1 - u_n(x) \geq C \lambda_n^\frac{\beta}{2} (d(x)^{\alpha})^{\frac{1}{2}} d_n(x)^{\frac{3}{2}}
\]

for any $x \in \Omega$ and $n \geq N_0$, for some $C > 0$ and $N_0 \in \mathbb{N}$ large, where $d(x)^{\alpha} = \min\{|x - p_i|^{\alpha_i} : i = 1, \ldots, k\}$ and $d_n(x) = \min\{|x - x_n^i| : i = 1, \ldots, m\}$. Therefore, we get the following bounds in $\Omega$:

\[
0 \leq \frac{\lambda_n f(x)}{(1 - u_n)^2} \leq C \frac{f(x)}{(d(x)^{\alpha})^{\frac{1}{2}}} \frac{\lambda_n^\frac{3}{4}}{d_n(x)^{\frac{3}{2}}},
\]

for some $C > 0$. Since by (1.2)

\[
\left|\frac{f(x)}{(d(x)^{\alpha})^{\frac{1}{2}}}\right| \leq |x - p_i|^{\alpha_i} \|f_i\|_{\infty} \leq C
\]

for $|x - p_i| \leq \delta$ and $f_i$ as in (3.8), we get that $\frac{f(x)}{(d(x)^{\alpha})^{\frac{1}{2}}}$ is a bounded function on $\Omega$. Hence, by (3.30)

$\frac{\lambda_n f(x)}{(1 - u_n)^2}$ is uniformly bounded in $L^s(\Omega)$, for any $1 < s < \frac{3N}{4}$. By elliptic regularity theory and Sobolev embeddings, up to a limit function $u_0 \in C(\Omega) \cap H^1_0(\Omega)$ as $n \to +\infty$. In particular, it holds that $\sup u_0 = 1$, by means of the uniform convergence of $u_n$ to $u_0$. Since $u_0 = 0$ on $\partial \Omega$, the maximum value 1 of $u_0$ is achieved in $\Omega$ and then, $S = \{x \in \Omega : u_0(x) = 1\}$ is a non empty set.

If $\lambda = \lim_{n \to +\infty} \lambda_n = 0$, by (3.30) $\frac{\lambda_n f(x)}{(1 - u_n)^2} \to 0$ in $L^s(\Omega)$ as $n \to +\infty$, for any $1 < s < \frac{3N}{4}$. So, $u_0 \in H^1_0(\Omega)$ is a
weak harmonic function and then, it should vanish identically, in contradiction to \( u_0 = 0 \).

Hence, we have that \( \lambda = \lim_{n \to +\infty} \lambda_n > 0 \), and by (3.29) we get that \( u_0 < 1 \) in \( \Omega \setminus \{ x^1, \ldots, x^m, p_1, \ldots, p_k \} \). In particular, the set \( S \) is finite because \( S \subset \{ x^1, \ldots, x^m, p_1, \ldots, p_k \} \).

Since \( \frac{\lambda_n f(x)}{(1 - u_0)^2} \) is uniformly bounded in \( L^s(\Omega) \) for any \( 1 < s < \frac{3N}{4} \) and \( \frac{\lambda_n f(x)}{(1 - u_0)^2} \to \frac{\lambda f(x)}{(1 - u_0)^2} \) uniformly on compact sets in \( \Omega \setminus \{ x^1, \ldots, x^m, p_1, \ldots, p_k \} \), we get that

\[
\frac{\lambda_n f(x)}{(1 - u_0)^2} \to \frac{\lambda f(x)}{(1 - u_0)^2} \quad \text{weakly in } L^s(\Omega), \quad 1 < s < \frac{3N}{4},
\]

Taking now the limit of the equation satisfied by \( u_n \), by (3.31) we get that \( u_0 \in C(\Omega) \) is a \( H^1(\Omega) \)-weak solution of:

\[
\begin{cases}
-\Delta u_0 = \frac{\lambda f(x)}{(1 - u_0)^2} & \text{in } \Omega, \\
u_0 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Since

\[
\int_\Omega \left( |\nabla \phi|^2 - \frac{2\lambda_n f(x)}{(1 - u_0)^2} \phi^2 \right) \to \int_\Omega \left( |\nabla \phi|^2 - \frac{2\lambda f(x)}{(1 - u_0)^2} \phi^2 \right)
\]

for any \( \phi \in C_c^\infty(\Omega) \) in view of (3.31), by (1.3) we get that \( u_0 \) has a finite Morse index according to definition (2.13). Since the set \( S = \{ x \in \Omega : u(x) = 1 \} \) is a non empty finite set, by Theorem 2.3 such a solution \( u_0 \) cannot exist and we reach a contradiction. Hence, (1.4) holds.

If we also assume that \( \mu_{1,n} < 0 \), then \( \lambda > 0 \). Indeed, if \( \lambda_n \to 0 \), then by compactness and elliptic regularity theory, we would get \( u_n \to u_0 \) in \( C^1(\Omega) \), where \( u_0 \) is an harmonic function so that \( u_0 = 0 \) on \( \partial \Omega \). Then, \( u_0 = 0 \). A contradiction.

\section*{4 Some consequences}

In the last Section, we derive some consequences of Theorem 1.1. First, let us prove the characterization of blow up stated in Theorem 1.2.

\textbf{Proof (of Theorem 1.2):}

(1) \( \Rightarrow \) (2) Assume that \( \max_{\Omega} u_n \to 1 \) as \( n \to +\infty \). If \( f \left( \frac{f(x)}{(1 - u_0)^2} \right) \to C < \infty \) along a subsequence, the R.H.S. of (1.1) would be uniformly bounded in \( L^{\frac{4N}{N-2}} \). By elliptic regularity theory and Sobolev embeddings, \( u_n \to u_0 \) weakly in \( H^1_0(\Omega) \) and strongly in \( C(\Omega) \), where \( u_0 \) is a \( H^1(\Omega) \)-weak solution of (1.1) with \( \lambda = \lim_{n \to +\infty} \lambda_n \) so that

\[
\int_\Omega \left( \frac{f(x)}{(1 - u_0)^2} \right)^{\frac{N}{2}} < \infty \quad \text{and} \quad 0 \leq u_0 \leq 1.
\]

By Proposition 2.1 we get \( \| u_0 \|_\infty < 1 \) and, by uniform convergence, \( \| u_n \|_\infty \to \| u_0 \|_\infty < 1 \) as \( n \to +\infty \). A contradiction. Hence, necessarily \( \int_\Omega \frac{f(x)}{(1 - u_0)^2} \to +\infty \) as \( n \to +\infty \).

(2) \( \Rightarrow \) (1) The viceversa is trivial as it follows by the following inequality:

\[
\int_\Omega \left( \frac{f(x)}{(1 - u_0)^2} \right)^{\frac{N}{2}} \leq \frac{\| f \|_\infty^{\frac{N}{2}}}{(1 - \| u_0 \|_\infty)^{\frac{N}{2}}} |\Omega|,
\]

where \( \| \cdot \| \) stands for the Lebesgue measure.

(1) \( \Rightarrow \) (3) Assume that \( \max_{\Omega} u_n \to 1 \) as \( n \to +\infty \). By Theorem 1.1 \( m(u_n, \lambda_n) \to +\infty \) as \( n \to +\infty \).
(3) ⇒ (1) Since as before \( \frac{f(x)}{(1-u_n)^+} \leq \frac{f_{\infty}}{(1-\|u_n\|_\infty)^+} \), by the variational characterization of the eigenvalues we get that

\[
\mu_{k,\lambda_n}(u_n) \geq \mu_k(L_n), \quad L_n := -\Delta - \frac{2\lambda_n f_{\infty}}{(1-\|u_n\|_\infty)^3},
\]

where \( \mu_k(L_n) \) stands for the \( k \)-th eigenvalue of the operator \( L_n \). Indeed, for operator \( L \) in the form \( L = -\Delta - c(x) \),\( c(x) \in L^s(\Omega) \) for some \( s > \frac{N}{2} \), let us recall that:

\[
\mu_1(L) = \inf_{\phi \in H_0^1(\Omega), \phi \neq 0} \frac{\langle L\phi, \phi \rangle}{\int_\Omega \phi^2},
\]

\[
\mu_k(L) = \sup \left\{ \inf_{\phi \in M^-, \phi \neq 0} \frac{\langle L\phi, \phi \rangle}{\int_\Omega \phi^2} : M \subset H_0^1(\Omega) \text{ linear, } \dim(M) = k-1 \right\} \quad \forall \ k \geq 2,
\]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( H_0^1(\Omega) \) and \( M^+ \) is the hortogonal space of \( M \) in \( H_0^1(\Omega) \) with respect to this inner product.

Therefore, point (3) implies that the Morse index of \( L_n \), the number of negative eigenvalues of \( L_n \), blows up as \( n \to +\infty \). Hence, the constant function \( \frac{2\lambda_n f_{\infty}}{(1-\|u_n\|_\infty)^3} \to +\infty \) as \( n \to +\infty \) and then, the validity of point (1) is established.

Now, we establish the uniqueness result contained in Theorem 1.3.

**Proof (of Theorem 1.3):**

(1) Let \( \lambda_n \to 0 \) as \( n \to +\infty \) and associated solutions \( u_n \) of (1.1) so that \( m(u_n, \lambda_n) \leq k \), \( k \in \mathbb{N} \). Theorem 1.1 implies that \( \mu_{1,n} \geq 0 \) for \( n \) large. By the characterization of the minimal solution \( u_1 \) as the only semi-stable solution, we get that \( u_n = u_{\lambda_n} \) for \( n \) large. Hence, necessarily there exists \( \delta = \delta_k > 0 \) so that \( u_\lambda \) is the unique solution of (1.1) with \( m(u, \lambda) \leq k \) for any \( \lambda \in (0, \delta) \).

(2) Let \( \lambda_n \to \lambda^* \) as \( n \to +\infty \) and associated solutions \( u_n \) with \( m(u_n, \lambda_n) \leq k \), for some \( k \in \mathbb{N} \). By Theorem 1.1 we get that \( \sup_{n \in \mathbb{N}} \|u_n\|_\infty < 1 \). By elliptic regularity theory, \( u_n \) is uniformly bounded in \( C^{1,\beta}(\Omega) \) for any \( \beta \in (0, 1) \). Up to a subsequence, \( u_n \to u_0 \) in \( C^1(\bar{\Omega}) \) as \( n \to +\infty \), where \( u_0 \) is a \( C^1(\bar{\Omega}) \)-solution of (1.1) with \( \lambda = \lambda^* \) so that \( \max_{\Omega} u_0 < 1 \). In [8] it is proved that equation (1.1) admits for \( \lambda = \lambda^* \) an unique solution, the extremal solution \( u^* \). Then, \( u_n \to u^* \) in \( C^1(\bar{\Omega}) \) as \( n \to +\infty \). By [4], in a \( C^1 \)-small neighborhood of \( u^* \) problem (1.1) has only the two solutions \( u_\lambda, U_\lambda \) for \( \lambda \) close to \( \lambda^* \). Hence, either \( u_n = u_{\lambda_n} \) or \( u_n = U_{\lambda_n} \) and the uniqueness result follows.

Finally, we conclude this Section by showing the existence of a solutions sequence whose Morse index blows up.

**Proof (of Theorem 1.4):** Let us define the solution set \( \mathbb{V} \) as

\[
\mathbb{V} = \{(\lambda, u) \in [0, +\infty) \times E : u \text{ is a solution of (1.1)}\},
\]

where \( E = \{u \in C^1(\bar{\Omega}) : u = 0 \text{ on } \partial\Omega\} \) is endowed with the standard norm. By contradiction and in view of the equivalence of Theorem 1.2, let us assume that

\[
\sup_{(\lambda, u) \in \mathbb{V}} \max_\Omega u \leq 1 - 2\delta,
\]

for some \( \delta \in (0, \frac{1}{2}) \). Hence, \( \mathbb{V} \) is a compact set in \([0, +\infty) \times E \). By Theorem 1.3 we can fix \( \lambda_1, \lambda_2 \in (0, \lambda^*) \), \( \lambda_1 < \lambda_2 \), so that (1.1) possesses:
- for $\lambda_1$ only the (non degenerate) minimal solution $u_{\lambda_1}$ with $m(u_{\lambda_1}, \lambda_1) = 0$;
- for $\lambda_2$ has only the two (non degenerate) solutions $u_{\lambda_2}, U_{\lambda_2}$ with $m(u_{\lambda_2}, \lambda_2) = 0, m(U_{\lambda_2}, \lambda_2) = 1$.

Let us define the projection of $V$ onto $E$:

$$U = \{ u \in E : \exists \lambda \text{ so that } (\lambda, u) \in U \},$$

and let us consider a $\delta$-neighborhood of $U$ in $E$:

$$U_\delta = \{ u \in E : \text{dist}_E(u, U) \leq \delta \}.$$

Let us remark that by (4.1) we get:

$$\sup_{u \in U_\delta} \max_{\Omega} u \leq 1 - \delta.$$

Let us regularize the nonlinearity $(1-u)^{-2}$ in the following way:

$$g_\delta(u) = \begin{cases} (1-u)^{-2} & \text{if } u \leq 1 - \delta \\ \delta^{-2} & \text{if } u \geq 1 - \delta, \end{cases}$$

in such a way that, for any fixed $\lambda$, problem (1.1) in $U_\delta$ is equivalent to find a zero of the map $T_\lambda = Id - K_\lambda$:

$$E \rightarrow E,$$

where $K_\lambda(u) = -\Delta^{-1}(\lambda f(x)g_\delta(u))$ is a compact operator and $\Delta^{-1}$ is the laplacian resolvent with homogeneous Dirichlet boundary condition. We can define the Leray-Schauder degree $d_\lambda$ of $T_\lambda$ on $U_\delta$ with respect to zero, since by definition of $U$ (the set of all solutions) $\partial U_\delta$ does not contain any solution of (1.1) for any value of $\lambda$. Since $d_\lambda$ is well defined for any $\lambda \in [0, \lambda^*]$, by omotopy $d_{\lambda_1} = d_{\lambda_2}$.

To get a contradiction, let us now compute $d_{\lambda_1}, d_{\lambda_2}$. Since the only zero of $T_{\lambda_1}$ in $U_\delta$ is $u_{\lambda_1}$ with Morse index zero, we have that $d_{\lambda_1} = 1$. While, $T_{\lambda_2}$ has in $U_\delta$ exactly two zeroes $u_{\lambda_2}, U_{\lambda_2}$ with Morse index zero, one respectively, and hence, $d_{\lambda_2} = 1 - 1 = 0$. This contradicts $d_{\lambda_1} = d_{\lambda_2}$. The proof is complete. □

5 Appendix

First of all, we give a sketch of proof of Theorem 3.2 and we refer to [5] for the details.

**Proof (of Theorem 3.2):** By contradiction, we assume $\mu_1(U) \geq 0$ and then,

$$\int |\nabla \phi|^2 \geq 2 \int \frac{|y|^\gamma}{U^3} \phi^2, \quad \forall \phi \in D^{1,2}(\mathbb{R}^N).$$

(5.1)

In particular, by (5.1) we get that

$$\int \frac{|y|^\gamma}{(1 + |y|^2)^{\frac{2}{2} + \delta} U^3} \leq C \int \frac{1}{(1 + |y|^2)^{\frac{\beta}{2} + 3}} < +\infty,$$

(5.2)

for any $\delta > 0$.

**Step 1.** We want to show that (5.1) allows us to perform the following Moser-type iteration scheme: for any $0 < q < 4 + 2\sqrt{6}$ and $\beta$ there holds

$$\int \frac{1}{(1 + |y|^2)^{\beta - 1 - \frac{2}{2} U^q + 3}} \leq C_q \left(1 + \int \frac{1}{(1 + |y|^2)^{\beta U^q}}\right),$$

(5.3)

(provided the second integral is finite).
Indeed, let $R > 0$ and consider a smooth radial cut-off function $\eta$ so that: $0 \leq \eta \leq 1$, $\eta = 1$ in $B_R(0)$, $\eta = 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$. Multiplying (3.6) by $\frac{\eta^2}{(1 + |y|^2)^{\beta -1} U + 3}$, $q \geq 0$, integrating by parts and using (5.2) we get:

$$
\int \frac{|y|^\gamma \eta^2}{(1 + |y|^2)^{\beta -1} U + 3} \geq \frac{8(q + 1)}{q^2} \int \frac{|y|^\gamma \eta^2}{(1 + |y|^2)^{\beta -1} U + 3} - \frac{2q}{q} \int \frac{1}{U^q} \left| \frac{\eta}{(1 + |y|^2)^{\beta -1} U + 3} \right|^2 \Delta \left( \frac{\eta}{(1 + |y|^2)^{\beta -1} U + 3} \right)
$$

Since $8q + 8 > q^2 > 0$ for any $0 < q \leq q_+$, assuming that $R|\nabla \eta| + R^2|\Delta \eta| \leq C$ we get that:

$$
\int \frac{|y|^\gamma \eta^2}{(1 + |y|^2)^{\beta -1} U + 3} \leq C_q \int \frac{1}{(1 + |y|^2)^{\beta} U^q},
$$

where $C_q$ does not depend on $R > 0$. Taking the limit as $R \rightarrow +\infty$, we get the validity of (5.3).

**Step 2.** Let now $1 \leq N \leq 7$ or $N \geq 8$, $\gamma > \frac{3N - 14 - 4\sqrt{6}}{4 + 2\sqrt{6}}$. We want to show that

$$
\int \frac{1}{(1 + |y|^2)^{U^q}} < +\infty
$$

for some $0 < q < q_+ = 4 + 2\sqrt{6}$.

Given $\delta > 0$, set $\beta_i = \beta_0 - i(1 + \frac{\gamma}{2})$ and $q_i = q_0 + 3i$, $i \in \mathbb{N}$. By (5.2) and iterating Step 1 two times in view of $q_0 < q < q_+ = 4 + 2\sqrt{6} < q_2$, we get that:

$$
\int \frac{1}{(1 + |y|^2)^{\beta_2} U^{q_2}} < +\infty
$$

where $\beta_2 = \frac{N - 6 - 3\gamma}{2} + \delta$, $q_2 = 9$. Fix now $q$: $0 < q < q_+ = 4 + 2\sqrt{6} < 9$. By (5.5) and Hölder inequality we get that

$$
\int \frac{1}{(1 + |y|^2)^{U^q}} < +\infty
$$

provided $-\frac{2q}{4(q + 1)} \beta_2 + \frac{18}{q} > N$ or equivalently

$$
q > \frac{9N - 18}{6 - 2\delta + 3\gamma}.
$$

To have (5.6) for some $\delta > 0$ small and $q < q_+$ at the same time, we need to require $\frac{3N - 6}{2 + \gamma} < q_+$ or equivalently $1 \leq N \leq 7$ or $N \geq 8$, $\gamma > \frac{3N - 14 - 4\sqrt{6}}{4 + 2\sqrt{6}}$. Our assumptions then provide the existence of some $0 < q < q_+ = 4 + 2\sqrt{6}$ such that (5.4) holds.

**Step 3.** To obtain a contradiction, fix $0 < q < 4 + 2\sqrt{6}$ such that (5.4) holds. Letting $\eta$ as before, using equation (3.6) we compute:

$$
\int |\nabla \left( \frac{\eta}{U^{q/2}} \right)|^2 - \int \frac{2|y|^\gamma \eta}{U^q} \left( \frac{\eta}{U^{q/2}} \right)^2 = - \frac{8q + 8 - q^2}{4(q + 1)} \int |y|^\gamma \eta^2 \left( \frac{U^q}{U^{q/3}} + \int \frac{|y|^\gamma \eta^2}{U^q} - \frac{q + 2}{4(q + 1)} \int \frac{\Delta \eta^2}{U^q}.
$$

Since $8q + 8 - q^2 > 0$, by (5.4) we get that:

$$
\int |\nabla \left( \frac{\eta}{U^{q/2}} \right)|^2 - \int \frac{2|y|^\gamma \eta}{U^q} \left( \frac{\eta}{U^{q/2}} \right)^2 \leq - \frac{8q + 8 - q^2}{4(q + 1)} \int_{B_2(0)} \frac{|y|^\gamma \eta}{U^{q/3} U^{q/2}} + O(\int_{|y| \geq R} \frac{1}{(1 + |y|^2)^{U^q}} < 0
$$

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for $R$ large, contradicting (5.1). To complete the proof, in [5] it is proved that $\lambda_* = \frac{(2+\gamma)(3N+\gamma-4)}{9}$ and $u^*(x) = 1 - |x|^\frac{3+\gamma}{2}$ are the extremal value and solution, respectively, of (1.1) on the unit ball with $f(x) = |x|^\gamma$ and $N \geq 8$, $0 \leq \gamma \leq \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. The second part of Theorem 3.2 follows by considering the limit profile function around zero as $\lambda \to \lambda^*$ for the minimal solution $u_{\lambda}$ for (1.1) on the unit ball with $f(x) = |x|^\gamma$.

Finally, we prove Lemma 3.4:

**Proof (of Lemma 3.4):** Assume that $\partial_\nu W_n > 0$ on $\partial A_n \cap B_2(0)$. Let $G(y)$ be the Green function at 0 of the operator $-\Delta$ in $B_2(0)$ with homogeneous Dirichlet boundary condition. Since $\partial_\nu G < 0$ on $\partial T \cap B_2(0)$ and $\partial A_n \to \partial T$, we get that $\partial_\nu G < 0$ on $\partial A_n \cap B_2(0)$ for $n$ large and $\int_{\partial A_n \cap B_2(0)} \partial_\nu G d\sigma \to \int_{\partial T \cap B_2(0)} \partial_\nu G d\sigma < 0$. Since $G > 0$ in $B_2(0)$, $\partial_\nu G < 0$ on $\partial B_2(0)$ and the assumptions on $W_n$, by the representation formula we get:

$$1 = W_n(0) \geq -\int_{A_n \cap B_2(0)} \frac{h_n(y)}{W_n^2} G - \min_{\partial A_n \cap B_2(0)} W_n \int_{\partial A_n \cap B_2(0)} \partial_\nu G d\sigma.$$

But $|\int_{A_n \cap B_2(0)} \frac{h_n(y)}{W_n^2} G| \leq C$, and then, $1 \geq -C + C^{-1} \left( \min_{\partial A_n \cap B_2(0)} W_n \right)$ for some $C > 0$ large enough. Hence, $\min_{\partial A_n \cap B_2(0)} W_n$ is uniformly bounded providing the validity of (3.18). The proof is complete.

**References**


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