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COMPACTNESS OF A NONLINEAR EIGENVALUE PROBLEM WITH A SINGULAR NONLINEARITY

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We study the Dirichlet boundary value problem $-\Delta u = \frac{\lambda f(x)}{(1-u)^2}$ on a bounded domain $\Omega \subset \mathbb{R}^N$. For $2 \le N \le 7$, we characterize compactness for solutions sequence in terms of spectral informations. As a by-product, we give an uniqueness result for λ close to 0 and λ^* in the class of all solutions with finite Morse index, λ^* being the extremal value associated to the nonlinear eigenvalue problem.

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1. Introduction

Let us consider the following problem:

$$\begin{cases} -\Delta u = \dfrac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda \ge 0$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $f \in C(\bar{\Omega})$ is a nonnegative function. We will say that u is a solution of (1.1) if $u \in C^1(\bar{\Omega}) \cap W^{2,2}(\Omega)$ satisfies the equation a.e. in Ω with $u = 0$ on $\partial\Omega$ and $0 < u < 1$ in Ω.

The equation models the stationary regime of a simple electrostatic Micro-Electromechanical System (MEMS device), consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid plate located at +1 immersed in an external electric field, where u is the (normalized) deflection of the elastic membrane (see [7, 15] for a detailed discussion on MEMS devices). More generally, the model is described by a nonlinear parabolic problem which has been considered by Ghoussoub and Guo in [9].

Problems as in (1.1) with smooth nonlinearities (for example of the form e^u or $(1+u)^p$ for $p > 1$) have been largely studied in the last thirty years and fine properties of the branch of minimal solutions have been established. We refer to the seminal works [4, 11, 12] and to [3] for a complete account on the topic.

In [8], Ghoussoub and Guo extended this analysis to problem (1.1) (see [10, 14] for some interesting numerical results). Given $\lambda^* \in (0, +\infty)$ the so-called extremal value:

$$\lambda^* = \sup\{\lambda > 0 : \exists\, u \text{ solution of (1.1)}\},$$

for any $\lambda \in (0, \lambda^*)$, they proved the existence of the minimal solution u_λ, namely:

$$u_\lambda \le u \quad \text{for any solution } u \text{ of (1.1)}.$$

The map $\lambda \to u_\lambda$ is continuous and pointwise increasing for $\lambda \in (0, \lambda^*)$. Moreover, the minimal solution u_λ is characterized as the unique semi-stable solution of (1.1) (in the sense that the linearized operator is a positive operator, see also [13]). Finally, they raised out the special role of dimension $N = 7$ for problem (1.1) by means of some energy estimates: the minimal branch satisfies

$$\sup_{\lambda \in (0, \lambda^*)} \|u_\lambda\|_\infty < 1$$

for $1 \le N \le 7$ and in general, this is not true anymore for $N \ge 8$. For $1 \le N \le 7$, as $\lambda \to \lambda^*$, the minimal branch u_λ converges to u^*, the so-called extremal solution, the unique solution of (1.1) with $\lambda = \lambda^*$. By [4], λ^* is a turning point and a second branch U_λ of solutions for (1.1) comes out from u^* for λ close to λ^* (U_λ is a nondegenerate solution with Morse index 1).

Unless semi-stable solutions are concerned, it is in general very difficult to show *a priori* bounds on the solutions energy and, for example, we were not able to establish energy estimates along U_λ. In [5], we exploited that the Morse index is 1 along the second branch, by developing a different approach to face noncompactness phenomena based on this spectral information. Since, in general, it is relatively much easier to construct solutions satisfying good spectral information (for example, by variational methods), assumption (1.3) below seems to be more natural than energy bounds in the study of (1.1).

The approach in [5], based on some fine asymptotic analysis, provided compactness along the second branch U_λ in the same low dimensions (compactness of the minimal branch was also recovered).

The paper is a continuation and a strong improvement of the results in [5]. In [5], the Morse index 1 of a solutions sequence u_n was used to ensure that blow up at 1 can occur "essentially" only along the maximum points x_n of u_n. Boundedness on the Morse index allows the presence of multiple blow up points. By a careful asymptotic analysis, we are able to overcome the related technical difficulties and, by a non existence result for singular solutions of (1.1), to show:

Theorem 1.1. *Assume* $2 \le N \le 7$. *Let* $f \in C(\bar{\Omega})$ *be such that*:

$$f(x) = \left(\prod_{i=1}^{k} |x - p_i|^{\alpha_i} \right) g(x), \quad g(x) \ge C > 0 \ in\ \Omega, \tag{1.2}$$

for some points $p_i \in \Omega$ *and exponents* $\alpha_i \ge 0$. *Let* $\{\lambda_n\}_{n\in\mathbb{N}}$ *be a sequence such that* $\lambda_n \to \lambda \in [0, \lambda^*]$ *and let* u_n *be an associated solution such that*:

$$\sup_{n\in\mathbb{N}} m(u_n, \lambda_n) < +\infty. \tag{1.3}$$

Then,

$$\sup_{n\in\mathbb{N}} \|u_n\|_\infty < 1. \tag{1.4}$$

Moreover, if in addition $\mu_{1,n} := \mu_{1,\lambda_n}(u_n) < 0$, *then necessarily* $\lambda > 0$.

Here and in the sequel, $\mu_{k,\lambda}(u)$ *denotes the kth eigenvalue of* $L_{u,\lambda} = -\Delta - \frac{2\lambda f(x)}{(1-u)^3}$ *with the convention that eigenvalues are repeated according to their multiplicities, and the Morse index* $m(u, \lambda)$ *is the number of negative eigenvalues of* $L_{u,\lambda}$.

Estimate (1.4) will be sometimes referred to as a "compactness property" of the solutions set of (1.1). Indeed, by elliptic regularity theory, for any $k \in \mathbb{N}$ the set $\{u :\ u \text{ is a solution of (1.1)},\ m(u, \lambda) \le k\}$ is a compact set in $C^m(\bar{\Omega})$-norm, where $m \ge 1$ depends on the regularity of $f(x)$.

Let us do some comments. For $2 \le N \le 7$ and $f(x) = 1$, Joseph and Lundgren in [11] showed that the bifurcation diagram of (1.1) on the ball has exactly the following form:

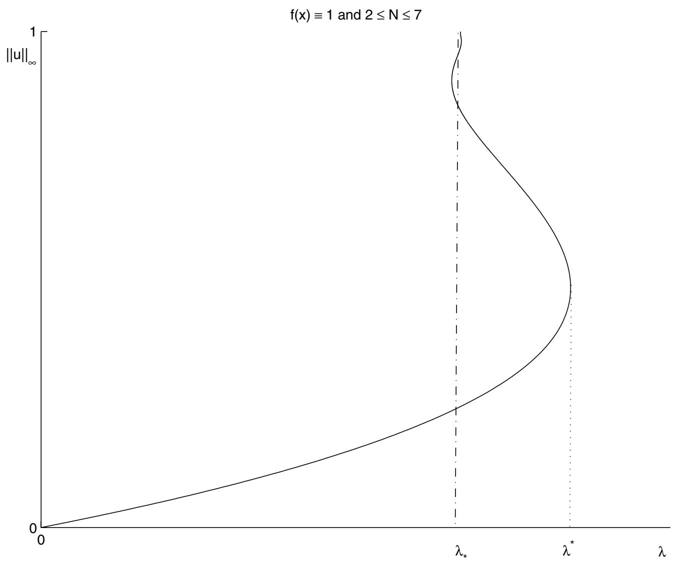

Fig. 1. Plots of $\|u\|_\infty$ versus λ in the case $f(x) = 1$ on the unit ball and $2 \le N \le 7$, where $\lambda_* = \frac{2(3N-4)}{9}$.

Namely, there are infinitely many turning points oscillating around the value $\lambda_* = \frac{2(3N-4)}{9}$, the solutions number of (1.1) going to $+\infty$ as λ approaches λ_*. For $2 \le N \le 7$ and $f(x) = |x|^\alpha$, $\alpha \ge 0$, numerically the diagram above still holds on the ball for $\lambda_* = \frac{(2+\alpha)(3N+\alpha-4)}{9}$ (see the thorough discussion in [8]).

Problem (1.1) presents in general a rich structure of the solutions set. The main goal now should be an existence theory for branches different from the first two, with Morse index higher than 1. Compactness properties are in general useful to establish existence results and Theorem 1.1 is a first step in the direction of an existence theory. An hopeful approach could be based on the analysis directly along the bifurcation diagram: any branch is characterized by a fixed Morse index and, when an eigenvalue of the linearized operator along the branch crosses zero, we have a "turning point" and the diagram turns into a new branch of higher Morse index (by [4], this is the case for example of the first turning point λ^*).

In view of Theorem 1.1, we can show *a posteriori* the equivalence among energy bounds and Morse index bounds. Indeed, we provide the following characterization of blow up sequences u_n (in the sense of blow up of $(1-u_n)^{-1}$), to be compared with [1, 2] in the context of polinomial subcritical nonlinearities:

Theorem 1.2. *Assume* $2 \le N \le 7$. *Let* $f \in C(\bar{\Omega})$ *be as in* (1.2). *Let* $\{\lambda_n\}_{n\in\mathbb{N}}$ *be a sequence such that* $\lambda_n \to \lambda \in [0, \lambda^*]$ *and let* u_n *be an associated solution. Then,*

(1) $\max_\Omega u_n \to 1$ *as* $n \to +\infty$,

(2) $\int_\Omega \left(\frac{f(x)}{(1-u_n)^3}\right)^{\frac{N}{2}} \to +\infty$ *as* $n \to +\infty$,

(3) $m(u_n, \lambda_n) \to +\infty$ *as* $n \to +\infty$,

are equivalent.

As a direct consequence of Theorem 1.1, Theorems 1.3–1.4 below show that some features of the bifurcation diagram on the ball hold for general domains. The following uniqueness result was strongly expected to be true:

Theorem 1.3. *Assume* $2 \le N \le 7$. *Let* $f \in C(\bar{\Omega})$ *be as in* (1.2). *For any fixed* $k \in \mathbb{N}$ *there exists* $\delta > 0$ *small so that*

(1) *for* $\lambda \in (0, \delta)$ *the minimal solution* u_λ *is the unique solution* u *of* (1.1) *with* $m(u, \lambda) \le k$;

(2) *for* $\lambda \in (\lambda^* - \delta, \lambda^*)$ u_λ *and* U_λ *are the unique solutions* u *of* (1.1) *with* $m(u, \lambda) \le k$.

As far as point (1) in Theorem 1.3 is concerned, in [6], the authors show that problem (1.1) on a two-dimensional annulus with $f(x) = 1$ has exactly two radial solutions for any $\lambda \in (0, \lambda^*)$. The second solution — the non minimal one — has Morse index unbounded in a neighborhood of $\lambda = 0$.

Finally, based on a degree argument, we get the existence of a solutions sequence u_n whose Morse index blows up (equivalently, by Theorem 1.2 the sequence blows up pointwise: $\max_\Omega u_n \to 1$ as $n \to +\infty$):

Theorem 1.4. *Assume* $2 \le N \le 7$. *Let* $f \in C(\bar{\Omega})$ *be as in* (1.2). *There exist a sequence* $\{\lambda_n\}_{n\in\mathbb{N}}$ *and associated solution* u_n *of* (1.1) *so that*

$$m(u_n, \lambda_n) \to +\infty \quad \textit{as } n \to +\infty.$$

Let us point out that the equivalence among points (1) and (2) in Theorem 1.2 was already proven in [8] (even if it is not stated). Moreover, a weaker form of Theorem 1.3, point (1), was already shown in [8] as uniqueness, for λ small, in the class of solutions of bounded energy: $\int_\Omega \left(\frac{f(x)}{(1-u)^3}\right)^{\frac{N}{2}} \le k$ for some $k > 0$.

The paper is organized as follows. In [8], a regularity result for finite energy solutions of (1.1) was proven. In Sec. 2, we extend it to discuss a nonhomogeneous Dirichlet version of (1.1), and to show a nonexistence result for solutions of (1.1) with finite Morse index and finite singular set (where the solutions touch the value 1). Improving the approach of [5] for the second branch, in Sec. 3, we describe the asymptotic behavior of a general blowing up sequence u_n (i.e. $\max_\Omega u_n \to 1$ as $n \to +\infty$) to get a strong pointwise estimate on the right-hand side of (1.1). This provides the uniform convergence of u_n in Ω to a limit singular solution u_0 of (1.1) having finite Morse index and finite singular set, which does not exist according to the regularity statements of Sec. 2. In Sec. 4, we give proofs of Theorems 1.2–1.4. For reader's convenience, in Appendix A we briefly sketch the proof of some results already proven in [5].

2. Regularity Properties

In this section, we establish some basic regularity results for the following boundary value problem:

$$\begin{cases} -\Delta u = \dfrac{f(x)}{(1-u)^2} & \text{in } \Omega, \\ u = \bar{u} & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f \in C(\bar{\Omega})$ satisfies (1.2) and $\bar{u} \in C^1(\bar{\Omega})$ is such that $0 \le \bar{u} \le \|\bar{u}\|_\infty < 1$. Solutions u of (2.1) are to be considered in the following $H^1(\Omega)$-weak sense: $u - \bar{u} \in H_0^1(\Omega)$, $\frac{f(x)}{(1-u)^2} \in H^{-1}(\Omega)$ and $-\Delta u = \frac{f(x)}{(1-u)^2}$ in $H^{-1}(\Omega)$, where $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

The first regularity result we give is already contained in [8] for $\bar{u} = 0$. We extend it to cover nonhomogeneous boundary values and we slightly improve the original statement (for $N = 2$). The following holds:

Proposition 2.1. *Let* $N \ge 2$. *Let* u *be a* $H^1(\Omega)$*-weak solution of* (2.1) *so that*

$$\frac{f}{(1-u)^3} \in L^{\frac{N}{2}}(\Omega). \tag{2.2}$$

Then, $u \in C^1(\bar{\Omega})$ *and* $0 < u \le \|u\|_\infty < 1$.

Proof. First of all, by (2.2) the right-hand side of (2.1) is in $L^{\frac{3N}{4}}(\Omega)$. Standard regularity theory implies that $u \in C^{0,\frac{2}{3}}(\bar{\Omega})$. If $u(x_0) = 1$ at some $x_0 \in \Omega \backslash \{p_1, \ldots, p_k\}$, the Hölder continuity of u implies $|1-u(x)| = |u(x_0)-u(x)| \le C|x-x_0|^{\frac{2}{3}}$. Then, for δ small

$$\left(\inf_{B_\delta(x_0)} f\right)^{-\frac{N}{2}} \int_{B_\delta(x_0)} \left|\frac{f}{(1-u)^3}\right|^{\frac{N}{2}} \ge \int_{B_\delta(x_0)} \frac{1}{|1-u|^{\frac{3N}{2}}} \ge \frac{1}{C}\int_{B_\delta(x_0)} \frac{1}{|x-x_0|^N} = +\infty,$$

in contradiction with (2.2). By continuity of u, $\|u\|_\infty \le 1$ and $\{x \in \Omega : u(x) = 1\} \subset \{p_1, \dots, p_k\}$.

We want to show now that (2.2) implies:

$$(1-u)^{-1} \in L^p(\Omega) \quad \forall\, p > 1. \tag{2.3}$$

Fix $p > 1$. Introduce $T_k u = \min\{u, 1-k\}$, the truncated function of u at level $1-k$, $0 < k < 1$.

Let us first discuss the case $N = 2$. For k small, take $(1-T_k u)^{-1} - (1-\bar{u})^{-1} \in H_0^1(\Omega)$ as a test function for (2.1):

$$\begin{aligned}\int_\Omega \frac{|\nabla T_k u|^2}{(1-T_k u)^2} &= \int_\Omega \frac{\nabla u \nabla \bar{u}}{(1-\bar{u})^2} + \int_\Omega \frac{f(x)}{(1-u)^2}((1-T_k u)^{-1} - (1-\bar{u})^{-1}) \\ &\le \int_\Omega \frac{\nabla u \nabla \bar{u}}{(1-\bar{u})^2} + \int_\Omega \frac{f(x)}{(1-u)^3} < +\infty,\end{aligned} \tag{2.4}$$

because of $(1-T_k u)^{-1} \le (1-u)^{-1}$ for $u \le 1$ and (2.2). Classical consequence of the Moser–Trudinger inequality is the following: there exists $C > 0$ so that

$$\int_\Omega e^{pv} \le C \exp\left(\frac{p^2}{16\pi}\|v\|^2_{H_0^1(\Omega)}\right) \quad \forall\, v \in H_0^1(\Omega), \quad p > 1. \tag{2.5}$$

Since $\log\left(\frac{1-\bar{u}}{1-T_k u}\right) \in H_0^1(\Omega)$ for k small, by (2.4) and (2.5) we get that for any $p > 1$:

$$\int_\Omega (1-T_k u)^{-p} \le C\int_\Omega \left(\frac{1-\bar{u}}{1-T_k u}\right)^p \le C\exp\left(\frac{p^2}{16\pi}\int_\Omega \left|\nabla \log\left(\frac{1-\bar{u}}{1-T_k u}\right)\right|^2\right) \le C,$$

where C denotes various positive constants depending only on p. Taking the limit as $k \to 0$, by $u \le 1$ we get the validity of (2.3).

The case $N \ge 3$ is more involved. Since $\{u(x) = 1\}$ is a finite set and u is continuous, we get that $|\{1-u \le \varepsilon\}| \to 0$ as $\varepsilon \to 0^+$, and by (2.2):

$$\int_{\{1-u\le\varepsilon\}} \left(\frac{f(x)}{(1-u)^3}\right)^{\frac{N}{2}} \le \left(\frac{p+1}{2p^2} S_N\right)^{\frac{N}{2}}, \tag{2.6}$$

for some $\varepsilon > 0$ small, where $|\cdot|$ stands for the Lebesgue measure and S_N is the Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$.

For k small, take $(1-T_ku)^{-p-1}-(1-\bar{u})^{-p-1}\in H_0^1(\Omega)$ as a test function for (2.1), and by (2.2) we get:

$$\begin{aligned}(p+1)\int_\Omega \frac{|\nabla T_k u|^2}{(1-T_ku)^{p+2}} &= (p+1)\int_\Omega \frac{\nabla u\nabla\bar{u}}{(1-\bar{u})^{p+2}}\\ &\quad+\int_\Omega \frac{f(x)}{(1-u)^2}((1-T_ku)^{-p-1}-(1-\bar{u})^{-p-1})\\ &\le \int_\Omega \frac{f(x)}{(1-u)^3}(1-T_ku)^{-p}+C. \end{aligned}\tag{2.7}$$

In view of $(a+b)^2=a^2+b^2+2ab\le(1+\delta)a^2+\frac{1+\delta}{\delta}b^2$ for $a,b\in\mathbb{R}$ and $\delta>0$, we deduce the following estimate:

$$\begin{aligned}(1-T_ku)^{-p}&\le(1+\delta)((1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}})^2\\ &\quad+\frac{1+\delta}{\delta}(1-\bar{u})^{-p},\quad \delta>0.\end{aligned}\tag{2.8}$$

Inserting (2.8) with $\delta=1$ into (2.7), we get:

$$(p+1)\int_\Omega\frac{|\nabla T_ku|^2}{(1-T_ku)^{p+2}}\le 2\int_\Omega\frac{f(x)}{(1-u)^3}((1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}})^2+C.\tag{2.9}$$

By (2.9) we get that:

$$\begin{aligned}&\int_\Omega\left|\nabla((1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}})\right|^2\\ &\quad\le 2\int_\Omega\left|\nabla((1-T_ku)^{-\frac{p}{2}})\right|^2+C=\frac{p^2}{2}\int_\Omega\frac{|\nabla T_ku|^2}{(1-T_ku)^{p+2}}+C\\ &\quad\le\frac{p^2}{p+1}\int_\Omega\frac{f(x)}{(1-u)^3}((1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}})^2+C\\ &\quad\le\frac{p^2}{p+1}\int_{\{1-u\le\varepsilon\}}\frac{f(x)}{(1-u)^3}((1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}})^2+C,\end{aligned}$$

where C denotes various positive constants depending only on ε and p. By Hölder inequality, (2.6) and the Sobolev embedding on $(1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}}\in H_0^1(\Omega)$, finally we get that

$$\begin{aligned}&\int_\Omega\left|\nabla((1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}})\right|^2\\ &\quad\le\frac{p^2}{p+1}\left(\int_{\{1-u\le\varepsilon\}}\left(\frac{f(x)}{(1-u)^3}\right)^{\frac{N}{2}}\right)^{\frac{2}{N}}\\ &\qquad\times\left(\int_\Omega\left|(1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}}\right|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}+C\\ &\quad\le\frac{1}{2}\int_\Omega\left|\nabla((1-T_ku)^{-\frac{p}{2}}-(1-\bar{u})^{-\frac{p}{2}})\right|^2+C.\end{aligned}$$

Hence, by Sobolev embedding:

$$\left(\int_\Omega |(1-T_k u)^{-\frac{p}{2}} - (1-\bar u)^{-\frac{p}{2}}|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}$$
$$\le S_N^{-1} \int_\Omega \left|\nabla((1-T_k u)^{-\frac{p}{2}} - (1-\bar u)^{-\frac{p}{2}})\right|^2 \le C,$$

and in turn

$$\int_\Omega (1-T_k u)^{-\frac{pN}{N-2}} \le C,$$

where $C > 0$ does not depend on k. Taking the limit as $k \to 0$, as before we get the validity of (2.3).

Now property (2.3) implies $\|u\|_\infty < 1$. Indeed, if $u(x_0) = 1$ for some $x_0 \in \Omega$, then $|1-u(x)| = |u(x_0) - u(x)| \le C|x-x_0|^{\frac{2}{3}}$, as already remarked. This is in contradiction with (2.3) for p large. Since $\|u\|_\infty < 1$ implies that the right-hand side of (2.1) is in $L^p(\Omega)$ for any $p > 1$, by elliptic regularity theory $u \in C^1(\bar\Omega)$ and then, we can conclude by maximum principle (in a weak form) that $0 < u \le \|u\|_\infty < 1$. $\square$

To cover nonhomogeneous boundary values, we adapt now the argument in [8] to show energy estimates for semi-stable solutions of (2.1). Proposition 2.1 applies to provide:

Proposition 2.2. *Let $2 \le N \le 7$. Let u be a $H^1(\Omega)$-weak solution of (2.1) so that $\|u\|_\infty \le 1$ and*

$$\int_\Omega \left(|\nabla\phi|^2 - \frac{2f(x)}{(1-u)^3}\phi^2\right) \ge 0, \quad \forall\, \phi \in H_0^1(\Omega). \tag{2.10}$$

Then, $u \in C^1(\bar\Omega)$ and $0 < u \le \|u\|_\infty < 1$.

Proof. We will show that (2.10) gives energy estimates sufficiently good for $2 \le N \le 7$. First of all, let us remark that (2.1) on $u - \bar u \in H_0^1(\Omega)$ gives:

$$\int_\Omega \frac{f(x)}{(1-u)^2} \le C\int_\Omega \frac{f(x)}{(1-u)^2}(1-\bar u)$$
$$= C\left(\int_\Omega \nabla u \nabla(u-\bar u) + \int_\Omega \frac{f(x)}{1-u}\right)$$
$$\le C\left(\|u\|_{H_0^1}\|u-\bar u\|_{H_0^1} + \epsilon\int_\Omega \frac{f(x)}{(1-u)^2} + \frac{1}{4\epsilon}\int_\Omega f(x)\right)$$

for any $\epsilon > 0$, because of the inequality $ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2$. Hence, for $\epsilon = \frac{1}{2C}$ we get:

$$\int_\Omega \frac{f(x)}{(1-u)^2} \le \frac{1}{2}\int_\Omega \frac{f(x)}{(1-u)^2} + C'$$

for some $C' > 0$, and then, $\int_\Omega \frac{f(x)}{(1-u)^2} < +\infty$. Now, (2.10) on $u - \bar{u} \in H_0^1(\Omega)$ gives that

$$\begin{aligned}\int_\Omega \frac{f(x)}{(1-u)^3} &\le 2C \int_\Omega \frac{f(x)}{(1-u)^3}(1-\bar{u})^2 \\ &\le C\left(\int_\Omega |\nabla(u-\bar{u})|^2 + \int_\Omega \frac{2f(x)}{1-u} + \int_\Omega \frac{4f(x)}{(1-u)^2}(u-\bar{u})\right) \\ &= C\left(\int_\Omega |\nabla(u-\bar{u})|^2 + \int_\Omega \frac{2f(x)}{1-u} + 4\int_\Omega \nabla u \nabla(u-\bar{u})\right) \\ &\le C.\end{aligned}$$

Fix $0 < p < 4 + 2\sqrt{6}$ in order to have $2 - \frac{p^2}{4(p+1)} > 0$. Introduce, as in the previous proof, $T_k u = \min\{u, 1-k\}$, $0 < k < 1$. For k small, taking $(1-T_k u)^{-p-1} - (1-\bar{u})^{-p-1} \in H_0^1(\Omega)$ as a test function in (2.1) we get:

$$\begin{aligned}&(p+1)\int_\Omega \left(\frac{|\nabla T_k u|^2}{(1-T_k u)^{p+2}} - \frac{\nabla u \nabla \bar{u}}{(1-\bar{u})^{p+2}}\right) \\ &\quad = \int_\Omega \frac{f(x)}{(1-u)^2}((1-T_k u)^{-p-1} - (1-\bar{u})^{-p-1}).\end{aligned} \tag{2.11}$$

Moreover, by (2.10) and the simple inequality $(a+b)^2 \le (1+\delta)a^2 + \frac{1+\delta}{\delta}b^2$ we get:

$$\begin{aligned}&2\int_\Omega \frac{f(x)}{(1-u)^3}((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}})^2 \\ &\quad \le \int_\Omega \left|\nabla((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}})\right|^2 \\ &\quad \le \frac{p^2}{4}(1+\delta)\int_\Omega \frac{|\nabla T_k u|^2}{(1-T_k u)^{p+2}} + C \\ &\quad \le \frac{p^2}{4}(1+\delta)\int_\Omega \left(\frac{|\nabla T_k u|^2}{(1-T_k u)^{p+2}} - \frac{\nabla u \nabla \bar{u}}{(1-\bar{u})^{p+2}}\right) + C,\end{aligned} \tag{2.12}$$

for some $C > 0$ depending on p and $\delta > 0$. Inserting (2.11) into (2.12) and using $(1-T_k u)^{-1} \le (1-u)^{-1}$ for $u \le 1$, we get that

$$\begin{aligned}&2\int_\Omega \frac{f(x)}{(1-u)^3}((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}})^2 \\ &\quad \le \frac{p^2(1+\delta)}{4(p+1)}\int_\Omega \frac{f(x)}{(1-u)^2}((1-T_k u)^{-p-1} - (1-\bar{u})^{-p-1}) + C \\ &\quad \le \frac{p^2(1+\delta)}{4(p+1)}\int_\Omega \frac{f(x)}{(1-u)^3}(1-T_k u)^{-p} + C \\ &\quad \le \frac{p^2(1+\delta)^2}{4(p+1)}\int_\Omega \frac{f(x)}{(1-u)^3}((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}})^2 + C\end{aligned}$$

in view of (2.8), where $C > 0$ does not depend on k. Since $p < 4+2\sqrt{6}$, $2-\frac{p^2(1+\delta)^2}{4(p+1)} > 0$ for δ small and then,

$$\int_\Omega \frac{f(x)}{(1-u)^3}(1-T_k u)^{-p} \le 2\int_\Omega \frac{f(x)}{(1-u)^3}((1-T_k u)^{-\frac{p}{2}} - (1-\bar{u})^{-\frac{p}{2}})^2 + C \le C$$

for some $C > 0$ not depending on k. Taking the limit as $k \to 0$, we get that

$$\int_\Omega \frac{f(x)}{(1-u)^{3+p}} \le C,$$

and then, $\frac{f}{(1-u)^3} \in L^{\frac{3+p}{3}}(\Omega)$ for any $0 < p < 4+2\sqrt{6}$. Since $\frac{N}{2} < \frac{1}{3}\left(3+(4+2\sqrt{6})\right)$ for $2 \le N \le 7$, we get the validity of (2.2) and hence, applying Proposition 2.1 the proof is complete. □

We conclude the section providing a non existence result for solutions of (2.1) with finite Morse index and finite singular set. We have that:

Theorem 2.3. *Let $2 \le N \le 7$. Let $u \in C(\bar{\Omega})$ be a $H^1(\Omega)$-weak solution of* (2.1) *so that $\|u\|_\infty \le 1$ and the singular set $S = \{x \in \Omega : u(x) = 1\}$ is a nonempty set. Assume that u has finite Morse index: there exists a finite dimensional subspace $T \subset H_0^1(\Omega)$ so that*

$$\int_\Omega \left(|\nabla\phi|^2 - \frac{2f(x)}{(1-u)^3}\phi^2\right) \ge 0,$$
$$\text{for any } \phi \in T^\perp = \left\{\phi \in H_0^1(\Omega) : \int_\Omega \nabla\phi\nabla\psi = 0 \,\forall\, \psi \in T\right\}. \tag{2.13}$$

Then, the singular set S has no isolated points.

Proof. Assume by contradiction that $x_0 \in S$ is an isolated point of S. Let δ_0 be such that $B_{2\delta_0}(x_0) \cap S = \{x_0\}$. We want to show that:

$$\int_{B_\delta} \left(|\nabla\phi|^2 - \frac{2f(x)}{(1-u)^3}\phi^2\right) \ge 0, \quad \text{for any } \phi \in H_0^1(B_\delta), \tag{2.14}$$

for some $0 < \delta \le \delta_0$ small, where $B_\delta := B_\delta(x_0)$.

By contradiction, assume that (2.14) is false for any $0 < \delta \le \delta_0$. Then, there exists $\phi_0 \in C_0^\infty(B_{\delta_0})$ such that

$$\int_{B_{\delta_0}} \left(|\nabla\phi_0|^2 - \frac{2f(x)}{(1-u)^3}\phi_0^2\right) < 0. \tag{2.15}$$

We can assume that $\phi_0 = 0$ in B_δ for some $0 < \delta < \delta_0$ small. Indeed, let us replace ϕ_0 with a truncated function ϕ_δ, $\delta > 0$ small, so that (2.15) is still true while $\phi_\delta = 0$

in B_δ. Set $\phi_\delta = \chi_\delta \phi_0$, where χ_δ is a cut-off function defined as:

$$\chi_\delta(x) = \begin{cases} 0 & |x-x_0| \le \delta, \\ 2\left(1 - \dfrac{\log|x-x_0|}{\log\delta}\right) & \delta \le |x-x_0| \le \sqrt{\delta}, \\ 1 & |x-x_0| \ge \sqrt{\delta}. \end{cases}$$

By Fatou's Lemma, we have:

$$\int_{B_{\delta_0}} \frac{2f(x)}{(1-u)^3}\phi_0^2 \le \liminf_{\delta\to 0} \int_{B_{\delta_0}} \frac{2f(x)}{(1-u)^3}\phi_\delta^2. \tag{2.16}$$

For the gradient term, we have the expansion:

$$\int_{B_{\delta_0}} |\nabla\phi_\delta|^2 = \int_{B_{\delta_0}} \phi_0^2 |\nabla\chi_\delta|^2 + \int_{B_{\delta_0}} \chi_\delta^2|\nabla\phi_0|^2 + 2\int_{B_{\delta_0}} \chi_\delta\phi_0\nabla\chi_\delta\nabla\phi_0.$$

The following estimates hold:

$$0 \le \int_{B_{\delta_0}} \phi_0^2|\nabla\chi_\delta|^2 \le 4\|\phi_0\|_\infty^2 \int_{\delta\le|x-x_0|\le\sqrt{\delta}} \frac{1}{|x-x_0|^2\log^2\delta} \le \frac{C}{\log\frac{1}{\delta}}$$

and

$$\left|2\int_{B_{\delta_0}} \chi_\delta\phi_0\nabla\chi_\delta\nabla\phi_0\right| \le \frac{4\|\phi_0\|_\infty\|\nabla\phi_0\|_\infty}{\log\frac{1}{\delta}} \int_{B_1(0)} \frac{1}{|x|},$$

and provide by Lebesgue's Theorem:

$$\int_{B_{\delta_0}} |\nabla\phi_\delta|^2 \to \int_{B_{\delta_0}} |\nabla\phi_0|^2 \quad \text{as } \delta\to 0. \tag{2.17}$$

Combining (2.16) and (2.17), we get that:

$$\int_{B_{\delta_0}} \left(|\nabla\phi_\delta|^2 - \frac{2f(x)}{(1-u)^3}\phi_\delta^2\right) < 0$$

for $\delta > 0$ sufficiently small.

In this way, we find $0 < \delta_1 < \delta_0$ small and $\phi_0 \in C_0(B_{\delta_0}\backslash B_{\delta_1}) \cap H_0^1(\Omega)$ such that (2.15) holds. Since by contradiction we are assuming that (2.14) is false for any $\delta > 0$, we can iterate now the argument to find a strictly decreasing sequence δ_n and $\phi_n \in C_0(B_{\delta_n}\backslash B_{\delta_{n+1}}) \cap H_0^1(\Omega)$ such that:

$$\int_{B_{\delta_n}} \left(|\nabla\phi_n|^2 - \frac{2f(x)}{(1-u)^3}\phi_n^2\right) < 0.$$

Since $\{\phi_n\}_{n\in\mathbb{N}}$ are mutually ortogonal having disjoint supports, we have found an infinite dimensional set $M = \mathrm{Span}\{\phi_n : n\in\mathbb{N}\} \subset H_0^1(\Omega)$ so that

$$\int_\Omega \left(|\nabla\phi|^2 - \frac{2f(x)}{(1-u)^3}\phi^2\right) < 0 \quad \forall\,\phi\in M.$$

Since M is an infinite dimensional subspace of $H_0^1(\Omega)$, we have that $M \cap T^\perp \neq \emptyset$, in contradiction with (2.13). Hence, (2.14) holds for some $\delta = \delta(x_0) \leq \delta_0$.

By elliptic regularity theory, we get that $u \in C^1_{\mathrm{loc}}(B_{2\delta_0} \setminus \{x_0\})$. Since $u \in C^1(\partial B_\delta)$ and $\max_{\partial B_\delta} u < 1$ in view of $0 < \delta \leq \delta_0$, we extend it on B_δ as a function $\bar{u} \in C^1(\bar{B}_\delta)$ satisfying $0 \leq \bar{u} \leq \|\bar{u}\|_{\infty, B_\delta} < 1$. Since (2.14) holds on B_δ, we can apply Proposition 2.2 to get that $\|u\|_{\infty, B_\delta} < 1$, contradicting $u(x_0) = 1$. Hence, S has no isolated points. □

3. Compactness Issues

In this section, we turn to the compactness result stated in Theorem 1.1. We follow the approach developed in [5] to prove compactness of the second branch of solutions. To deal with higher branches, we improve the argument to discuss multiple blow up (for the second branch the blow up occurs only at the maximum point).

Let $2 \leq N \leq 7$. Assume that $f \in C(\bar{\Omega})$ is in the form (1.2), and let $(u_n)_n$ be a solutions sequence of (1.1) associated to $\lambda_n \to \lambda \in [0, \lambda^*]$. Since we want to show that $\sup_{n \in \mathbb{N}} \|u_n\|_\infty < 1$, by contradiction and up to a subsequence, we will assume all along the section that $u_n(x_n) = \max_\Omega u_n \to 1^-$ as $n \to +\infty$, x_n being a maximum point of u_n.

3.1. *A blow-up approach*

Let $y_n \in \Omega$ be a sequence of points so that $u_n(y_n) \to 1^-$ as $n \to +\infty$. Set $\mu_n = 1 - u_n(y_n)$. As we will see later, for our purposes it is not restrictive to assume that $\mu_n^3 \lambda_n^{-1} \to 0$ and $y_n \to p \in \bar{\Omega}$ as $n \to +\infty$. Depending on the location of p and the rate of $|y_n - p|$, the length scale to see around y_n some nontrivial limit profile is the following:

$$r_n = \begin{cases} \mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} & \text{if } p \notin Z \\ \mu_n^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |y_n - p_i|^{-\frac{\alpha_i}{2}} & \text{if } p = p_i,\ \mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha_i + 2} \to +\infty \text{ as } n \to +\infty \\ \mu_n^{\frac{3}{2+\alpha_i}} \lambda_n^{-\frac{1}{2+\alpha_i}} & \text{if } \limsup\limits_{n \to +\infty} \mu_n^{-3} \lambda_n |y_n - p_i|^{\alpha_i + 2} < +\infty, \end{cases} \tag{3.1}$$

where $Z = \{p_1, \ldots, p_k\}$ is the zero set of the potential $f(x)$ and $\alpha_1, \ldots, \alpha_k$ are the related multiplicities given by (1.2). Let us remark that $\mu_n^3 \lambda_n^{-1} \to 0$ implies $r_n \to 0$ as $n \to +\infty$.

Only to give an idea, let us establish the following rough correspondence: the first situation in the definition of r_n corresponds to a blow up at some point outside Z, the second one to a "slow" blow up at some $p_i \in Z$, while the third one is a "fast" blow at some $p_i \in Z$. Let us now introduce the following rescaled function around y_n:

$$U_n(y) = \frac{1 - u_n(r_n y + y_n)}{\mu_n}, \quad y \in \Omega_n = \frac{\Omega - y_n}{r_n}.$$

Since $U_n(0) = 1$ by construction, in order to get a limit profile equation we should add a condition avoiding vanishing on compact sets of Ω_n. Let us remark that, for x_n the maximum point of u_n and $\varepsilon_n = 1 - u_n(x_n)$, the associated rescaled function U_n satisfies: $U_n \geq U_n(0) = 1$ in Ω_n.

Proposition 3.1. *Assume that*

$$\mu_n^3 \lambda_n^{-1} (\operatorname{dist}(y_n, \partial\Omega))^{-2} \to 0 \quad \text{as } n \to +\infty \tag{3.2}$$

and

$$U_n \geq C > 0 \quad \text{in } \Omega_n \cap B_{R_n}(0), \tag{3.3}$$

for some $R_n \to +\infty$ as $n \to +\infty$. Then, up to a subsequence, $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where U is a solution of the equation:

$$\begin{cases} \Delta U = s\dfrac{|y + y_0|^\gamma}{U^2} & \text{in } R^N, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.4}$$

for some $s > 0$, $\gamma \in \{0, \alpha_1, \ldots, \alpha_k\}$ and $y_0 \in \mathbb{R}^N$ (depending on the type of blow up). Moreover, there exists a function $\phi_n \in C_0^\infty(\Omega)$ such that:

$$\int_\Omega \left(|\nabla \phi_n|^2 - \frac{2\lambda_n f(x)}{(1 - u_n)^3} \phi_n^2 \right) < 0 \tag{3.5}$$

and Supp $\phi_n \subset B_{Mr_n}(y_n)$ for some $M > 0$.

To establish property (3.5), it will be crucial the knowledge of the linear instability for solutions of (3.4) in low dimensions:

Theorem 3.2 ([5]). *Assume either $1 \leq N \leq 7$ or $N \geq 8$, $\gamma > \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. Let U be a solution of*

$$\begin{cases} \Delta U = \dfrac{|y|^\gamma}{U^2} & \text{in } \mathbb{R}^N, \\ U(y) \geq C > 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{3.6}$$

Then,

$$\mu_1(U) = \inf \left\{ \int_{\mathbb{R}^N} \left(|\nabla \phi|^2 - \frac{2|y|^\gamma}{U^3} \phi^2 \right); \phi \in C_0^\infty(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} \phi^2 = 1 \right\} < 0. \tag{3.7}$$

Moreover, if $N \geq 8$ and $0 \leq \gamma \leq \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, then there exists at least a solution U of (3.6) *such that $\mu_1(U) \geq 0$.*

For the sake of completeness, we will sketch the proof of Theorem 3.2 in Appendix A.

Proof of Proposition 3.1. First of all, let us remark that (3.2) implies $\mu_n^3\lambda_n^{-1} \to 0$, and then $r_n \to 0$ as $n \to +\infty$. If $p \in \partial\Omega$, we have that $r_n = \mu_n^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}$ and, by (3.2)

$$\mathrm{dist}(0,\partial\Omega_n) = \frac{\mathrm{dist}(y_n,\partial\Omega)}{r_n} = \left(\mu_n^3\lambda_n^{-1}(\mathrm{dist}(y_n,\partial\Omega))^{-2}\right)^{-\frac{1}{2}} \to +\infty$$

as $n \to +\infty$. Arguing in a simpler way if $p \in \Omega$, we get that $\Omega_n \to \mathbb{R}^N$ as $n \to +\infty$. Introduce the following notation

$$f_i(x) = \left(\prod_{j=1,\, j\neq i}^{k} |x-p_j|^{\alpha_j}\right) g(x). \tag{3.8}$$

The function U_n satisfies $\Delta U_n = \frac{f_n(y)}{U_n^2}$ in Ω_n, where $f_n(y)$ is given by:

$$f_n = \begin{cases} f(r_n y + y_n) & \text{if } p \notin Z \\ \left|\dfrac{r_n}{|y_n-p_i|}y + \dfrac{y_n-p_i}{|y_n-p_i|}\right|^{\alpha_i} f_i(r_n y + y_n) & \text{if } p = p_i,\ \mu_n^{-3}\lambda_n|y_n-p_i|^{\alpha_i+2} \to +\infty \\ & \qquad \text{as } n \to +\infty \\ \left|y + \dfrac{y_n-p_i}{r_n}\right|^{\alpha_i} f_i(r_n y + y_n) & \text{if } \limsup\limits_{n\to+\infty} \mu_n^{-3}\lambda_n|y_n-p_i|^{\alpha_i+2} < +\infty, \end{cases} \tag{3.9}$$

and $p = \lim_{n\to+\infty} y_n$. Only in the latter situation $\limsup_{n\to+\infty} \mu_n^{-3}\lambda_n|y_n - p_i|^{\alpha_i+2} < +\infty$, up to a subsequence assume that

$$\frac{y_n-p_i}{r_n} \to y_0 \quad \text{as } n \to +\infty. \tag{3.10}$$

Let $R > 0$. For n large, decompose $U_n = U_{n,1} + U_{n,2}$, where $U_{n,2}$ satisfies:

$$\begin{cases} \Delta U_{n,2} = \Delta U_n & \text{in } B_R(0), \\ U_{n,2} = 0 & \text{on } \partial B_R(0). \end{cases}$$

Since (3.3) implies $0 \le \Delta U_n \le C_R$ on $B_R(0)$, by elliptic regularity theory we get that $U_{n,2}$ is uniformly bounded in $C^{1,\beta}(B_R(0))$, $\beta \in (0,1)$. Since $U_{n,1} = U_n \ge C$ on $\partial B_R(0)$, by harmonicity $U_{n,1} \ge C$ in $B_R(0)$. Since $U_n(0) = 1$, by Harnack inequality we get:

$$\begin{aligned} \sup_{B_{R/2}(0)} U_{n,1} &\le C_R \inf_{B_{R/2}(0)} U_{n,1} \le C_R U_{n,1}(0) \\ &= C_R\left(1 - U_{n,2}(0)\right) \le C_R\left(1 + \sup_{n\in\mathbb{N}}|U_{n,2}(0)|\right) < \infty. \end{aligned}$$

Hence, $U_{n,1}$ is uniformly bounded in $C^{1,\beta}(B_{R/4}(0))$, $\beta \in (0,1)$. Since $U_n = U_{n,1} + U_{n,2}$ is uniformly bounded in $C^{1,\beta}(B_{R/4}(0))$ for any $R > 0$, by a diagonal process and up to a subsequence, we get that $U_n \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$. According to the three situations described in the definition (3.9) of f_n, the function $U \ge C > 0$ is a solution of (3.4) with: $s = f(p)$, $\gamma = 0$ in the first case; $s = f_i(p)$, $\gamma = 0$

in the second case; $s = f_i(p)$, $\gamma = \alpha_i$ and y_0 as in (3.10) in the third case. Set $f_\infty(y) := \lim_{n\to+\infty} f_n(y) = s|y+y_0|^\gamma$.

Since $2 \le N \le 7$ and $s > 0$, by Theorem 3.2 we get that $\mu_1(U) < 0$ and then, we find $\phi \in C_0^\infty(\mathbb{R}^N)$ so that:

$$\int \left(|\nabla\phi|^2 - \frac{2f_\infty(y)}{U^3}\phi^2\right) < 0.$$

Define now $\phi_n(x) = r_n^{-\frac{N-2}{2}}\phi(\frac{x-y_n}{r_n})$. We have that:

$$\begin{aligned}&\int_\Omega \left(|\nabla\phi_n|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3}\phi_n^2\right)\\ &\quad = \int \left(|\nabla\phi|^2 - \frac{2f_n(y)}{U_n^3}\phi^2\right) \to \int \left(|\nabla\phi|^2 - \frac{2f_\infty(y)}{U^3}\phi^2\right) < 0\end{aligned}$$

as $n \to +\infty$, since ϕ has compact support and $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R})$. □

Remark 3.1. In case of fast blow up at p_i: $\limsup_{n\to+\infty} \mu_n^{-3}\lambda_n|y_n - p_i|^{\alpha_i+2} < +\infty$, Proposition 3.1 is still true if, instead of condition (3.3), we assume:

$$U_n \ge C\left|y + \frac{y_n - p_i}{r_n}\right|^{\frac{\alpha_i}{3}} \quad \text{in } \Omega_n \cap B_{R_n}(0), \tag{3.11}$$

for some $R_n \to +\infty$ as $n \to +\infty$ and $C > 0$. Recall that in this situation $r_n = \mu_n^{\frac{3}{2+\alpha_i}}\lambda_n^{-\frac{1}{2+\alpha_i}}$. By (3.11), we get easily that on $\Omega_n \cap B_{R_n}(0)$:

$$0 \le \Delta U_n \le C\left|y + \frac{y_n - p_i}{r_n}\right|^{\frac{\alpha_i}{3}}.$$

Given $R > 0$, then $0 \le \Delta U_n \le C_R$ on $B_R(0)$ for n large. Arguing as in the proof of Proposition 3.1, up to a subsequence, we get that $U_n \to U$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, where $U \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \backslash \{-y_0\})$ is a solution of the equation

$$\begin{cases} \Delta U = |y+y_0|^{\alpha_i}\dfrac{f_i(p_i)}{U^2} & \text{in } \mathbb{R}^N\backslash\{-y_0\},\\ U(y) \ge C|y+y_0|^{\frac{\alpha_i}{3}} & \text{in } \mathbb{R}^N. \end{cases}$$

By Hopf Lemma, we have that $U(-y_0) > 0$. Indeed, let B some ball so that $-y_0 \in \partial B$ and assume by contradiction that $U(-y_0) = 0$. Since

$$-\Delta U + c(y)U = 0 \quad \text{in } B, \quad U \in C^2(B) \cap C(\bar{B}), \quad U(y) > U(-y_0) \quad \text{in } B,$$

and $c(y) = f_i(p_i)\frac{|y+y_0|^{\alpha_i}}{U^3} \ge 0$ is a bounded function, by Hopf Lemma we get that $\partial_\nu U(-y_0) < 0$, where ν is the unit outward normal of B. Hence, along the outward normal direction of B at $-y_0$ U becomes negative in contradiction with the positivity of U. Hence, $U(-y_0) > 0$ and $U(y) \ge C := \inf_{R^N} U(y) > 0$ in $\mathbb{R}^N$. The argument now goes as in the proof of Proposition 3.1.

3.2. *A pointwise estimate*

Let us assume now the validity of (1.3), namely $m(u_n, \lambda_n) \le k$ for any $n \in \mathbb{N}$ and some $k \in \mathbb{N}$. This information, combined with Proposition 3.1, will permit us to control the blow up behavior of u_n. Indeed, the following pointwise estimate on u_n is available:

Theorem 3.3. *Assume* $2 \le N \le 7$. *Let* $f \in C(\bar\Omega)$ *be as in* (1.2). *Let* u_n *be a solution of* (1.1) *associated to* $\lambda_n \in [0, \lambda^*]$. *Assume that* $\lambda_n \to \lambda$ *and* $u_n(x_n) = \max_\Omega u_n \to 1$ *as* $n \to +\infty$. *Then, up to a subsequence, there exist constants* $C > 0$, $N_0 \in \mathbb{N}$ *and* m*-sequences* $x_n^1, \dots, x_n^m$, $m \le k$, *such that*

$$1 - u_n(x) \ge C\lambda_n^{\frac{1}{3}} \left(d(x)^\alpha\right)^{\frac{1}{3}} d_n(x)^{\frac{2}{3}}, \quad \forall\, x \in \Omega, \quad \forall\, n \ge N_0, \tag{3.12}$$

where $d(x)^\alpha := \min\{|x - p_i|^{\alpha_i} : i = 1, \dots, k\}$ *is a "distance function" from* Z *and* $d_n(x) = \min\{|x - x_n^i| : i = 1, \dots, m\}$ *is the distance function from* $\{x_n^1, \dots, x_n^m\}$.

More precisely, letting r_n^i *be associated to* x_n^i *by means of* (3.1), *for any* $i, j = 1, \dots, m$, $i \ne j$, *there holds*:

$$(\varepsilon_n^i)^3\lambda_n^{-1} \to 0, \quad U_n^i(y) = \frac{1 - u_n(r_n^i y + x_n^i)}{\varepsilon_n^i} \to U^i(y) \text{ in } C^1_{\mathrm{loc}}(\mathbb{R}^N), \quad \frac{r_n^i + r_n^j}{|x_n^i - x_n^j|} \to 0 \tag{3.13}$$

as $n \to +\infty$, *where* $\varepsilon_n^i := 1 - u_n(x_n^i)$ *and* U^i *satisfies an equation of type* (3.4). *In addition, there exist* m*-sequences of test functions* $\phi_n^1, \dots, \phi_n^m \in C_0^\infty(\Omega)$ *so that*

$$\int_\Omega \left(|\nabla \phi_n^i|^2 - \frac{2\lambda_n f(x)}{(1 - u_n)^3}(\phi_n^i)^2 \right) < 0, \quad \text{Supp } \phi_n^i \subset B_{Mr_n^i}(x_n^i), \quad \forall\, i = 1, \dots, m, \tag{3.14}$$

for some $M > 0$ *large.*

Proof. Let $\varepsilon_n = 1 - u_n(x_n)$, where x_n is a maximum point of u_n. By the inequality

$$0 \le \frac{\lambda_n f(x)}{(1 - u_n)^2} \le \frac{\lambda_n}{\varepsilon_n^2}\|f\|_\infty,$$

we get that:

$$\varepsilon_n^3 \lambda_n^{-1} \to 0 \quad \text{as } n \to +\infty. \tag{3.15}$$

□

By contradiction, if $\varepsilon_n^3\lambda_n^{-1} \ge \delta > 0$ along a subsequence, the right-hand side of (1.1) would converge uniformly to zero as $n \to +\infty$ and, by elliptic regularity theory, $u_n \to u$ in $C^1(\bar\Omega)$ (up to a further subsequence), where u is an harmonic function so that $u = 0$ on $\partial\Omega$, $\max_\Omega u = 1$. A contradiction. Hence, (3.15) must hold.

As needed in Proposition 3.1, (3.15) now implies:

$$\varepsilon_n^3\lambda_n^{-1}(\mathrm{dist}(x_n, \partial\Omega))^{-2} \to 0 \quad \text{as } n \to +\infty. \tag{3.16}$$

In order to prove it, we will use the following lemma and we refer to Appendix A for the proof:

Lemma 3.4. *Let A_n be a bounded domain in $\mathbb{R}^N$ so that $A_n \to T$ as $n \to +\infty$, where T is an hyperspace so that $0 \in T$ and $\text{dist}(0, \partial T) = 1$. Let h_n be a function on A_n and W_n be a solution of*:

$$\begin{cases} \Delta W_n = \dfrac{h_n(y)}{W_n^2} & \text{in } A_n, \\ W_n(y) \geq C > 0 & \text{in } A_n, \\ W_n(0) = 1, \end{cases} \tag{3.17}$$

for some $C > 0$. Assume that $\sup_{n\in\mathbb{N}} \|h_n\|_\infty < +\infty$ and $\partial A_n \cap B_2(0)$ is smooth. Then, either

$$\min_{\partial A_n \cap B_2(0)} W_n \leq C \tag{3.18}$$

or

$$\min_{\partial A_n \cap B_2(0)} \partial_\nu W_n \leq 0, \tag{3.19}$$

where ν is the unit outward normal of A_n.

Assume by contradiction that (3.16) is false, namely, up to a subsequence, $\varepsilon_n^3 \lambda_n^{-1} d_n^{-2} \to \delta > 0$ as $n \to +\infty$, where $d_n := \text{dist}(x_n, \partial\Omega)$. In view of (3.15), we get $d_n \to 0$ as $n \to +\infty$. Introduce a rescaled function W_n:

$$W_n(y) = \frac{1 - u_n(d_n y + x_n)}{\varepsilon_n}, \quad y \in A_n = \frac{\Omega - x_n}{d_n}.$$

Since $d_n \to 0$ and

$$\text{dist}(0, \partial\Omega_n) = \frac{\text{dist}(x_n, \partial\Omega)}{d_n} = 1,$$

we get that $A_n \to T$ as $n \to +\infty$, where T is an hyperspace so that $0 \in T$ and $\text{dist}(0, \partial T) = 1$. The function W_n solves problem (3.17) with $h_n(y) = \frac{\lambda_n d_n^2}{\varepsilon_n^3} f(d_n y + x_n)$ and $C = W_n(0) = 1$. We have that for n large:

$$\|h_n\|_\infty \leq \frac{\lambda_n d_n^2}{\varepsilon_n^3} \|f\|_\infty \leq \frac{2}{\delta} \|f\|_\infty.$$

Since $W_n = \frac{1}{\varepsilon_n} \to +\infty$ on ∂A_n, by Lemma 3.4 we get that (3.19) must hold. A contradiction to Hopf Lemma applied to u_n. Hence, the validity of (3.16).

Let r_n be associated to x_n according to (3.1). Up to a subsequence, Proposition 3.1 gives:

$$\frac{1 - u_n(r_n y + x_n)}{\varepsilon_n} \to U(y) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N) \text{ as } n \to +\infty,$$

where U satisfies an equation of type (3.4), and provides the existence of $\phi_n \in C_0^\infty(\Omega)$ such that (3.5) holds with $\text{Supp}\, \phi_n \subset B_{Mr_n}(x_n)$, $M > 0$.

Let now $x_n^1 = x_n$, $\varepsilon_n^1 = \varepsilon_n$, $r_n^1 = r_n$, $U^1 = U$ and $\phi_n^1 = \phi_n$. If (3.12) is true for some subsequence of u_n with x_n^1, we take $m = 1$ and the proof is done. Otherwise, we proceed by an inductive method. Indeed, assume that, up to a subsequence, we have already found l-sequences $x_n^1, \ldots, x_n^l$, associated $r_n^1, \ldots, r_n^l$ (defined by (3.1)) and test functions $\phi_n^1, \ldots, \phi_n^l \in C_0^\infty(\Omega)$ so that (3.13) and (3.14) hold at lth step. If (3.12) holds for some subsequence of u_n with $x_n^1, \ldots, x_n^l$, we take $m = l$ and the proof is done. Otherwise, up to a subsequence, we will show the existence of x_n^{l+1}, r_n^{l+1} and ϕ_n^{l+1} so that (3.13) and (3.14) are still true at $(l+1)$th step. Since (3.13) and (3.14) at lth step imply that $\phi_n^1, \ldots, \phi_n^l$ have mutually disjoint compact supports, we get that $m(u_n, \lambda_n) \geq l$. Then, by (1.3) the inductive process must stop after a finite number of steps, say m steps, with $m \leq k$, and (3.12) holds with $x_n^1, \ldots, x_n^m$.

In order to complete the proof, we need to show how the induction process works. Assume that (3.13) and (3.14) hold at lth step and (3.12) is not true for any subsequence of u_n with $x_n^1, \ldots, x_n^l$. Let $x_n^{l+1} \in \Omega$ be such that

$$\begin{aligned}
&\lambda_n^{-\frac{1}{3}} \left(d(x_n^{l+1})^\alpha\right)^{-\frac{1}{3}} d_n(x_n^{l+1})^{-\frac{2}{3}} (1 - u_n(x_n^{l+1})) \\
&\quad = \lambda_n^{-\frac{1}{3}} \min_{x \in \Omega} ((d(x)^\alpha)^{-\frac{1}{3}} d_n(x)^{-\frac{2}{3}} (1 - u_n(x))) \to 0 \qquad (3.20)
\end{aligned}$$

as $n \to +\infty$, where $d_n(x)$ is the distance function from $\{x_n^1, \ldots, x_n^l\}$. Let $\varepsilon_n^{l+1} := 1 - u_n(x_n^{l+1})$.

Formula (3.20) gives a lot of informations about the blow up around x_n^{l+1}. First of all, it can be rewritten in the more convenient form:

$$\frac{(\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^i|\, |x_n^{l+1} - p_j|^{\frac{\alpha_j}{2}}} \to 0 \quad \text{as } n \to +\infty, \quad \forall\, i = 1, \ldots, l,\ j = 1, \ldots, k. \qquad (3.21)$$

The inductive assumption gives $\frac{r_n^i + r_n^j}{|x_n^i - x_n^j|} \to 0$ as $n \to +\infty$ for any $i, j = 1, \ldots, l$, $i \neq j$. Then, by definition of r_n^j we get for $|y| \leq R$ and $n \geq n_R$:

$$\begin{aligned}
&\lambda_n^{-\frac{1}{3}} (d(r_n^j y + x_n^j)^\alpha)^{-\frac{1}{3}} d_n(r_n^j y + x_n^j)^{-\frac{2}{3}} (1 - u_n(r_n^j y + x_n^j)) \\
&= \begin{cases}
(d(r_n^j y + x_n^j)^\alpha)^{-\frac{1}{3}} |y|^{-\frac{2}{3}} U_n^j(y) & \text{if } x_n^j \to p \notin Z \\[2ex]
\left| \frac{r_n^j}{|x_n^j - p_i|} y + \frac{x_n^j - p_i}{|x_n^j - p_i|} \right|^{-\frac{\alpha_i}{3}} |y|^{-\frac{2}{3}} U_n^j(y) & \text{if } x_n^j \to p_i \in Z, \\
& \quad (\varepsilon_n^j)^{-3} \lambda_n |x_n^j - p_i|^{\alpha_i + 2} \to +\infty \\[2ex]
|y + (r_n^j)^{-1}(x_n^j - p_i)|^{-\frac{\alpha_i}{3}} |y|^{-\frac{2}{3}} U_n^j(y) & \text{if } (\varepsilon_n^j)^{-3} \lambda_n |x_n^j - p_i|^{\alpha_i + 2} \leq C
\end{cases}
\end{aligned}$$

for any $j = 1, \ldots, l$. By inductive assumption, we have that $U_n^j(y) = \frac{1 - u_n(r_n^j y + x_n^j)}{\varepsilon_n^j} \to U^j(y)$ in $C^1_{\text{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ for any $j = 1, \ldots, l$. Associating (eventually) to x_n^j the limit point y_0 as in (3.10), we get that:

$$\lambda_n^{-\frac{1}{3}}\left(d(r_n^j y + x_n^j)^\alpha\right)^{-\frac{1}{3}} d_n(r_n^j y + x_n^j)^{-\frac{2}{3}}(1 - u_n(r_n^j y + x_n^j))$$
$$\to \begin{cases} (d(p)^\alpha)^{-\frac{1}{3}}|y|^{-\frac{2}{3}}U^j(y) & \text{if } x_n^j \to p \notin Z \\ |y|^{-\frac{2}{3}}U^j(y) & \text{if } x_n^j \to p_i \in Z,\ (\varepsilon_n^j)^{-3}\lambda_n|x_n^j - p_i|^{\alpha_i+2} \to +\infty \\ |y + y_0|^{-\frac{\alpha_i}{3}}|y|^{-\frac{2}{3}}U^j(y) & \text{if } (\varepsilon_n^j)^{-3}\lambda_n|x_n^j - p_i|^{\alpha_i+2} \le C \end{cases}$$

uniformly for $|y| \le R$ as $n \to +\infty$. Since U^j is bounded away from zero, then (3.20) gives also that x_n^{l+1} cannot asymptotically lie in balls centered at x_n^i of radius $\approx r_n^i$, $i = 1, \ldots, l$, namely:

$$\frac{r_n^i}{|x_n^{l+1} - x_n^i|} \to 0 \quad \text{as } n \to +\infty, \quad \forall\, i = 1, \ldots, l. \tag{3.22}$$

Finally, the choice of x_n^{l+1} as a minimum point in (3.20) gives that:

$$\frac{1 - u_n(\beta_n y + x_n^{l+1})}{\varepsilon_n^{l+1}} \ge \left(\frac{d(\beta_n y + x_n^{l+1})^\alpha}{d(x_n^{l+1})^\alpha}\right)^{\frac{1}{3}} \left(\frac{d_n(\beta_n y + x_n^{l+1})}{d_n(x_n^{l+1})}\right)^{\frac{2}{3}}, \tag{3.23}$$

for any sequence β_n. Indeed, by the following chain of estimates:

$$\begin{aligned} \varepsilon_n^{l+1} &\le (d(x_n^{l+1})^\alpha)^{\frac{1}{3}} d_n(x_n^{l+1})^{\frac{2}{3}} \min_{x\in\Omega}((d(x)^\alpha)^{-\frac{1}{3}} d_n(x)^{-\frac{2}{3}}(1 - u_n(x))) \\ &\le (d(x_n^{l+1})^\alpha)^{\frac{1}{3}} d_n(x_n^{l+1})^{\frac{2}{3}} (d(\beta_n y + x_n^{l+1})^\alpha)^{-\frac{1}{3}} \\ &\quad \times d_n(\beta_n y + x_n^{l+1})^{-\frac{2}{3}}(1 - u_n(\beta_n y + x_n^{l+1})), \end{aligned}$$

the validity of (3.23) follows. Here and in the sequel of the proof, the crucial point to establish the validity of (3.3) (or (3.11)) for suitable rescaled functions around x_n^{l+1} is exactly given by the validity of (3.23). By (3.21), we get that in particular $(\varepsilon_n^{l+1})^3\lambda_n^{-1} \to 0$ as $n \to +\infty$. We need now to discuss all the possible types of blow up at x_n^{l+1}.

1st Case. Assume that $x_n^{l+1} \to q \notin Z$. Then, $|x_n^{l+1} - p_j| \ge C > 0$ for any $j = 1, \ldots, k$ which reduces (3.21) to:

$$\frac{(\varepsilon_n^{l+1})^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^i|} \to 0 \quad \text{as } n \to +\infty, \quad \forall\, i = 1, \ldots, l. \tag{3.24}$$

In order to apply Proposition 3.1 to x_n^{l+1}, first of all we need to show that (3.2) holds for x_n^{l+1}:

$$(\varepsilon_n^{l+1})^3\lambda_n^{-1}(\mathrm{dist}(x_n^{l+1}, \partial\Omega))^{-2} \to 0 \quad \text{as } n \to +\infty.$$

We proceed exactly as in the proof of (3.16). By contradiction, up to a subsequence, assume that $(\varepsilon_n^{l+1})^3\lambda_n^{-1}d_n^{-2} \to \delta > 0$ as $n \to +\infty$, where $d_n = \mathrm{dist}(x_n^{l+1}, \partial\Omega)$ (do not confuse d_n with $d_n(x)$), and then by (3.24):

$$\frac{d_n}{|x_n^{l+1}-x_n^i|} = \frac{d_n}{(\varepsilon_n^{l+1})^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}} \frac{(\varepsilon_n^{l+1})^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}{|x_n^{l+1}-x_n^i|}$$
$$\leq \frac{2}{\sqrt{\delta}} \frac{(\varepsilon_n^{l+1})^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}}{|x_n^{l+1}-x_n^i|} \to 0 \quad \text{as } n \to +\infty, \quad \forall\, i = 1,\dots,l.$$

Let $M_n = \left(\frac{d_n(x_n^{l+1})}{d_n}\right)^{\frac{1}{2}} \to +\infty$ as $n \to +\infty$. We introduce the rescaling W_n of u_n in the form:

$$W_n(y) = \frac{1-u_n(d_n y + x_n^{l+1})}{\varepsilon_n^{l+1}} \quad \text{for} \quad y \in A_n = \frac{\Omega - x_n^{l+1}}{d_n} \cap B_{M_n}(0).$$

Since $d_n = \operatorname{dist}(x_n^{l+1}, \partial\Omega) \to 0$ and $M_n \to +\infty$, we get that $A_n \to T$ as $n \to +\infty$, where T is an hyperspace so that $0 \in T$ and $\operatorname{dist}(0, \partial T) = 1$. Since $\{d_n y + x_n^{l+1} : y \in A_n\}$ is uniformly far away from $Z = \{p_1, \dots, p_k\}$, by (3.23) we get for W_n (here, β_n is exactly d_n):

$$W_n(y) \geq C_0 \left(1 - \frac{d_n M_n}{d_n(x_n^{l+1})}\right)^{\frac{2}{3}} \geq \frac{C_0}{2}$$

for any n large and $y \in A_n$. We have used here the following estimate:

$$\frac{d_n(\beta_n y + x_n^{l+1})}{d_n(x_n^{l+1})} = \min\left\{ \left| \frac{x_n^{l+1}-x_n^i}{d_n(x_n^{l+1})} + \frac{\beta_n}{d_n(x_n^{l+1})} y \right| : i = 1,\dots,l \right\}$$
$$\geq 1 - \frac{\beta_n}{d_n(x_n^{l+1})}|y|. \tag{3.25}$$

Hence, the function W_n solves problem (3.17) with $h_n(y) = \frac{\lambda_n d_n^2}{(\varepsilon_n^{l+1})^3} f(d_n y + x_n^{l+1})$ and $C = \frac{C_0}{2}$. Since

$$\|h_n\|_\infty \leq \frac{\lambda_n d_n^2}{(\varepsilon_n^{l+1})^3}\|f\|_\infty \leq \frac{2}{\delta}\|f\|_\infty$$

and $W_n = \frac{1}{\varepsilon_n^{l+1}} \to +\infty$ on $\partial A_n \cap B_2(0)$, Lemma 3.4 provides that (3.19) must hold, contradicting Hopf Lemma for u_n. Hence, (3.2) holds for x_n^{l+1}.

Associated to x_n^{l+1}, let $r_n^{l+1} = (\varepsilon_n^{l+1})^{\frac{3}{2}}\lambda_n^{-\frac{1}{2}}$ be defined according to (3.1). By (3.24) we get that

$$\frac{r_n^{l+1}}{|x_n^{l+1}-x_n^i|} \to 0 \quad \text{as } n \to +\infty, \quad \forall\, i = 1,\dots,l, \tag{3.26}$$

and then, $R_n = \left(\frac{d_n(x_n^{l+1})}{r_n^{l+1}}\right)^{\frac{1}{2}} \to +\infty$ as $n \to +\infty$. Since $\{r_n^{l+1} y + x_n^{l+1} : |y| \leq R_n\}$ is uniformly far away from Z, by (3.23) and (3.25) we get that:

$$U_n^{l+1}(y) := \frac{1-u_n(r_n^{l+1} y + x_n^{l+1})}{\varepsilon_n^{l+1}} \geq C_0 \left(1 - \frac{r_n^{l+1} R_n}{d_n(x_n^{l+1})}\right)^{\frac{2}{3}} \geq \frac{C_0}{2}$$

for n large and $y \in \frac{\Omega - x_n^{l+1}}{r_n^{l+1}} \cap B_{R_n}(0)$. Up to a subsequence, Proposition 3.1 provides $U_n^{l+1} \to U^{l+1}$ in $C^1_{\rm loc}(\mathbb{R}^N)$ as $n \to +\infty$, U^{l+1} being a solution of an equation of type (3.4), and some $\phi_n^{l+1} \in C_0^\infty(\Omega)$ such that (3.5) holds with Supp $\phi_n^{l+1} \subset B_{Mr_n^{l+1}}(x_n^{l+1})$, $M > 0$. By (3.26), combined with (3.22), we get that (3.13) and (3.14) are still true at $(l+1)$th step, as needed.

2nd Case. Assume that $x_n^{l+1} \to p_j$ with the following rate:

$$(\varepsilon_n^{l+1})^{-3}\lambda_n |x_n^{l+1} - p_j|^{\alpha_j+2} \to +\infty \quad \text{as } n \to +\infty.$$

Let $r_n^{l+1} = (\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}} |x_n^{l+1} - p_j|^{-\frac{\alpha_j}{2}}$ according to (3.1). By (3.21) we get that (3.26) still holds and then, $R_n = \left(\min\left\{\frac{|x_n^{l+1}-p_j|}{r_n^{l+1}}, \frac{d_n(x_n^{l+1})}{r_n^{l+1}}\right\}\right)^{\frac{1}{2}} \to +\infty$ as $n \to +\infty$. Since $\{r_n^{l+1} y + x_n^{l+1} : |y| \le R_n\}$ is uniformly close to $p_j \in Z$, estimates (3.23) and (3.25) imply:

$$\begin{aligned} U_n^{l+1}(y) := \frac{1 - u_n(r_n^{l+1}y + x_n^{l+1})}{\varepsilon_n^{l+1}} &\ge \left(\frac{|r_n^{l+1}y + x_n^{l+1} - p_j|}{|x_n^{l+1} - p_j|}\right)^{\frac{\alpha_j}{3}} \left(\frac{d_n(r_n^{l+1}y + x_n^{l+1})}{d_n(x_n^{l+1})}\right)^{\frac{2}{3}} \\ &\ge \left(1 - \frac{r_n^{l+1} R_n}{|x_n^{l+1} - p_j|}\right)^{\frac{\alpha_j}{3}} \left(1 - \frac{r_n^{l+1} R_n}{d_n(x_n^{l+1})}\right)^{\frac{2}{3}} \ge \frac{1}{2} \end{aligned}$$

for n large and $|y| \le R_n$. Up to a subsequence, Proposition 3.1 provides $U_n^{l+1} \to U^{l+1}$ in $C^1_{\rm loc}(\mathbb{R}^N)$ as $n \to +\infty$, where U^{l+1} solves an equation of type (3.4), and the existence of $\phi_n^{l+1} \in C_0^\infty(\Omega)$ such that (3.5) holds, Supp $\phi_n^{l+1} \subset B_{Mr_n^{l+1}}(x_n^{l+1})$ for some $M > 0$. Finally, (3.22) with (3.26) gives that (3.13) and (3.14) are still true at $(l+1)$th step, also in this second case.

3rd Case. Assume that $x_n^{l+1} \to p_j$ and

$$(\varepsilon_n^{l+1})^{-3}\lambda_n |x_n^{l+1} - p_j|^{\alpha_j+2} \le C.$$

By (3.21) $x_n^{l+1} \ne p_j$ and for any $i = 1, \ldots, l$ there holds:

$$\begin{aligned} \frac{|x_n^{l+1} - p_j|}{|x_n^{l+1} - x_n^i|} &= \frac{(\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^i|\,|x_n^{l+1} - p_j|^{\frac{\alpha_j}{2}}} \\ &\quad \times ((\varepsilon_n^{l+1})^{-3}\lambda_n |x_n^{l+1} - p_j|^{\alpha_j+2})^{\frac{1}{2}} \to 0 \quad \text{as } n \to +\infty. \end{aligned} \tag{3.27}$$

Let $r_n^{l+1} = (\varepsilon_n^{l+1})^{\frac{3}{2+\alpha_j}} \lambda_n^{-\frac{1}{2+\alpha_j}}$ according to (3.1). By (3.21) and (3.27) we get that for any $i = 1, \ldots, l$:

$$\begin{aligned} \frac{r_n^{l+1}}{|x_n^{l+1} - x_n^i|} &= \left(\frac{(\varepsilon_n^{l+1})^{\frac{3}{2}} \lambda_n^{-\frac{1}{2}}}{|x_n^{l+1} - x_n^i|\,|x_n^{l+1} - p_j|^{\frac{\alpha_j}{2}}}\right)^{\frac{2}{2+\alpha_j}} \\ &\quad \times \left(\frac{|x_n^{l+1} - p_j|}{|x_n^{l+1} - x_n^i|}\right)^{\frac{\alpha_j}{2+\alpha_j}} \to 0 \quad \text{as } n \to +\infty, \end{aligned} \tag{3.28}$$

providing the validity of (3.26). Let $R_n = (\frac{d_n(x_n^{l+1})}{r_n^{l+1}})^{\frac{1}{2}} \to +\infty$ as $n \to +\infty$. Since $\{r_n^{l+1} y + x_n^{l+1} : |y| \le R_n\}$ is uniformly close to $p_j \in Z$, by (3.23) and (3.25) we get:

$$\begin{aligned} U_n^{l+1}(y) := \frac{1 - u_n(r_n^{l+1} y + x_n^{l+1})}{\varepsilon_n^{l+1}} &\ge \left(\frac{|r_n^{l+1} y + x_n^{l+1} - p_j|}{|x_n^{l+1} - p_j|} \right)^{\frac{\alpha_j}{3}} \left(1 - \frac{r_n^{l+1} R_n}{d_n(x_n^{l+1})} \right)^{\frac{2}{3}} \\ &\ge \frac{1}{2} \left(\frac{|x_n^{l+1} - p_j|}{r_n^{l+1}} \right)^{-\frac{\alpha_j}{3}} \left| y + \frac{x_n^{l+1} - p_j}{r_n^{l+1}} \right|^{\frac{\alpha_j}{3}} \\ &\ge C \left| y + \frac{x_n^{l+1} - p_j}{r_n^{l+1}} \right|^{\frac{\alpha_j}{3}} \end{aligned}$$

for n large and $|y| \le R_n$, where $C > 0$ is a constant. We have used that $\frac{|x_n^{l+1} - p_j|}{r_n^{l+1}} \le C$, which is true for assumption in this case. We use now Proposition 3.1 in combination with Remark 3.1 to get that, up to a subsequence, $U_n^{l+1} \to U^{l+1}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $n \to +\infty$ and U^{l+1} is a solution of an equation of type (3.4).

Moreover, we find $\phi_n^{l+1} \in C_0^\infty(\Omega)$ such that (3.5) holds and $\mathrm{Supp}\, \phi_n^{l+1} \subset B_{Mr_n^{l+1}}(x_n^{l+1})$, $M > 0$. Since (3.22) together with (3.28) gives the validity of (3.13) and (3.14) at $(l+1)$th step, the induction scheme also works in this last case and the proof of Theorem 3.3 is complete. □

3.3. *Compactness of unstable branches*

We are now in position to give the proof of Theorem 1.1. The essential ingredient will be the pointwise estimate of Theorem 3.3. The contradiction will come out from the non existence result of Theorem 2.3.

Proof of Theorem 1.1. By contradiction, up to a subsequence, let us assume that $\max_\Omega u_n \to 1$ as $n \to +\infty$. Up to a further subsequence, Theorem 3.3 gives the existence of m-sequences $x_n^1, \ldots, x_n^m$ so that $x_n^i \to x^i \in \bar{\Omega}$ as $n \to +\infty$ and the following pointwise estimate holds:

$$1 - u_n(x) \ge C \lambda_n^{\frac{1}{3}} \left(d(x)^\alpha\right)^{\frac{1}{3}} d_n(x)^{\frac{2}{3}} \tag{3.29}$$

for any $x \in \Omega$ and $n \ge N_0$, for some $C > 0$ and $N_0 \in \mathbb{N}$ large, where $d(x)^\alpha = \min\{|x - p_i|^{\alpha_i} : i = 1, \ldots, k\}$ and $d_n(x) = \min\{|x - x_n^i| : i = 1, \ldots, m\}$. Therefore, we get the following bounds in Ω:

$$0 \le \frac{\lambda_n f(x)}{(1 - u_n)^2} \le C \frac{f(x)}{(d(x)^\alpha)^{\frac{2}{3}}} \frac{\lambda_n^{\frac{1}{3}}}{d_n(x)^{\frac{4}{3}}}, \tag{3.30}$$

for some $C > 0$. Since by (1.2)

$$\left| \frac{f(x)}{(d(x)^\alpha)^{\frac{2}{3}}} \right| \le |x - p_i|^{\frac{\alpha_i}{3}} \|f_i\|_\infty \le C$$

for $|x - p_i| \le \delta$ and f_i as in (3.8), we get that $\frac{f(x)}{(d(x)^\alpha)^{\frac{2}{3}}}$ is a bounded function on Ω. Hence, by (3.30) $\frac{\lambda_n f(x)}{(1-u_n)^2}$ is uniformly bounded in $L^s(\Omega)$, for any $1 < s < \frac{3N}{4}$.

By elliptic regularity theory and Sobolev embeddings, up to a subsequence, we get that u_n converges weakly in $H_0^1(\Omega)$ and strongly in $C(\bar{\Omega})$ to a limit function $u_0 \in C(\bar{\Omega}) \cap H_0^1(\Omega)$ as $n \to +\infty$. In particular, it holds that $\sup_\Omega u_0 = 1$, by means of the uniform convergence of u_n to u_0. Since $u_0 = 0$ on $\partial\Omega$, the maximum value 1 of u_0 is achieved in Ω and then, $S = \{x \in \Omega : u_0(x) = 1\}$ is a non empty set.

If $\lambda = \lim_{n\to+\infty} \lambda_n = 0$, by (3.30) $\frac{\lambda_n f(x)}{(1-u_n)^2} \to 0$ in $L^s(\Omega)$ as $n \to +\infty$, for any $1 < s < \frac{3N}{4}$. So, $u_0 \in H_0^1(\Omega)$ is a weak harmonic function and then, it should vanish identically, in contradiction to $\max_\Omega u_0 = 1$.

Hence, we have that $\lambda = \lim_{n\to+\infty} \lambda_n > 0$, and by (3.29) we get that $u_0 < 1$ in $\Omega \backslash \{x^1, \ldots, x^m, p_1, \ldots, p_k\}$. In particular, the set S is finite because $S \subset \{x^1, \ldots, x^m, p_1, \ldots, p_k\}$.

Since $\frac{\lambda_n f(x)}{(1-u_n)^2}$ is uniformly bounded in $L^s(\Omega)$ for any $1 < s < \frac{3N}{4}$ and $\frac{\lambda_n f(x)}{(1-u_n)^2} \to \frac{\lambda f(x)}{(1-u_0)^2}$ uniformly on compact sets in $\bar{\Omega} \backslash \{x^1, \ldots, x^m, p_1, \ldots, p_k\}$, we get that

$$\frac{\lambda_n f(x)}{(1-u_n)^2} \to \frac{\lambda f(x)}{(1-u_0)^2} \quad \text{weakly in } L^s(\Omega), \quad 1 < s < \frac{3N}{4}. \tag{3.31}$$

Taking now the limit of the equation satisfied by u_n, by (3.31) we get that $u_0 \in C(\bar{\Omega})$ is a $H^1(\Omega)$-weak solution of:

$$\begin{cases} -\Delta u_0 = \dfrac{\lambda f(x)}{(1-u_0)^2} & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

Since

$$\int_\Omega \left(|\nabla\phi|^2 - \frac{2\lambda_n f(x)}{(1-u_n)^3}\phi^2 \right) \to \int_\Omega \left(|\nabla\phi|^2 - \frac{2\lambda f(x)}{(1-u_0)^3}\phi^2 \right)$$

for any $\phi \in C_0^\infty(\Omega)$ in view of (3.31), by (1.3) we get that u_0 has a finite Morse index according to definition (2.13). Since the set $S = \{x \in \Omega : u_0(x) = 1\}$ is a nonempty finite set, by Theorem 2.3 such a solution u_0 cannot exist and we reach a contradiction. Hence, (1.4) holds.

If we also assume that $\mu_{1,n} < 0$, then $\lambda > 0$. Indeed, if $\lambda_n \to 0$, then by compactness and elliptic regularity theory, we would get $u_n \to u_0$ in $C^1(\bar{\Omega})$, where u_0 is an harmonic function so that $u_0 = 0$ on $\partial\Omega$. Then, $u_0 = 0$ and $u_n \to 0$ in $C^1(\bar{\Omega})$. Hence, $\mu_{1,n} = \mu_{1,\lambda_n}(u_n) \to \mu_{1,0}(0) > 0$ as $n \to +\infty$. A contradiction. □

4. Some Consequences

In the last section, we derive some consequences of Theorem 1.1. First, let us prove the characterization of blow up stated in Theorem 1.2.

Proof of Theorem 1.2. (1) ⇒ (2). Assume that $\max_\Omega u_n \to 1$ as $n \to +\infty$. If $\int_\Omega \left(\frac{f(x)}{(1-u_n)^3}\right)^{\frac{N}{2}} \le C < \infty$ along a subsequence, the right-hand side of (1.1) would be uniformly bounded in $L^{\frac{3N}{4}}$. By elliptic regularity theory and Sobolev embeddings,

$u_n \to u_0$ weakly in $H_0^1(\Omega)$ and strongly in $C(\bar{\Omega})$, where u_0 is a $H^1(\Omega)$-weak solution of (1.1) with $\lambda = \lim_{n\to+\infty} \lambda_n$ so that $\int_\Omega \left(\frac{f(x)}{(1-u_0)^3}\right)^{\frac{N}{2}} < \infty$ and $0 \le u_0 \le 1$. By Proposition 2.1, we get $\|u_0\|_\infty < 1$ and, by uniform convergence, $\|u_n\|_\infty \to \|u_0\|_\infty < 1$ as $n \to +\infty$. A contradiction. Hence, necesssarily $\int_\Omega \left(\frac{f(x)}{(1-u_n)^3}\right)^{\frac{N}{2}} \to +\infty$ as $n \to +\infty$.

(2) $\Rightarrow$ (1). The vice versa is trivial as it follows by the following inequality:

$$\int_\Omega \left(\frac{f(x)}{(1-u_n)^3}\right)^{\frac{N}{2}} \le \frac{\|f\|_\infty^{\frac{N}{2}}}{(1-\|u_n\|_\infty)^{\frac{3N}{2}}}|\Omega|,$$

where $|\cdot|$ stands for the Lebesgue measure.

(1) $\Rightarrow$ (3). Assume that $\max_\Omega u_n \to 1$ as $n \to +\infty$. By Theorem 1.1 $m(u_n, \lambda_n) \to +\infty$ as $n \to +\infty$.

(3) $\Rightarrow$ (1). Since as before $\frac{f(x)}{(1-u_n)^3} \le \frac{\|f\|_\infty}{(1-\|u_n\|_\infty)^3}$, by the variational characterization of the eigenvalues we get that

$$\mu_{k,\lambda_n}(u_n) \ge \mu_k(L_n), \quad L_n := -\Delta - \frac{2\lambda_n \|f\|_\infty}{(1-\|u_n\|_\infty)^3},$$

where $\mu_k(L_n)$ stands for the kth eigenvalue of the operator L_n. Indeed, for operator L in the form $L = -\Delta - c(x)$, $c(x) \in L^s(\Omega)$ for some $s > \frac{N}{2}$, let us recall that:

$$\mu_1(L) = \inf_{\phi \in H_0^1(\Omega),\, \phi \ne 0} \frac{\langle L\phi, \phi\rangle}{\int_\Omega \phi^2},$$

$$\mu_k(L) = \sup\left\{ \inf_{\phi \in M^\perp,\, \phi \ne 0} \frac{\langle L\phi, \phi\rangle}{\int_\Omega \phi^2} : \ M \subset H_0^1(\Omega) \text{ linear, } \dim(M) = k-1 \right\} \quad \forall\, k \ge 2,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $H_0^1(\Omega)$ and $M^\perp$ is the ortogonal space of M in $H_0^1(\Omega)$ with respect to this inner product.

Therefore, point (3) implies that the Morse index of L_n, the number of negative eigenvalues of L_n, blows up as $n \to +\infty$. Hence, the constant function $\frac{2\lambda_n\|f\|_\infty}{(1-\|u_n\|_\infty)^3} \to +\infty$ as $n \to +\infty$ and then, the validity of point (1) is established. $\square$

Now, we establish the uniqueness result contained in Theorem 1.3.

Proof of Theorem 1.3. (1) Let $\lambda_n \to 0$ as $n \to +\infty$ and associated solutions u_n of (1.1) so that $m(u_n, \lambda_n) \le k$, $k \in \mathbb{N}$. Theorem 1.1 implies that $\mu_{1,n} \ge 0$ for n large. By the characterization of the minimal solution u_λ as the only semi-stable solution, we get that $u_n = u_{\lambda_n}$ for n large. Hence, necessarily there exists $\delta = \delta_k > 0$ so that u_λ is the unique solution u of (1.1) with $m(u,\lambda) \le k$ for any $\lambda \in (0,\delta)$.

(2) Let $\lambda_n \to \lambda^*$ as $n \to +\infty$ and associated solutions u_n with $m(u_n,\lambda_n) \le k$, for some $k \in \mathbb{N}$. By Theorem 1.1 we get that $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < 1$. By elliptic regularity theory, u_n is uniformly bounded in $C^{1,\beta}(\bar{\Omega})$ for any $\beta \in (0,1)$. Up to

a subsequence, $u_n \to u_0$ in $C^1(\bar\Omega)$ as $n \to +\infty$, where u_0 is a $C^1(\bar\Omega)$-solution of (1.1) with $\lambda = \lambda^*$ so that $\max_\Omega u_0 < 1$. In [8] it is proven that Eq. (1.1) admits for $\lambda = \lambda^*$ an unique solution, the extremal solution u^*. Then, $u_n \to u^*$ in $C^1(\bar\Omega)$ as $n \to +\infty$. By [4], in a C^1-small neighborhood of u^* problem (1.1) has only the two solutions u_λ, U_λ for λ close to λ^*. Hence, either $u_n = u_{\lambda_n}$ or $u_n = U_{\lambda_n}$ and the uniqueness result follows. □

Finally, we conclude this section by showing the existence of a solutions sequence whose Morse index blows up.

Proof of Theorem 1.4. Let us define the solution set $\mathcal{V}$ as

$$\mathcal{V} = \{(\lambda, u) \in [0, +\infty) \times E : u \text{ is a solution of (1.1)}\},$$

where $E = \{u \in C^1(\bar\Omega) : u = 0 \text{ on } \partial\Omega\}$ is endowed with the standard norm. By contradiction and in view of the equivalence of Theorem 1.2, let us assume that

$$\sup_{(\lambda,u)\in\mathcal{V}} \max_\Omega u \le 1 - 2\delta, \tag{4.1}$$

for some $\delta \in (0, \frac{1}{2})$. Hence, $\mathcal{V}$ is a compact set in $[0, +\infty) \times E$. By Theorem 1.3 we can fix $\lambda_1, \lambda_2 \in (0, \lambda^*)$, $\lambda_1 < \lambda_2$, so that (1.1) possesses:

- for λ_1 only the (non degenerate) minimal solution u_{λ_1} with $m(u_{\lambda_1}, \lambda_1) = 0$;
- for λ_2 has only the two (non degenerate) solutions u_{λ_2}, U_{λ_2} with $m(u_{\lambda_2}, \lambda_2) = 0$, $m(U_{\lambda_2}, \lambda_2) = 1$.

Let us define the projection of $\mathcal{V}$ onto E:

$$\mathcal{U} = \{u \in E : \exists\, \lambda \text{ so that } (\lambda, u) \in \mathcal{U}\},$$

and let us consider a δ-neighborhood of $\mathcal{U}$ in E:

$$\mathcal{U}_\delta = \{u \in E : \operatorname{dist}_E(u, \mathcal{U}) \le \delta\}.$$

Let us remark that by (4.1) we get:

$$\sup_{u\in\mathcal{U}_\delta} \max_\Omega u \le 1 - \delta.$$

Let us regularize the nonlinearity $(1-u)^{-2}$ in the following way:

$$g_\delta(u) = \begin{cases} (1-u)^{-2} & \text{if } u \le 1 - \delta \\ \delta^{-2} & \text{if } u \ge 1 - \delta, \end{cases}$$

in such a way that, for any fixed λ, problem (1.1) in $\mathcal{U}_\delta$ is equivalent to find a zero of the map $T_\lambda = Id - K_\lambda : E \to E$, where $K_\lambda(u) = -\Delta^{-1}(\lambda f(x) g_\delta(u))$ is a compact operator and Δ^{-1} is the laplacian resolvent with homogeneous Dirichlet boundary condition. We can define the Leray–Schauder degree d_λ of T_λ on $\mathcal{U}_\delta$ with respect to zero, since by definition of $\mathcal{U}$ (the set of all solutions) $\partial\mathcal{U}_\delta$ does not contain any solution of (1.1) for any value of λ. Since d_λ is well defined for any $\lambda \in [0, \lambda^*]$, by omotopy $d_{\lambda_1} = d_{\lambda_2}$.

To get a contradiction, let us now compute d_{λ_1}, d_{λ_2}. Since the only zero of T_{λ_1} in $\mathcal{U}_\delta$ is u_{λ_1} with Morse index zero, we have that $d_{\lambda_1} = 1$. While, T_{λ_2} has in $\mathcal{U}_\delta$ exactly two zeroes u_{λ_2}, U_{λ_2} with Morse index zero, one respectively, and hence, $d_{\lambda_2} = 1 - 1 = 0$. This contradicts $d_{\lambda_1} = d_{\lambda_2}$. The proof is complete. □

Appendix A

First of all, we give a sketch of proof of Theorem 3.2 and we refer to [5] for the details.

Proof of Theorem 3.2. By contradiction, we assume $\mu_1(U) \geq 0$ and then,

$$\int |\nabla\phi|^2 \geq 2\int \frac{|y|^\gamma}{U^3}\phi^2\,, \quad \forall\,\phi \in D^{1,2}(\mathbb{R}^N). \tag{A.1}$$

In particular, by (A.1) we get that

$$\int \frac{|y|^\gamma}{(1+|y|^2)^{\frac{N-2}{2}+\delta}U^3} \leq C\int \frac{1}{(1+|y|^2)^{\frac{N}{2}+\delta}} < +\infty\,, \tag{A.2}$$

for any $\delta > 0$.

Step 1. We want to show that (A.1) allows us to perform the following Moser-type iteration scheme: for any $0 < q < 4 + 2\sqrt{6}$ and β there holds

$$\int \frac{1}{(1+|y|^2)^{\beta-1-\frac{\gamma}{2}}U^{q+3}} \leq C_q\left(1 + \int \frac{1}{(1+|y|^2)^\beta U^q}\right) \tag{A.3}$$

(provided the second integral is finite).

Indeed, let $R > 0$ and consider a smooth radial cut-off function η so that: $0 \leq \eta \leq 1$, $\eta = 1$ in $B_R(0)$, $\eta = 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$. Multiplying (3.6) by $\frac{\eta^2}{(1+|y|^2)^{\beta-1}U^{q+1}}$, $q > 0$, integrating by parts and using (A.2) we get:

$$\begin{aligned}\int \frac{|y|^\gamma\eta^2}{(1+|y|^2)^{\beta-1}U^{q+3}} \geq{}& \frac{8(q+1)}{q^2}\int \frac{|y|^\gamma\eta^2}{(1+|y|^2)^{\beta-1}U^{q+3}}\\ &- \frac{2}{q}\int \frac{1}{U^q}\left|\nabla\left(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\right)\right|^2\\ &+ \frac{2(q+2)}{q^2}\int \frac{1}{U^q}\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\Delta\left(\frac{\eta}{(1+|y|^2)^{\frac{\beta-1}{2}}}\right).\end{aligned}$$

Since $8q + 8 - q^2 > 0$ for any $0 < q < q_+$, assuming that $R|\nabla\eta| + R^2|\Delta\eta| \leq C$ we get that:

$$\int \frac{|y|^\gamma\eta^2}{(1+|y|^2)^{\beta-1}U^{q+3}} \leq C_q\int \frac{1}{(1+|y|^2)^\beta U^q},$$

where C_q does not depend on $R > 0$. Taking the limit as $R \to +\infty$, we get the validity of (A.3).

Step 2. Let now $1 \le N \le 7$ or $N \ge 8$, $\gamma > \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. We want to show that

$$\int \frac{1}{(1+|y|^2)U^q} < +\infty \tag{A.4}$$

for some $0 < q < q_+ = 4 + 2\sqrt{6}$.

Given $\delta > 0$, set $\beta_i = \beta_0 - i(1 + \frac{\gamma}{2})$ and $q_i = q_0 + 3i$, $i \in \mathbb{N}$. By (A.2) and iterating Step 1 two times in view of $q_0 < q_1 < q_+ = 4 + 2\sqrt{6} < q_2$, we get that:

$$\int \frac{1}{(1+|y|^2)^{\beta_2} U^{q_2}} < +\infty \tag{A.5}$$

where $\beta_2 = \frac{N-6-3\gamma}{2} + \delta$, $q_2 = 9$. Fix now q: $0 < q < q_+ = 4 + 2\sqrt{6} < 9$. By (A.5) and Hölder inequality we get that $\int \frac{1}{(1+|y|^2)U^q} < +\infty$ provided $-\frac{2q}{9-q}\beta_2 + \frac{18}{9-q} > N$ or equivalently

$$q > \frac{9N-18}{6-2\delta+3\gamma}. \tag{A.6}$$

To have (A.6) for some $\delta > 0$ small and $q < q_+$ at the same time, we need to require $\frac{3N-6}{2+\gamma} < q_+$ or equivalently $1 \le N \le 7$ or $N \ge 8$, $\gamma > \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. Our assumptions then provide the existence of some $0 < q < q_+ = 4 + 2\sqrt{6}$ such that (A.4) holds.

Step 3. To obtain a contradiction, fix $0 < q < 4 + 2\sqrt{6}$ such that (A.4) holds. Letting η as before, using equation (3.6) we compute:

$$\begin{aligned}\int \left|\nabla\left(\frac{\eta}{U^{\frac{q}{2}}}\right)\right|^2 - \int \frac{2|y|^\gamma}{U^3}\left(\frac{\eta}{U^{\frac{q}{2}}}\right) &= -\frac{8q+8-q^2}{4(q+1)}\int \frac{|y|^\gamma \eta^2}{U^{q+3}} + \int \frac{|\nabla \eta|^2}{U^q} \\ &\quad - \frac{q+2}{4(q+1)}\int \frac{\Delta \eta^2}{U^q}.\end{aligned}$$

Since $8q + 8 - q^2 > 0$, by (A.4) we get that:

$$\begin{aligned}&\int \left|\nabla\left(\frac{\eta}{U^{q/2}}\right)\right|^2 - \int \frac{2|y|^\gamma}{U^3}\left(\frac{\eta}{U^{q/2}}\right)^2 \\ &\quad \le -\frac{8q+8-q^2}{4(q+1)}\int_{B_1(0)} \frac{|y|^\gamma}{U^{q+3}} + O\left(\int_{|y|\ge R} \frac{1}{(1+|y|^2)U^q}\right) < 0\end{aligned}$$

for R large, contradicting (A.1). To complete the proof, in [5] it is proven that $\lambda_* = \frac{(2+\gamma)(3N+\gamma-4)}{9}$ and $u^*(x) = 1 - |x|^{\frac{2+\gamma}{3}}$ are the extremal value and solution, respectively, of (1.1) on the unit ball with $f(x) = |x|^\gamma$ and $N \ge 8$, $0 \le \gamma \le \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. The second part of Theorem 3.2 follows by considering the limit profile function around zero as $\lambda \to \lambda^*$ for the minimal solution u_λ for (1.1) on the unit ball with $f(x) = |x|^\gamma$. □

Finally, we prove Lemma 3.4:

Proof of Lemma 3.4. Assume that $\partial_\nu W_n > 0$ on $\partial A_n \cap B_2(0)$. Let $G(y)$ be the Green function at 0 of the operator $-\Delta$ in $B_2(0)$ with homogeneous Dirichlet boundary condition. Since $\partial_\nu G < 0$ on $\partial T \cap B_2(0)$ and $\partial A_n \to \partial T$, we get that $\partial_\nu G < 0$ on $\partial A_n \cap B_2(0)$ for n large and $\int_{\partial A_n \cap B_2(0)} \partial_\nu G d\sigma \to \int_{\partial T \cap B_2(0)} \partial_\nu G d\sigma < 0$. Since $G > 0$ in $B_2(0)$, $\partial_\nu G < 0$ on $\partial B_2(0)$ and the assumptions on W_n, by the representation formula we get:

$$1 = W_n(0) \geq -\int_{A_n \cap B_2(0)} \frac{h_n(y)}{W_n^2} G - \left(\min_{\partial A_n \cap B_2(0)} W_n \right) \int_{\partial A_n \cap B_2(0)} \partial_\nu G d\sigma.$$

But $\left| \int_{A_n \cap B_2(0)} \frac{h_n(y)}{W_n^2} G \right| \leq C$, and then, $1 \geq -C + C^{-1} (\min_{\partial A_n \cap B_2(0)} W_n)$ for some $C > 0$ large enough. Hence, $\min_{\partial A_n \cap B_2(0)} W_n$ is uniformly bounded providing the validity of (3.18). The proof is complete. □

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