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Perturbations of Paneitz-Branson operators on S^n.

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ABSTRACT - We prove the existence of solutions on the standard unit sphere (S^n, h) for the equation $P_h^n u = d_n |u|^{\frac{8}{n-4}} u + (\varepsilon K + o(\varepsilon)) |u|^{q-1} u$, ε small, and $1 \leq q \leq \frac{n+4}{n-4}$, where P_g^n is the fourth order conformally invariant Paneitz-Branson operator. We will approach this problem via a finite dimensional reduction which lead us to consider the «stable» critical points of the «Melnikov function»: in the case $q = \frac{n+4}{n-4}$ a more subtle analysis will be carried out by means of a Morse relation for functions on manifolds with boundary which are quite degenerate on the boundary.

1. Introduction and main results.

Given (M, g) a smooth compact Riemannian manifold of dimension $n \geq 5$, let S_g be the scalar curvature of g and let Rc_g be the Ricci curvature of g.

The Paneitz-Branson operator P_g^n, introduced by Branson in [Bra] as the n-dimensional generalization of the Paneitz operator on 4-manifolds (see [Pan]), is the fourth-order operator defined by

$$P_g^n u = \Delta_g^2 u - div_g[(a_n S_g g + b_n Rc_g)(\cdot, du^{\#})] + \frac{n-4}{2} Q_g^n u$$

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where $\Delta_g = div_g(\nabla)$ is the Laplace-Beltrami operator and

$$Q_g^n = -\frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}S_g^2 - \frac{2}{(n-2)^2}|Rc_g|_g^2$$

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)} \qquad b_n = -\frac{4}{n-2}\,.$$

For $u \in C^1(M)$, we denote by $du^\#$ the vector field $C_1^1 g^{-1} \otimes \nabla u$.

The operator P_g^n is conformally invariant in the sense that if $\tilde{g} = u^{4/(n-4)}g$ is a conformal metric to g, then for all $\varphi \in C^\infty(M)$

$$P_g^n(u\varphi) = u^{\frac{n+4}{n-4}} P_{\tilde{g}}^n \varphi\,.$$

In particular, by taking $\varphi \equiv 1$

$$P_g^n u = \frac{n-4}{2} Q_{\tilde{g}}^n u^{\frac{n+4}{n-4}}.$$

On the standard unit sphere (S^n, h) with $n \geqslant 5$, the expression of P_h^n reduces to

$$P_h^n u = \Delta_h^2 u - c_n \Delta_h u + d_n u$$

where

$$c_n = \frac{1}{2}(n^2-2n-4) \qquad d_n = \frac{n-4}{16}n(n^2-4)\,.$$

In this paper we study perturbations on (S^n, h) of the equation

$$\Delta_h^2 u - c_n \Delta_h u + d_n u = d_n u^{\frac{n+4}{n-4}}. \tag{1}$$

We consider perturbations of the form $g_\varepsilon |u|^{q-1}u$ with $q \in \left[1, \frac{n+4}{n-4}\right]$ and $g_\varepsilon = \varepsilon K + o(\varepsilon)$.

The special case of linear perturbations, e.g. the case $q = 1$, was considered in [Esp] for the conformal Laplacian on the sphere, motivated by uniqueness result in case of constant negative perturbations (see [BVV]).

Even if we don't know whether a similar phenomenon occurs also for the fourth order conformally invariant Paneitz-Branson operator, we extend our previous multiplicity result to the present situation.

THEOREM 1.1. *Let $g_\varepsilon \in C(S^n)$ be a function of the form $g_\varepsilon = \varepsilon K + o(\varepsilon)$ where $o(\varepsilon)$ is in the uniform norm of $C(S^n)$ and let q be an exponent in $\left[1, \frac{n+4}{n-4}\right)$, $n \geq 5$.*

Then, for ε small, the equation

$$\Delta_h^2 u - c_n \Delta_h u + d_n u = d_n u^{\frac{n+4}{n-4}} + g_\varepsilon u^q \tag{2}$$

admits two smooth positive solutions if either K changes sign and $q \geq \frac{4}{n-4}$ or $q < \frac{4}{n-4}$ and $\int_{S^n} K = 0$.

The case $q = \frac{n+4}{n-4}$ corresponds to a «prescribed scalar curvature problem» for the Paneitz-Branson operator.

It is of special interest because of obstructions similar to the Kazdan-Warner obstructions for the Nirenberg problem (see [DHL]).

As for the «scalar curvature type equation» we prove an existence result under a Bahri-Coron index count condition (see [CY1], [BaCo], [CY2], [CGY] and [AAP]).

THEOREM 1.2. *Let $n \geq 5$, let K be a smooth Morse function on S^n such that $\Delta_h K(x) \neq 0$ for all $x \in \mathrm{Crit}(K)$ and let g_ε as above.*

Let $m(K, x)$ be the Morse index of K in $x \in \mathrm{Crit}(K)$ on S^n.

If $\sum_{p \in \mathrm{Crit}\, K,\, \Delta_h K(p) < 0} (-1)^{m(K,p)} \neq (-1)^n$, then for ε small there exists a smooth positive solution of the equation

$$\Delta_h^2 u - c_n \Delta_h u + d_n u = (d_n + g_\varepsilon)^{\frac{n+4}{n-4}} \tag{3}$$

The proofs rely on a finite dimensional reduction (see [AmBa]), exploiting the «non degeneracy» of the manifold of solutions of the unperturbed equation (1): we will describe this approach in the section 3.

The second result is obtained by means of a Morse relation for the «Melnikov function» associated to (3).

Morse theory has been used in [Mal] to solve the scalar curvature equation on the sphere, in the perturbative case.

The main novelty here is the use of Morse theory for functions, on compact manifolds with boundary, which are «degenerate» on boundary points.

Usual Morse theory on manifolds with boundary does not apply in this case and, since we didn't find in the literature the result we need, we will give a proof in the next section.

For convenience, we state here the Morse relation we will use in the proof of Theorem 1.2.

LEMMA 1.3. *Let* $\Psi \in C^2(\overline{B^{n+1}}, \mathbb{R})$ *be a Morse function such that* $\frac{\partial \Psi}{\partial \varrho} = 0$ *on* ∂B^{n+1} *and* $\Psi_0 = \Psi|_{\partial B^{n+1}}$ *be the trace of* Ψ *on the boundary of* B^{n+1}.

Then

$$1 = \chi(\overline{B^{n+1}}) = \sum_{p \in \mathrm{Crit}\, \Psi \cap B^{n+1}} (-1)^{m(\Psi, p)} + \sum_{p \in \mathrm{Crit}\, \Psi_0 \cap \partial B^{n+1},\ \frac{\partial^2 \Psi}{\partial \varrho^2}(p) > 0} (-1)^{m(\Psi_0, p)}. \tag{4}$$

REMARK 1.4. *We have learned from Prof. Hebey that the above result, concerning the «prescribed scalar curvature» problem for the Paneitz-Branson operator, has been recently obtained with a somehow different approach by Ahmedou, Djadli and Malchiodi.*

While submitting the paper, we got to know a preprint by Felli, where, among other things, similar results are proved.

We wish to thank Prof. Hebey for introducing us to the study of the Paneitz-Branson operator.

2. A Morse relation.

In this section we give a proof of Lemma 1.3 as stated in the introduction.

PROOF. (Lemma 1.3).

We consider $\overline{B^{n+1}}$ with the standard metric δ of $\mathbb{R}^n$ and, in a neighborhood of the boundary, we can define the following local coordinates system

$$z_\sigma \colon x \in \left\{ x \in \mathbb{R}^{n+1} \colon \frac{1}{2} < |x| \leq 1,\ \frac{x}{|x|} \neq \sigma \right\} \to \left(\pi_\sigma\left(\frac{x}{|x|} \right), 1 - |x| \right) \in \mathbb{R}^n \times \left[0, \frac{1}{2} \right)$$

and similarly $z_{-\sigma}$ via stereographic projection through the pole $-\sigma$.

In these local coordinates systems the metric δ decomposes in such a way that

$$g_{n+1j} = 0 \quad \forall j = 1, \dots, n \qquad g_{n+1n+1} = 1 .$$

If the functional F is such that, for every point $p \in \partial B^{n+1}$ $\frac{\partial F}{\partial \varrho}(p) = 0$, then it is always possible to define the flow $\varphi_t(q)$ associated to $dF^\# = C_1^1 g^{-1} \otimes \nabla F$, where $\varphi_t(q)$ is the unique solution of

$$\begin{cases} \dfrac{\partial \varphi_t(q)}{\partial t} = -dF^\#(\varphi_t(q)) \quad \text{for all } t \geq 0 \\ \varphi_0(q) = q \end{cases}$$

since

$$(dF^\#)_{n+1} = -g^{n+1n+1} \frac{\partial F}{\partial \varrho} = 0$$

on ∂B^{n+1} with respect to z_σ or $z_{-\sigma}$.

We define $M_F^c := F^{-1}(-\infty, c]$ and, following exactly the proof of the same result in Milnor [Mil], we have that $\forall c$, $\varepsilon > 0$ $M_F^{c-\varepsilon}$ is a deformation retract of $M_F^{c+\varepsilon}$ if $F^{-1}[c-\varepsilon, c+\varepsilon]$ contains no critical points of F, since we can construct deformations along gradient lines of F.

Now, we proceed in the following way.

STEP 1) We prove a Morse lemma for critical point on the boundary (see [Mil]): given $p \in \partial B^{n+1}$ a non degenerate critical point for Ψ and $z := z_{-p}$ a local coordinates system, using $\frac{\partial \Psi}{\partial \varrho} = 0$ on ∂B^{n+1}, we will construct a local coordinates system $y = (y_1, \dots, y_{n+1})$ in a small neighborhood U of p in $\overline{B^{n+1}}$ such that $U \cap \partial B^{n+1} = \{y_{n+1} = 0\}$ and

$$\Psi(q) = \Psi(p) - (y_1(q))^2 - \dots - (y_\eta(q))^2 + \\ + (y_{\eta+1}(q))^2 + \dots + (y_n(q))^2 + \operatorname{sign}\left(\frac{\partial^2 \Psi}{\partial \varrho^2}(p)\right)(y_{n+1}(q))^2$$

holds true for all $q \in U$, where $\eta = m(\Psi_0, p)$.

In particular, we will get that y_i does not depend on $z_{n+1} = 1 - \varrho$ and $y_{n+1}(q) = \alpha(q)(1-\varrho)$ for some $\alpha > 0$.

STEP 2) Following [Mil], we construct a functional F, which is a small perturbation of Ψ in a neighborhood of p, such that for ε small $F^{-1}[c-\varepsilon, c+\varepsilon]$ contains no critical points, the sublevels of F verify $M_F^{c+\varepsilon} = M_\Psi^{c+\varepsilon}$ and $M_\Psi^{c-\varepsilon} \subseteq M_F^{c-\varepsilon}$ and there holds $\frac{\partial F}{\partial \varrho} = 0$ on ∂B^{n+1}, where $c = \Psi(p)$.

For the third statement we will use $\frac{\partial \Psi}{\partial \varrho} = 0$ on ∂B^{n+1} and the link between the local coordinates systems y and z.

STEP 3) We can choose, modulo rearrangement, the local coordinates system $(y_1, \ldots, y_{n+1})$ in the neighborhood U of p such that for all $q \in U$

$$\Psi(q) = \Psi(p) - y_1(q)^2 - \ldots - y_\lambda(q)^2 + y_{\lambda+1}(q)^2 + \ldots + y_{n+1}(q)^2$$

where

$$\lambda = \begin{cases} m(\Psi_0, p) & \text{if } \dfrac{\partial^2 \Psi}{\partial \varrho^2}(p) > 0 \\ m(\Psi_0, p) + 1 & \text{if } \dfrac{\partial^2 \Psi}{\partial \varrho^2}(p) < 0 \end{cases}$$

and $\partial B^{n+1} \cap U = \{y_i = 0\}$ for some $i \in \{1, \ldots, n+1\}$.

We denote $e^\lambda = \{q \in B^{n+1}: y_1(q)^2 + \ldots + y_\lambda(q)^2 \leqslant \varepsilon, y_{\lambda+1}(q) = \ldots = y_{n+1}(q) = 0\}$ and $\dot{e}^\lambda = \{q \in B^{n+1}: y_1(q)^2 + \ldots + y_\lambda(q)^2 = \varepsilon, y_{\lambda+1}(q) = \ldots = y_{n+1}(q) = 0\}$.

Then, using the deformations induced by F,

$$M_\Psi^{c+\varepsilon} = M_F^{c+\varepsilon} \simeq M_F^{c-\varepsilon} = M_\Psi^{c-\varepsilon} \cup H$$

where $H = \overline{M_F^{c-\varepsilon} \setminus M_\Psi^{c-\varepsilon}}$ and $\simeq$ denotes omotopy equivalence.

We denote by $M_\Psi^{c-\varepsilon} \cup e^\lambda$ the attachement of e^λ to $M_\Psi^{c-\varepsilon}$ along $\dot{e}^\lambda = e^\lambda \cap M_\Psi^{c-\varepsilon}$.

We can deform H onto e^λ in a similar way as Milnor [Mil], taking into account that the deformations map $\{y_i \geqslant 0\}$ in itself, and we get

$$M_\Psi^{c+\varepsilon} \simeq M_\Psi^{c-\varepsilon} \cup H \simeq M_\Psi^{c-\varepsilon} \cup e^\lambda.$$

We note that if $\frac{\partial^2 \Psi}{\partial \varrho^2}(p) < 0$, Ψ increases along the direction y_i and e^λ is homeomorphic to an half ball and in $M_\Psi^{c-\varepsilon} \cup e^\lambda$ it is attached just for an half sphere to $M_\Psi^{c-\varepsilon}$.

Hence $M_\Psi^{c-\varepsilon} \simeq M_\Psi^{c+\varepsilon}$.

If $\frac{\partial^2\Psi}{\partial\varrho^2}(p)>0$, then e^λ is a full ball in ∂B^{n+1} entirely attached to $M_\Psi^{c-\varepsilon}$. Hence $M_\Psi^{c-\varepsilon}\simeq M_\Psi^{c+\varepsilon}\cup e^\lambda$ where $\lambda=m(\Psi_0, p)$.

STEP 4) More generally suppose that there are k non degenerate critical points $p_1, \ldots, p_k$ in $\Psi^{-1}(c)$ and suppose that

$$\{p_1, \ldots, p_s\}=\left\{p_i\in\partial B^{n+1}\colon \frac{\partial^2\Psi}{\partial\varrho^2}(p_i)>0\right\}, \quad \{p_{s+1}, \ldots, p_m\}=\{p_i\in B^{n+1}\}$$

and

$$\lambda_i=\begin{cases} m(\Psi_0, p_i) & \text{if } i=1, \ldots, s\\ m(\Psi, p_i) & \text{if } i=s+1, \ldots, m\end{cases}.$$

We remark that for points not on the boundary, the classical Morse theory is true, provided we can define deformations along gradient lines of F.

Then

$$M_\Psi^{c+\varepsilon}\simeq M_\Psi^{c-\varepsilon}\cup e^{\lambda_1}\cup\ldots\cup e^{\lambda_m}.$$

Hence

$$1=\chi(\overline{B^{n+1}})=\sum_{p\in\operatorname{Crit}\Psi\cap B^{n+1}}(-1)^{m(\Psi, p)}+\sum_{p\in\operatorname{Crit}\Psi_0\cap\partial B^{n+1},\ \frac{\partial^2\Psi}{\partial\varrho^2}(p)>0}(-1)^{m(\Psi_0, p)}.$$

For the first step, we sketch the proof of this special version of Morse lemma, just explaining the differences with [Mil].

We will denote $\frac{\partial}{\partial\sigma_i}$ as the derivative with respect to z_i for all $i=1, \ldots, n$. We choose U_1 such that $z(U_1)$ is convex and, by letting $\bar z(q)=(z_1(q),\ldots,z_n(q))$, we can write, since $\frac{\partial\Psi}{\partial\varrho}=0$ on ∂B^{n+1}, for all $q\in U_1$

$$\Psi(q)-\Psi(p)=\sum_{i=1}^n z_i\int_0^1\frac{\partial\Psi}{\partial\sigma_i}(t\bar z, 0)\,dt-(1-\varrho)\int_0^1\frac{\partial\Psi}{\partial\varrho}(\bar z, t(1-\varrho))\,dt=$$

$$=\sum_{i,j=1}^n z_iz_j\int_{[0,1]^2}tdtds\,\frac{\partial^2\Psi}{\partial\sigma_i\partial\sigma_j}(ts\bar z, 0)+$$

$$+(1-\varrho)^2\int_{[0,1]^2}tdtds\,\frac{\partial^2\Psi}{\partial\varrho^2}(\bar z, ts(1-\varrho))=:\sum_{i,j}z_iz_jH_{ij}(z)$$

where

$$H(p) = (H_{ij}(p)) = \frac{1}{2} D^2 \Psi(p) = \frac{1}{2} \begin{pmatrix} \dfrac{\partial^2 \Psi}{\partial \sigma^2}(p) & 0 \\ 0 & \dfrac{\partial^2 \Psi}{\partial \varrho^2}(p) \end{pmatrix}.$$

Following Milnor [Mil], from the non degeneracy of the critical point p for Ψ, it is always possible to find a neighborhood $U_2 \subseteq U_1$ of p and a local coordinates system $(y_1, \ldots, y_n, (1-\varrho))$ such that y_i does not depend on ϱ for all $i = 1, \ldots, n$ and

$$\Psi(q) - \Psi(p) = \sum_{i=1}^{n} \pm y_i(q)^2 + (1-\varrho)^2 H_{n+1n+1}(y_1, \ldots, y_n, (1-\varrho)).$$

This can be done essentially for the fact that H_{ij} does not depend on ϱ for all $i, j = 1, \ldots n$ and $H_{in+1} \equiv 0$ for all $i < n+1$.

Since $H_{n+1n+1}(p) = \frac{1}{2} \frac{\partial^2 \Psi}{\partial \varrho^2}(p) \neq 0$, we can find a neighborhood $U \subseteq U_1$ of p in which it is possible to define

$$y_{n+1} = (1-\varrho)\sqrt{|H_{n+1n+1}(y_1, \ldots, y_n, (1-\varrho))|}.$$

The local coordinates system $(y_1, \ldots, y_{n+1})$ in U is what we are looking for.

For the second step, we remark that in $U \cap \partial B^{n+1} = \{y_{n+1} = 0\}$

$$dF^{\#}(dy_{n+1}) = \sum_j dF^{\#}(dz_j) \frac{\partial z_{n+1}}{\partial y_j} =$$

$$= -\frac{1}{\sqrt{|H_{n+1n+1}|}} \frac{\partial F}{\partial \varrho} = -\frac{1}{\sqrt{|H_{n+1n+1}|}} \frac{\partial \Psi}{\partial \varrho} = 0$$

since F differs from Ψ for a function depending on the square of the local coordinates y_j for all j, while outside U we have $F \equiv \Psi$.

Remark 2.1. *It is possible to extend this result to any manifold (M, g) with boundary ∂M whenever there exist local coordinates systems on the boundary for which the normal derivative of Ψ vanishes on ∂M and the tensor metric g splits between the normal and the boundary directions on ∂M, i.e. $g_{n+1j} = 0$ on ∂M.*

3. Proof of Theorems 1.1 and 1.2.

Weak solutions of (2) and (3) are critical points of the energy functional

$$E_\varepsilon(u) := E_0(u) + G(\varepsilon, u), \qquad u \in H_2^2(S^n)$$

where

$$E_0(u) := \frac{1}{2}\int_{S^n} (\Delta_h u)^2 \, dv_h +$$

$$+ \frac{c_n}{2}\int_{S^n} |\nabla u|^2 dv_h + \frac{d_n}{2}\int_{S^n} u^2 dv_h - d_n \frac{n-4}{2n}\int_{S^n} |u|^{\frac{2n}{n-4}} dv_h$$

$$G(\varepsilon, u) := -\frac{1}{q+1}\int_{S^n} g_\varepsilon |u|^{q+1} dv_h .$$

The homoteties $y \to ty$, through stereographic projection from $\sigma \in S^n$, induce conformal diffeomorphisms $\varphi_{\sigma, t}$ on the sphere and isomorphisms

$$T_{\sigma, t} u(y) := (u \circ \varphi_{\sigma, t})(y) \, |\det d\varphi_{\sigma, t}(y)|^{\frac{n-4}{2n}}, \qquad u \in H_2^2(S^n).$$

For conformal invariance of E_0, we have that

$$Z := \left\{ T_{\sigma, (1-\varrho)^{-1}} 1 = |\det d\varphi_{\sigma, (1-\varrho)^{-1}}(y)|^{\frac{n-4}{2n}} : \varrho\sigma \in B^{n+1} \right\}$$

is a *critical manifold* for E_0 at the fixed energy level $b = E_0(1) = = \frac{(n-4)(n^2-4)}{8}\omega_n \neq 0$, which is the image through the map

$$\Phi : \varrho\sigma \in B^{n+1} \to T_{\sigma, (1-\varrho)^{-1}} 1 . \tag{5}$$

The kernel of the linearized operator at $\bar{u} \equiv 1$ is the set S of u such that

$$\Delta_h^2 u - c_n \Delta_h u + d_n u = d_n \frac{n+4}{n-4} u \Leftrightarrow \left(-\Delta_h + \frac{c_n}{2}\right)^2 u = \left(n + \frac{c_n}{2}\right)^2 u .$$

By choosing $\varphi \in \Lambda_k := \{u : -\Delta_h u = \lambda_k u\}$ as a test function, where $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_k < \ldots < +\infty$ is the ordered sequence of the eigenvalues of $-\Delta_h$ on S^n, we get

$$\left(n + \frac{c_n}{2}\right)^2 \int_{S^n} u\varphi = \int_{S^n} \varphi \left(-\Delta_h + \frac{c_n}{2}\right)^2 u =$$

$$= \int_{S^n} u \left(-\Delta_h + \frac{c_n}{2}\right)^2 \varphi = \left(\lambda_k + \frac{c_n}{2}\right)^2 \int_{S^n} u\varphi\,.$$

Since $\lambda_k \neq n$ for all $k \neq 1$, we get $\int_{S^n} u\varphi = 0$ for $K \neq 1$: hence $u \in \Lambda_1 = \left(\bigoplus_{k \neq 1} \Lambda_k\right)^{\perp}$ and $S \subseteq \Lambda_1$.

Since $\Lambda_1 \subseteq S$ by direct computations, we see that $S = \Lambda_1$.

The eigenspace of the Laplace-Beltrami operator corresponding to the first eigenvalue $\lambda_1 = n$ has dimension $n+1$ (see [Ber]) and by conformal invariance

$$\ker(D^2 E_0(T_{\sigma,t} 1)) = T_{\sigma,t} \ker(D^2 E_0(1))\,.$$

Hence the manifold Z satisfies the non degeneracy assumption $T_z Z = \ker(D^2 E_0(z))$ for all $z \in Z$, being $\dim \ker(D^2 E_0(T_{\sigma,t} 1)) = n+1$ and the inclusion $T_z Z \subseteq \ker(D^2 E_0(z))$ always true.

So, a finite dimensional reduction can be performed (see [AmBa]) and for all $q \in \left[1, \frac{n+4}{n-4}\right]$ we are lead to consider the «stable» critical points of the «Melnikov» function

$$\Gamma(\varrho\sigma) := \lim_{\varepsilon \to 0} \frac{G(\varepsilon, \Phi(\varrho\sigma))}{\varepsilon} = \tag{6}$$

$$= -\frac{1}{q+1} \int_{S^n} K |\det d\varphi_{\sigma,(1-\varrho)^{-1}}|^{\frac{(n-4)(q+1)}{2n}} dv_h.$$

In this context, it is true that $\|w\|$ and $\|\partial_i w\|$ go to zero as $\varepsilon \to 0$ uniformly on compact subsets, $\Gamma \in C^2(B^{n+1})$ and

$$\nabla[E_\varepsilon(z + w(\varepsilon, z))](z) = \varepsilon \nabla\Gamma(z) + o(\varepsilon) \tag{7}$$

where for ε small $w(\varepsilon, \cdot) : B^{n+1} \to H_2^2(S^n)$ is a suitable map and there holds the following result (see [AmBa]).

THEOREM 3.3. *Suppose that there exist* $\bar{z} \in Z$ *and* U *open neighborhood in* Z *of* $\bar{z}$ *such that either* $\min_{\partial U} \Gamma(z) > \Gamma(\bar{z})$ *or* $\max_{\partial U} \Gamma(z) < \Gamma(\bar{z})$.

Then, for ε *small enough, the functional* E_ε *has a critical point* $u_\varepsilon = z_\varepsilon + w(\varepsilon, z_\varepsilon)$ *with* $z_\varepsilon \in U$.

Now we need an expansion around boundary points of the function Γ, see the Appendix for the proof.

LEMMA 3.2. *Let* q *be in* $\left(0, \frac{n+4}{n-4}\right]$ *and* $n \geq 5$.
It results that as $\varrho \to 1$ *uniformly in* $\sigma \in S^n$:

$q > \frac{4}{n-4}$) $\Gamma(\varrho\sigma) =$

$$= -\frac{2^n}{q+1}(1-\varrho)^{n - \frac{n-4}{2}(q+1)}\left[K(-\sigma)\int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n-4}{2}(q+1)}} + o(1)\right]$$

$q = \frac{4}{n-4}$) $\Gamma(\varrho\sigma) = \frac{n-4}{n} 2^n \omega_{n-1}(1-\varrho)^{\frac{n}{2}} \ln(1-\varrho)[K(-\sigma) + o(1)]$

$q < \frac{4}{n-4}$) $\Gamma(\varrho\sigma) =$

$$= -\frac{2^{\frac{n-4}{2}(q+1)}}{q+1}(1-\varrho)^{\frac{n-4}{2}(q+1)}\left[\int_{S^n} \frac{K(y)}{(1+\cos d(\sigma, y))^{\frac{n-4}{2}(q+1)}} dy + o(1)\right].$$

Now, we can prove the existence result in the subcritical case $1 \leq q < \frac{n+4}{n-4}$.

PROOF. (Theorem 1.1).
We see from Lemma 3.2 that Γ vanishes at ∂B^{n+1} since $q < \frac{n+4}{n-4}$.

Moreover, Γ changes sign near the boundary if either $q \geqslant \frac{4}{n-4}$ and K changes sign or $q < \frac{4}{n-4}$ and $\int_{S^n} K = 0$ since

$$\int_{S^n} d\sigma \left(\int_{S^n} \frac{K(y)}{(1+\cos d(\sigma, y))^{\frac{n-4}{2}(q+1)}} dy \right) =$$

$$= \int_{S^n} K(y) \left(\int_{S^n} \frac{d\sigma}{(1+\cos d(\sigma, y))^{\frac{n-4}{2}(q+1)}} \right) dy = 0$$

for the independence on y of $\int_{S^n} \frac{d\sigma}{(1+\cos d(\sigma, y))^{\frac{n-4}{2}(q+1)}}$.

Then Γ has a positive maximum and a negative minimum and satisfies in these points the assumptions of Theorem 3.1 for U with $\operatorname{dist}(U, \partial B^{n+1}) = \eta > 0$.

Hence, from Theorem 3.1, for ε small we find two weak solutions of the equation

$$\Delta_h^2 u - c_n \Delta_h u + d_n u = d_n |u|^{\frac{8}{n-4}} u + g_\varepsilon |u|^{q-1} u$$

which are of the form $u_\varepsilon = \Phi(\xi_\varepsilon) + w(\varepsilon, \Phi(\xi_\varepsilon))$ with $\xi_\varepsilon \in U$ and Φ as in (5).

These functions are smooths, following the regularity result Lemma 2.1 in [DHL] based on ideas in [VDV].

We want to show that for ε small the functions u_ε must be positive.

Let us set $z_\varepsilon = \Phi(\xi_\varepsilon)$.

First, we remark that there exist δ and M positive constant such that $\delta \leqslant z_\varepsilon(x) \leqslant M$ for all $x \in S^n$ and ε small, because for $\operatorname{dist}(\xi_\varepsilon, \partial B^{n+1}) \geqslant \eta > 0$ the functions z_ε are uniformly far from the infinite concentration corresponding to the boundary ∂B^{n+1}.

Moreover, by uniform convergence on compact subsets of B^{n+1} we have

$$\|w(\varepsilon, z_\varepsilon)\|_{H_2^2} \to 0 .$$

We need to show that the convergence to zero in the $H_2^2(S^n)$-norm of $w(\varepsilon, z_\varepsilon)$ implies the convergence to zero in the uniform norm: this will prove that the functions u_ε are positive.

By using the equation for u_ε and $z_\varepsilon \in Z$, we can get for $w_\varepsilon = w(\varepsilon, z_\varepsilon)$

$$(8) \qquad \Delta_h^2 w_\varepsilon - c_n \Delta_h w_\varepsilon + d_n w_\varepsilon =$$

$$= d_n \Big(|z_\varepsilon + w_\varepsilon|^{\frac{8}{n-4}} (z_\varepsilon + w_\varepsilon) - z_\varepsilon^{\frac{n+4}{n-4}} \Big) + g_\varepsilon |z_\varepsilon + w_\varepsilon|^{q-1} (z_\varepsilon + w_\varepsilon).$$

We follow the ideas developed in [VDV] and [DHL] to get a sort of bootstrap for w_ε.

We write

$$P_h^n w_\varepsilon = (-\Delta_h + \alpha_1) \circ (-\Delta_h + \alpha_2) w_\varepsilon = a_\varepsilon w_\varepsilon + b_\varepsilon$$

where $\alpha_{1,2} = \dfrac{c_n \pm \sqrt{c_n^2 - 4d_n}}{2}$, $b_\varepsilon = g_\varepsilon |z_\varepsilon + w_\varepsilon|^{q-1} (z_\varepsilon + w_\varepsilon)$ with

$$(9) \qquad |b_\varepsilon|_{L^p} = O(\varepsilon)$$

for some $p > \dfrac{2n}{n+4}$ and

$$a_\varepsilon = d_n \int_0^1 |z_\varepsilon + t w_\varepsilon|^{\frac{8}{n-4}} dt \in L^{\frac{n}{4}}(S^n).$$

We define $K_\varepsilon = \{x \in S^n : |w_\varepsilon(x)| \leq \sqrt{\|w_\varepsilon\|_{H_2^2}}\}$, $c_\varepsilon = a_\varepsilon \chi_{S^n \backslash K_\varepsilon}$ and $f_\varepsilon = a_\varepsilon w_\varepsilon \chi_{K_\varepsilon}$.

We remark that

$$(10) \qquad |c_\varepsilon|_{L^{\frac{n}{4}}} = O\left(\int_{S^n \backslash K_\varepsilon} (1 + |w_\varepsilon|^{\frac{2n}{n-4}}) \, dx \right)^{\frac{4}{n}} =$$

$$= O\Big(\mathrm{vol}(S^n \backslash K_\varepsilon)^{\frac{4}{n}} + \|w_\varepsilon\|_{H_2^2}^{\frac{8}{n-4}} \Big) \to 0$$

because

$$\mathrm{vol}(S^n \backslash K_\varepsilon) \sqrt{\|w_\varepsilon\|_{H_2^2}} \leq \int_{S^n \backslash K_\varepsilon} |w_\varepsilon| \leq C \|w_\varepsilon\|_{H_2^2}$$

while clearly

$$(11) \qquad |f_\varepsilon|_{L^\infty} \to 0\,.$$

We rewrite (8) in the form

$$(Id - H_\varepsilon)\, w_\varepsilon = (P_h^n)^{-1}(f_\varepsilon) + (P_h^n)^{-1}(b_\varepsilon) \tag{12}$$

where $H_\varepsilon(v) = (P_h^n)^{-1}(c_\varepsilon v)$.

Let us remark that for any $f \in L^s(M)$ and $s > 1$, there exists one and only one $u \in H_4^s(M)$ such that $P_h^n u = f$.

Then, from elliptic theory, for any $s \geq \frac{2n}{n-4}$ and $v \in L^s(S^n)$ there holds

$$|H_\varepsilon v|_{L^s} = |(P_h^n)^{-1}(c_\varepsilon v)|_{L^s} = O(|c_\varepsilon v|_{L^h}) = O(|c_\varepsilon|_{L^{\frac{n}{4}}}\, |v|_{L^s})$$

where $h = \dfrac{ns}{n+4s}$.

Hence $H_\varepsilon \colon L^s(S^n) \to L^s(S^n)$ is an operator with norm $\|H_\varepsilon\| = O(|c_\varepsilon|_{L^{\frac{n}{4}}}) \to 0$ because of (10) and then for ε small the operator $Id - H_\varepsilon \colon L^s(S^n) \to L^s(S^n)$ is invertible.

From (12), (11) and (9), we obtain

$$|w_\varepsilon|_{L^{\bar{s}}} \leq C|(P_h^n)^{-1}(f_\varepsilon) + (P_h^n)^{-1}(b_\varepsilon)|_{L^{\bar{s}}} \leq \tilde{C}(|f_\varepsilon|_{L^\infty} + \varepsilon) \to 0 \tag{13}$$

where $\bar{s} = \dfrac{np}{n-4p} > \dfrac{2n}{n-4}$.

We remark that for $b_\varepsilon \in L^p(S^n)$ with $p > \dfrac{2n}{n+4}$, the standard elliptic estimates give estimates for $(P_h^n)^{-1}(b_\varepsilon)$ in $L^{\bar{s}}(S^n)$.

From (8) and standard bootstrap arguments, the estimate (13) gives for all $s \geq 1$

$$|w_\varepsilon|_{L^s} \to 0\,.$$

Then we infer that

$$P_h^n w_\varepsilon = a_\varepsilon w_\varepsilon + b_\varepsilon \in L^s(S^n)$$

for all $s \geq 1$ with $|a_\varepsilon w_\varepsilon + b_\varepsilon|_{L^s} \to 0$.

Then $w_\varepsilon \to 0$ in $H_4^s(S^n)$ for all $s \geq 1$ which implies that w_ε tends to zero in the uniform norm.

This ends the proof of Theorem 1.1.

To prove Theorem 1.2 we need to know the behaviour the derivatives of Γ up to the boundary: this is performed in Lemma 3.3 below, see the Appendix for the proof. We will extend Γ on $\overline{B^{n+1}}$ to have a C^2 functional for which we will find critical points via Morse theory.

LEMMA 3.3. *Let $n \geq 5$ and K a smooth function on S^n. Then as $\varrho \to 1$*

$$\frac{\partial \Gamma}{\partial \varrho}(\varrho\sigma) \to 0\,; \quad \frac{\partial \Gamma}{\partial \sigma_i}(\varrho\sigma) \to \frac{n-4}{n} 2^{n-1} c_0 \frac{\partial K}{\partial \sigma_i}(-\sigma); \quad \frac{\partial^2 \Gamma}{\partial \sigma_i \partial \varrho}(\varrho\sigma) \to 0;$$

$$\frac{\partial^2 \Gamma}{\partial \sigma_i \partial \sigma_j}(\varrho\sigma) \to -\frac{n-4}{n} 2^{n-1} c_0 \frac{\partial^2 K}{\partial \sigma_i \partial \sigma_j}(-\sigma);$$

$$\frac{\partial^2 \Gamma}{\partial \varrho^2}(\varrho\sigma) \to -\frac{n-4}{n} 2^n c_1 \Delta_h K(-\sigma)$$

uniformly in $\sigma \in S^n$, where $c_0 = \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^n}$ *and* $c_1 = \int_{\mathbb{R}^n} |x|^2 \frac{(n-1)|x|^4 - 2(n+2)|x|^2 + (n+1)}{(1+|x|^2)^{n+2}} dx$ *are positive constants.*

The derivatives in σ_i of Γ in σ and K in $-\sigma$ are respectively evaluated in $z_{-\sigma}$ and stereographic projection through σ.

Now, we can deal with the critical case $q = \frac{n+4}{n-4}$.

PROOF. (Theorem 1.2).

We want to show that there exists ε_0 such that for all ε with $|\varepsilon| \leq \varepsilon_0$

$$\Gamma_\varepsilon(z) := E_\varepsilon(z + w(\varepsilon, z)) = b + \varepsilon\Gamma(z) + o(\varepsilon) \tag{14}$$

possesses a critical point, where

$$E_0(u) := \frac{1}{2}\int_{S^n} (\Delta_h u)^2 dv_h + $$

$$+ \frac{c_n}{2}\int_{S^n} |\nabla u|^2 dv_h + \frac{d_n}{2}\int_{S^n} u^2 dv_h - d_n \frac{n-4}{2n}\int_{S^n} u_+^{\frac{2n}{n-4}} dv_h$$

$$G(\varepsilon, u) = -\frac{n-4}{2n}\int_{S^n} g_\varepsilon u_+^{\frac{2n}{n-4}} dv_h.$$

The critical points of Γ_ε will produce free critical points of E_ε: see [AmBa] for this property and for the expansion (14).

We consider Γ as a C^2-functional on $\overline{B^{n+1}}$, since Z is the image of B^{n+1} through Φ and by Lemma 3.3 Γ can be smoothly extented up to the boundary by setting $\Gamma|_{\partial B^{n+1}}(\sigma) = -\frac{n-4}{n}2^{n-1}c_0 K(-\sigma)$.

Arguing by contradiction, we suppose that there exists a sequence $\varepsilon \to 0$ such that Γ_ε and equivalently $\Phi_\varepsilon := \frac{\Gamma_\varepsilon - b}{\varepsilon} = \Gamma + o(1)$ possess no critical points in B^{n+1}.

Since Γ on ∂B^{n+1} has the same critical points of K, $p_1, \ldots p_k$, non degenerates and then isolated, we can choose δ small such that Γ has no critical points in $B^{n+1} \backslash B_{1-2\delta}$ and $|\nabla \Gamma| \geqslant \tau > 0$ in $\overline{B_{1-\delta}} \backslash B_{1-2\delta}$, where $B_r = \{x \in \mathbb{R}^{n+1} \colon |x| < r\}$.

We can take a cut-off function $\eta \in C_0^\infty(\overline{B_{1-\delta}})$ with $0 \leqslant \eta \leqslant 1$ and $\eta \equiv 1$ in $\overline{B_{1-2\delta}}$ and we can define $\Psi_\varepsilon = \eta \Phi_\varepsilon + (1-\eta)\Gamma = \Gamma + o(1)$.

Since $\nabla \Psi_\varepsilon = \nabla \Gamma + o(1)$ by (7), the functional Ψ_ε has critical points just on the boundary for ε small enough.

Then

$$\chi(\overline{B^{n+1}}) = 1 = \sum_{p \in \mathrm{Crit}\, \Gamma \cap \partial B^{n+1},\ \frac{\partial^2}{\partial \varrho^2}\Gamma(p) > 0} (-1)^{m(\Gamma, p)} = \\ = \sum_{p \in \mathrm{Crit}\, K,\ \Delta_h K(p) < 0} (-1)^{n - m(K, p)}.$$

Then

$$\sum_{p \in \mathrm{Crit}\, K,\ \Delta_h K(p) < 0} (-1)^{m(K, p)} = (-1)^n$$

which is in contradiction with our assumption.

So, for ε small we find u_ε critical point of E_ε constrained on Z_ε and therefore free critical point.

From a regularity result as Lemma 2.1 in [DHL] based on [VDV], it follows that u_ε is a smooth function and, since $(d_n + \varepsilon K)\, u_+^{\frac{n+4}{n-4}} \geqslant 0$ for ε small, from a double application of the maximum principle to $P_h^n = (-\Delta_h + \alpha_1) \circ (-\Delta_h + \alpha_2)$ with $\alpha_{1,2} = \frac{c_n \pm \sqrt{c_n^2 - 4d_n}}{2}$ positive constants, we obtain $u_\varepsilon > 0$ positive solution of (3).

Appendix B.

Proof. (Lemma 3.2).

It's an easy generalization of Lemma 3.1 in [Esp] and we refer to that paper for the details. Using

$$|\det d\varphi_{\sigma,t}|^{\frac{n-4}{2n}}(y) = \left(t\frac{1+|\pi_\sigma(y)|^2}{1+t^2|\pi_\sigma(y)|^2} \right)^{\frac{n-4}{2}}$$

and integrating in stereographic coordinates, if $q > \frac{4}{n-4}$, by dominated convergence, as $t = (1-\varrho)^{-1} \to +\infty$ we get

$$I := \int_{S^n} K |\det d\varphi_{\sigma,t}|^{\frac{n-4}{2n}(q+1)} dv_h =$$

$$= \frac{2^n}{t^{n-\frac{n-4}{2}(q+1)}} \left[K(-\sigma) \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^{\frac{n-4}{2}(q+1)}} + o(1) \right].$$

If $q = \frac{4}{n-4}$, it is enough to split I into two integrals: the former on $B(0, \delta t)$ behaves as $2^n \omega_{n-1} K(-\sigma) \frac{\ln t}{t^{n/2}}$ for δ small and $t \geq \bar{t}(\delta)$ large while the latter on $\mathbb{R}^n \setminus B(0, \delta t)$ as t^2.

If $q < \frac{4}{n-4}$, the function $|x|^{-(n-4)(q+1)}$ is locally integrable and, recalling that $\pi_\sigma^{-1}\left(\frac{z}{|z|^2}\right) = \pi_{-\sigma}^{-1}(z)$ and $1 + |\pi_{-\sigma}(y)|^2 = \frac{2}{1+\cos d(\sigma, y)}$, we get

$$I = \frac{2^{\frac{n-4}{2}(q+1)}}{t^{\frac{n-4}{2}(q+1)}} \left[\int_{S^n} \frac{K(y)}{(1+\cos d(\sigma, y))^{\frac{n-4}{2}(q+1)}} dy + o(1) \right].$$

Proof. (Lemma 3.3).

We set $t = (1-\varrho)^{-1}$. For the first derivative in ϱ, we get

$$\frac{\partial \Gamma}{\partial \varrho}(\varrho\sigma) = -(n-4)2^{n-1} t \int_{\mathbb{R}^n} K\left(\pi_\sigma^{-1}\frac{x}{t}\right) \frac{1-|x|^2}{(1+|x|^2)^{n+1}} dx \to 0$$

because of Taylor expansion of K at the second order, oddness properties and $\int_{\mathbb{R}^n} \frac{1-|x|^2}{(1+|x|^2)^{n+1}} = 0$.

For the first derivative in σ_i, evaluated with respect to $z_{-\sigma}$ in a neighborhood of the boundary, we remark that

$$\frac{\partial}{\partial r_i}\left(K \circ \pi_{\bar{r}}^{-1} \frac{x}{t}\right)\Big|_{r=0} =$$

$$= -2 \sum_{j \neq i} \frac{\partial K}{\partial \sigma_j}\left(\pi_\sigma^{-1} \frac{x}{t}\right) \frac{x_i x_j}{t^2} - \frac{\partial K}{\partial \sigma_i}\left(\pi_\sigma^{-1} \frac{x}{t}\right) \frac{t^2 + 2x_i^2 - |x|^2}{t^2}$$

where $\bar{r} = \pi_{-\sigma}^{-1}(r)$, since the relation

$$\pi_{\bar{s}}^{-1} \frac{x}{t} = A_s^{-1} \circ \pi_\sigma^{-1} \frac{x}{t}$$

for $A_s \in O(n+1)$ choosen as the shorter rotation in the plane $\langle e_i, e_{n+1}\rangle$ which maps $\bar{s} = \pi_{-\sigma}^{-1}(se_i)$ in σ, lead us to compute exactly the derivative in s of the expression $\pi_\sigma \circ A_s^{-1} \circ \pi_\sigma^{-1} \frac{x}{t}$.

Hence

$$\frac{\partial \Gamma}{\partial \sigma_i}(\varrho\sigma) = -\frac{n-4}{n} 2^{n-1} \int_{\mathbb{R}^n} \frac{\partial}{\partial r_i}\left[K \circ \pi_{\bar{r}}^{-1} \frac{x}{t}\right]\Big|_{r=0} \frac{dx}{(1+|x|^2)^n} \to$$

$$\to \frac{n-4}{n} 2^{n-1} c_0 \frac{\partial K}{\partial \sigma_i}(-\sigma).$$

Similarly, for the second derivatives we have

$$\frac{\partial^2 \Gamma}{\partial \varrho \partial \sigma_i}(\varrho\sigma) \to 0$$

and

$$\frac{\partial^2 \Gamma}{\partial \sigma_i \partial \sigma_j}(\varrho\sigma) \to -\frac{n-4}{n} 2^{n-1} \frac{\partial^2 K}{\partial \sigma_i \partial \sigma_j}(-\sigma)$$

because

$$\int_{\mathbb{R}^n} \frac{1-|x|^2}{(1+|x|^2)^{n+1}} dx = \int_{\mathbb{R}^n} x_j \frac{1-|x|^2}{(1+|x|^2)^{n+1}} dx = 0\,.$$

Finally, the second derivative in ϱ gives

$$\frac{\partial^2 \Gamma}{\partial \varrho^2}(\varrho\sigma) \to -\frac{n-4}{n} 2^{n-2} c_1 \left(\sum_i \frac{\partial^2 K}{\partial \sigma_i^2}(-\sigma)\right) = -\frac{n-4}{n} 2^n c_1 \Delta_h K(-\sigma)$$

because

$$\int_{\mathbb{R}^n} t^{n+1} \frac{1-t^2|x|^2}{(1+t^2|x|^2)^{n+1}} = 0 \ \forall t \Rightarrow \int_{\mathbb{R}^n} \frac{(n-1)|x|^4 - 2(n+2)|x|^2 + (n+1)}{(1+|x|^2)^{n+2}} dx = 0 .$$

The constant c_1, defined as in statement of Lemma 3.3, verifies

$$c_1 = \frac{8(n+1)}{n(n-2)} \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^{n+1}} dx$$

and then is positive.

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