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SYMMETRIES AND BLOW-UP PHENOMENA FOR A DIRICHLET PROBLEM WITH A LARGE PARAMETER

Pierpaolo Esposito and Maristella Petralla

Dipartimento di Matematica, Università degli Studi "Roma Tre"
Largo S. Leonardo Murialdo 1, Rome, 00146, Italy

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Abstract. For the Dirichlet problem $-\Delta u+\lambda V(x)u=u^p$ in $\Omega\subset\mathbb{R}^N$, $N\geq 3$, in the regime $\lambda\to+\infty$ we aim to give a description of the blow-up mechanism. For solutions with symmetries an uniform bound on the "invariant" Morse index provides a localization of the blow-up orbits in terms of c.p.'s of a suitable modified potential. The main difficulty here is related to the presence of fixed points for the underlying group action.

1. **Introduction.**

We study the Dirichlet problem

$$\begin{cases} -\Delta u+\lambda V u=u^p & \text{in } \Omega \\ u>0 & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, $N\geq 2$, $p>1$, V is a positive potential and λ is a large parameter.

Under the transformation $u(x)\to\lambda^{-\frac{1}{p-1}}u(x)$, $\lambda\to\varepsilon=\frac{1}{\sqrt{\lambda}}$, notice that problem (1.1) reads equivalently as a singularly perturbed Dirichlet equation. Both with Dirichlet and Neumann boundary condition, singularly perturbed problems have been widely investigated in literature, as they arise as steady state equation in several biological and physical models, such as population dynamics, pattern formation theories and chemical reactor theory.

The main feature of problem (1.1) is the intrinsic non-compactness as $\lambda\to+\infty$. To be more precise, it is well known that

$$\|u_n\|_\infty\to\infty \qquad \text{as } n\to+\infty,$$

where u_n is a sequence of solutions of (1.1) with $\lambda_n\to+\infty$ as $n\to\infty$ (see for example [16]). An energy or a Morse index bound forces the blow-up set to be finite, and an accurate description of the asymptotic behavior for ground-state solutions is available in the Dirichlet [31, 38] and the Neumann [29, 30] case. More generally, in the Dirichlet case energy and Morse index bounds give an equivalent asymptotic information [16], and as a by-product a non-degeneracy result can be obtained. The construction of solutions with pointwise blow-up – the so-called spike-layers– has been subject of an extensive investigation in the past [5, 6, 8, 9, 11, 13, 19, 21, 34, 36, 37, 38].

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Higher dimensional blow-up (on curves, surfaces,...) had been conjectured by Wei Ming Ni [28] (in the case of Neumann boundary condition): for every $k = 1, \dots, N-1$ there exist solutions that blow-up at a suitable $k-$dimensional subset of Ω. For domains with symmetries positive constructive results were available [1, 2, 3, 4, 10, 12, 26, 27]. The general case has been recently proved [22], while the case $k = N-1$ and $k = 1$ with $N = 3$ were treated previously in [24, 25] and [23], respectively.

For radial solutions (on the annulus and the ball) an interesting result –due to A. Ambrosetti, A. Malchiodi and W.-M. Ni [1]– identifies the crucial role played by the modified potential $M(r) = r^{N-1}V^{\theta}(r)$, $\theta = \frac{p+1}{p-1} - \frac{1}{2}$: they construct families of radial solutions which blow-up on spheres whose radii are non-degenerate c.p.'s of M. From the asymptotical point of view very few is known. In the Dirichlet case on an annulus, an asymptotic analysis has been firstly performed by E.N. Dancer [7] by means of ODE techniques, showing that, for $V \equiv 1$ and p sub-critical, the only positive radial solution is the radial ground state with its unique maximum on a sphere whose radius goes to 1 as $\lambda \to +\infty$. Notice that the radial ground state solution has both energy and Morse index very large, and the asymptotic techniques based on a bound for the energy (see for example [14]) do not work. An alternative asymptotic approach has been developed [15] by the first author in collaboration with G. Mancini, S. Santra and P.N. Srikanth so to deal with radial solutions of uniformly bounded radial Morse indices and general V's and to rigorously establish the correspondence between c.p.'s of M and blow-up radii.

The aim of the paper is to continue the analysis of [15] and exploit partial symmetries in describing the asymptotic behavior of solutions to (1.1). To be more precise, given a $k-$dimensional subgroup $G \subset O(N)$, let Ω be a $G-$invariant set and V a $G-$invariant function: for every $x \in \Omega$ and $g \in G$ there holds $gx \in \Omega$ and $V(gx) = V(x)$. We deal with $G-$invariant solutions u of problem (1.1) and look for a localization of the blow-up set. As we will discuss, the presence of a non-trivial $G_0 = \{x \in \Omega : gx = x\}$ –the set of fixed points under the action of G– is generally responsible for a degeneration of the blow-up $G-$orbits onto points of G_0. To establish high dimensional blow-up, in [35] the authors explicitly construct in $\mathbb{R}^4$ a $1-$parameter group action with $G_0 = \emptyset$, and then carry over an asymptotic analysis for ground-state solutions on an annulus with $V = 1$ which are invariant under this action.

The main point here is to allow general groups G (possibly with $G_0 \neq \emptyset$), general dimensions N and solutions which are not ground states. Since every smooth action on a sphere of even dimension has fixed points, notice that in odd dimensions N we always have $G_0 \neq \emptyset$. We will consider the group G as generated by the rotations in the planes $\{x_1, x_{k+1}\}, \dots, \{x_k, x_{2k}\}$. Letting $s = (x_{2k+1}, \dots, x_N) \in \mathbb{R}^{N-2k}$ (with the agreement that $N \geq 2k$ and s is disregarded when $N = 2k$), we have that Ω and G_0 are generated by $\Omega_0 = \{(r, s) \in [0, +\infty)^k \times \mathbb{R}^{N-2k} : (r, 0, s) \in \Omega\}$ and $\Omega_0 \cap \{r = 0\}$ under the action of G, respectively. The main tool in the asymptotic approach we propose is given by uniform bounds on the reduced Morse index $m_G(u)$ for a $G-$invariant solution u of (1.1). Let us define

$$H^G = \{u \in H_0^1(\Omega) : u \text{ is } G\text{-invariant a.e.}\},$$

and let $m_G(u)$ be the maximal dimension of subspaces $W \subset H^G$ for which the quadratic form associated to $-\Delta + \lambda V - pu^{p-1}$ is strictly negative in $W \setminus \{0\}$.

Introduce the Sobolev exponent

$$p_S(N) = \begin{cases} +\infty & \text{if } N = 2 \\ \frac{N+2}{N-2} & \text{if } N \geq 3 \end{cases}$$

and the Joseph-Lundgren exponent

$$p_{JL}(N) = \begin{cases} +\infty & \text{if } N \leq 10 \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11. \end{cases}$$

By an asymptotic approach based on the assumption $\sup\limits_{n \in \mathbb{N}} m_G(u_n) < \infty$, we have the following description of the blow-up mechanism along u_n (see Theorem 2.2 for a more refined statement):

Theorem 1.1. *Let $1 < p < p_{JL}(N)$ with $p \notin \{p_S(j) : j = 3, \dots, N\}$. Let u_n be a positive $G-$invariant solution of (1.1) with $\lambda = \lambda_n \to +\infty$ as $n \to +\infty$ and $\sup\limits_n m_G(u_n) < +\infty$. Up to a sub-sequence, there exist $(r_n^1, s_n^1), \dots, (r_n^h, s_n^h) \in \Omega_0$, $h \leq \sup\limits_n m_G(u_n)$, so that for all $i, j = 1, \dots, h,$, $i \neq j$,*

$$\lambda_n |P_n^i - P_n^j|^2 \to +\infty\,, \ \ \lambda_n\, d(P_n^i, \partial\Omega)^2 \to +\infty\,, \ \ \lambda_n V(P_n^i) \sim u_n^{p-1}(P_n^i) \quad \text{as } n \to +\infty,$$

and

$$u_n(P_n^i) = (1 + o_n(1)) \max_{\Omega \cap B_{R_n \lambda_n^{-\frac{1}{2}}}(P_n^i)} u_n$$

for some $R_n \to +\infty$ and $o_n(1) \to 0$ as $n \to +\infty$, where $P_n^i = (r_n^i, 0, s_n^i) \in \Omega$. Moreover, there holds

$$u_n(r, 0, s) \leq C(\lambda_n)^{\frac{1}{p-1}} \sum_{i=1}^{h} e^{-\gamma\, \lambda_n^{\frac{1}{2}} |(r,0,s) - P_n^i|} \quad \forall\, (r, s) \in \Omega_0, \ \ n \in \mathbb{N}$$

for some C, $\gamma > 0$.

Just to comment the assumption on p in Theorem 1.1, let us recall that [17] finite Morse index solutions of $-\Delta U = U^p$ do not exist nor in $\mathbb{R}^j$ neither in the half-space as long as $1 < p < p_{JL}(j)$, $p \neq p_S(j)$. Even though the G-invariant problem (1.1) might be studied as an equation in Ω_0 with the operator Δ re-written in cylindrical coordinates, we will not pursue this approach so to better exploit the information on $m_G(u)$ which, in our opinion, seems more readable in Ω.

A careful expansion of Pohozaev-type identities now provides a localization of the blow-up set.

Main Theorem. *Let u_n be a positive $G-$invariant solution of (1.1) with $\lambda = \lambda_n \to +\infty$ as $n \to +\infty$ and $\sup\limits_n m_G(u_n) < +\infty$. Assume that $x \cdot \nu(x) \neq 0$ for all $x \in \partial\Omega$. Letting P_n^i, $i = 1, \dots, h$, be the points given by Theorem 1.1, set $P^i = \lim\limits_{n \to +\infty} P_n^i$ (up to a sub-sequence). Letting $\nu = (\nu_r, 0, \nu_s)$ be the unit outward normal at $(r, 0, s) \in \partial\Omega$ and $V(r, s) := V(r, 0, s)$, we have that*

- *if $P^i \in \Omega$, then $\nabla_s V(P^i) = 0$ and*

$$\left(\sum_{j=1}^{k} r_j \partial_{r_j} V + \Theta_i V \right)(P^i) = 0,$$

where $\Theta_i = \Theta(P^i)$ is given by (3.3);

- *if* $P^i \in \partial\Omega$, *then there exists* $\mu_i \geq 0$ *so that* $\nabla_s V(P^i) = -\mu_i \nu_s(P^i)$ *and*

$$\left(\sum_{j=1}^{k} r_j \partial_{r_j} V + \mu_i^{'} r \cdot \nu_r + \Theta_i V \right)(P^i) = 0$$

where

$$\mu_i^{'} = \begin{cases} \mu_i & \text{if } \nu_s(P^i) \neq 0 \\ \geq 0 & \text{if } \nu_s(P^i) = 0. \end{cases}$$

The paper rises from partial results contained, among other things, in [32]. Section 2 will be devoted to give a global asymptotic description for a blowing-up sequence u_n provided $\sup_n m_G(u_n) < +\infty$ does hold. In Section 3 an expansion of some Pohozev identities will follow from all the previous analysis, providing the localization of the blow-up set $S = \{P^i : i = 1, \dots, h\}$ as given in Theorem 1.

2. **Asymptotic analysis and blow-up profile.** Let u_n be a positive $G-$invariant solution of

$$\begin{cases} -\Delta u_n + \lambda_n V u_n = u_n^p & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\lambda_n \to +\infty$. Assuming that $\sup_n m_G(u_n) < +\infty$, we aim to obtain a global description of the asymptotic behavior of u_n as $n \to +\infty$. By a blow-up procedure, the first step is to study the local asymptotic profile of u_n around local maximum points Q_n, usually described in terms of an entire solution (in the whole space or the half-space) of a limiting equation. Depending on the distance of Q_n from G_0 w.r.t. the blow-up rate, the asymptotic profile keeps $k - k_0$ of the original symmetries and becomes constant in k_0 directions. The main difficulty is to describe correctly the different situations.

Recalling the definiton of the Sobolev exponent

$$p_S(N) = \begin{cases} +\infty & \text{if } N = 2 \\ \frac{N+2}{N-2} & \text{if } N \geq 3 \end{cases}$$

and the Joseph-Lundgren exponent

$$p_{JL}(N) = \begin{cases} +\infty & \text{if } N \leq 10 \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11, \end{cases}$$

let us notice that $p_S(N) < p_{JL}(N)$ for $N \geq 3$ and $p_S(N)$, $p_{JL}(N)$ are strictly decreasing in N for $N \geq 3$, $N \geq 11$, respectively. The result we have is:

Theorem 2.1. *Let* $p > 1$ *and* u_n *be a positive* $G-$*invariant solution of (2.1) with* $\sup_n m_G(u_n) < +\infty$. *Let* $Q_n = (\tilde{r}_n, 0, s_n) \in \Omega$, $(\tilde{r}_n, s_n) \in \Omega_0$, *be so that for some* $R_n \to +\infty$

$$u_n(Q_n) = \max_{\Omega \cap B_{R_n \mu_n}(Q_n)} u_n \to +\infty$$

as $n \to +\infty$, *where* $\mu_n = u_n(Q_n)^{-\frac{p-1}{2}}$. *Letting*

$$J = \left\{ j = 1, \dots, k : \frac{\tilde{r}_{n,j}}{\mu_n} \to +\infty \text{ as } n \to +\infty \right\}, \quad \bar{J} = \{1, \dots, k\} \setminus J,$$

we define $k_0 = card\ J$, $G_{\bar{J}} = Span\ \{rotation\ in\ x_j, x_{k+j}\ :\ j \in \bar{J}\}$ *and* r_n *as*

$$r_{n,j} = \begin{cases} \tilde{r}_{n,j} & \text{if } j \in J \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $1 < p < p_{JL}(N-k_0)$ *and* $p \neq p_S(N-k_0)$. *Setting* $P_n = (r_n, 0, s_n)$ *and* $\varepsilon_n = \lambda_n^{-\frac{1}{2}} V(P_n)^{-\frac{1}{2}}$, *we introduce* $U_n(y) = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n y + P_n)$ *in* $\Omega_n := \frac{\Omega - P_n}{\varepsilon_n}$. *Up to a sub-sequence, then we have that* $1 < p < p_S(N-k_0)$ *and*

- $\lambda_n d^2(P_n, \partial\Omega) \to +\infty$ *as* $n \to +\infty$;
- $\lambda_n |P_n - Q_n|^2 \to 0$ *as* $n \to +\infty$;
- $u_n(P_n) = (1 + o_n(1)) \max\limits_{\Omega \cap B_{R_n \epsilon_n}(P_n)} u_n$ *for some* $R_n \to +\infty$ *and* $o_n(1) \to 0$ *as* $n \to +\infty$;
- $\lambda_n V(P_n) u_n(P_n)^{-(p-1)} \to \tilde{\lambda} \in (0, +\infty)$ *as* $n \to +\infty$;
- $U_n \to U_0$ *in* $C^1_{loc}(\mathbb{R}^N)$, *where* U_0 *is constant in* y_{k+j}, $j \in J$, *and in the remaining variables coincides with the unique radial solution of*

$$-\Delta U_0 + U_0 = U_0^p \qquad \text{in } \mathbb{R}^{N-k_0}. \tag{2.2}$$

Moreover, there exists a $G-$*invariant function* $\phi_n \in C^1_0(\Omega)$ *with* $\operatorname{supp} \phi_n \subset A_{R\epsilon_n}(P_n)$, *where for* $R > 0$

$$A_{R\epsilon_n}(P_n) := \Big\{ x \in \mathbb{R}^N : \sum_{j=1}^{k} \Big(\sqrt{|x_j|^2 + |x_{k+j}|^2} - r_{n,j} \Big)^2 + |(x_{2k+1}, \ldots, x_N) - s_n|^2 \le R^2 \varepsilon_n^2 \Big\},$$

so that

$$\int_\Omega |\nabla \phi_n|^2 + (\lambda_n V - p u_n^{p-1}) \phi_n^2 < 0 \tag{2.3}$$

for all n *large.*

Proof. By a blow-up procedure, we aim to describe the asymptotic profile of u_n around Q_n (at distance μ_n from Q_n) in terms of non-trivial entire solutions for a limiting equation. When the point Q_n is sufficiently close to the fixed points set of $G_{\bar{J}}$, we expect that, up to a translation, the limiting profile is a $G_{\bar{J}}$-invariant entire solution. In order to re-absorb this translation, we replace Q_n with P_n, and for the blow-up argument it is crucial to have a-priori $\mu_n \sim \tilde{\epsilon}_n := u_n(P_n)^{-\frac{p-1}{2}}$. Since as we will see $\tilde{\epsilon}_n \sim \epsilon_n$, it is more convenient to replace $\tilde{\epsilon}_n$ with ϵ_n in order to get a unified form for the limiting equation (2.2). For simplicity in the notations, assume that $\bar{J} = \{1, \ldots, k - k_0\}$.

Let $\hat{d}_n$ denote $d(Q_n, \partial\Omega)$. Up to a sub-sequence, suppose that $\frac{\mu_n}{\hat{d}_n} \to L \in [0, +\infty]$ as $n \to +\infty$. Then $\hat{\Omega}_n = \frac{\Omega - Q_n}{\mu_n} \to H$, with H an half-space so that $0 \in \overline{H}$ and $d(0, \partial H) = \frac{1}{L}$. The function $W_n(y) = \mu_n^{\frac{p-1}{2}} u_n(\mu_n y + Q_n)$ satisfies

$$\begin{cases} -\Delta W_n + \lambda_n \mu_n^2 V(\mu_n y + Q_n) W_n = W_n^p & \text{in } \hat{\Omega}_n \\ 0 < W_n \le W_n(0) = 1 & \text{in } \hat{\Omega}_n \cap B_{R_n}(0) \\ W_n = 0 & \text{on } \partial\hat{\Omega}_n. \end{cases}$$

Since Q_n is a point of local maximum of u_n, we have

$$0 \le -\Delta W_n(0) = 1 - \lambda_n \mu_n^2 V(Q_n),$$

and then, up to a sub-sequence,

$$\lambda_n \mu_n^2 V(Q_n) \to \hat{\lambda} \in [0, 1]$$

as $n \to +\infty$. Since $W_n^p - \lambda_n \mu_n^2 V(\mu_n y + Q_n) W_n$ is uniformly bounded in $\hat{\Omega}_n \cap B_{R_n}(0)$, up to a further sub-sequence, by elliptic regularity theory [18] we get that

$W_n \to W$ in $C^1_{loc}(\overline{H})$ as $n \to +\infty$, where W solves

$$\begin{cases} -\Delta W + \hat\lambda W = W^p & \text{in } H \\ 0 < W \le W(0) = 1 & \text{in } H \\ W = 0 & \text{on } \partial H. \end{cases}$$

Since $W(0) = 1$ and $W = 0$ on ∂H, we deduce that $0 \in H$ and $L < +\infty$.

Given $J = \{k - k_0 + 1, \dots, k\}$, we want to show now that H contains all the lines $y_{k+j} = t$, $j \in J$, passing through points in H : $y_t = (y_1, \dots, y_{k+j-1}, t, y_{k+j+1}, \dots, y_N) \in H$ for all $y \in H$, $t \in \mathbb{R}$ and $j \in J$. For n large, we have that $y \in \hat\Omega_n$, and then $\mu_n y + Q_n \in \Omega$. Since Ω is invariant under rotation in the $\{x_j, x_{k+j}\}-$plane, we have that

$$\mu_n y + Q_n + (0, \dots, 0, R_n \cos\theta - (\mu_n y_j + \tilde r_{n,j}), 0, \dots, 0, R_n \sin\theta - \mu_n y_{k+j}, 0, \dots, 0) \in \Omega$$

for all $\theta \in \mathbb{R}$ and n large, where $R_n = \sqrt{(\mu_n y_j + \tilde r_{n,j})^2 + \mu_n^2 y_{k+j}^2}$. Going back to $\hat\Omega_n$, we have that

$$P_\theta := y + \Big(0, \dots, 0, \frac{R_n}{\mu_n}(\cos\theta - 1) - y_j + \frac{R_n - \tilde r_{n,j}}{\mu_n}, 0, \dots, 0, \frac{R_n}{\mu_n}\sin\theta - y_{k+j}, 0, \dots, 0\Big) \in \hat\Omega_n$$

for all $\theta \in \mathbb{R}$ and n large. For $j \in J$ we have that $\frac{\tilde r_{n,j}}{\mu_n} \to +\infty$ and then $\frac{R_n - \tilde r_{n,j}}{\mu_n} \to y_j$ as $n \to +\infty$. Choosing $\theta = \theta_n := \frac{\mu_n}{R_n} t$ for a given $t \in \mathbb{R}$, we get that

$$\lim_{n\to\infty} P_{\theta_n} = y_t \in \bar H$$

in view of $\theta_n \to 0$ as $n \to +\infty$. Since H is an half-space, a straight line in $\bar H$ lies either in H or ∂H. Since $y \in H$, then $y_t \in H$ for all $t \in \mathbb{R}$, as claimed. Since P_θ and y are connected through a rotation in the original variables $\{x_j, x_{k+j}\}$, we have that $W_n(P_{\theta_n}) = W_n(y)$ for n large, and then as $n \to +\infty$

$$W(y_t) = W(y)$$

for all $t \in \mathbb{R}$. Since W does not depend on y_{k+j} for all $j \in J$, W is a solution of

$$\begin{cases} -\Delta W + \hat\lambda W = W^p & \text{in } H' \\ 0 < W \le W(0) = 1 & \text{in } H' \\ W = 0 & \text{on } \partial H', \end{cases}$$

where $H' = H \cap \{y_{k+j} = 0 : \forall\, j \in J\}$ is either an half-space or $\mathbb{R}^{N-k_0}$. Since W is non-trivial, in case H is an half-space by Theorem 12-[17] we have $\hat\lambda > 0$ as long as $1 < p < p_{JL}(N - k_0 - 1)$, and this is a contradiction in view of Theorem 1.1-[16]. Since we assume $1 < p < p_{JL}(N - k_0)$, we necessarily have that $H' = \mathbb{R}^{N-k_0}$ and $H = \mathbb{R}^N$ (i.e. $L = 0$).

Since

$$|P_n - Q_n| = \Big(\sum_{j \in \bar J} (\tilde r_n)_j^2\Big)^{\frac12} = O(\mu_n),$$

up to a sub-sequence we get that $\frac{P_n - Q_n}{\mu_n} \to Z$, and then

$$\frac{u_n(P_n)}{u_n(Q_n)} = W_n\Big(\frac{P_n - Q_n}{\mu_n}\Big) \to W(Z) > 0$$

as $n \to +\infty$, in view of $W_n \to W$ in $C^1_{loc}(\mathbb{R}^N)$. In particular, we have shown that

$$\frac{\mu_n}{\tilde\epsilon_n} \ge \delta_0 > 0. \tag{2.4}$$

Since $d(Q_n, \partial\Omega) >> \mu_n$ as $n \to +\infty$ in view of $L = 0$, by (2.4) we get that

$$\frac{d(P_n, \partial\Omega)}{\tilde{\epsilon}_n} \geq \delta_0 \frac{d(Q_n, \partial\Omega) - |P_n - Q_n|}{\mu_n} \to +\infty \tag{2.5}$$

as $n \to +\infty$.

We are now in position to replace Q_n, μ_n with P_n, $\tilde{\epsilon}_n$. Since (up to a subsequence) $\lambda_n \tilde{\epsilon}_n^2 V(P_n) \to \tilde{\lambda} \in [0, +\infty)$ in view of (2.4), by (2.5) we have that $\tilde{\Omega}_n = \frac{\Omega - P_n}{\tilde{\epsilon}_n} \to \mathbb{R}^N$, and $\tilde{U}_n(y) = \tilde{\epsilon}_n^{\frac{p-1}{2}} u_n(\tilde{\epsilon}_n y + P_n)$ converges in $C^1_{loc}(\mathbb{R}^N)$ to a $G_{\bar{J}}-$invariant solution $\tilde{U}$ of

$$\begin{cases} -\Delta \tilde{U} + \tilde{\lambda}\tilde{U} = \tilde{U}^p & \text{in } \mathbb{R}^N \\ 0 < \tilde{U} \leq \tilde{U}(\tilde{Z}) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.6}$$

where $\tilde{Z} = -ZW^{\frac{p-1}{2}}(Z)$. Arguing as before, the function $\tilde{U}$ does not depend on y_{k+j} for all $j \in J$ and does solve (2.6) in $\mathbb{R}^{N-k_0}$. As we will see in the next Proposition, we have that $m_{G_{\bar{J}}}(\tilde{U}) < +\infty$. The argument in [17] for the case $m(\tilde{U}) < +\infty$ still works in our context: notice that the test functions $\eta \tilde{U}^q$ with η a radial cut-off function, used in [17] to get estimates, are $G_{\bar{J}}-$invariant as long as $\tilde{U}$ is (see [32] for the details). In this way we see that $\tilde{\lambda} > 0$ whenever $1 < p < p_{JL}(N - k_0)$, $p \neq p_S(N - k_0)$, i.e. $\tilde{\epsilon}_n \sim \epsilon_n := \lambda_n^{-\frac{1}{2}} V^{-\frac{1}{2}}(P_n)$. In particular, by (2.5) we deduce that

$$\lambda_n d^2(P_n, \partial\Omega) \to +\infty \qquad \text{as } n \to +\infty.$$

Finally, let us replace $\tilde{\epsilon}_n$ with ϵ_n. The function $U_n(y) = \epsilon_n^{\frac{p-1}{2}} u_n(\epsilon_n y + P_n)$, $y \in \Omega_n = \frac{\Omega - P_n}{\epsilon_n}$, converges in $C^1_{loc}(\mathbb{R}^N)$ to a $G_{\bar{J}}-$invariant solution U of

$$\begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^{N-k_0} \\ 0 < U \leq U(Z_0) & \text{in } \mathbb{R}^{N-k_0} \end{cases} \tag{2.7}$$

where $Z_0 = -Z\tilde{\lambda}^{\frac{1}{2}} W^{\frac{p-1}{2}}(Z)$ (U is constant in y_{k+j} for all $j \in J$). As already explained for [17], the argument in [16, 33] for the case $m(U) < +\infty$ works as well when $m_{G_{\bar{J}}}(U) < \infty$ (see also [32]). Since $m_{G_{\bar{J}}}(U) < +\infty$ as we will see in the next Proposition, by [33] we get that $1 < p < p_S(N - k_0)$, and by [16] we conclude that U coincides with the unique radial solution U_0 of (2.7), according to [20]. Since U_0 achieves its maximum at 0, we get that $Z_0 = 0$, i.e.

$$\lim_{n \to +\infty} \lambda_n |P_n - Q_n|^2 = \lim_{n \to +\infty} \frac{|P_n - Q_n|^2}{\epsilon_n^2} = Z = 0.$$

Since

$$\lim_{n \to +\infty} \frac{u_n(P_n)}{u_n(Q_n)} = W(0) = 1,$$

we get that

$$u_n(P_n) = (1 + o_n(1)) u_n(Q_n) = (1 + o_n(1)) \max_{\Omega \cap B_{\delta R_n \epsilon_n}(P_n)} u_n$$

where $o_n(1) \to 0$ as $n \to +\infty$, in view of $B_{\delta R_n \epsilon_n}(P_n) \subset B_{R_n \mu_n}(Q_n)$ for some $\delta > 0$ small. □

The last part of Theorem 2.1 follows now by

Proposition 1. *Under the assumptions of Theorem 2.1, there exists a $G-$invariant function $\phi_n \in C_0^1(\Omega)$ so that supp $\phi_n \subset A_{R\epsilon_n}(P_n)$, for some $R>0$ large, and*

$$\int_\Omega |\nabla \phi_n|^2 + (\lambda_n V - p\, u_n^{p-1})\phi_n^2 dx < 0$$

for n large.

Proof. Assume for simplicity that $\bar J = \{1,\dots,k-k_0\}$. We have established that $U_n \to U$ in $C^1_{loc}(\mathbb{R}^{N-k_0})$, where U is a $G_{\bar J}-$invariant solution of (2.7) in $\mathbb{R}^{N-k_0}$. Let Φ be a $G_{\bar J}-$invariant function in $\mathbb{R}^{N-k_0}$ so that supp $\Phi \subset B_R(0)$ and

$$\int_{\mathbb{R}^{N-k_0}} |\nabla \Phi|^2 + \int_{\mathbb{R}^{N-k_0}} \Phi^2 - p\int_{\mathbb{R}^{N-k_0}} U^{p-1}\Phi^2 < 0 \tag{2.8}$$

does hold. Setting $r_j = \sqrt{x_j^2 + x_{k+j}^2}$, define ϕ_n as

$$\phi_n(x) \\ =\Big(\frac{\epsilon_n^{-(N-k_0-2)}}{\prod_{j\in J} r_{n,j}}\Big)^{\frac12}\Phi\left(\frac{r_1-r_{n,1}}{\epsilon_n},\dots,\frac{r_k-r_{n,k}}{\epsilon_n},0,\dots,0,\frac{x_{2k+1}-s_{n,1}}{\epsilon_n},\dots,\frac{x_N-s_{n,N-2k}}{\epsilon_n}\right).$$

Since supp $\Phi \subset B_R(0)$, we get that ϕ_n is $G-$invariant function such that supp $\phi_n \subset A_{R\epsilon_n}(P_n)$, where $A_{R\epsilon_n}(P_n) \subset \Omega$ for n large in view of $d(P_n,\partial\Omega) >> \epsilon_n$. Let us stress that ϕ_n is a smooth function: for $j \in \bar J$ the quantity $r_j - r_{n,j}$ reduces to $r_j = \sqrt{x_j^2 + x_{k+j}^2}$ and $\Phi(\dots,\frac{r_j}{\epsilon_n},\dots,0,\dots) = \Phi(\dots,\frac{x_j}{\epsilon_n},\dots,\frac{x_{k+j}}{\epsilon_n},\dots)$ is smooth in x_j, x_{k+j} by the $G_{\bar J}-$invariance of Φ; for $j\in J$ the set $A_{R\epsilon_n}(P_n)$ does not touch $\{r_j=0\}$ (where ϕ_n might be singular), in view of $\frac{r_{n,j}}{\epsilon_n} \to +\infty$ as $n\to+\infty$, and then ϕ_n is smooth also in x_j, x_{k+j}. Since Φ is $G_{\bar J}-$invariant, for $j\in\bar J$ and $h\neq j$ let us remark that

$$\begin{aligned}
&\Phi(\dots,y_j,\dots,y_{k+j},\dots) = \Phi(\dots,\sqrt{y_j^2+y_{k+j}^2},\dots,0,\dots)\\
&\partial_{y_h}\Phi(\dots,y_j,\dots,y_{k+j},\dots) = \partial_{y_h}\Phi(\dots,\sqrt{y_j^2+y_{k+j}^2},\dots,0,\dots)\\
&(\partial_{y_j}\Phi)^2 + (\partial_{y_{k+j}}\Phi)^2](\dots,y_j,\dots,y_{k+j},\dots) = (\partial_{y_j}\Phi)^2(\dots,\sqrt{y_j^2+y_{k+j}^2},\dots,0,\dots).
\end{aligned} \tag{2.9}$$

Since Φ has compact support, through cylindrical coordinates and the change of variables $(r,s)\to(r_n+\epsilon_n r, s_n+\epsilon_n s)$ we have that

$$\begin{aligned}
&\int_\Omega |\nabla\phi_n|^2 + (\lambda_n V - p\,u_n^{p-1})\phi_n^2 = (2\pi)^k\frac{\epsilon_n^{-(N-k_0)}}{\prod_{j\in J} r_{n,j}}\times\\
&\times\int_{\{|(r-r_n,s-s_n)|\le R\epsilon_n,\, r\in[0,+\infty)^k\}}\prod_{j=1}^k r_j\left[\Big(\sum_{h=1}^k+\sum_{h=2k+1}^N\Big)(\partial_{y_h}\Phi)^2\Big(\frac{r-r_n}{\epsilon_n},\hat 0,\frac{s-s_n}{\epsilon_n}\Big)\right.\\
&\left.+\left(\frac{V(r,0,s)}{V(P_n)} - p\epsilon_n^2 u_n^{p-1}(r,0,s)\right)\Phi^2\Big(\frac{r-r_n}{\epsilon_n},\hat 0,\frac{s-s_n}{\epsilon_n}\Big)\right]drds\\
=&(2\pi)^k\int_{\{|(r,s)|\le R, r_j\ge -\frac{r_{n,j}}{\epsilon_n}\forall j=1,\dots,k\}}\prod_{j\in\bar J} r_j\prod_{j\in J}\Big(\frac{\epsilon_n}{r_{n,j}}r_j+1\Big)\left[\Big(\sum_{h=1}^k+\sum_{h=2k+1}^N\Big)(\partial_{y_h}\Phi)^2(r,\hat 0,s)\right.\\
&\left.+\left(\frac{V(\epsilon_n(r,0,s)+P_n)}{V(P_n)} - pU_n^{p-1}(r,0,s)\right)\Phi^2(r,\hat 0,s)\right]drds
\end{aligned}$$

converges to

$$(2\pi)^k \int_{\{|(r,s)|\le R,\ r_j\ge 0\ \forall\ j\in \bar J\}} \prod_{j\in\bar J} r_j \Big[(\sum_{h=1}^{k} + \sum_{h=2k+1}^{N})(\partial_{y_h}\Phi)^2(r,\hat 0,s) + \big(1-pU^{p-1}(r,0,s)\big)\,\Phi^2(r,\hat 0,s)\Big]\, drds$$

as $n\to+\infty$ in view of $\frac{\epsilon_n}{r_{n,j}}\to 0$ for all $j\in J$. We use the notation 0, $\hat 0$ to denote the origin in $\mathbb{R}^k$, $\mathbb{R}^{k-k_0}$, respectively. Since U is $G_{\bar J}-$invariant and is constant in y_j for $j=2k-k_0+1,\dots,2k$, by (2.9) we deduce that

$$\begin{aligned}
&(2\pi)^k \int_{\{|(r,s)|\le R,\ r_j\ge 0\ \forall\ j\in \bar J\}} \prod_{j\in\bar J} r_j \Big[(\sum_{h=1}^{k} + \sum_{h=2k+1}^{N})(\partial_{y_h}\Phi)^2(r,\hat 0,s) \\
&\quad + \big(1-pU^{p-1}(r,0,s)\big)\,\Phi^2(r,\hat 0,s)\Big]\, drds \\
=&(2\pi)^k \int_{\{|(r,\hat 0,s)|\le R,\ (r,\hat 0,s)\in\mathbb{R}^{N-k_0},\ r_j\ge 0\ \forall\ j\in \bar J\}} \prod_{j\in\bar J} r_j \Big[(\sum_{h=1}^{k} + \sum_{h=2k+1}^{N})(\partial_{y_h}\Phi)^2(r,\hat 0,s) \\
&\quad + \big(1-pU^{p-1}(r,\hat 0,s)\big)\,\Phi^2(r,\hat 0,s)\Big]\, drds \\
=&(2\pi)^{k_0}\int_{\mathbb{R}^{N-k_0}} \big[|\nabla\Phi|^2+(1-pU^{p-1})\Phi^2\big]<0
\end{aligned}$$

in view of supp $\Phi\subset B_R(0)$. If Φ_1, Φ_2 are $G_{\bar J}-$invariant functions with compact support so that (2.8) does hold and $\int_{\mathbb{R}^{N-k_0}}\Phi_1\Phi_2=0$, then the corresponding $\phi_{1,n}$, $\phi_{2,n}$ satisfy

$$\int_\Omega \frac{\phi_{1,n}}{(\int_\Omega \phi_{1,n}^2)^{\frac12}}\frac{\phi_{2,n}}{(\int_\Omega \phi_{2,n}^2)^{\frac12}} \to \int_{\mathbb{R}^{N-k_0}} \frac{\Phi_1}{(\int_{\mathbb{R}^{N-k_0}}\Phi_1^2)^{\frac12}}\frac{\Phi_2}{(\int_{\mathbb{R}^{N-k_0}}\Phi_2^2)^{\frac12}}=0$$

as $n\to+\infty$. In this way, we show, as already claimed in the previous proof, that $m_{G_{\bar J}}(U)\le \sup\limits_{n\in\mathbb{N}} m_G(u_n)$. Since the same does hold for the solution $\tilde U$, the arguments here fill the missing points in the previous proof.

Then, we have that $1<p<p_S(N-k_0)$ and $U_n\to U_0$ in $C^1_{loc}(\mathbb{R}^N)$, where U_0 is the radial solution of (2.7) in $\mathbb{R}^{N-k_0}$. Since U_0 decays exponentially fast at infinity, we have that U_0 satisfies

$$\int_{\mathbb{R}^{N-k_0}}|\nabla U_0|^2+\int_{\mathbb{R}^{N-k_0}}U_0^2-p\int_{\mathbb{R}^{N-k_0}}U_0^{p+1}=-(p-1)\int_{\mathbb{R}^{N-k_0}}U_0^{p+1}<0.$$

Through a radial cut-off function χ so that $\chi=1$ in $B_{\frac R2}(0)$ and $\chi=0$ outside $B_R(0)$, we have that $\Phi=\chi U_0$ is radial and satisfies

$$\int_{\mathbb{R}^{N-k_0}}|\nabla \Phi|^2+\int_{\mathbb{R}^{N-k_0}}\Phi^2-p\int_{\mathbb{R}^{N-k_0}}U_0^{p-1}\Phi^2<0$$

for R large. From Φ we can construct a function ϕ_n which satisfies (2.3), as desired. □

Once the limiting problem has been identified and the local behavior has been described, we can control the global behavior.

Theorem 2.2. *Let $1<p<p_{JL}(N)$ with $p\notin\{p_S(j):j=3,\dots,N\}$ and u_n be a positive $G-$invariant solution of (2.1) with $\sup\limits_n m_G(u_n)<+\infty$. Up to a subsequence, there exist $P_n^1=(r_n^1,0,s_n^1),\dots,P_n^h=(r_n^h,0,s_n^h)$, $h\le\sup\limits_n m_G(u_n)$, with*

$(r_n^i, s_n^i) \in \Omega_0$ *for* $i = 1, \dots, h$, *so that for all* i, $j = 1, \dots, h$ *with* $i \neq j$ *as* $n \to +\infty$:

$$\lambda_n |P_n^i - P_n^j|^2 \to +\infty\,, \quad \lambda_n\, d(P_n^i, \partial\Omega)^2 \to +\infty\,, \quad \frac{\lambda_n V(P_n^i)}{u_n(P_n^i)^{(p-1)}} \to \lambda_i > 0, \tag{2.10}$$

$$u_n(P_n^i) = (1 + o_n(1)) \max_{\Omega \cap B_{R_n \lambda_n^{-\frac{1}{2}}}(P_n^i)} u_n \tag{2.11}$$

for some $R_n \to +\infty$ *and* $o_n(1) \to 0$ *as* $n \to +\infty$, *and there exists* $J_i \subset \{1, \dots, k\}$ *so that*

$$U_n^i(y) := \lambda_n^{-\frac{1}{p-1}} V^{-\frac{1}{p-1}}(P_n^i) u_n(\lambda_n^{-\frac{1}{2}} V^{-\frac{1}{2}}(P_n^i) y + P_n) \to U_{0,i}(y) \text{ in } C^1_{loc}(\mathbb{R}^N) \tag{2.12}$$

where $U_{0,i}$ *is constant in* y_{k+j}, $j \in J_i$, *and in the remaining variables coincides with the unique radial solution of (2.2) in* $\mathbb{R}^{N-k_i}$, $k_i = card\ J_i$. *Moreover, there holds*

$$u_n(r, 0, s) \leq C(\lambda_n)^{\frac{1}{p-1}} \sum_{i=1}^{h} e^{-\gamma\, \lambda_n^{\frac{1}{2}} |(r,0,s) - P_n^i|} \qquad \forall\, (r,s) \in \Omega_0\,,\ n \in \mathbb{N} \tag{2.13}$$

for some C, $\gamma > 0$.

Proof. The proof is by now rather standard (see for example [15, 16]) and proceeds in two steps. In the sequel, the notation $(r, 0, s)$ implicitly means that $(r, s) \in \Omega_0$.

$1^{\mathbf{st}}$ **Step** There exist sequences $P_n^1 = (r_n^1, 0, s_n^1), \dots, P_n^h = (r_n^h, 0, s_n^h)$, $h \leq \sup_n m_G(u_n)$, satisfying (2.10)-(2.12) so that

$$\lim_{R \to +\infty} \left(\limsup_{n \to +\infty} \left[\lambda_n^{-\frac{1}{p-1}} \max_{\{d_n(r,s) \geq R \lambda_n^{-\frac{1}{2}}\}} u_n(r, 0, s) \right] \right) = 0 \tag{2.14}$$

where $d_n(r, s) = \min\{|(r, s) - (r_n^i, s_n^i)| \,:\, i = 1, \dots, h\}$ is the distance function in Ω_0 from $\{(r_n^1, s_n^1), \dots, (r_n^h, s_n^h)\}$.

Let $Q_n^1 = (\tilde{r}_n^1, 0, s_n^1)$ be a point of global maximum of u_n: $u_n(Q_n^1) = \max_\Omega u_n$. By Theorem 2.1 the second and third in (2.10), (2.11) and (2.12) do hold for the corresponding $P_n^1 = (r_n^1, 0, s_n^1)$. In particular, we also have that

$$\lambda_n V(P_n^1)(\max_\Omega u_n)^{-(p-1)} \to \lambda_1 > 0 \qquad \text{as } n \to +\infty. \tag{2.15}$$

If (2.14) does hold too, then we take $h = 1$ and the Claim is proved. In order to apply Theorem 2.1, notice that $1 < p < p_{JL}(N) \leq p_{JL}(N - k_1)$ and $p \neq p_S(N - k_1)$. If (2.14) does not hold, set $\epsilon_n^1 = \lambda_n^{-\frac{1}{2}} V(P_n^1)^{-\frac{1}{2}}$ and suppose by contradiction that

$$\limsup_{R \to +\infty} \left(\limsup_{n \to +\infty} [(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{|(r,0,s) - P_n^1| \geq R\varepsilon_n^1\}} u_n(r, 0, s)] \right) = 4\delta > 0.$$

By (2.12) we have that $U_{0,1}$ is constant in y_{k+j}, $j \in J_1$, and in the remaining variables y' coincides with the unique radial solution of (2.2) in $\mathbb{R}^{N-k_1}$, $k_1 = \text{card } J_1$. Since $U_{0,1}(y') \to 0$ as $|y'| \to \infty$ we can find R large so that

$$U_{0,1}(y) \leq \delta \qquad \forall\, y : \ |y| \geq R, \tag{2.16}$$

and, up to a sub-sequence, we can assume that

$$(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{|(r,0,s) - P_n^1| \geq R\varepsilon_n^1\}} u_n(r, 0, s) \geq 2\delta. \tag{2.17}$$

Since $u_n = 0$ on $\partial\Omega$, then we have that there exists $Q_n^2 = (\tilde r_n^2, 0, s_n^2) \in \{|(r,0,s) - P_n^1| \ge R\varepsilon_n^1\} \cap \Omega$ so that

$$u_n(Q_n^2) = \max_{\{|(r,0,s)-P_n^1|\ge R\,\varepsilon_n^1\}} u_n(r,0,s).$$

By (2.12) and (2.16) we have that $\frac{|Q_n^2 - P_n^1|}{\varepsilon_n^1} \to +\infty$. Indeed, if $\frac{Q_n^2 - P_n^1}{\varepsilon_n^1} \to Z$, $|Z| \ge R$, were true along a sub-sequence, we would get

$$(\varepsilon_n^1)^{\frac{2}{p-1}}\, u_n(Q_n^2) = U_n^1\Big(\frac{Q_n^2 - P_n^1}{\varepsilon_n^1}\Big) \to U_{0,1}(Z) \le \delta,$$

in contradiction with (2.17). Setting now $\mu_n^2 = u_n(Q_n^2)^{-\frac{p-1}{2}}$ and $R_n^2 = \frac{1}{2}\frac{|Q_n^2 - P_n^1|}{\mu_n^2}$, by (2.17) we get

$$\mu_n^2 \le (2\,\delta)^{-\frac{p-1}{2}}\, \varepsilon_n^1$$

and then

$$R_n^2 \ge \frac{(2\,\delta)^{\frac{p-1}{2}}}{2}\, \frac{|Q_n^2 - P_n^1|}{\varepsilon_n^1} \to +\infty \ \text{ as } n \to +\infty.$$

For $x \in \Omega \cap B_{R_n^2\,\mu_n^2}(Q_n^2)$ we have that $u_n(x) = u_n(r_x, 0, s_x)$, where

$$(r_x, 0, s_x) := \Big(\sqrt{x_1^2 + x_{k+1}^2}, \ldots, \sqrt{x_k^2 + x_{2k}^2}, 0, \ldots, 0, x_{2k+1}, \ldots, x_N\Big)$$

belongs to the set $\{|(r,0,s) - Q_n^2| \le R_n^2\mu_n^2\}$, and then

$$u_n(Q_n^2) \le \max_{\Omega\cap B_{R_n^2\,\mu_n^2}(Q_n^2)} u_n \le \max_{\{|(r,0,s)-Q_n^2|\le R_n^2\mu_n^2\}} u_n.$$

Since $\varepsilon_n^1 << |Q_n^2 - P_n^1|$, for all $(r,0,s) \in \{|(r,0,s) - Q_n^2| \le R_n^2\mu_n^2\}$ we have

$$|(r,0,s) - P_n^1| \ge |Q_n^2 - P_n^1| - |(r,0,s) - Q_n^2| \ge \frac{1}{2}|Q_n^2 - P_n^1| \ge R\,\varepsilon_n^1.$$

The inclusion $\{|(r,0,s) - Q_n^2| \le R_n^2\mu_n^2\} \subset \{|(r,0,s) - P_n^1| \ge R\varepsilon_n^1\}$ leads to

$$u_n(Q_n^2) \le \max_{\Omega\cap B_{R_n^2\,\mu_n^2}(Q_n^2)} u_n \le \max_{\{|(r,0,s)-Q_n^2|\le R_n^2\mu_n^2\}} u_n \le \max_{\{|(r,0,s)-P_n^1|\ge R\epsilon_n^1\}} u_n = u_n(Q_n^2),$$

implying that

$$u_n(Q_n^2) = \max_{\Omega\cap B_{R_n^2\,\mu_n^2}(Q_n^2)} u_n.$$

Let associate the set J_2 to Q_n^2 according to Theorem 2.1 and set $k_2 = \mathrm{card}\, J_2$. Since $R_n^2 \to +\infty$ as $n \to +\infty$ and $1 < p < p_{JL}(N - k_2)$ with $p \ne p_S(N - k_2)$, by Theorem 2.1 we can replace Q_n^2 with $P_n^2 = (r_n^2, 0, s_n^2)$ so that the second and third in (2.10), (2.11) and (2.12) do hold for P_n^2. Moreover, the first in (2.10) does hold too for $\{\,P_n^1,\, P_n^2\,\}$ as it follows by

$$\lambda_n^{\frac12}|P_n^1 - P_n^2| \ge \lambda_n^{\frac12}|P_n^1 - Q_n^2| + \lambda_n^{\frac12}|Q_n^2 - P_n^2| \to +\infty$$

as $n \to +\infty$, in view of $\dfrac{|Q_n^2 - P_n^1|}{\varepsilon_n^1} \to +\infty$ and $\lambda_n|P_n^2 - Q_n^2|^2 \to 0$ as $n \to +\infty$. If (2.14) does hold for $\{\,P_n^1, P_n^2\,\}$ we are done.

Otherwise, we iterate the above argument: let $P_n^1 = (r_n^1, 0, s_n^1), \ldots, P_n^s = (r_n^s, 0, s_n^s)$ sequences so that (2.10)-(2.12) do hold true, but (2.14) is not satisfied. As before, we can find $R > 0$ large and a sub-sequence so that

$$(\varepsilon_n^1)^{\frac{2}{p-1}} \max_{\{\,d_n(r,s)\ge R\varepsilon_n^1\,\}} u_n(r,0,s) \ge 2\,\delta,$$

where $d_n(r,s)=\min\{\,|(r,s)-(r_n^i,s_n^i)|: i=1,\dots,s\,\}$. Up to a further sub-sequence, for all $i=1,\dots,s$ we can assume that

$$\frac{\varepsilon_n^1}{\varepsilon_n^i}=\Big(\frac{V(P_n^i)}{V(P_n^1)}\Big)^{\frac12}\to\theta_i\in(0,+\infty)\qquad\text{as } n\to+\infty, \tag{2.18}$$

so that by (2.12) we find that

$$(\varepsilon_n^1)^{\frac{2}{p-1}}u_n(\,\varepsilon_n^1 y+P_n^i)=\Big(\frac{\varepsilon_n^1}{\varepsilon_n^i}\Big)^{\frac{2}{p-1}}U_n^i\Big(\frac{\varepsilon_n^1}{\varepsilon_n^i}y\Big)\to\theta_i^{\frac{2}{p-1}}U_{0,i}(\theta_i\,y)$$

in $C^1_{loc}(\mathbb{R}^N)$ as $n\to+\infty$. The function $U_{0,i}$ is constant in y_{k+j}, $j\in J_i$, and in the remaining variables y' coincides with the unique radial solution of (2.2) in $\mathbb{R}^{N-k_i}$, $k_i=\text{card } J_i$. Since $U_{0,i}(y')\to 0$ as $|y'|\to+\infty$, we can find R large so that $\theta_i^{\frac{2}{p-1}}U_{0,i}(\theta_i\,y)\le\delta$ for $|y|\ge R$ and all $i=1,\dots,s$. As before, let $Q_n^{s+1}=(\tilde r_n^{s+1},0,s_n^{s+1})$ be so that

$$u_n(Q_n^{s+1})=\max_{\{d_n(r,s)\ge R\,\varepsilon_n^1\}}u_n\ge 2\,\delta\,(\varepsilon_n^1)^{-\frac{2}{p-1}}. \tag{2.19}$$

By (2.18) and $\theta_i^{\frac{2}{p-1}}U_{0,i}(\theta_i y)\le\delta$ for $|y|\ge R$, we deduce as before that $\frac{|Q_n^{s+1}-P_n^i|}{\varepsilon_n^1}\to+\infty$ as $n\to+\infty$ for all $i=1,\dots,s$. Setting $\mu_n^{s+1}=u_n(Q_n^{s+1})^{-\frac{p-1}{2}}$ and $R_n^{s+1}=\frac12\frac{d_n(\tilde r_n^{s+1},s_n^{s+1})}{\mu_n^{s+1}}$, we still have by (2.19)

$$\mu_n^{s+1}\le(2\,\delta)^{-\frac{p-1}{2}}\,\varepsilon_n^1,$$

and then $R_n^{s+1}\to+\infty$ as $n\to+\infty$. Since as before

$$u_n(Q_n^{s+1})=\max_{\Omega\cap B_{R_n^{s+1}\,\mu_n^{s+1}}(Q_n^{s+1})}u_n$$

with $R_n^{s+1}\to+\infty$ as $n\to+\infty$, by Theorem 2.1 we replace Q_n^{s+1} with $P_n^{s+1}=(r_n^{s+1},0,s_n^{s+1})$ so that (2.10)-(2.12) do hold for $\{P_n^1,\dots,P_n^{s+1}\}$.

For P_n^i, $i=1,\dots,s+1$, Theorem 2.1 also provides a $G-$invariant function $\phi_n^i\in C_0^\infty(\Omega)$ with $\operatorname{supp}\phi_n^i\subset A_{R\lambda_n^{-\frac12}}(P_n^i)$, $R>0$, which satisfies (2.3). By (2.10) the functions ϕ_n^i, $i=1,\dots,s+1$, have disjoint compact supports for n large yielding to $s+1\le\sup_n m_G(u_n)$. Then the iterative procedure must stop after h steps, $h\le\sup_n m_G(u_n)$, providing sequences $P_n^1,\dots,P_n^h$ so that (2.10)-(2.12) and (2.14) do hold.

$2^{\textbf{nd}}$ **Step** There exists γ, $C>0$ so that

$$u_n(r,0,s)\le C\,\lambda_n^{\frac{1}{p-1}}\sum_{i=1}^h e^{-\gamma\,\lambda_n^{\frac12}\,|(r,0,s)-P_n^i|}\qquad\forall\,(r,s)\in\Omega_0,\ n\in\mathbb{N}.$$

By (2.14) for $R>0$ large and $n\ge n(R)$, there holds

$$(\lambda_n)^{-\frac{1}{p-1}}\max_{\{\,d_n(r,s)\ge R\lambda_n^{-\frac12}\,\}}u_n(r,0,s)\le\Big(\frac{1}{2\,p}\inf_\Omega V\Big)^{\frac{1}{p-1}}.$$

Hence in $\{d_n(r,s)\ge R\,\lambda_n^{-\frac12}\,\}$ for $n\ge n(R)$ we have

$$\lambda_n\,V-p\,u_n^{p-1}\ge\frac{\lambda_n}{2}\inf_\Omega V.$$

Recalling the notation

$$(r_x, 0, s_x) := \left(\sqrt{x_1^2 + x_{k+1}^2}, \ldots, \sqrt{x_k^2 + x_{2k}^2}, 0, \ldots, 0, x_{2k+1}, \ldots, x_N\right)$$

for every $x \in \Omega$, we would like to use $\lambda_n^{\frac{1}{p-1}} \sum_{i=1}^{h} e^{-\gamma\, \lambda_n^{\frac{1}{2}} |(r_x,0,s_x)-P_n^i|}$ as a barrier function in Ω. However, the function is singular on the set $\{r_j = 0\}$ whenever $j \in \cup_{i=1}^h J_i$ (i.e. when $(r_n^i)_j \neq 0$ for some $i = 1, \ldots, h$). To explain how to overcome the problem, we can think (2.1) as a differential problem in Ω_0 with Δ replaced by $\Delta_{(r,s)} + \sum_{j=1}^{k} \frac{1}{r_j}\partial_{r_j}$. On $\partial\Omega_0$ we have a mixed Neumann-Dirichlet boundary condition: $\partial_{r_j} u = 0$ on $\partial\Omega_0 \cap \{r_j = 0\}$ for all $j = 1, \ldots, k$, and $u = 0$ on $\partial\Omega_0 \cap \{r_j > 0 : \forall\, j = 1, \ldots, k\}$. Inspired by [14], in order to deal with the Neumann b.c. we use a very simple idea. When (r_n, s_n) is a blow-up sequence of u_n, the Neumann boundary condition on $\{r_j = 0!\}$ creates a sort of additional mirror blow-up sequence given by the reflection of (r_n, s_n) w.r.t. to $\{r_j = 0\}$. For an asymptotic control of u_n we have to consider both r_n and its reflection (simply obtained by reversing the sign of $r_{n,j}$), and then pull back this idea onto the original problem in Ω.

To this aim, for $j = 1, \ldots, k$ let

$$\Sigma = \{\sigma : \{1, \ldots, k\} \to \{+1, -1\}\}, \quad \Sigma_j = \{\sigma \in \Sigma \text{ s.t. } \sigma(j) = +1\},$$

and, for $r = (r_1, \ldots, r_k) \in [0, +\infty)^k$ and $\sigma \in \Sigma$, define $r^\sigma = (\sigma(1)r_1, \ldots, \sigma(k)r_k)$. If $P_n^i = (r_n^i, 0, s_n^i)$, define $\psi_n^i = \sum_{\sigma \in \Sigma} \psi_n^{i,\sigma}$, where

$$\psi_n^{i,\sigma}(x) = e^{-\gamma\, \lambda_n^{\frac{1}{2}} |(r_x - (r_n^i)^\sigma, s_x - s_n^i)|}.$$

For the first derivatives we have that

$$\partial_{x_j} \psi_n^i = -\gamma\, \lambda_n^{\frac{1}{2}} \sum_{\sigma \in \Sigma} \psi_n^{i,\sigma} \frac{(r_x - (r_n^i)^\sigma)_j}{|(r_x - (r_n^i)^\sigma, s_x - s_n^i)|} \frac{x_j}{\sqrt{x_j^2 + x_{k+j}^2}}$$

for $j = 1, \ldots, k$, and a similar formula does hold for the derivative in x_{k+j} with the numerator x_j replaced by x_{k+j}. Since for all $\sigma \in \Sigma_j$ there exists a unique $\hat\sigma$ so that $\hat\sigma(l) = \sigma(l)$ for $l \neq j$ and $\hat\sigma(j) = -1$, when $r_j = 0$ we have that $\psi_n^{i,\sigma} = \psi_n^{i,\hat\sigma}$, and then

$$\begin{aligned}
&\lim_{r_j \to 0} \partial_{x_j} \psi_n^i \\
&= \lim_{r_j \to 0} \sum_{\sigma \in \Sigma_j} \partial_{x_j} [\psi_n^{i,\sigma} + \psi_n^{i,\hat\sigma}] \\
&= -\gamma\, \lambda_n^{\frac{1}{2}} \sum_{\sigma \in \Sigma_j} \frac{\psi_n^{i,\sigma}}{|(r_x - (r_n^i)^\sigma, s_x - s_n^i)|}\Big|_{r_j=0} \lim_{r_j \to 0} \frac{x_j[(r_x - (r_n^i)^\sigma)_j + (r_x - (r_n^i)^{\hat\sigma})_j]}{\sqrt{x_j^2 + x_{k+j}^2}} \\
&= -2\gamma\, \lambda_n^{\frac{1}{2}} \sum_{\sigma \in \Sigma_j} \frac{\psi_n^{i,\sigma}}{|(r_x - (r_n^i)^\sigma, s_x - s_n^i)|}\Big|_{r_j=0} \lim_{r_j \to 0} x_j = 0.
\end{aligned}$$

Hence, the first derivatives of ψ_n^i are continuous in $\Omega \setminus \{P_n^i\}$ with $\partial_{x_j} \psi_n^i\Big|_{r_j=0} = \partial_{x_{k+j}} \psi_n^i\Big|_{r_j=0} = 0$ for all $j = 1, \ldots, k$. Compute now the second derivatives for

$j=1,\dots,k$:

$$\partial_{x_j x_j}\psi_n^i+\partial_{x_{k+j}x_{k+j}}\psi_n^i=\gamma^2\,\lambda_n\sum_{\sigma\in\Sigma}\psi_n^{i,\sigma}\Bigg\{\frac{(r_x-(r_n^i)^\sigma)_j^2}{|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|^2}$$
$$-\frac{1}{\gamma\lambda_n^{\frac12}}\left[\frac{1}{|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|}-\frac{(r_x-(r_n^i)^\sigma)_j^2}{|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|^3}\right.$$
$$\left.+\frac{(r_x-(r_n^i)^\sigma)_j}{|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|}\frac{1}{\sqrt{x_j^2+x_{k+j}^2}}\right]\Bigg\},$$

and then

$$\begin{aligned}\Delta\psi_n^i &= \gamma^2\,\lambda_n\sum_{\sigma\in\Sigma}\psi_n^{i,\sigma}\left[1-\frac{1}{\gamma\lambda_n^{\frac12}}\frac{N-k-1}{|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|}\right]\\ &\quad -\gamma\lambda_n^{\frac12}\sum_{j=1}^k\sum_{\sigma\in\Sigma}\psi_n^{i,\sigma}\frac{(r_x-(r_n^i)^\sigma)_j}{|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|}\frac{1}{\sqrt{x_j^2+x_{k+j}^2}}.\end{aligned}$$

Arguing as before, we can show that $\Delta\psi_n^i$ is a continuous function in $\Omega\setminus\{P_n^i\}$, and then by elliptic regularity theory [18] we have that $\psi_n^i\in W^{2,q}_{loc}(\Omega\setminus\{P_n^i\})\cap C^1_{loc}(\bar\Omega\setminus\{P_n^i\})$ for all $q>1$. Let us stress that in general ψ_n^i is not a C^2-function.

We aim to show now uniform (in n) bounds for $\Delta\psi_n^i$. Since $(r_n^i)_j$, $(r_x)_j\ge 0$ for all $j=1,\dots,k$, notice that

$$|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|\ge|(r_x-r_n^i,s_x-s_n^i)|\qquad\forall\,\sigma\in\Sigma. \tag{2.20}$$

For $j\in\bar J_i$, we have that $(r_n^i)_j=0$, and then by (2.20)

$$\frac{(r_x-(r_n^i)^\sigma)_j}{|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|}\frac{1}{\sqrt{x_j^2+x_{k+j}^2}}\le\frac{1}{|(r_x-r_n^i,s_x-s_n^i)|}=O(\lambda_n^{\frac12})$$

in $\{d_n(r,s)\ge R\,\lambda_n^{-\frac12}\}$. Given $j\in J_i$ and $\sigma\in\Sigma_j$, let us focus now on estimating the term

$$h:=\frac{1}{\sqrt{x_j^2+x_{k+j}^2}}\left[\psi_n^{i,\sigma}\frac{(r_x-(r_n^i)^\sigma)_j}{|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|}+\psi_n^{i,\hat\sigma}\frac{(r_x-(r_n^i)^{\hat\sigma})_j}{|(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|}\right].$$

When $(r_x)_j\ge\frac12(r_n^i)_j$ we have that

$$h\le\frac{2}{(r_n^i)_j}\left[\psi_n^{i,\sigma}+\psi_n^{i,\hat\sigma}\right]=O(\lambda_n^{\frac12})\left(\psi_n^{i,\sigma}+\psi_n^{i,\hat\sigma}\right)$$

as $n\to+\infty$ in view of $\lambda_n^{\frac12}(r_n^i)_j\to+\infty$. When $(r_x)_j\le\frac12(r_n^i)_j$, we can use

$$|(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|\ge|(r_x-(r_n^i)^\sigma,s_x-s_n^i)|\ge|(r_x)_j-(r_n^i)_j|\ge\frac12(r_n^i)_j$$

to obtain the two estimates:

$$\frac{1}{|(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|}-\frac{1}{|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|}$$
$$=\frac{|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|^2-|(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|^2}{|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)||(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|[|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|+|(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|]}$$
$$=O\Big(\frac{(r_x)_j}{(r_n^i)_j^2}\Big)$$

and

$$\begin{aligned}\Big|1-\frac{\psi_n^{i,\hat\sigma}}{\psi_n^{i,\sigma}}\Big| =&1-\exp\ \Big[-\gamma\,\lambda_n^{\frac12}\left(|(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|-|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|\right)\Big]\\ \le&\gamma\,\lambda_n^{\frac12}\frac{|(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|^2-|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|^2}{|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|+|(r_x-(r_n^i)^{\hat\sigma},s_x-s_n^i)|}\\ \le&4\gamma\,\lambda_n^{\frac12}(r_x)_j.\end{aligned}$$

When $(r_x)_j\le\frac12(r_n^i)_j$ the two estimates above then yield to

$$\begin{aligned} h\ =&\ \frac{1}{\sqrt{x_j^2+x_{k+j}^2}}\frac{\psi_n^{i,\sigma}}{|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|}\left[(r_x-(r_n^i)^{\sigma})_j+(r_x-(r_n^i)^{\hat\sigma})_j\right]\\ &+O\Big(\frac{1}{(r_n^i)_j}+\gamma\lambda_n^{\frac12}\Big)\psi_n^{i,\sigma}\\ =&\ 2\frac{\psi_n^{i,\sigma}}{|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|}+[o(1)+O(\gamma)]\lambda_n^{\frac12}\psi_n^{i,\sigma}=O(\lambda_n^{\frac12})\psi_n^{i,\sigma}\end{aligned}$$

in $\{d_n(r,s)\ge R\,\lambda_n^{-\frac12}\ \}$, in view of (2.20) and $\lambda_n^{\frac12}(r_n^i)_j\to+\infty$ as $n\to+\infty$. Resuming the two cases above, we have shown that

$$\gamma\lambda_n^{\frac12}\sum_{j=1}^k\sum_{\sigma\in\Sigma}\psi_n^{i,\sigma}\frac{(r_x-(r_n^i)^{\sigma})_j}{|(r_x-(r_n^i)^{\sigma},s_x-s_n^i)|}\frac{1}{\sqrt{x_j^2+x_{k+j}^2}}=O(\gamma)\lambda_n\sum_{\sigma\in\Sigma}\psi_n^{i,\sigma}$$

in $\{d_n(r,s)\ge R\,\lambda_n^{-\frac12}\ \}$. Since $\lambda_n V-p\,u_n^{p-1}\ge\frac{\lambda_n}{2}\inf_\Omega V$, the linear operator $L_n:=-\Delta+(\lambda_n V-p\,u_n^{p-1})$ on ψ_n^i gives in $\{\,d_n(r,s)\ge R\,\lambda_n^{-\frac12}\ \}$:

$$L_n\psi_n^i\ge\lambda_n\sum_{\sigma\in\Sigma}\psi_n^{i,\sigma}\Big[O(\gamma^2+\gamma)+\frac12\inf_\Omega V\Big]\ge0$$

for n large, provided $\gamma>0$ is small. By (2.12) the function $\psi_n=e^{\gamma R}\sum_{i=1}^h\psi_n^i$ satisfies for R large

$$\begin{aligned}&\Big(\psi_n(x)-\lambda_n^{-\frac{1}{p-1}}u_n(x)\Big)|_{\partial A_{R\lambda_n^{-\frac12}}(P_n^i)}\\ \ge&\Big(e^{\gamma R-\gamma\,\lambda_n^{\frac12}\,|(r-r_n^i,s-s_n^i)|}-\lambda_n^{-\frac{1}{p-1}}u_n(r,0,s)\Big)|_{\{|(r,s)-(r_n^i,s_n^i)|=R\,\lambda_n^{-\frac12}\}}\\ \to&\,1-s_i^{\frac{2}{p-1}}U_{0,i}(r,0,s)\Big|_{\{|(r,0,s)|=Rs_i\}}>0\end{aligned}$$

as $n \to +\infty$, where $s_i = \lim_{n \to +\infty} V(P_n^i)^{\frac{1}{2}}$. Since by (2.10)

$$\partial\{d_n(r,s) \geq R\,\lambda_n^{-\frac{1}{2}}\} = \cup_{i=1}^h \partial A_{R\lambda_n^{-\frac{1}{2}}}(P_n^i) \cup \partial\Omega$$

and $L_n(\psi_n - \lambda_n^{-\frac{1}{p-1}} u_n) = L_n \psi_n \geq 0$ in $\{d_n(r,s) \geq R\,\lambda_n^{-\frac{1}{2}}\}$, by the weak maximum principle (recall that ψ_n is $C^1 \cap W^{2,2}$ in this set) we get that

$$u_n \leq \lambda_n^{\frac{1}{p-1}} \psi_n \leq \mathrm{card}\Sigma\, e^{\gamma\, R} (\lambda_n)^{\frac{1}{p-1}} \sum_{i=1}^k e^{-\gamma\, \lambda_n^{\frac{1}{2}}\, |(r,0,s) - P_n^i|}$$

in $\{d_n(r,s) \geq R\ \lambda_n^{-\frac{1}{2}}\}$, if R is large, γ small and $n \geq n(R)$, in view of (2.20). Since by (2.15)

$$u_n \leq \max_{\Omega} u_n \leq C\lambda_n^{\frac{1}{p-1}} \leq C\, e^{\gamma\, R}\, \lambda_n^{\frac{1}{p-1}} \sum_{i=1}^h e^{-\gamma\, \lambda_n^{\frac{1}{2}}\, |(r,0,s) - P_n^i|}$$

for some $C > 0$ if $d_n(r,s) \leq R\,\lambda_n^{-\frac{1}{2}}$, we have that (2.13) holds true in Ω for a suitable constant C_R and $n \geq n(R)$. Up to take a larger constant C, we have the validity of (2.13) in Ω for every $n \in \mathbb{N}$. □

3. **Classification of blow-up points.** Let u_n be a positive $G-$invariant solution of (2.1) with $\sup_n m_G(u_n) < +\infty$. According to the notations of Theorem 2.2 and up to a sub-sequence, let us define the blow-up set S as $S = \{\lim_{n \to +\infty} P_n^i\}$, and for a given $P_0 = (r_0, 0, s_0) \in S$ let us set $L = \{i = 1, \dots, h\ :\ P_n^i \to P_0 \text{ as } n \to +\infty\}$. Introduce the notation

$$A_\delta(P_0) = \{\, x \in \mathbb{R}^N\ :\ |(r_x - r_0, s_x - s_0)| \leq \delta\,\},$$

and fix $\delta > 0$ small so that $I_\delta := A_\delta(P_0) \cap \Omega$ satisfies $\overline{I_{2\delta}} \cap S = \{P_0\}$.

We have the following integral expansion:

Lemma 3.1. *Let $P_0 \in S$ and g be a continuous $G-$invariant function in $\bar{\Omega}$. Assume that $1 < p < p_{JL}(N)$ with $p \notin \{p_S(j) : j = 3, \dots, N\}$. For every $q > 1$ there holds*

$$\int_{I_\delta} g\, u_n^q = g(P_0) \sum_{i \in L} (1 + o_n(1)) \left[(2\pi)^{k_i} (\epsilon_n^i)^{N - k_i - 2\frac{q}{p-1}} \prod_{j \in J_i} (r_n^i)_j \int_{\mathbb{R}^{N-k_i}} U_{0,i}^q \right],$$

where $o_n(1) \to 0$ as $n \to +\infty$, $\epsilon_n^i = \lambda_n^{-\frac{1}{2}} V^{-\frac{1}{2}}(P_n^i)$ and $U_{0,i}$ are given by (2.12).

Proof. Given $i \in L$, through cylindrical coordinates and the change of variables $(r,s) \to (r_n^i + \epsilon_n^i r, s_n^i + \epsilon_n^i s)$ we have that

$$\int_{A_{R\epsilon_n^i}(P_n^i)} g u_n^q = (2\pi)^k \int_{\{|(r-r_n^i, s-s_n^i)| \le R\epsilon_n^i,\ r\in[0,+\infty)^k\}} \prod_{j=1}^k r_j\, g(r,s) u_n^q(r,0,s) drds$$

$$= (2\pi)^k (\epsilon_n^i)^{N-k-\frac{2q}{p-1}} \times$$

$$\times \int_{\{|(r,s)|\le R, r_j \ge -\frac{(r_n^i)_j}{\epsilon_n^i} \forall j=1,\dots,k\}} \prod_{j=1}^k (r_n^i + \epsilon_n^i r)_j g(r_n^i + \epsilon_n^i r, s_n^i + \epsilon_n^i s)(U_n^i)^q(r,0,s) drds$$

$$= (2\pi)^k g(P_0)(1+o_n(1))(\epsilon_n^i)^{N-k_i-\frac{2q}{p-1}} \times$$

$$\times \int_{\{|(r,s)|\le R,\ r_j \ge -\frac{(r_n^i)_j}{\epsilon_n^i}\ \forall\, j=1,\dots,k\}} \prod_{j\in \bar J_i} r_j \prod_{j\in J_i} (r_n^i + \epsilon_n^i r)_j\, U_{0,i}^q(r,0,s) drds$$

$$= (2\pi)^k g(P_0)(1+o_n(1))(\epsilon_n^i)^{N-k_i-\frac{2q}{p-1}} \prod_{j\in J_i} (r_n^i)_j \times$$

$$\times \int_{\{|(r,s)|\le R,\ r_i \ge 0\ \forall j \in \bar J_i\}} \prod_{j\in \bar J_i} r_j\, U_{0,i}^q(r,0,s) drds$$

in view of (2.12), $(r_n^i)_j = 0$ for $j \in \bar J_i$ and $\frac{(r_n^i)_j}{\epsilon_n^i} \to +\infty$ as $n \to +\infty$ for all $j \in J_i$. Recall that $U_{0,i}$ is constant in y_{k+j}, $j \in J_i$, and in the remaining variables coincides with the unique radial solution of (2.2) in $\mathbb{R}^{N-k_i}$, $k_i = \text{card } J_i$. Since in particular $U_{0,i}$ is invariant by the rotations in the planes $\{y_j, y_{k+j}\}$ for all $j \in J_i$, we get that

$$\begin{aligned}\int_{A_{R\epsilon_n^i}(P_n^i)} g u_n^q &= (2\pi)^{k_i} g(P_0)(1+o_n(1))(\epsilon_n^i)^{N-k_i-\frac{2q}{p-1}} \prod_{j\in J_i} (r_n^i)_j \times \\ &\quad \times \int_{\{y\in\mathbb{R}^{N-k_i}:\ |y|\le R\}} U_{0,i}^q(y) dy \\ &= (2\pi)^{k_i} g(P_0)(1+o_{n,R}(1))(\epsilon_n^i)^{N-k_i-\frac{2q}{p-1}} \prod_{j\in J_i} (r_n^i)_j \int_{\mathbb{R}^{N-k_i}} U_{0,i}^q,\end{aligned}$$

where $o_{n,R}(1)$ is small when R is large and $n \ge n(R)$ is large. In view of (2.10) we have that $A_{R\epsilon_n^i}(P_n^i)$, $i \in L$, are disjoint sets included in Ω. Since by Theorem 2.2

$$u_n^q(r,0,s) \le C(\lambda_n)^{\frac{q}{p-1}} \sum_{i=1}^h e^{-q\,\gamma\,\lambda_n^{\frac12}\,|(r,0,s)-P_n^i|}$$

for all $(r,s) \in \Omega_0$, by the previous expansions on $A_{R\epsilon_n^i}(P_n^i)$, $i \in L$, we can then write

$$\begin{aligned}\int_{I_\delta} g\, u_n^q &= \sum_{i\in L} \int_{A_{R\epsilon_n^i}(P_n^i)} g\, u_n^q + \int_{I_\delta \setminus \cup_{i\in L} A_{R\epsilon_n^i}(P_n^i)} g\, u_n^q \\ &= g(P_0) \sum_{i\in L} \Bigg[(1+o_{n,R}(1)) \left((2\pi)^{k_i} (\epsilon_n^i)^{N-k_i-2\frac{q}{p-1}} \prod_{j\in J_i} (r_n^i)_j \int_{\mathbb{R}^{N-k_i}} U_{0,i}^q \right) \\ &\quad + O\left((\lambda_n)^{\frac{q}{p-1}} \int_{A_{2\delta}(P_n^i)\setminus A_{R\epsilon_n^i}(P_n^i)} e^{-q\gamma\lambda_n^{\frac12}|(r,0,s)-P_n^i|} \right) \Bigg] + O\left((\lambda_n)^{\frac{q}{p-1}} e^{-\gamma'\lambda_n^{\frac12}} \right).\end{aligned}$$

Since $\lambda_n^{\frac{1}{2}}(r_n^i)_j \to +\infty$ as $n \to +\infty$ for all $j \in J_i$, we deduce that

$$(\lambda_n)^{\frac{q}{p-1}} e^{-\gamma' \lambda_n^{\frac{1}{2}}} = o_n \left(\sum_{i \in L} (\epsilon_n^i)^{N-k_i-2\frac{q}{p-1}} \prod_{j \in J_i} (r_n^i)_j \right)$$

as $n \to +\infty$. Moreover, arguing as before, for $\delta > 0$ small we get that

$$\begin{aligned}
&(\lambda_n)^{\frac{q}{p-1}} \int_{A_{2\delta}(P_n^i) \setminus A_{R\epsilon_n^i}(P_n^i)} e^{-q\,\gamma\,\lambda_n^{\frac{1}{2}}\,|(r,0,s)-P_n^i|} \\
&= O\Big((\epsilon_n^i)^{N-k_i-2\frac{q}{p-1}} \prod_{j \in J_i} (r_n^i)_j \Big) \int_{\mathbb{R}^{N-k_i} \setminus B_{\delta R}(0)} e^{-q\gamma|y|} \\
&= o_R\Big((\epsilon_n^i)^{N-k_i-2\frac{q}{p-1}} \prod_{j \in J_i} (r_n^i)_j \Big)
\end{aligned}$$

as $R \to +\infty$. In conclusion, taking first R large and then n large (depending on R), we show that $o_n(1)$, $o_R(1)$ and $o_{n,R}$ are small quantities for n large, and the validity of the Lemma follows. □

Far away from the the $G-$orbits emanating from the blow-up set S we have a very strong decay:

Lemma 3.2. *For every $K > 0$ there exists $C_K > 0$ so that*

$$u_n + |\nabla u_n| = O(\lambda_n^{-K})$$

on $\partial I_\delta \cap \Omega$.

Proof. In $I_{2\delta} \setminus I_{\frac{\delta}{2}}$ we decompose u_n as $u_n = u_{n,1} + u_{n,2}$, where $u_{n,2}$ is an harmonic function in $I_{2\delta} \setminus I_{\frac{\delta}{2}}$ so that $u_{n,2} = u_n$ on $\partial(I_{2\delta} \setminus I_{\frac{\delta}{2}})$. By (2.13) we have that $u_{n,2} = O(\lambda_n^{-K-1})$ uniformly on $\partial(I_{2\delta} \setminus I_{\frac{\delta}{2}})$, and by the mean value theorem we deduce that $u_{n,2} + |\nabla u_{n,2}| = O(\lambda_n^{-K-1})$ in $I_{\frac{3}{2}\delta} \setminus I_{\frac{3}{4}\delta}$.

As far as $u_{n,1}$, by (2.13) we have that $-\Delta u_{n,1} = u_n^p - \lambda_n V u_n = O(\lambda_n^{-K})$ in $\overline{I}_{2\delta} \setminus I_{\frac{\delta}{2}}$. By elliptic regularity theory, we then have that $u_{n,1} + |\nabla u_{n,1}| = O(\lambda_n^{-K})$ in $I_{2\delta} \setminus I_{\frac{\delta}{2}}$. The result then follows. □

The asymptotic analysis we have developed so far can be used to study the behavior of a $G-$invariant ground-state solution u_λ as $\lambda \to +\infty$. Given $1 < p < p_S(N)$, u_λ is found as the positive minimizer of the corresponding Rayleigh quotient I_λ in $H^G \setminus \{0\}$, where $I_\lambda(u)$ is given by

$$I_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 + \lambda \int_\Omega V u^2}{(\int_\Omega |u|^{p+1})^{\frac{2}{p+1}}}\,, \qquad u \in H^G.$$

Since u_λ solves (1.1) (up to a scaling factor to re-absorb the Lagrange multiplier), we have that

$$I_\lambda(u_\lambda) = (\int_\Omega u_\lambda^{p+1})^{\frac{p-1}{p+1}} = \inf_{u \in H^G \setminus \{0\}} I_\lambda(u).$$

Since it is easy to see that $m_G(u_\lambda) = 1$, by Theorem 2.2 we see that u_λ has just one blow-up sequence $Q_\lambda = (\tilde{r}_\lambda, 0, s_\lambda)$ given by the maximum point of u_λ: $u_\lambda(Q_\lambda) = \sup_\Omega u_\lambda$. In order to make $I_\lambda(u_\lambda) = (\int_\Omega u_\lambda^{p+1})^{\frac{p-1}{p+1}}$ as small as possible, by Lemma 3.1 we see that Q_λ prefers to converge very fast to G_0 because $(\epsilon_\lambda)^{N-k-2\frac{p+1}{p-1}} \prod_{j \in J} (r_\lambda)_j$

is asymptotically bigger than $(\epsilon_\lambda)^{N-2\frac{p+1}{p-1}}$ (here $k=\text{card } J$). The argument can be made rigorous so to show that $\lambda \text{dist } (Q_\lambda, G_0) \to 0$ as $\lambda \to +\infty$ or, equivalently, the corresponding $P_\lambda \in G_0$. A better localization in G_0 of the limiting point of P_λ as $\lambda \to +\infty$ follows as a by-product of the next Theorem, where a classification is also provided for the blow-up points outside G_0 (which do not arise for the $G-$invariant ground-state solution u_λ but might possibly arise for other solutions).

We have the following localization for P_0.

Theorem 3.3. *Assume that $x \cdot \nu(x) \neq 0$ for all $x \in \partial\Omega$. The blow-up point $P_0 = (r_0, 0, s_0) \in S$ satisfies*

- *if $P_0 \in \Omega$, then $\nabla_s V(P_0) = 0$ and*
$$\left(\sum_{j=1}^k r_j \partial_{r_j} V + \Theta_0 V\right)(P_0) = 0,$$
where Θ_0 is given by (3.3);
- *if $P_0 \in \partial\Omega$, then there exists $\mu \geq 0$ so that $\nabla_s V(P_0) = -\mu \nu_s(P_0)$ and*
$$\left(\sum_{j=1}^k r_j \partial_{r_j} V + \mu' r \cdot \nu_r + \Theta_0 V\right)(P_0) = 0$$
where
$$\mu' = \begin{cases} \mu & \text{if } \nu_s(P_0) \neq 0 \\ \geq 0 & \text{if } \nu_s(P_0) = 0. \end{cases}$$

Proof. Multiplying (2.1) by $\partial_{s_m} u_n$ and integrating by parts in I_δ we get that
$$\begin{aligned} \frac{\lambda_n}{2} \int_{I_\delta} \partial_{s_m} V\, u_n^2 \quad &= \int_{\partial I_\delta \cap \Omega} \Big[\frac{\lambda_n}{2} V\, u_n^2 - \frac{1}{p+1} u_n^{p+1}\Big] \nu_{s_m} \\ &\quad + \int_{\partial I_\delta} \Big[\frac{1}{2} |\nabla u_n|^2 \nu_{s_m} - \partial_\nu u_n \partial_{s_m} u_n\Big], \end{aligned} \tag{3.1}$$
where ν_{s_m} is the $(2k+m)-$th component of the unit outward normal vector ν. By Lemma 3.2 and (3.1) we then deduce that
$$\frac{\lambda_n}{2} \int_{I_\delta} \partial_{s_m} V\, u_n^2 = O(\lambda_n^{-K}) - \frac{1}{2} \int_{I_\delta \cap \partial\Omega} (\partial_\nu u_n)^2 \nu_{s_m}$$
for all $K > 0$. When $P_0 \in \Omega$ we have that $I_\delta \cap \partial\Omega = \emptyset$ for $\delta > 0$ small. Since $\partial_{s_m} V$ is a $G-$invariant function, by Lemma 3.1 we deduce that
$$\partial_{s_m} V(P_0) \sum_{i \in L} (1 + o_n(1)) \left[(2\pi)^{k_i} (V(P_0))^{\frac{2}{p-1} - \frac{N-k_i}{2}} (\lambda_n)^{\frac{p+1}{p-1} - \frac{N-k_i}{2}} \prod_{j \in J_i} (r_n^i)_j \int_{\mathbb{R}^{N-k_i}} U_{0,i}^2\right]$$
$$= O(\lambda_n^{-K})$$
for all $K > 0$. Since $\lambda_n^{\frac{1}{2}} (r_n^i)_j \to +\infty$ as $n \to +\infty$ for all $j \in J_i$, we divide by $\max_{i \in L,\, j \in J_i} \Big\{(\lambda_n)^{\frac{p+1}{p-1} - \frac{N-k_i}{2}} \prod_{j \in J_i} (r_n^i)_j\Big\}$, and letting $n \to +\infty$ we get that $\nabla_s V(P_0) = 0$ (for K sufficiently large).

When $P_0 \in \partial\Omega$, if $\nu_{s_m}(P_0) \neq 0$ we just obtain an inequality in the form $\partial_{s_m} V(P_0) \nu_{s_m}(P_0) \leq 0$. Let us stress that it clearly holds also when $\nu_{s_m}(P_0) = 0$.

Given $Q=(0,0,\hat s)$, multiplying (2.1) by $(x-Q)\cdot\nabla u_n$ and integrating by parts in I_δ we get that

$$\begin{aligned}&\frac{\lambda_n}{2}\int_{I_\delta}(x-Q)\cdot\nabla V\,u_n^2+\Big(\frac{N-2}{2}-\frac{N}{p+1}\Big)\int_{I_\delta}u_n^{p+1}+\lambda_n\int_{I_\delta}V\,u_n^2\\&=\int_{\partial I_\delta\cap\Omega}(x-Q)\cdot\nu\Big[\frac{\lambda_n}{2}V\,u_n^2-\frac{u_n^{p+1}}{p+1}\Big]-\frac{N-2}{2}\int_{\partial I_\delta\cap\Omega}u_n\partial_\nu u_n\\&\quad+\int_{\partial I_\delta}\Big[\frac12(x-Q)\cdot\nu|\nabla u_n|^2-(x-Q)\cdot\nabla u_n\partial_\nu u_n\Big].\end{aligned}$$

By Lemma 3.2 we then deduce that

$$\begin{aligned}&\frac{\lambda_n}{2}\int_{I_\delta}(x-Q)\cdot\nabla V\,u_n^2+\Big(\frac{N-2}{2}-\frac{N}{p+1}\Big)\int_{I_\delta}u_n^{p+1}+\lambda_n\int_{I_\delta}V\,u_n^2\\&=O(\lambda_n^{-K})-\frac12\int_{I_\delta\cap\partial\Omega}(x-Q)\cdot\nu(\partial_\nu u_n)^2.\end{aligned}\tag{3.2}$$

As before, when $P_0\in\Omega$ let su fix $\delta>0$ small so that $I_\delta\cap\partial\Omega=\emptyset$. Since V is a $G-$invariant function, let us notice that

$$(x-Q)\cdot\nabla V(x)=\sum_{j=1}^{k}r_j\partial_{r_j}V+\sum_{j=1}^{N-2k}(s_j-\hat s_j)\partial_{s_j}V$$

is a $G-$invariant function too. We can then use Lemma 3.1 to deduce by (3.2) that

$$\begin{aligned}&\sum_{i\in L}(1+o_n(1))\,(2\pi)^{k_i}(\lambda_n)^{\frac{p+1}{p-1}-\frac{N-k_i}{2}}(V(P_0))^{\frac{p+1}{p-1}-\frac{N-k_i}{2}}\prod_{j\in J_i}(r_n^i)_j\int_{\mathbb{R}^{N-k_i}}U_{0,i}^2\times\\&\times\left[\frac{(P_0-Q)\cdot\nabla V(P_0)}{2V(P_0)}+\Big(\frac{N-2}{2}-\frac{N}{p+1}\Big)\frac{\int_{\mathbb{R}^{N-k_i}}U_{0,i}^{p+1}}{\int_{\mathbb{R}^{N-k_i}}U_{0,i}^2}+1\right]=O(\lambda_n^{-K}).\end{aligned}$$

Since $U_{0,i}$ solves (2.2) in $\mathbb{R}^{N-k_i}$ and decays exponentially fast at infinity, we can multiply by $y\cdot\nabla U_{0,i}$ and get by integration by parts that

$$\Big(\frac{N-k_i}{p+1}-\frac{N-k_i-2}{2}\Big)\int_{\mathbb{R}^{N-k_i}}U_{0,i}^{p+1}=\int_{\mathbb{R}^{N-k_i}}U_{0,i}^2.$$

Up to a sub-sequence, we can let

$$\lambda_i:=\lim_{n\to+\infty}\frac{(\lambda_n)^{\frac{p+1}{p-1}-\frac{N-k_i}{2}}\prod_{j\in J_i}(r_n^i)_j}{\max_{h\in L,\,j\in J_h}(\lambda_n)^{\frac{p+1}{p-1}-\frac{N-k_h}{2}}\prod_{j\in J_h}(r_n^h)_j}$$

and $L_0=\{i\in L:\ \lambda_i>0\}\neq\emptyset$, and get by (3.2) that

$$\begin{aligned}&\sum_{i\in L_0}\lambda_i(2\pi)^{k_i}(V(P_0))^{\frac{k_i}{2}}\int_{\mathbb{R}^{N-k_i}}U_{0,i}^2\left[\frac{(P_0-Q)\cdot\nabla V(P_0)}{2V(P_0)}\right.\\&\left.+\Big(\frac{N-2}{2}-\frac{N}{p+1}\Big)\Big(\frac{N-k_i}{p+1}-\frac{N-k_i-2}{2}\Big)^{-1}+1\right]=0\end{aligned}$$

as $n\to+\infty$, in view of $\lambda_n^{\frac12}(r_n^i)_j\to+\infty$ as $n\to+\infty$ for all $j\in J_i$. Setting

$$\begin{aligned}\Theta_0&=\Theta(P_0)\\&:=\frac{\sum_{i\in L_0}\lambda_i(2\pi)^{k_i}(V(P_0))^{\frac{k_i}{2}}\int_{\mathbb{R}^{N-k_i}}U_{0,i}^2\left[(N-2-2\frac{N}{p+1})(\frac{N-k_i}{p+1}-\frac{N-k_i-2}{2})^{-1}+2\right]}{\sum_{i\in L_0}\lambda_i(2\pi)^{k_i}(V(P_0))^{\frac{k_i}{2}}\int_{\mathbb{R}^{N-k_i}}U_{0,i}^2},\end{aligned}\tag{3.3}$$

we deduce that

$$(P_0-Q)\cdot\nabla V(P_0)+\Theta_0 V(P_0)=\left(\sum_{j=1}^k r_j\partial_{r_j}V+\Theta_0 V\right)(P_0)=0$$

in view of $\nabla_s V(P_0)=0$. Notice that $\Theta_0>0$ unless $k_i=0$ for all $i\in L_0$.

If $P_0\in\partial\Omega$, we have to distinguish whether $r_0\cdot\nu_r(P_0)=0$ or not. When $r_0\cdot\nu_r(P_0)\neq 0$, we can take $\hat{s}=s_0-t\tau$, where τ is orthogonal to $\nu_s(P_0)$. For $|t|$ large, we can find $\delta>0$ sufficiently small so that $(x-Q)\cdot\nu$ has the same sign of $r_0\cdot\nu_r(P_0)$ in $I_\delta\cap\partial\Omega$. By (3.2) we then get that

$$\left(\sum_{j=1}^k r_j\partial_{r_j}V+(s_0-\hat{s})\cdot\nabla_s V+\Theta_0 V\right)(P_0)=\left(\sum_{j=1}^k r_j\partial_{r_j}V+t\tau\cdot\nabla_s V+\Theta_0 V\right)(P_0)$$

has a given sign (the opposite one of $r_0\cdot\nu_r(P_0)$). Since this this is true for all t large, we get that $\tau\cdot\nabla_s V(P_0)=0$ for all τ so that $\tau\cdot\nu_s(P_0)$, i.e. $\nabla_s V(P_0)$ and $\nu_s(P_0)$ are proportional. Since $\partial_{s_m}V(P_0)\nu_{s_m}(P_0)\leq 0$ for all m, we have that $\nabla_s V(p_0)=-\mu\nu_s(P_0)$ for some $\mu\geq 0$.

If $\nu_s(P_0)\neq 0$, for $\hat{s}=s_0-t\nu_s(P_0)$ as $t\to t_0^\pm$, $t_0:=-\frac{r_0\cdot\nu_r(P_0)}{|\nu_s(P_0)|^2}$, by (3.2) we get that

$$\left(\sum_{j=1}^k r_j\partial_{r_j}V+t_0\nu_s\cdot\nabla_s V+\Theta_0 V\right)(P_0)=\left(\sum_{j=1}^k r_j\partial_{r_j}V+\mu r\cdot\nu_r+\Theta_0 V\right)(P_0)$$

has to be non-positive and non-negative, respectively, yielding to

$$\left(\sum_{j=1}^k r_j\partial_{r_j}V+\Theta_0 V\right)(P_0)=-\mu r_0\cdot\nu_r(P_0). \tag{3.4}$$

When $\nu_s(P_0)=0$, we have that $\nabla_s V(P_0)=0$, and then the L.H.S. in (3.4) has the opposite sign w.r.t $r_0\cdot\nu_r(P_0)$, yielding to (3.4) with μ replaced by $\mu'\geq 0$. We are left with the case $r_0\cdot\nu_r(P_0)=0$. Since we assume $x\cdot\nu(x)\neq 0$ for all $x\in\partial\Omega$, we have that

$$P_0\cdot\nu(P_0)=(r_0,0,s_0)\cdot(\nu_r(P_0),0,\nu_s(P_0))=s_0\cdot\nu_s(P_0)\neq 0$$

implies $\nu_s(P_0)\neq 0$. Given τ so that $\tau\cdot\nu_s(P_0)>0$, the choice $\hat{s}=s_0-t\tau$ for $t\to+\infty$ gives that $\tau\cdot\nabla_s V(P_0)\leq 0$. Letting τ approach the orthogonal space of $\nu_s(P_0)$, we deduce that $\tau\cdot\nabla_s V(P_0)\leq 0$ for all τ with $\tau\cdot\nu_s(P_0)=0$. Applying it for τ and $-\tau$, we still get that $\tau\cdot\nabla_s V(P_0)=0$ for all τ so that $\tau\cdot\nu_s(P_0)$, i.e. $\nabla_s V(p_0)=-\mu\nu_s(P_0)$ for some $\mu\geq 0$. With always the same choice of τ, as $t\to 0^\pm$ the inequalities $\leq$ and $\geq$, respectively, have to hold for

$$\left(\sum_{j=1}^k r_j\partial_{r_j}V+\Theta_0 V\right)(P_0),$$

yielding to the validity of (3.4) also in this case. □

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E-mail address: esposito@mat.uniroma3.it
E-mail address: petralla@mat.uniroma3.it