CONCENTRATING SOLUTIONS FOR THE HÉNON EQUATION IN $\mathbb{R}^2$

By

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Abstract. We consider the boundary value problem $\Delta u + |x|^{2\alpha} u^p = 0$, $\alpha > 0$, in the unit ball B with homogeneous Dirichlet boundary condition and p a large exponent. We find a condition which ensures the existence of a positive solution u_p concentrating outside the origin at k symmetric points as p goes to $+\infty$. The same techniques lead also to a more general result on general domains. In particular, we find that concentration at the origin is always possible, provided $\alpha \notin \mathbb{N}$.

1 Introduction and statement of main results

In this paper, we consider the following so-called Hénon equation ([16])

$$
\begin{cases} \Delta u + |x|^{2\alpha} u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $\alpha > 0$, Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) containing the origin, and $p > 1$.

Problem (1.1) has attracted considerable attention in recent years. In [18], Ni showed the existence of a radially symmetric solution when $p < \frac{N+2+2\alpha}{N-2-2\alpha}$ for $N \geq 3$ and $\Omega = B_1(0)$. When $\Omega = B_1(0) \subset \mathbb{R}^2$, numerical computations by Chen, Ni and Zhou [8] suggest that for some parameters (α, p), the ground state solutions are nonradial. This was partially confirmed recently by Smets, Su and William in [24], in which it was proved that for each $2 < p + 1 < 2^* (= \frac{2N}{N-2}$ if $N \geq 3; = +\infty$ if $N = 2)$, there exists α^* such that for $\alpha > \alpha^*$ the ground states are nonradial.

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They also showed that for fixed α, the ground state solution must be radial if p is close to 1. When $N \geq 2$, the asymptotic behavior of (radial or nonradial) ground state solutions as $\alpha \to +\infty$ is studied by Byeon and Wang in [3, 4], in which it is shown that the ground state solution develop boundary concentrations. In another direction, when $N \geq 3$, α is fixed, $\Omega = B_1$, and $p + 1 \to \frac{N+2}{N-2}$, Cao and Peng [5] showed that the ground state solution develops a boundary bubble (hence must be nonradial). In [19] and [20], multiple boundary concentrations have been constructed when $N \geq 3, \Omega = B_1$ and $p \to \frac{N+2}{N-2}$.

In this paper, we consider the problem (1.1) when $N = 2$ and p is large, i.e., the boundary value problem

$$\begin{cases} \Delta u + |x|^{2\alpha} u^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

where $\alpha > 0$, $B = B(0, 1)$ is the unit ball in $\mathbb{R}^2$ and p is a large exponent. Unlike [5], as $p \to +\infty$, there are no boundary concentration solutions. The proof of this fact follows from the proof of Proposition 5 of [17]. One of the main results of this paper is to show the presence of solutions concentrating at the origin or outside the origin as long as $\alpha \notin \mathbb{N}$ and Ω contains the origin.

Let $K_\alpha = \max\{k \in \mathbb{N} : k < \alpha + 1\}$. Concerning concentration outside the origin, the main result we obtain for (1.2) is the following.

Theorem 1.1. *There exists $p_0 > 0$ large such that for any $1 \leq k \leq K_\alpha$ and $p \geq p_0$, the problem* (1.2) *has a solution u_p which concentrates at k (symmetric) different points of $B \setminus \{0\}$, i.e., as p goes to $+\infty$,*

$$p|x|^{2\alpha} u_p^{p+1} \rightharpoonup 8\pi e \sum_{i=1}^{k} \delta_{\xi_i} \text{ weakly in the sense of measure in } \overline{B}$$

for some $\xi = (\xi_1, \ldots, \xi_k)$. More precisely, for any $\delta > 0$,

$$\max_{B \setminus \bigcup_{i=1}^k B(\xi_i, \delta)} u_p \to 0\,, \qquad \sup_{B(\xi_i, \delta)} u_p \to \sqrt{e}$$

as $p \to +\infty$.

Theorem 1.1 is based on a constructive method which works also for the more general problem

$$\begin{cases} \Delta u + a(x) u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth, bounded domain in $\mathbb{R}^2$, p is a large exponent and $a(x) \geq 0$ is a potential.
We make the following assumption on $a(x)$. For any $q \in \Omega$ such that $a(q) = 0$, there exists $\alpha_q > 0$ such that $a_q(x) := a(x)|x - q|^{-2\alpha_q}$ is a strictly positive continuous function in a neighborhood of q. Set $Z := \{q \in \Omega : a(q) = 0\}$. We observe that Z could be an empty set.

Let $G(x, y)$ be the Green's function, i.e., the solution of the problem

$$\begin{cases} -\Delta_x G(x, y) = \delta_y(x) & x \in \Omega, \\ G(x, y) = 0 & x \in \partial\Omega, \end{cases}$$

and let $H(x, y)$ be the regular part

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$

Let $q_1, \ldots, q_m \in Z$ be distinct points so that $\alpha_i := \alpha_{q_i} \notin \mathbb{N}$ for any $i = 1, \ldots, m$. In order to find a solution concentrating at $q_1, \ldots, q_m$ and at $\xi_1, \ldots, \xi_k \in \Omega \setminus Z$, the location of the concentration points $\xi_1, \ldots, \xi_k$ should be a critical point of the function

$$\Phi(\xi) = \sum_{i=1}^{k} \left[H(\xi_i, \xi_i) + \sum_{\substack{j=1 \\ j \neq i}}^{k} G(\xi_i, \xi_j) + \frac{1}{4\pi} \log a(\xi_i) + 2 \sum_{j=1}^{m} (1 + \alpha_j) G(\xi_i, q_j) \right], \tag{1.4}$$

where

$$\xi := (\xi_1, \ldots, \xi_k) \in \mathcal{M} := \left\{ (\xi_1, \ldots, \xi_k) \in (\Omega \setminus Z)^k \ : \ \xi_i \neq \xi_j \text{ for } i \neq j \right\}.$$

The role of the function Φ in concentration phenomena was already shown for (1.3) with $a(x) = 1$ in [14] (see also [2, 9, 13] in the context of the mean field equation). Considering changing sign solutions of (1.2) (u^p replaced by $|u|^{p-1}u$ in the equation), we can also allow negative concentration phenomena; and the function responsible for locating the concentration points is "essentially" Φ, as already shown in [15] for $a(x) = 1$. To understand the role of Φ in the presence of some concentration point in Z, we refer to [11, 12], where blowing up solutions are constructed in the context of the mean field equation.

The result we have is the following.

Theorem 1.2. *Let m, k be nonnegative integers. If $m \geq 1$, take $q_1, \ldots, q_m \in Z$ to be different points so that $\alpha_i = \alpha_{q_i} \notin \mathbb{N}$ for any $i = 1, \ldots, m$. If $k \geq 1$, assume that $(\xi_1^*, \ldots, \xi_k^*) \in \mathcal{M}$ is a C^0-stable critical point of Φ (according to Definition 3.1).*

Then, there exists $p_0 > 0$ such that for any $p \geq p_0$, problem (1.3) *has a solution u_p which concentrates at $m+k$ different points of Ω, i.e., as p goes to $+\infty$,*

$$pa(x)u_p^{p+1} \rightharpoonup 8\pi e \sum_{i=1}^{m}(\alpha_i+1)\delta_{q_i} + 8\pi e \sum_{i=1}^{k}\delta_{\xi_i} \tag{1.5}$$

weakly in the sense of measure in $\overline{\Omega}$ for some $\xi \in \mathcal{M}$ such that $\Phi(\xi_1,\ldots,\xi_k) = \Phi(\xi_1^*,\ldots,\xi_k^*)$. More precisely, for any $\delta > 0$ as p goes to $+\infty$,

$$u_p \to 0 \ \ \textit{uniformly in } \Omega \setminus (\textstyle\bigcup_{i=1}^m B(q_i,\delta)) \cup \left(\bigcup_{i=1}^k B(\xi_i,\delta)\right) \tag{1.6}$$

and

$$\sup_{x\in B(q_i,\delta)} u_p(x) \to \sqrt{e}\,, \qquad \sup_{x\in B(\xi_j,\delta)} u_p(x) \to \sqrt{e} \tag{1.7}$$

for any $i = 1,\ldots,m$ and $j = 1,\ldots,k$.

Theorem 1.2 implies always the existence of solutions for (1.2) concentrating at points $q_1,\ldots,q_m$ provided $\alpha_{q_i} \notin I\!N$ for any $i = 1,\ldots,m$. Moreover, Theorem 1.2 holds even if $m = 0$, yielding solutions concentrating at k different points in $\Omega \setminus Z$, whose location depends on the critical point of the function Φ given in (1.4), which reduces to

$$\Phi(\xi) = \sum_{i=1}^{k}\left[H(\xi_i,\xi_i) + \sum_{\substack{j=1\\ j\neq i}}^{k} G(\xi_i,\xi_j) + \frac{1}{4\pi}\log a(\xi_i)\right].$$

As in the mean field equation, it is possible to identify a limit profile problem of Liouville-type (for $a(x) = 1$, see the asymptotic analysis in [1, 10, 22, 23]):

$$\begin{cases} \Delta u + |x|^{2\alpha}e^u = 0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |x|^{2\alpha}e^u < \infty \end{cases} \tag{1.8}$$

with $\alpha \geq 0$. Problem (1.8) possesses exactly a three-parameter family of solutions:

$$U_{\delta,\xi}(x) = \log\frac{8\delta^2}{(\delta^2+|x-\xi|^2)^2}, \quad \delta > 0,\ \xi \in \mathbb{R}^2 \tag{1.9}$$

if $\alpha = 0$ (see [7]), and a one-parameter family of solutions

$$U_\delta(x) = \log\frac{8(\alpha+1)^2\delta^2}{(\delta^2+|x|^{2(\alpha+1)})^2}, \quad \delta > 0 \tag{1.10}$$

if $\alpha \notin I\!N$ (see [21]).

We build solutions for problem (1.3) which, up to a suitable normalization, look like a sum of concentrated solutions for the limit profile problem (1.8) centered at several points $q_1, \dots, q_m, \xi_1, \dots, \xi_k$ as $p \to \infty$. We use arguments and ideas introduced in [14, 15].

The paper is organized as follows. In Section 2, we describe exactly the Ansatz for the solution we are looking for and rewrite the problem in terms of a linear operator L (for which a solvability theory is performed in Appendix C). In Section 3, we solve an auxiliary nonlinear problem and prove Theorem 1.2. In Section 4, we prove Theorem 1.1 in a radial setting. Appendices A, B, and C contain the proofs of various auxiliary results. Displayed formulas in these sections are numbered (A.1), (A.2), etc.

2 Approximating solutions

Let us consider the problem

$$\begin{cases} -\Delta u = a(x) g_p(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here $g_p(s) = (s^+)^p$. Let $q_1, \dots, q_m \in Z$ and set $\alpha_i = \alpha_{q_i}$, $a_i(x) = a_{q_i}(x)$, for any $i = 1, \dots, m$. Assume that $\alpha_i \notin I\!N$ and $|q_i - q_j| \geq 2\varepsilon$ for any $i \neq j$, for some $\varepsilon > 0$. Take a $k-$tuple $\xi = (\xi_1, \dots, \xi_k) \in \mathcal{O}_\varepsilon$, where

$$\mathcal{O}_\varepsilon = \left\{ \xi \in \Omega^k : \ \text{dist}\,(\xi_i, \partial(\Omega \setminus Z)) \geq 2\varepsilon, \ |\xi_i - \xi_j| \geq 2\varepsilon, \ i \neq j \right\}.$$

Define $q_i = \xi_{i-m}$, $\alpha_i = 0$ and $a_i(x) = a(x)$ for any $i = m+1, \dots, m+k$.

Let $i = 1, \dots, m+k$. Let us set $U^i(y) := \log \frac{8(\alpha_i+1)^2}{(1+|y|^{2(\alpha_i+1)})^2}$. Let f^{0i}, f^{1i} be defined in (A.1), (A.2) and V^i, W^i be the solutions of (A.1), (A.2) with $\alpha = \alpha_i$, for any $i = 1, \dots, m+k$. Define

$$U_{\delta_i, q_i}(x) = U^i\left(\delta_i^{-\frac{1}{\alpha+1}}(x - q_i)\right) - 2\log\delta_i = \log \frac{8(\alpha_i+1)^2 \delta_i^2}{(\delta_i^2 + |x - q_i|^{2(\alpha_i+1)})^2}$$

and

$$V_{\delta_i, q_i}(x) = V^i\left(\delta_i^{-\frac{1}{\alpha+1}}(x - q_i)\right), \quad W_{\delta_i, q_i}(x) = W^i\left(\delta_i^{-\frac{1}{\alpha+1}}(x - q_i)\right).$$

Set

$$U_\xi(x) := \sum_{i=1}^{m+k} \frac{1}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} P\left(U_{\delta_i, q_i} + \frac{1}{p} V_{\delta_i, q_i} + \frac{1}{p^2} W_{\delta_i, q_i}\right),$$

where

$$\gamma := p^{\frac{p}{p-1}} e^{-\frac{p}{2(p-1)}}$$

and the concentration parameters satisfy

$$\delta_i = \mu_i e^{-p/4} \tag{2.2}$$

(with μ_i to be chosen below). Here $P : H^1(\Omega) \to H_0^1(\Omega)$ denotes the projection operator onto $H_0^1(\Omega)$, i.e., $\Delta Pu = \Delta u$ in Ω, $u = 0$ on $\partial\Omega$.

By Lemmas B.1–B.2, we have that for $|x - q_i| \le \varepsilon$,

$$\begin{aligned} U_\xi(x) = \frac{1}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} \Bigg(& p + U^i(y) - \log 8(\alpha_i+1)^2 \mu_i^4 + \frac{1}{p} V^i(y) + \frac{1}{p^2} W^i(y) \\ & + 8\pi H(x,q_i) \Big(\alpha_i + 1 - \frac{C_0(\alpha_i)}{4p} - \frac{C_1(\alpha_i)}{4p^2} \Big) + \frac{1}{\alpha_i+1} \frac{\log \delta_i}{p} \Big(C_0(\alpha_i) + \frac{C_1(\alpha_i)}{p} \Big) \\ & + 8\pi \sum_{j \ne i} \Big(\frac{\mu_i^2 a_i(q_i)}{\mu_j^2 a_j(q_j)} \Big)^{\frac{1}{p-1}} G(x,q_j) \Big(\alpha_j + 1 - \frac{C_0(\alpha_j)}{4p} - \frac{C_1(\alpha_j)}{4p^2} \Big) + O(e^{-\frac{p}{4}}) \Bigg), \end{aligned}$$

where $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x - q_i)$.

Let us choose $\{\mu_i\}$ as the solution of the system

$$\begin{aligned} \log(8(\alpha_i+1)^2 \mu_i^4) &= 8\pi H(q_i,q_i) \Big(\alpha_i + 1 - \frac{C_0(\alpha_i)}{4p} - \frac{C_1(\alpha_i)}{4p^2} \Big) + \frac{\log \delta_i}{p(\alpha_i+1)} \Big(C_0(\alpha_i) + \frac{C_1(\alpha_i)}{p} \Big) \\ & + 8\pi \sum_{j \ne i} \Big(\frac{\mu_i^2 a_i(q_i)}{\mu_j^2 a_j(q_j)} \Big)^{\frac{1}{p-1}} G(q_i,q_j) \Big(\alpha_j + 1 - \frac{C_0(\alpha_j)}{4p} - \frac{C_1(\alpha_j)}{4p^2} \Big), \end{aligned} \tag{2.3}$$

in order to have

$$\begin{aligned} U_\xi(x) = & \frac{1}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} \Big(p + U^i(y) + \frac{1}{p} V^i(y) + \frac{1}{p^2} W^i(y) \Big) \\ & + O\Big(\frac{e^{-\frac{p}{4(\alpha_i+1)}} |y|}{\gamma} + \frac{e^{-\frac{p}{4}}}{\gamma} \Big) \end{aligned} \tag{2.4}$$

for $|y| \le \varepsilon \delta_i^{-\frac{1}{\alpha_i+1}}$.

Remark 2.1. Since $|V^i| + |W^i| \le C \log(|y|+2)$ in view of (A.3), by (2.4) we have

$$U_\xi(x) = \frac{1}{\gamma \mu_i^{2/(p-1)} a_i(q_i)^{1/(p-1)}} \big(p + U^i(y) \big) + O(1/\gamma)$$

for $|x - q_i| \le \varepsilon$, where $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x - q_i)$. Since $U^i \le C$, $\max_{B(q_i,\delta)} U_\xi(x) \le \frac{1}{\gamma}(p + O(1))$ for any $0 < \delta \le \varepsilon$. Since $U^i(0) = \log(8(\alpha_i+1)^2)$, we also have the reversed inequality: $\max_{B(q_i,\delta)} U_\xi(x) \geq U_\xi(0) = \frac{1}{\gamma}(p + O(1))$. Hence,

$$\lim_{p\to+\infty} \max_{B(q_i,\delta)} U_\xi(x) = \lim_{p\to+\infty} \frac{p}{\gamma} = \sqrt{e} \tag{2.5}$$

for any $0 < \delta \leq \varepsilon$ and $i = 1, \ldots, m+k$.

For p large, μ_i bifurcates from the solution of (2.3) with $p = +\infty$:

$$\mu_i = e^{-\frac{3}{4}} e^{2\pi(\alpha_i+1)H(q_i,q_i)+2\pi\sum_{j\neq i}(\alpha_j+1)G(q_j,q_i)}\big(1 + O(\tfrac{1}{p})\big), \tag{2.6}$$

in view of the choice of δ_i (see (2.2)) and of the value of $C_0(\alpha_i)$ (see (A.5)).

Remark 2.2. Let us remark that U_ξ is a positive function. Since

$$p + U^i + \frac{1}{p}V^i + \frac{1}{p^2}W^i \geq \log \frac{2(\alpha_i+1)^2\mu_i^4}{\varepsilon^{4(\alpha_i+1)}} - C$$

in $|y| \leq \varepsilon\delta_i^{-\frac{1}{\alpha_i+1}}$, by (2.4) we see that U_ξ is positive in $B(q_i,\varepsilon)$ for any $i = 1, \ldots, m+k$ for ε sufficiently small. Moreover, by elliptic regularity theory, Lemmas B.1–B.2 imply that for any $i = 1, \ldots, m+k$,

$$P\left(U_{\delta_i,q_i} + \frac{1}{p}V_{\delta_i,q_i} + \frac{1}{p^2}W_{\delta_i,q_i}\right) \to 8\pi(\alpha_i+1)G(\cdot,q_i)$$

in C^1-norm on $|x - q_i| \geq \varepsilon$. Hence, since $\frac{\partial G}{\partial n}(\cdot,q_i) < 0$ on $\partial\Omega$, U_ξ is a positive function in Ω.

We seek solutions u of problem (2.1) in the form $u = U_\xi + \phi$, where ϕ represents a higher order term in the expansion of u. In terms of ϕ, the problem (2.1) becomes

$$\begin{cases} L(\phi) = -[R + N(\phi)] & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$L(\phi) := \Delta\phi + a(x)g_p'(U_\xi)\phi, \tag{2.7}$$

$$R := \Delta U_\xi + a(x)g_p(U_\xi), \tag{2.8}$$

$$N(\phi) = a(x)[g_p(U_\xi + \phi) - g_p(U_\xi) - g_p'(U_\xi)\phi]. \tag{2.9}$$

For any $h \in L^\infty(\Omega)$, define

$$\|h\|_* = \sup_{x\in\Omega} \left| \left(\sum_{i=1}^{m+k} \frac{\delta_i |x - q_i|^{2\alpha_i}}{(\delta_i^2 + |x - q_i|^{2(\alpha_i+1)})^{\frac{3}{2}}} \right)^{-1} h(x) \right|. \tag{2.10}$$

We conclude this section by proving an estimate on R in $\|\cdot\|_*$.

Proposition 2.1. *There exist* $C > 0$ *and* $p_0 > 0$ *such that for any* $\xi \in \mathcal{O}_\varepsilon$ *and* $p \geq p_0$,

$$\|\Delta U_\xi + a(x) U_\xi^p\|_* \leq C/p^4. \tag{2.11}$$

Proof. Observe that by equations (A.1)–(A.2),

$$\begin{aligned}\Delta U_\xi(x) &= \sum_{i=1}^{m+k} \frac{\delta_i^{-\frac{2}{\alpha_i+1}}}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} \Big(-|y_i|^{2\alpha_i} e^{U^i(y_i)} + \frac{1}{p} \Delta V^i(y_i) + \frac{1}{p^2} \Delta W^i(y_i) \Big) \\ &= \sum_{i=1}^{m+k} \frac{\delta_i^{-\frac{2}{\alpha_i+1}}}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} |y_i|^{2\alpha_i} \Big(- e^{U^i(y_i)} + \frac{1}{p} f^{0i}(y_i) + \frac{1}{p^2} f^{1i}(y_i) \\ &\qquad - \frac{1}{p} e^{U^i(y_i)} V^i(y_i) - \frac{1}{p^2} e^{U^i(y_i)} W^i(y_i) \Big),\end{aligned} \tag{2.12}$$

where $y_i = \delta_i^{-\frac{1}{\alpha_i+1}}(x - q_i)$. If $|x - q_i| \geq \varepsilon$ for any $i = 1, \dots, m+k$, by (B.2) and (B.4), formula (2.12) gives

$$\begin{aligned}&\Big| \Big(\sum_{j=1}^{m+k} \frac{\delta_j |x - q_j|^{2\alpha_j}}{(\delta_j^2 + |x - q_j|^{2(\alpha_j+1)})^{3/2}} \Big)^{-1} \Big(\Delta U_\xi + a(x) U_\xi^p \Big)(x) \Big| \\ &\qquad \leq C e^{p/4} \Big(\Big(\frac{C}{p} \Big)^p + p e^{p/2} \Big) = O(p e^{-p/4}).\end{aligned} \tag{2.13}$$

If, on the other hand, $|x - q_i| \leq \varepsilon$ for some $i = 1, \dots, m+k$,

$$\begin{aligned}|\Delta U_\xi + a(x) U_\xi^p| = \Big| &\frac{\delta_i^{-\frac{2}{\alpha_i+1}}}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} |y|^{2\alpha_i} \\ &\times \Big(- e^{U^i} + \frac{1}{p} f^{0i} + \frac{1}{p^2} f^{1i} - \frac{1}{p} e^{U^i} V^i - \frac{1}{p^2} e^{U^i} W^i \Big) \\ &+ \delta_i^{\frac{2\alpha_i}{\alpha_i+1}} |y|^{2\alpha_i} a_i(\delta_i^{\frac{1}{\alpha_i+1}} y + q_i) U_\xi^p(\delta_i^{\frac{1}{\alpha_i+1}} y + q_i) + O(p e^{-\frac{p}{2}}) \Big|\end{aligned}$$

where $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x - q_i)$. By (2.4), we deduce that for $x = \delta_i^{\frac{1}{\alpha_i+1}} y + q_i$,

$$\begin{aligned}U_\xi^p = \Big(\frac{p}{\gamma \mu_i^{2/(p-1)} a_i(q_i)^{1/(p-1)}} \Big)^p \Big(1 + \frac{1}{p} U^i(y) + \frac{1}{p^2} V^i(y) + \frac{1}{p^3} W^i(y) \\ + O\Big(\frac{e^{-\frac{p}{4(\alpha_i+1)}}}{p} |y| + \frac{e^{-\frac{p}{4}}}{p} \Big) \Big)^p.\end{aligned} \tag{2.14}$$

By Taylor expansions of the exponential and logarithmic functions, we have for $|y| \le Ce^{\frac{p}{8(\alpha+1)}}$,

$$\Big(1+\frac{a}{p}+\frac{b}{p^2}+\frac{c}{p^3}\Big)^p = e^a\Big[1+\frac{1}{p}(b-\frac{a^2}{2})+\frac{1}{p^2}\Big(c-ab+\frac{a^3}{3} + \frac{b^2}{2}+\frac{a^4}{8}-\frac{a^2b}{2}\Big)+O\Big(\frac{\log^6(|y|+2)}{p^3}\Big)\Big] \tag{2.15}$$

provided $-5(\alpha+1)\log(|y|+2) \le a(y) \le C$ and $|b(y)|+|c(y)| \le C\log(|y|+2)$.

Since $\Big(\frac{p}{\gamma\mu_i^{2/(p-1)}}\Big)^p = \frac{1}{\gamma\delta_i^2\mu_i^{2/(p-1)}}$, we have by (2.15) that for $|x-q_i| \le \varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}}$,

$$\begin{aligned} U_\xi^p(x) = &\frac{1}{\gamma\delta_i^2\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{p}{p-1}}}e^{U^i(y)}\Big[1+\frac{1}{p}\Big(V^i-\frac{1}{2}(U^i)^2\Big)(y) \\ &+\frac{1}{p^2}\Big(W^i-U^iV^i+\frac{1}{3}(U^i)^3+\frac{1}{2}(V^i)^2+\frac{1}{8}(U^i)^4-\frac{1}{2}V^i(U^i)^2\Big)(y) \\ &+O\Big(\frac{\log^6(|y|+2)}{p^3}+e^{-\frac{p}{4(\alpha_i+1)}}|y|+e^{-p/4}\Big)\Big], \end{aligned}$$

where $y=\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. Since

$$\delta_i^{\frac{2\alpha_i}{\alpha_i+1}}|y|^{2\alpha_i}\frac{a_i(\delta_i^{\frac{1}{\alpha_i+1}}y+q_i)}{a_i(q_i)}U_\xi^p(\delta_i^{\frac{1}{\alpha_i+1}}y+q_i) = O\Big(p^2\delta_i^{-\frac{1}{\alpha_i+1}}|y|^{2\alpha_i+1}e^{U^i(y)}\Big),$$

we have in this region

$$\begin{aligned} &|\Delta U_\xi + a(x)U_\xi^p| \\ &=\Big|\frac{\delta_i^{-\frac{2}{\alpha_i+1}}}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}|y|^{2\alpha_i}\Big(-e^{U^i}+\frac{1}{p}f^{0i}+\frac{1}{p^2}f^{1i}-\frac{1}{p}e^{U^i}V^i-\frac{1}{p^2}e^{U^i}W^i\Big) \\ &\quad+\delta_i^{\frac{2\alpha_i}{\alpha_i+1}}|y|^{2\alpha_i}a_i(q_i)U_\xi^p(\delta_i^{\frac{1}{\alpha_i+1}}y+q_i)+O\Big(p^2\delta_i^{-\frac{1}{\alpha_i+1}}|y|^{2\alpha_i+1}e^{U^i(y)}+pe^{-\frac{p}{2}}\Big)\Big| \\ &=\frac{1}{\delta_i^{\frac{2}{\alpha_i+1}}}|y|^{2\alpha_i}e^{U^i(y)}O\Big(\frac{1}{p^4}\log^6(|y|+2)+p^2\delta_i^{\frac{1}{\alpha_i+1}}|y|\Big)+O(pe^{-\frac{p}{2}}). \end{aligned} \tag{2.16}$$

Hence, in this region we obtain that

$$\begin{aligned} &\Big|\Big(\sum_{j=1}^{m+k}\frac{\delta_j|x-q_j|^{2\alpha_j}}{(\delta_j^2+|x-q_j|^{2(\alpha_j+1)})^{3/2}}\Big)^{-1}\Big(\Delta U_\xi+a(x)U_\xi^p\Big)(x)\Big| \\ &\le C\delta_i^{\frac{2}{\alpha_i+1}}\frac{(1+|y|^{2(\alpha_i+1)})^{3/2}}{|y|^{2\alpha_i}}\frac{1}{\delta_i^{\frac{2}{\alpha_i+1}}}|y|^{2\alpha_i}e^{U^i(y)}\Big(\frac{1}{p^4}\log^6(|y|+2) \\ &\quad+p^2\delta_i^{\frac{1}{\alpha_i+1}}|y|\Big)+Cpe^{-p/4} \le \frac{C}{p^4}, \end{aligned} \tag{2.17}$$

where $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. Let us remark that if $m+k=1$, the weighted $\|\cdot\|_*$-norm has a singular weight at q_1. However, the expression for $\Delta U_\xi + a(x)U_\xi^p$ in (2.16) reduces to take the form

$$\Delta U_\xi + a(x)U_\xi^p = \frac{1}{\delta_1^{2/(\alpha_1+1)}}|y|^{2\alpha_1}e^{U^1(y)}\Big(\frac{1}{p^4}\log^6(|y|+2) + p^2\delta_1^{\frac{1}{\alpha_1+1}}|y|\Big),$$

since the term $O(pe^{-\frac{p}{2}})$ comes out from the interaction with all the other concentration points. Hence, the estimate (2.17) does not present any problem.

On the other hand, if $\varepsilon\delta_i^{1/(2(\alpha_i+1))} \le |x-q_i| \le \varepsilon$, we have by (2.12),

$$|\Delta U_\xi| = O\left(pe^{-\frac{p}{2}} + p\delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)}\right)$$

and by (2.14),

$$a(x)U_\xi^p(x) = O\left(\frac{1}{\gamma}\delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)}\right),$$

since $(1+\frac{s}{p})^p \le e^s$, where $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. Thus, in this region,

$$\left|\left(\sum_{j=1}^{m+k}\frac{\delta_j|x-q_j|^{2\alpha_j}}{(\delta_j^2+|x-q_j|^{2(\alpha_j+1)})^{3/2}}\right)^{-1}\left(\Delta U_\xi + a(x)U_\xi^p\right)(x)\right| \tag{2.18}$$
$$\le Cpe^{-p/4} + \frac{Cp}{(1+|y|^{2(\alpha_i+1)})^{1/2}} \le Cpe^{-p/8}, \quad y = \delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i).$$

By (2.13), (2.17) and (2.18), we obtain the desired result. □

3 The finite dimensional reduction

First of all, we solve the following linear problem. Given $h \in C(\bar\Omega)$, find a function $\phi \in W^{2,2}(\Omega)$ such that

$$\begin{cases} L(\phi) = h + \sum_{j=1}^{2}\sum_{i=m+1}^{m+k} c_{ij}e^{U_{\delta_i,q_i}}Z_{ij} & \text{in } \Omega,\\ \phi = 0 & \text{on } \partial\Omega,\\ \int_\Omega e^{U_{\delta_i,q_i}}Z_{ij}\phi = 0 \quad j=1,2,\ i=m+1,\dots,m+k, \end{cases} \tag{3.1}$$

for some coefficients c_{ij}, $j=1,2$ and $i=m+1,\dots,m+k$. Here and in the sequel, for any $i=1,\dots,m+k$,

$$Z_{i0}(x) := \frac{|x-q_i|^{2(\alpha_i+1)} - \delta_i^2}{|x-q_i|^{2(\alpha_i+1)} + \delta_i^2}$$

and for any $j=1,2$, $i=m+1,\dots,m+k$,

$$Z_{ij}(x) := \frac{4\delta_i(x-\xi_i)_j}{\delta_i^2+|x-\xi_i|^2}.$$

Following the approach in [14, 15] for $a(x) = 1$ (see also [9, 13]), in Appendix C we prove

Proposition 3.1. *There exist $p_0 > 0$ and $C > 0$ such that for $h \in C(\bar\Omega)$, there is a unique solution to problem* (3.1) *for any $p > p_0$ and $\xi \in \mathcal{O}_\varepsilon$, which satisfies*

$$\|\phi\|_\infty \le Cp\|h\|_*. \tag{3.2}$$

Moreover,

$$\sum_{j=1}^{2}\sum_{i=m+1}^{m+k} |c_{ij}| \le C\left(\frac{1}{p}\|\phi\|_\infty + \|h\|_*\right) \tag{3.3}$$

and

$$\|\phi\| \le C\left(\|\phi\|_\infty + \|h\|_*\right). \tag{3.4}$$

Let us now introduce the following auxiliary nonlinear problem:

$$\begin{cases} \Delta(U_\xi + \phi) + a(x)g_p\left(U_\xi + \phi\right) = \sum_{j=1}^{2}\sum_{i=m+1}^{m+k} c_{ij}e^{U_{\delta_i,q_i}}Z_{ij} & \text{in } \Omega,\\ \phi = 0 & \text{on } \partial\Omega,\\ \int_\Omega e^{U_{\delta_i,q_i}}Z_{ij}\phi = 0 \quad j = 1,2,\ i = m+1,\dots,m+k. \end{cases} \tag{3.5}$$

Proposition 3.2. *Let $\varepsilon > 0$ be fixed. There exist $c > 0$ and $p_0 > 0$ such that for any $p > p_0$ and $\xi \in \mathcal{O}_\varepsilon$, the problem* (3.5) *has a unique solution $\phi_p(\xi)$ which satisfies $\|\phi_p(\xi)\|_\infty \le c/p^3$. Furthermore, the function $\xi \to \phi_p(\xi)$ is a C^1 function in $L^\infty(\Omega)$ and in $H_0^1(\Omega)$.*

Proof. Using (2.7)–(2.9), we can rewrite the problem (3.5) as

$$L(\phi) = -\left(R + N(\phi)\right) + \sum_{i,j} c_{ij}e^{U_{\delta_i,q_i}}Z_{ij}.$$

Denote by C_* the function space $C(\bar\Omega)$ endowed with the norm $\|\cdot\|_*$. Proposition 3.1 ensures that the unique solution $\phi = T(h)$ of (3.1) defines a continuous linear map from the Banach space C_* into $C(\bar\Omega)$ with norm bounded by a multiple of p. Problem (3.5) then becomes

$$\phi = \mathcal{A}(\phi) := -T\left[R + N(\phi)\right].$$

Let $\mathcal{B}_r := \left\{\phi \in \mathrm{C}(\overline{\Omega}) :\ \phi = 0 \text{ on } \partial\Omega,\ \|\phi\|_\infty \le r/p^3\right\}$ for $r > 0$. Arguing as in [14] and using Remark C.1, we can prove that for any $\phi, \phi_1, \phi_2 \in \mathcal{B}_r$,

$$\|N(\phi)\|_* \le cp\|\phi\|_\infty^2,\ \|N(\phi_1) - N(\phi_2)\|_* \le cp\max_{i=1,2}\|\phi_i\|_\infty\|\phi_1 - \phi_2\|_\infty. \tag{3.6}$$

By (3.6), Proposition 2.1 and Proposition 3.1, it follows that $\mathcal{A}$ is a contraction mapping of $\mathcal{B}_r$ for a suitable $r > 0$. Thus a unique fixed point of $\mathcal{A}$ exists in $\mathcal{B}_r$. The regularity of the map $\xi \to \phi_p(\xi)$ follows using standard arguments as in [14]. □

After problem (3.5) has been solved, we find a solution to problem (2.1), if we can find a point $\xi = (\xi_1, \dots, \xi_k)$ such that coefficients $c_{ij}(\xi)$ in (3.5) satisfy

$$c_{ij}(\xi) = 0 \quad \text{for } i = m+1, \dots, m+k,\ j = 1, 2.$$

Let us introduce the energy functional $J_p : \mathrm{H}^1_0(\Omega) \to \mathbb{R}$, given by

$$J_p(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p+1}\int_\Omega a(x)(u^+)^{p+1} dx,$$

whose critical points are solutions to (2.1). We also introduce the finite dimensional restriction $\widetilde{J}_p : \mathcal{M} \to \mathbb{R}$, given by

$$\widetilde{J}_p(\xi) := J_p\big(U_\xi + \phi_p(\xi)\big). \tag{3.7}$$

The following result can be proved using standard arguments, as in [14, 15].

Lemma 3.1. *For all p sufficiently large, if $\xi \in \mathcal{M}$ is a critical point of $\widetilde{J}_p$, then $U_\xi + \phi_p(\xi)$ is a critical point of J_p, namely a solution to the problem* (2.1) .

Next, we need to write the expansion of $\widetilde{J}_p$ as p goes to $+\infty$,

Lemma 3.2. *We have*

$$\widetilde{J}_p(\xi) = \frac{c_1}{p} + \frac{c_2}{p^2} - \frac{c_3}{p^2}\Phi(\xi) + R_p(\xi),$$

where $R_p = O(\frac{\log^2 p}{p^3})$ uniformly with respect to ξ in compact sets of $\mathcal{M}$. Here c_1, c_2 and $c_3 \neq 0$ are constants (depending only on $q_1, \dots, q_m$), and the function $\Phi : \mathcal{M} \to \mathbb{R}$ is defined by

$$\Phi(\xi_1, \dots, \xi_k) = \sum_{i=1}^k H(\xi_i, \xi_i) + \sum_{\substack{i,j=1\\ i\neq j}}^k G(\xi_i, \xi_j) + \frac{1}{4\pi}\sum_{i=1}^k \log a(\xi_i) + 2\sum_{i=1}^k\sum_{j=1}^m (\alpha_j + 1) G(\xi_i, q_j).$$

Proof. Multiplying equation in (3.5) by $U_\xi + \phi_p(\xi)$ and integrating by parts, we obtain

$$\int_\Omega a(x)\,(U_\xi + \phi_p(\xi))_+^{p+1} = -\int_\Omega |\nabla(U_\xi + \phi_p(\xi))|^2 + \sum_{i,j} c_{ij}(\xi)\int_\Omega e^{U_{\delta_i, q_i}} Z_{ij} U_\xi. \tag{3.8}$$

In particular, by (3.8) it follows that

$$\widetilde{J}_p(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |\nabla(U_\xi + \phi_p(\xi))|^2 - \frac{1}{p+1} \sum_{i,j} c_{ij}(\xi) \int_\Omega e^{U_{\delta_i,q_i}} Z_{ij} U_\xi.$$

Let us expand the leading term $\int_\Omega |\nabla U_\xi|^2$. In view of (2.4), we have

$$\begin{aligned}
&\int_\Omega |\nabla U_\xi|^2 \\
&= -\int_\Omega \Delta U_\xi(x) U_\xi(x) dx \\
&= \sum_{i=1}^{m+k} \frac{1}{\gamma \mu_i^{\frac{2}{p-1}} a_i(q_i)^{\frac{1}{p-1}}} \int_{B(q_i,\varepsilon)} \Big(|x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} - \tfrac{1}{p} \Delta V_{\delta_i,q_i} - \tfrac{1}{p^2} \Delta W_{\delta_i,q_i}\Big) U_\xi(x) dx \\
&\quad + O\big(e^{-(p/2)}\big) \qquad \text{(setting } x - q_i = \delta_i^{\frac{1}{\alpha_i+1}} y\text{)} \\
&= \sum_{i=1}^{m+k} \frac{1}{\gamma^2 \mu_i^{\frac{4}{p-1}} a_i(q_i)^{\frac{2}{p-1}}} \\
&\quad \times \int_{B\left(0,\varepsilon\delta_i^{-\frac{1}{\alpha_i+1}}\right)} \left(|y|^{2\alpha_i} e^{U^i} - \frac{\delta_i^{\frac{2}{\alpha_i+1}}}{p} \Delta V_{\delta_i,q_i} - \frac{\delta_i^{\frac{2}{\alpha_i+1}}}{p^2} \Delta W_{\delta_i,q_i}\right) \\
&\qquad\qquad \times \left(p + U^i(y) + \tfrac{1}{p} V^i(y) + \tfrac{1}{p^2} W^i(y)\right) dy \\
&\quad + O\big(\tfrac{1}{p^3}\big) \\
&= \sum_{i=1}^{m+k} \frac{1}{\gamma^2 \mu_i^{\frac{4}{p-1}} a_i(q_i)^{\frac{2}{p-1}}} \\
&\quad \times \int_{B\left(0,\varepsilon\delta_i^{-\frac{1}{\alpha_i+1}}\right)} |y|^{2\alpha_i} \left(e^{U^i} - \tfrac{1}{p} f^{0i} + \tfrac{1}{p} e^{U^i} V^i - \tfrac{1}{p^2} f^{1i} + \tfrac{1}{p^2} e^{U^i} W^i\right) \\
&\qquad\qquad \times \left(p + U^i(y) + \tfrac{1}{p} V^i(y) + \tfrac{1}{p^2} W^i(y)\right) dy \\
&\quad + O\big(\tfrac{1}{p^3}\big) \\
&= \sum_{i=1}^{m+k} \frac{1}{\gamma^2 \mu_i^{\frac{4}{p-1}} a_i(q_i)^{\frac{2}{p-1}}} \\
&\quad \times \Big(p \int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} dy + \int_{\mathbb{R}^2} |y|^{2\alpha_i} U^i e^{U^i} dy - \int_{\mathbb{R}^2} |y|^{2\alpha_i} f^{0i} dy + \int_{\mathbb{R}^2} |y|^{2\alpha_i} V^i e^{U^i} \\
&\qquad + O\big(\tfrac{1}{p}\big)\Big) \\
&\quad = \sum_{i=1}^{m+k} \Big[\tfrac{e}{p}\big(1 - 2\tfrac{\log p}{p} + \tfrac{1}{p} - \tfrac{2}{p} \log a_i(q_i)\big) A_i + \tfrac{e}{p^2} B_i - \tfrac{4e}{p^2} A_i \log \mu_i\Big] + O\big(\tfrac{\log^2 p}{p^3}\big),
\end{aligned}$$

where

$$A_i := \int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} dy = 8\pi(\alpha_i + 1)$$
$$B_i := \int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} \left(U^i - \frac{1}{2}(U^i)^2 + V^i \right) dy,$$

because

$$\mu_i^{-\frac{4}{p-1}} = 1 - \frac{4}{p}\log\mu_i + O\Big(\frac{1}{p^2}\Big),$$
$$a_i(q_i)^{-\frac{2}{p-1}} = 1 - \frac{2}{p}\log a_i(q_i) + O\Big(\frac{1}{p^2}\Big),$$
$$\frac{1}{\gamma^2} = \frac{e}{p^2}\left(1 - 2\frac{\log p}{p} + \frac{1}{p} + O\Big(\frac{\log^2 p}{p^2}\Big)\right).$$

Recalling the expansion of μ_i in (2.6), we get

(3.9)

$$\begin{aligned}
&\int_\Omega |\nabla U_\xi|^2 \\
&=\frac{8\pi e}{p}\Big(1 - 2\frac{\log p}{p} + \frac{4}{p}\Big)\sum_{i=1}^{m+k}(\alpha_i+1) + \frac{e}{p^2}\sum_{i=1}^{m+k} B_i \\
&\quad - \frac{16\pi e}{p^2}\sum_{i=1}^{m+k}(\alpha_i+1)\Big(\log a_i(q_i) + 4\pi(\alpha_i+1)H(q_i,q_i) + 4\pi\sum_{j\neq i}(\alpha_j+1)G(q_j,q_i)\Big) \\
&\quad + O\Big(\frac{\log^2 p}{p^3}\Big) \\
&=\frac{8\pi e}{p}\left(1 - 2\frac{\log p}{p} + \frac{4}{p}\right)\sum_{i=1}^{m+k}(\alpha_i+1) + \frac{e}{p^2}\sum_{i=1}^{m+k} B_i \\
&\quad - \frac{16\pi e}{p^2}\sum_{i=1}^{m}(\alpha_i+1)\Big(\log a_i(q_i) + 4\pi(\alpha_i+1)H(q_i,q_i) + 4\pi\sum_{\substack{j=1\\ j\neq i}}^{m}(\alpha_j+1)G(q_i,q_j)\Big) \\
&\quad - \frac{64\pi^2 e}{p^2}\Phi(\xi_1,\dots,\xi_k) + O\left(\frac{\log^2 p}{p^3}\right)
\end{aligned}$$

uniformly for ξ in a compact set of $\mathcal{M}$. In particular,

$$\int_\Omega |\nabla U_\xi|^2 = O\Big(\frac{1}{p}\Big). \tag{3.10}$$

Now, using Proposition 3.2 and estimates (2.11), (3.6), we deduce by (3.3)–(3.4) that

$$|c_{ij}(\xi)| = O\left(\frac{1}{p}||\phi_p(\xi)||_\infty + \|N(\phi_p(\xi))\|_* + \|R\|_*\right) = O\Big(\frac{1}{p^4}\Big)$$

and

$$\|\phi_p(\xi)\| = O\left(\|\phi_p(\xi)\|_\infty + \|N(\phi_p(\xi))\|_* + \|R\|_*\right) = O\Big(\frac{1}{p^3}\Big).$$

Therefore, by (3.10), we have

$$\widetilde{J}_p(\xi) = \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_\Omega |\nabla U_\xi|^2 + O\Big(\frac{1}{p^3}\Big);$$

and our claim follows with suitable constants c_1, c_2 and $c_3 = 32\pi^2 e \neq 0$. □

We introduce the following definition.

Definition 3.1. We say that ξ is a C^0-stable critical point of $\Phi : \mathcal{M} \to \mathbb{R}$ if for any sequence of functions $\Phi_n : \mathcal{M} \to \mathbb{R}$ such that $\Phi_n \to \Phi$ uniformly on compact sets of $\mathcal{M}$, Φ_n has a critical point ξ_n such that $\Phi_n(\xi_n) \to \Phi(\xi)$.

In particular, if ξ is a strict local minimum/maximum point of Φ, then ξ is a C^0-stable critical point.

Proof of Theorem 1.2. According to Lemma 3.1, we have a solution to the problem (1.3) if we find a critical point ξ_p of $\widetilde{J}_p$. This is equivalent to finding a critical point of the function $\Phi_p : \mathcal{M} \to \mathbb{R}$ defined by $\Phi_p(\xi) := \big(pc_1 + c_2 - p^2 \widetilde{J}_p(\xi)\big)/c_3$ (see Lemma 3.2). On the other hand, $\Phi_p \to \Phi$ uniformly on compact sets of $\mathcal{M}$ as p goes to $+\infty$, by Lemma 3.2. By Definition 3.1 we deduce that if p is large enough, there exists a critical point $\xi^p \in \mathcal{M}$ of Φ_p such that $\Phi_p(\xi^p) \to \Phi(\xi^*)$. Moreover, up to a subsequence, $\xi^p \to \xi$ as p goes to $+\infty$, with $\Phi(\xi) = \Phi(\xi^*)$. The function $u_p = U_{\xi^p} + \phi_{\xi^p}$ is therefore a positive solution to (1.3) (the proof of the positivity of u_p follows the lines of Remark 2.2). Moreover, the sequence u_p has the qualitative properties predicted by the theorem, as can be easily shown. For instance, for (1.5) consider (3.8)–(3.9) and (1.7) follows by (2.5), because ϕ_{ξ^p} is a higher order term in u_p. □

4 Proof of Theorem 1.1

Let $\Omega = \{x \in \mathbb{R}^2 \ : \ |x| < 1\}$ be the unit ball and let $a(x) = |x|^{2\alpha}$ for some $\alpha > 0$. Let $k \geq 1$ be a fixed integer and set

$$\xi_i^* := \left(\cos\frac{2\pi}{k}(i-1), \sin\frac{2\pi}{k}(i-1)\right) \qquad \text{for any } i = 1, \ldots, k.$$

We seek a solution to problem (2.1) as $u_p = U_\rho + \phi_p(\rho)$, where

$$U_\rho := \sum_{i=1}^k \frac{1}{\gamma \mu_i^{\frac{2}{p-1}} \rho^{\frac{2\alpha}{p-1}}} \left(PU_{\delta_i,\xi_i} + \frac{1}{p} PV_{\delta_i,\xi_i} + \frac{1}{p^2} PW_{\delta_i,\xi_i}\right)$$

and the concentration parameters δ_i are given in (2.2), μ_i are defined in (2.3) and the concentration points ξ are given, for any $i = 1, \dots, k$, by

$$\xi_i := \xi_i(\rho) = \rho\xi_i^* = \left(\rho\cos\frac{2\pi}{k}(i-1), \rho\sin\frac{2\pi}{k}(i-1)\right) \ , \quad \rho \in (0,1).$$

The rest term $\phi_p(\rho)$ can be found symmetric with respect to the variable x_2 and each line $\{t\xi_i^* \ : \ t \in \mathbb{R}\}$, for any $i = 1, \dots, k$.

Using results obtained in the previous sections and taking into account the symmetry of the domain and the function a, we reduce the problem of finding solutions to (2.1) to that of finding critical points of the function $\widetilde{J}_p : (0,1) \to \mathbb{R}$, defined as in (3.7) by $\widetilde{J}_p(\rho) := J_p(U_\rho + \phi_p(\rho))$. Using Lemma 3.2, it is not difficult to check that

$$\widetilde{J}_p(\rho) = \frac{c_1}{p} + \frac{c_2}{p^2} - \frac{c_3}{p^2}\Phi(\rho) + R_p(\rho),$$

where $R_p(\rho) = O\left(\frac{\log^2 p}{p^3}\right)$ uniformly for ρ in compact sets of $(0,1)$. Moreover, c_1, c_2 and $c_3 \neq 0$ are constants and

$$\Phi(\rho) := H(\rho\xi_1^*, \rho\xi_1^*) + \sum_{i=2}^{k} G(\rho\xi_1^*, \rho\xi_i^*) + \frac{\alpha}{2\pi}\log\rho, \ \rho \in (0,1).$$

In this case, we have

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} - \frac{1}{2\pi}\log\frac{1}{\sqrt{|x|^2|y|^2 + 1 - 2(x,y)}},$$
$$H(x,x) = -\frac{1}{2\pi}\log\frac{1}{1-|x|^2};$$

and so,

$$\Phi(\rho) = \frac{1}{2\pi}\log(1-\rho^2) + \frac{\alpha-(k-1)}{2\pi}\log\rho + \frac{1}{2\pi}\sum_{i=2}^{k}\log\frac{\sqrt{\rho^4 + 1 - 2\rho^2(\xi_1^*, \xi_i^*)}}{|\xi_1^* - \xi_i^*|}.$$

Now there exists $\rho_0 \in (0,1)$ such that $\Phi(\rho_0) = \max_{\rho\in(0,1)} \Phi(\rho)$, provided $\alpha - k + 1 > 0$, since $\lim_{\rho\to 1^-}\Phi(\rho) = \lim_{\rho\to 0^+}\Phi(\rho) = -\infty$. Then ρ_0 is a C^0-stable critical point of Φ, and so function $\widetilde{J}_p$ has a critical point ρ_p for p large enough. That proves our claim for any $k \leq K_\alpha$.

A Appendix

Let us recall the following basic result stated by Chae and Imanuvilov in [6]: for any $f(t) \in C^1[0, +\infty)$, there exists a smooth radial solution

$$w(r) = \frac{r^{2(\alpha+1)} - 1}{r^{2(\alpha+1)} + 1}\left(\int_0^r \frac{\phi_f(s) - \phi_f(1)}{(s-1)^2}ds + \phi_f(1)\frac{r}{1-r}\right)$$

for the equation

$$\Delta w + \frac{8(\alpha+1)^2|y|^{2\alpha}}{(1+|y|^{2(\alpha+1)})^2} w = |y|^{2\alpha} f(|y|),$$

where $\phi_f(s) = (\frac{s^{2(\alpha+1)}+1}{s^{2(\alpha+1)}-1})^2 \frac{(s-1)^2}{s} \int_0^s t^{2\alpha+1} \frac{t^{2(\alpha+1)}-1}{t^{2(\alpha+1)}+1} f(t)dt$ for $s \neq 1$ and $\phi_f(1) = \lim_{s\to 1} \phi_f(s)$.

Assume that $\int_0^\infty t^{2\alpha+1}|\log t||f|(t)dt < +\infty$. A straightforward computation shows that

$$w(r) = C_f \log r + D_f + O\bigg(\int_r^{+\infty} t^{2\alpha+1}|\log t||f|(t)dt + \frac{1}{r^{2(\alpha+1)}}\bigg) \quad \text{as } r \to +\infty,$$

where $C_f = \int_0^{+\infty} t^{2\alpha+1} \frac{t^{2(\alpha+1)}-1}{t^{2(\alpha+1)}+1} f(t)dt$. A similar computation can be performed for $\partial_r w(r)$. Therefore, up to replacing $w(r)$ with $w(r) - D_f \frac{r^{2(\alpha+1)}-1}{r^{2(\alpha+1)}+1}$, we have shown

Lemma A.1. *Let $f \in C^1[0,+\infty)$ be such that $\int_0^{+\infty} t^{2\alpha+1}|\log t||f|(t)dt < +\infty$. There exists a C^2 radial solution $w(r)$ of equation*

$$\Delta w + \frac{8(\alpha+1)^2|y|^{2\alpha}}{(1+|y|^{2(\alpha+1)})^2} w = |y|^{2\alpha} f(|y|) \quad \text{in } \mathbb{R}^2$$

such that as $r \to +\infty$,

$$w(r) = C_f \log r + O\bigg(\int_r^{+\infty} t^{2\alpha+1}|\log t||f|(t)dt + \frac{1}{r^{2(\alpha+1)}}\bigg)$$

and

$$\partial_r w(r) = \frac{C_f}{r} + O\bigg(\frac{1}{r}\int_r^{+\infty} t^{2\alpha+1}|f|(t)dt + \frac{|\log r|}{r^{2\alpha+3}}\bigg),$$

where $C_f = \Big(\int_0^{+\infty} t^{2\alpha+1} \frac{t^{2(\alpha+1)}-1}{t^{2(\alpha+1)}+1} f(t)dt\Big)$.

Now let $U(y) = \log \frac{8(\alpha+1)^2}{(1+|y|^{2(\alpha+1)})^2}$. Let V, W be radial solutions of

$$\Delta V + |y|^{2\alpha} e^U V = |y|^{2\alpha} f^0 \quad \text{in } \mathbb{R}^2, \quad f^0(y) := \frac{1}{2} e^{U(y)} U^2(y), \tag{A.1}$$

and

$$\Delta W + |y|^{2\alpha} e^U W = |y|^{2\alpha} f^1 \quad \text{in } \mathbb{R}^2, \tag{A.2}$$
$$f^1(y) := e^{U(y)}\left(VU - \frac{1}{2}V^2 - \frac{1}{3}U^3 - \frac{1}{8}U^4 + \frac{1}{2}VU^2\right)(y)$$

such that

$$V(y) = C_0(\alpha)\log|y| + O\Big(\frac{1}{|y|^{\alpha+1}}\Big), \tag{A.3}$$
$$W(y) = C_1(\alpha)\log|y| + O\Big(\frac{1}{|y|^{\alpha+1}}\Big),$$

as $|y| \to +\infty$, where $C_i(\alpha) = \int_0^{+\infty} t^{2\alpha+1} \frac{t^{2(\alpha+1)}-1}{t^{2(\alpha+1)}+1} f^i(t)dt$, $i = 1, 2$.

It is possible to construct W, since by (A.3), V has logarithmic growth at infinity. The exact expression for V, which is needed later, is

$$\begin{aligned} V(y) =& \frac{1}{2}U^2(y) + 6\log(|y|^{2(\alpha+1)} + 1) + \frac{2\log 8(\alpha+1)^2 - 10}{|y|^{2(\alpha+1)}+1} \\ &+ \frac{|y|^{2(\alpha+1)}-1}{|y|^{2(\alpha+1)}+1}\Bigg(2\log^2(|y|^{2(\alpha+1)}+1) - \frac{1}{2}\log^2 8(\alpha+1)^2 \\ &\qquad + 4\int_{|y|^{2(\alpha+1)}}^{+\infty} \frac{ds}{s+1}\log\frac{s+1}{s} \\ &\qquad - 8(\alpha+1)\log|y|\log(|y|^{2(\alpha+1)}+1)\Bigg), \end{aligned} \tag{A.4}$$

as can be seen by direct inspection. Moreover, it is easy to compute the value:

$$C_0(\alpha) = 12(\alpha+1) - 4(\alpha+1)\log 8(\alpha+1)^2 \tag{A.5}$$

B Appendix

Let $\alpha \geq 0$. Define

$$U_{\delta,\xi}(x) = \log\frac{8(\alpha+1)^2\delta^2}{(\delta^2 + |x-\xi|^{2(\alpha+1)})^2}, \quad \delta > 0,\ \xi \in \mathbb{R}^2,$$

which is a solution of $-\Delta U_{\delta,\xi} = |x-\xi|^{2\alpha}e^{U_{\delta,\xi}}$ in $\mathbb{R}^2$ (see (1.9)–(1.10)). The following expansions hold.

Lemma B.1. *As $\delta \to 0$,*

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \log 8(\alpha+1)^2\delta^2 + 8\pi(\alpha+1)H(x,\xi) + O\left(\delta^2\right) \tag{B.1}$$

in $C(\bar\Omega)$ and

$$PU_{\delta,\xi}(x) = 8\pi(\alpha+1)G(x,\xi) + O\left(\delta^2\right) \tag{B.2}$$

in $C_{loc}(\bar\Omega \setminus \{\xi\})$, uniformly for ξ away from $\partial\Omega$.

Proof. Since $PU_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \log 8(\alpha+1)^2\delta^2 = -4(\alpha+1)\log\frac{1}{|x-\xi|} + O(\delta^2)$ as $\delta \to 0$ uniformly for $x \in \partial\Omega$ and ξ away from $\partial\Omega$, (B.1) readily follows by harmonicity and the maximum principle.

On the other hand, away from ξ, we have $U_{\delta,\xi}(x) - \log 8(\alpha+1)^2\delta^2 = 4(\alpha+1)\log\frac{1}{|x-\xi|} + O(\delta^2)$. This fact, together with (B.1) gives (B.2). $\square$

Let V, W be the radial solutions of (A.1), (A.2) respectively, which satisfy (A.3):

$$V(y) = C_0(\alpha)\log|y| + O\Big(\frac{1}{|y|^{\alpha+1}}\Big),$$
$$W(y) = C_1(\alpha)\log|y| + O\Big(\frac{1}{|y|^{\alpha+1}}\Big) \quad \text{as } |y| \to +\infty,$$

for some constants $C_0(\alpha)$, $C_1(\alpha)$. For any $\delta > 0$ and $\xi\, in \mathbb{R}^2$, define

$$V_{\delta,\xi}(x) := V\left(\delta^{-\frac{1}{\alpha+1}}(x-\xi)\right),\ W_{\delta,\xi}(x) := W\left(\delta^{-\frac{1}{\alpha+1}}(x-\xi)\right)$$

for $x \in \Omega$. Then $V_{\delta,\xi}$ and $W_{\delta,\xi}$ satisfy

$$\Delta V_{\delta,\xi} + |x-\xi|^{2\alpha}e^{U_{\delta,\xi}}V_{\delta,\xi} = |x-\xi|^{2\alpha}f^0_{\delta,\xi} \quad \text{in } \mathbb{R}^2,$$

and

$$\Delta W_{\delta,\xi} + |x-\xi|^{2\alpha}e^{U_{\delta,\xi}}W_{\delta,\xi} = |x-\xi|^{2\alpha}f^1_{\delta,\xi} \quad \text{in } \mathbb{R}^2,$$

where

$$f^j_{\delta,\xi}(x) := \frac{1}{\delta^2}f^j\left(\frac{x-\xi}{\delta^{\frac{1}{\alpha+1}}}\right), \quad j = 0, 1.$$

By (A.3), we deduce the following expansions.

Lemma B.2. *As $\delta \to 0$,*

(B.3)
$$PV_{\delta,\xi}(x) = V_{\delta,\xi}(x) - 2\pi C_0(\alpha)H(x,\xi) + \frac{C_0(\alpha)}{\alpha+1}\log\delta + O(\delta)$$
$$PW_{\delta,\xi}(x) = W_{\delta,\xi}(x) - 2\pi C_1(\alpha)H(x,\xi) + \frac{C_1(\alpha)}{\alpha+1}\log\delta + O(\delta)$$

in $C(\bar\Omega)$ and

(B.4)
$$PV_{\delta,\xi}(x) = -2\pi C_0(\alpha)G(x,\xi) + O(\delta)$$
$$PW_{\delta,\xi}(x) = -2\pi C_1(\alpha)G(x,\xi) + O(\delta)$$

in $C_{loc}(\bar\Omega \setminus \{\xi\})$, uniformly for ξ away from $\partial\Omega$. In particular, for any $\varepsilon > 0$, there exists $c > 0$, such that for any small δ and $\xi \in \Omega$ with $\operatorname{dist}(\xi, \partial\Omega) \geq \varepsilon$, *we have*

$$\|PV_{\delta,\xi}\|_\infty + \|PW_{\delta,\xi}\|_\infty \leq c|\log\delta|.$$

Proof. The proof follows from the same argument used to prove Lemma B.1 and from estimates (A.3). □

C Appendix

In this section, we prove invertibility of the operator L and give a bound (uniformly in $\xi \in \mathcal{O}_\varepsilon$) on its inverse norm by using the L^∞-norms introduced in (2.10). Recall that $L(\phi) = \Delta\phi + a(x)W_\xi\phi$, where $W_\xi(x) = pU_\xi^{p-1}(x)$.

As in Proposition 2.1, we have for the potential $a(x)W_\xi(x)$ the following expansions. By (2.14), if $|x-q_i| \le \varepsilon$ for some $i = 1,\dots,m+k$, we have

$$\begin{aligned} a(x)W_\xi(x) =& p\Big(\frac{p}{\gamma\mu_i^{\frac{2}{p-1}}a_i(q_i)^{\frac{1}{p-1}}}\Big)^{p-1}a(x) \\ &\times\Big(1+\tfrac{1}{p}U^i(y)+\tfrac{1}{p^2}V^i(y)+\tfrac{1}{p^3}W^i(y)+O\Big(\frac{e^{-\frac{p}{4(\alpha_i+1)}}}{p}|y|+\frac{e^{-\frac{p}{4}}}{p}\Big)\Big)^{p-1} \\ =& \delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}\left(1+O\Big(\delta_i^{\frac{1}{\alpha_i+1}}|y|\Big)\right) \\ &\times\Big(1+\tfrac{1}{p}U^i(y)+\tfrac{1}{p^2}V^i(y)+\tfrac{1}{p^3}W^i(y)+O\Big(\frac{e^{-\frac{p}{4(\alpha_i+1)}}}{p}|y|+\frac{e^{-\frac{p}{4}}}{p}\Big)\Big)^{p-1}, \end{aligned}$$

where again we use the notation $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. In this region,

$$a(x)W_\xi(x) \le C\delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)} = O\left(|x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}(x)}\right).$$

Furthermore, by Taylor expansions of exponential and logarithmic functions as in (2.15), we obtain that, if $|x-q_i| \le \varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}}$ (and $|y| \le \varepsilon\delta_i^{-\frac{1}{2(\alpha_i+1)}}$),

$$\begin{aligned} a(x)W_\xi(x) =& \delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}\left(1+O\Big(\delta_i^{\frac{1}{\alpha_i+1}}|y|\Big)\right) \\ &\times\left(1+\tfrac{1}{p}U^i(y)+\tfrac{1}{p^2}V^i(y)\tfrac{1}{p^3}W^i(y)+O\Big(\frac{e^{-\frac{p}{4(\alpha_i+1)}}}{p}|y|+\frac{e^{-\frac{p}{4}}}{p}\Big)\right)^{p-1} \\ =& \delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)}\left[1+\tfrac{1}{p}\left(V^i-U^i-\tfrac{1}{2}(U^i)^2\right)+O\Big(\frac{\log^4(|y|+2)}{p^2}\Big)\right]. \end{aligned}$$

If $|x-q_i| \ge \varepsilon$ for any $i = 1,\dots,m+k$,

$$a(x)W_\xi(x) = O\left(p\Big(\frac{C}{p}\Big)^{p-1}\right).$$

Summing up, we have

Lemma C.1. *There exist $D_0 > 0$ and $p_0 > 0$ such that*

$$a(x)W_\xi(x) \le D_0\sum_{i=1}^{m+k}|x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}(x)}$$

for any $\xi \in \mathcal{O}_\varepsilon$ *and* $p \geq p_0$. *Furthermore,*

$$a(x)W_\xi(x) = \delta_i^{-\frac{2}{\alpha_i+1}}|y|^{2\alpha_i}e^{U^i(y)}\left[1 + \frac{1}{p}\left(V^i - U^i - \frac{1}{2}(U^i)^2\right) + O\Big(\frac{\log^4(|y|+2)}{p^2}\Big)\right]$$

for any $|x-q_i| \leq \varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}}$, *where* $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$.

Remark C.1. As for W_ξ, let us point out that if $|x-q_i| \leq \varepsilon$ for some $i = 1,\dots,m+k$,

$$pa(x)\left(U_\xi + O\Big(\frac{1}{p^3}\Big)\right)^{p-2} \leq Cp\Big(\frac{p}{\gamma}\Big)^{p-2}|x-q_i|^{2\alpha_i}e^{U^i(y)} = O\left(|x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}(x)}\right),$$

where $y = \delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)$. Since this estimate is true if $|x-q_i| \geq \varepsilon$ for any $i = 1,\dots,m+k$, we have

$$pa(x)\left(U_\xi + O\Big(\frac{1}{p^3}\Big)\right)^{p-2} \leq C\sum_{i=1}^{m+k}|x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}(x)}.$$

In an heuristic way, the operator L is close to $\tilde{L}$ defined by

$$\tilde{L}(\phi) = \Delta\phi + \left(\sum_{i=1}^{m+k}|x-q_i|^{2\alpha_i}e^{U_{\delta_i,q_i}}\right)\phi.$$

The operator $\tilde{L}$ is "essentially" a superposition of linear operators which, after a dilation and translation, approach, as $p \to \infty$, the linear operator in $\mathbb{R}^2$:

$$\phi \;\to\; \Delta\phi + \frac{8(\alpha_i+1)^2|y|^{2\alpha_i}}{(1+|y|^{2(\alpha_i+1)})^2}\phi, \quad i = 1,\dots,m+k,$$

namely the equation $\Delta v + |y|^{2\alpha_i}e^v = 0$ linearized around the radial solution $\log\frac{8(\alpha_i+1)^2}{(1+|y|^{2(\alpha_i+1)})^2}$.

Set $z_0^i(y) = \frac{|y|^{2(\alpha_i+1)}-1}{|y|^{2(\alpha_i+1)}+1}$ for any $i = 1,\dots,m+k$ and $z_j(y) = \frac{4y_j}{1+|y|^2}$, $j = 1,2$. The first ingredient in the desired solvability theory for L is the well-known fact that any bounded solution of $L(\phi) = 0$ in $\mathbb{R}^2$ is

- for $i = 1,\dots,m$ proportional to z_0^i;
- for $i = m+1,\dots,m+k$ a linear combination of z_0^i and z_j, $j = 1,2$.

The second ingredient is a detailed analysis of $L - \tilde{L}$. Let us rewrite the problem (3.1). Given $h \in C(\bar{\Omega})$, we consider the linear problem of finding a function

$\phi \in W^{2,2}(\Omega)$ such that

$$\text{(C.1)} \qquad L(\phi) = h + \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} c_{ij} e^{U_{\delta_i,q_i}} Z_{ij} \qquad \text{in } \Omega,$$

$$\text{(C.2)} \qquad \phi = 0 \qquad \text{on } \partial\Omega,$$

$$\text{(C.3)} \qquad \int_\Omega e^{U_{\delta_i,q_i}} Z_{ij} \phi = 0 \qquad j = 1, 2,\ i = m+1, \dots, m+k,$$

for some coefficients c_{ij}, $j = 1, 2$ and $i = m+1, \dots, m+k$. Here and in the sequel, we denote

$$Z_{i0}(x) := z_0^i \left(\delta_i^{-\frac{1}{\alpha_i+1}} (x - q_i) \right) = \frac{|x - q_i|^{2(\alpha_i+1)} - \delta_i^2}{|x - q_i|^{2(\alpha_i+1)} + \delta_i^2}$$

for any $i = 1, \dots, m+k$; and

$$Z_{ij}(x) := z_j \left(\delta_i^{-1} (x - \xi_i) \right) = \frac{4\delta_i (x - \xi_i)_j}{\delta_i^2 + |x - \xi_i|^2}.$$

for any $j = 1, 2$, $i = m+1, \dots, m+k$. Following some ideas in [14] for $a(x) = 1$, we give the proof of Proposition 3.1, which consists of six steps.

1st Step. The operator L satisfies the maximum principle in

$$\tilde{\Omega} := \Omega \setminus \bigcup_{i=1}^{m+k} B(q_i, R\delta_i^{\frac{1}{\alpha_i+1}})$$

for R large, independent on p. Specifically,

if $L(\psi) \leq 0$ in $\tilde{\Omega}$ and $\psi \geq 0$ on $\partial\tilde{\Omega}$, then $\psi \geq 0$ in $\tilde{\Omega}$.

In order to prove this fact, we show the existence of a positive function Z in $\tilde{\Omega}$ satisfying $L(Z) < 0$. Indeed, let

$$Z(x) = \sum_{i=1}^{m+k} z_0^i \left(a^{\frac{1}{\alpha_i+1}} \delta_i^{-\frac{1}{\alpha_i+1}} (x - q_i) \right), \quad a > 0.$$

First, observe that for $x \in \tilde{\Omega}$, if $R > \frac{1}{a^{1/(\alpha_i+1)}}$ for any $i = 1, \dots, m+k$, then $Z(x) > 0$. On the other hand,

$$a(x) W_\xi(x) \leq D_0 \left(\sum_{i=1}^{m+k} |x - q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}(x)} \right) \leq D_0 \sum_{i=1}^{m+k} \frac{8(\alpha_i + 1)^2 \delta_i^2}{|x - q_i|^{2\alpha_i+4}},$$

where D_0 is the constant in Lemma C.1. Further, by the definition of z_0^i, we have that for any $x \in \tilde{\Omega}$,

$$
\begin{aligned}
-\Delta Z(x) &= \sum_{i=1}^{m+k} a^2|x-q_i|^{2\alpha_i} \frac{8(\alpha_i+1)^2\delta_i^2(a^2|x-q_i|^{2(\alpha_i+1)}-\delta_i^2)}{(a^2|x-q_i|^{2(\alpha_i+1)}+\delta_i^2)^3} \\
&\geq \frac{1}{3}\sum_{i=1}^{m+k} \frac{8a^2(\alpha_i+1)^2\delta_i^2|x-q_i|^{2\alpha_i}}{(a^2|x-q_i|^{2(\alpha_i+1)}+\delta_i^2)^2} \\
&\geq \frac{4}{27}\sum_{i=1}^{m+k} \frac{8(\alpha_i+1)^2\delta_i^2}{a^2|x-q_i|^{2\alpha_i+4}},
\end{aligned}
$$

provided $R > (\frac{\sqrt{2}}{a})^{\frac{1}{\alpha_i+1}}$ for any $i = 1, \dots, m+k$. Hence,

$$
LZ(x) \leq \left(-\frac{4}{27a^2} + D_0(m+k)\right) \sum_{i=1}^{m+k} \frac{8(\alpha_i+1)^2\delta_i^2}{|x-q_i|^{2\alpha_i+4}} < 0,
$$

since $Z(x) \leq m+k$, provided that a is chosen sufficiently small (independent of p). The function $Z(x)$ is what we are looking for.

2$^{\text{nd}}$ Step. Let R be as before. We define the "inner norm" of ϕ as

$$
\|\phi\|_i = \sup_{x \in \bigcup_{i=1}^{m+k} B\left(q_i, R\delta_i^{\frac{1}{\alpha_i+1}}\right)} |\phi|(x)
$$

and claim that there is a constant $C > 0$ such that if $L(\phi) = h$ in Ω and $\phi = 0$ on $\partial\Omega$, then

$$
\|\phi\|_\infty \leq C[\|\phi\|_i + \|h\|_*]
$$

for any $h \in C^{0,\alpha}(\bar{\Omega})$. We establish this estimate with the use of suitable barriers. Let $M = 2$ diam Ω. Consider the solution $\psi_i(x)$ of the problem

$$
\begin{cases}
-\Delta\psi_i = \frac{2\delta_i}{|x-q_i|^{\alpha_i+3}} & \text{in } R\delta_i^{\frac{1}{\alpha_i+1}} < |x-q_i| < M \\
\psi_i(x) = 0 & \text{on } |x-q_i| = R\delta_i^{\frac{1}{\alpha_i+1}} \text{ and } |x-q_i| = M.
\end{cases}
$$

The function $\psi_i(x)$ is the positive function given by

$$
\psi_i(x) = (\alpha_i+1)^{-2}\Big(-\frac{2\delta_i}{|x-q_i|^{\alpha_i+1}} + A_i + B_i \log|x-q_i|\Big),
$$

where

$$
B_i = 2\Big(\frac{\delta_i}{M^{\alpha_i+1}} - \frac{1}{R^{\alpha_i+1}}\Big)\frac{1}{\log\big(\frac{M}{R\delta_i^{\frac{1}{\alpha_i+1}}}\big)} < 0
$$

and

$$A_i = \frac{2\delta_i}{M^{\alpha_i+1}} - B_i \log M.$$

Hence $\psi_i(x)$ is uniformly bounded from above by a constant independent of p, since for $R\delta_i^{\frac{1}{\alpha_i+1}} \le |x-q_i| \le M$,

$$\psi_i(x) \le (\alpha_i+1)^{-2}\Big(A_i + B_i \log\Big(R\delta_i^{\frac{1}{\alpha_i+1}}\Big)\Big) = (\alpha_i+1)^{-2}\left(\frac{2\delta_i}{M^{\alpha_i+1}} - B_i \log \frac{M}{R\delta_i^{\frac{1}{\alpha_i+1}}}\right)$$
$$= \frac{2}{R^{\alpha_i+1}}(\alpha_i+1)^{-2} \le \frac{2}{R}.$$

Now let

$$\tilde\phi(x) = 3\|\phi\|_i Z(x) + \|h\|_* \sum_{i=1}^{m+k} \psi_i(x),$$

where Z was defined in the previous step. Observe that, by the definition of Z,

$$\tilde\phi(x) \ge 3\|\phi\|_i Z(x) \ge \|\phi\|_i \ge |\phi|(x) \text{ for } |x-q_i| = R\delta_i^{\frac{1}{\alpha_i+1}}, \quad i = 1,\dots,m+k;$$

and, by the positivity of $Z(x)$ and $\psi_i(x)$,

$$\tilde\phi(x) \ge 0 = |\phi|(x) \quad \text{for } x \in \partial\Omega.$$

By the definition of $\|\cdot\|_*$,

$$\left(\sum_{i=1}^{m+k} \frac{\delta_i |x-q_i|^{2\alpha_i}}{(\delta_i^2 + |x-q_i|^{2(\alpha_i+1)})^{\frac{3}{2}}}\right)\|h\|_* \ge |h(x)|, \tag{C.4}$$

so we obtain

$$\begin{aligned}
L\tilde\phi &\le \|h\|_* \sum_{i=1}^{m+k} L\psi_i(x) = \|h\|_* \sum_{i=1}^{m+k}\left(-\frac{2\delta_i}{|x-q_i|^{\alpha_i+3}} + a(x)W(x)\psi_i(x)\right)\\
&\le \|h\|_* \sum_{i=1}^{m+k} |x-q_i|^{2\alpha_i}\left(-\frac{2\delta_i}{|x-q_i|^{3(\alpha_i+3)}} + \frac{2(m+k)D_0}{R} e^{U_{\delta_i,q_i}(x)}\right)\\
&\le -\|h\|_* \left(\sum_{i=1}^{m+k} \frac{\delta_i |x-q_i|^{2\alpha_i}}{(\delta_i^2 + |x-q_i|^{2(\alpha_i+1)})^{\frac{3}{2}}}\right)\\
&\le -|h(x)|\\
&\le -|L\phi|(x),
\end{aligned}$$

provided $R \ge 16(m+k)D_0(\alpha_i+1)^2$ for any $i = 1,\dots,m+k$ and p large enough. Hence, by the maximum principle in Step 1, we obtain

$$|\phi|(x) \le \tilde\phi(x) \quad \text{for } x \in \tilde\Omega;$$

and therefore, since $Z(x) \le m + k$ and $\psi_i(x) \le \frac{2}{R}$,

$$\|\phi\|_\infty \le C[\|\phi\|_i + \|h\|_*].$$

3rd Step. We prove uniform a priori estimates for solutions ϕ of the problem $L\phi = h$ in Ω, $\phi = 0$ on $\partial\Omega$, where $h \in C^{0,\alpha}(\bar\Omega)$ and ϕ satisfies (C.3) and in addition the orthogonality conditions

$$\int_\Omega |x - q_i|^{2\alpha_i} e^{U_{d e_i, q_i}} Z_{i0}\phi = 0, \quad \text{for } i = 1, \dots, m + k. \tag{C.5}$$

Namely, we prove that there exists a positive constant C such that for any $\xi \in \mathcal{O}_\varepsilon$ and $h \in C^{0,\alpha}(\bar\Omega)$,

$$\|\phi\|_\infty \le C\|h\|_*$$

for p sufficiently large. By contradiction, assume the existence of sequences $p_n \to \infty$, points $\xi^n \in \mathcal{O}_\varepsilon$, functions h_n and associated solutions ϕ_n such that $\|h_n\|_* \to 0$ and $\|\phi_n\|_\infty = 1$.

Since $\|\phi_n\|_\infty = 1$, Step 2 shows that $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$. Set $\hat\phi_i^n(y) = \phi_n\Big((\delta_i^n)^{\frac{1}{\alpha_i+1}} y + q_i^n\Big)$ for $i = 1, \dots, m + k$, where $q_i^n = q_i$ for $i = 1, \dots, m$ and $q_i^n = \xi_{i-m}^n$ for $i = m + 1, \dots, m + k$. By Lemma C.1 and (C.4), elliptic estimates readily imply that $\hat\phi_i^n$ converges uniformly over compact sets to a bounded solution $\hat\phi_i^\infty$ of the equation in $\mathbb{R}^2$

$$\Delta\phi + \frac{8(\alpha_i + 1)^2 |y|^{2\alpha_i}}{(1 + |y|^{2(\alpha_i+1)})^2}\phi = 0.$$

This implies that $\hat\phi_i^\infty$ is proportional to z_0^i if $i = 1, \dots, m$ and is a linear combination of the functions z_0^i and z_j, $j = 1, 2$, if $i = m + 1, \dots, m + k$. Since $\|\hat\phi_i^n\|_\infty \le 1$, the orthogonality conditions (C.3) and (C.5) on ϕ_n pass to the limit by Lebesgue's theorem and give rise to

$\int_{\mathbb{R}^2} \frac{8(\alpha_i+1)^2|y|^{2\alpha_i}}{(1+|y|^{2(\alpha_i+1)})^2} z_0^i(y)\hat\phi_i^\infty = 0$ for any $i = 1, \dots, m + k$;

$\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_j(y)\hat\phi_i^\infty = 0$ for any $j = 1, 2$ and $i = m + 1, \dots, m + k$.

Hence, $\hat\phi_i^\infty \equiv 0$ for any $i = 1, \dots, m + k$, contradicting $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$.

4th Step. We prove that there exists a constant $C > 0$ such that any solution ϕ of equation $L\phi = h$ in Ω, $\phi = 0$ on $\partial\Omega$, satisfies

$$\|\phi\|_\infty \le Cp\|h\|_*,$$

when $h \in C^{0,\alpha}(\bar\Omega)$ and we assume on ϕ only the orthogonality conditions (C.3). Proceeding by contradiction as in Step 3, we can suppose further that

$$p_n\|h_n\|_* \to 0 \quad \text{as } n \to +\infty \tag{C.6}$$

but we lose the condition $\int_{\mathbb{R}^2} \frac{8(\alpha_i+1)^2|y|^{2\alpha_i}}{(1+|y|^{2(\alpha_i+1)})^2} z_0^i(y)\hat{\phi}_i^\infty = 0$ in the limit. Hence, we have

$$\hat{\phi}_i^n \to C_i \frac{|y|^{2(\alpha_i+1)} - 1}{|y|^{2(\alpha_i+1)} + 1} \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^2) \tag{C.7}$$

for some constants C_i. To reach a contradiction, we have to show that $C_i = 0$ for any $i = 1, \dots, m+k$. We obtain this from the stronger condition (C.6) on h_n.

To this end, we perform the following construction. By Lemma A.1, we find radial solutions w_i and t_i of the equations $\Delta w_i + |y|^{2\alpha_i} e^{U^i} w_i = |y|^{2\alpha_i} e^{U^i} z_0^i$ and $\Delta t_i + |y|^{2\alpha_i} e^{U^i} t_i = |y|^{2\alpha_i} e^{U^i}$ in $\mathbb{R}^2$, such that as $|y| \to +\infty$,

$$w_i(y) = \frac{4}{3}(\alpha_i+1)\log|y| + O\Big(\frac{1}{|y|^{\alpha_i+1}}\Big) \ , \quad t_i(y) = O\Big(\frac{1}{|y|^{\alpha_i+1}}\Big),$$

since $\int_0^{+\infty} t^{2\alpha_i+1} \frac{(t^{2(\alpha_i+1)}-1)^2}{(t^{2(\alpha_i+1)}+1)^4} dt = \frac{1}{6(\alpha_i+1)}$ and $\int_0^{+\infty} t^{2\alpha_i+1} \frac{t^{2(\alpha_i+1)}-1}{(t^{2(\alpha_i+1)}+1)^3} dt = 0$.

For simplicity, from now on we omit the dependence on n. For $i = 1, \dots, m+k$, define

$$\begin{aligned} u_i(x) =& w_i\Big(\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)\Big) + \frac{4}{3}(\log\delta_i) Z_{i0}(x) \\ &+ \frac{8\pi}{3}(\alpha_i+1)H(q_i,q_i)t_i\Big(\delta_i^{-\frac{1}{\alpha_i+1}}(x-q_i)\Big) \end{aligned}$$

and denote by Pu_i the projection of u_i onto $H_0^1(\Omega)$. Since

$$u_i - Pu_i - \frac{4}{3}(\alpha_i+1)\log|\cdot - q_i| = O(\delta_i)$$

on $\partial\Omega$ (together with boundary derivatives), by harmonicity we get

$$Pu_i = u_i - \frac{8\pi}{3}(\alpha_i+1)H(\cdot,q_i) + O(e^{-\frac{p}{4}}) \text{ in } C^1(\bar{\Omega}),$$

$$Pu_i = -\frac{8\pi}{3}(\alpha_i+1)G(\cdot,q_i) + O(e^{-\frac{p}{4}}) \text{ in } C^1_{\text{loc}}(\bar{\Omega}\setminus\{q_i\}). \tag{C.8}$$

The function Pu_i solves

$$\begin{aligned} \Delta Pu_i + a(x)W_\xi(x)Pu_i =& |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} Z_{i0} \\ &+ \big(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}\big) Pu_i + R_i, \end{aligned} \tag{C.9}$$

where

$$R_i(x) = \left(Pu_i - u_i + \frac{8\pi}{3}(\alpha_i+1)H(q_i,q_i)\right)|x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}.$$

Multiply (C.9) by ϕ and integrate by parts to obtain

$$\begin{aligned} \int_\Omega |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} Z_{i0}\phi + \int_\Omega \big(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}\big) Pu_i\phi \\ = \int_\Omega Pu_i h - \int_\Omega R_i\phi. \end{aligned} \tag{C.10}$$

First of all, by Lebesgue's theorem and (C.7), we get

$$\int_\Omega |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} Z_{i0}\phi \to C_i \int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} (z_0^i)^2 = \frac{8\pi}{3}(\alpha_i+1)C_i. \tag{C.11}$$

The more delicate term is $\int_\Omega \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}\right) Pu_i\phi$. By Lemma C.1 and (C.8), we have

$$\begin{aligned}
&\int_\Omega \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}\right) Pu_i\phi \\
&\quad = \int_{B\left(q_i,\varepsilon\delta_i^{\frac{1}{2(\alpha_i+1)}}\right)} \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}\right) Pu_i\phi \\
&\qquad - \frac{8\pi}{3}(\alpha_i+1)\sum_{j\neq i} G(q_j,q_i) \int_{B\left(q_j,\varepsilon\delta_j^{\frac{1}{2(\alpha_j+1)}}\right)} a(x)W_\xi(x)\phi + O\Big(\frac{1}{p}\Big) \\
&\quad = \frac{4}{3}\frac{\log\delta_i}{p} \int_{B\left(0,\varepsilon\delta_i^{-\frac{1}{2(\alpha_i+1)}}\right)} |y|^{2\alpha_i} e^{U^i} \left(V^i - U^i - \frac{1}{2}(U^i)^2\right) z_0^i(y)\hat{\phi}_i \\
&\qquad - \frac{8\pi}{3}(\alpha_i+1)\sum_{j\neq i} G(q_j,q_i) \int_{B\left(0,\varepsilon\delta_j^{-\frac{1}{2(\alpha_j+1)}}\right)} |y|^{2\alpha_j} e^{U^j}\hat{\phi}_j + O\Big(\frac{1}{p}\Big) \\
&\quad = -\frac{C_i}{3}\int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} (z_0^i)^2 \left(V^i - U^i - \frac{1}{2}(U^i)^2\right)(y) + o(1)
\end{aligned}$$

since Lebesgue's theorem and (C.7) imply

$$\begin{aligned}
&\int_{B\left(0,\varepsilon\delta_i^{-\frac{1}{2(\alpha_i+1)}}\right)} |y|^{2\alpha_i} e^{U^i} \left(V^i - U^i - \frac{1}{2}(U^i)^2\right) z_0^i(y)\hat{\phi}_i \to \\
&\qquad\qquad C_i \int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} (z_0^i)^2 \left(V^i - U^i - \frac{1}{2}(U^i)^2\right)
\end{aligned}$$

and

$$\int_{B\left(0,\varepsilon\delta_j^{-\frac{1}{2(\alpha_j+1)}}\right)} |y|^{2\alpha_j} e^{U^j}\hat{\phi}_j \to C_j \int_{\mathbb{R}^2} |y|^{2\alpha_j} e^{U^j} z_0^j = 0.$$

In a straightforward but tedious way, by (A.4) we can compute

$$\int_{\mathbb{R}^2} |y|^{2\alpha_i} e^{U^i} (z_0^i)^2 \left(V^i - U^i - \frac{1}{2}(U^i)^2\right)(y) = -8\pi(\alpha_i+1),$$

so that we obtain

$$\int_\Omega \left(a(x)W_\xi(x) - |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}}\right) Pu_i\phi = \frac{8\pi}{3}(\alpha_i+1)C_i + o(1). \tag{C.12}$$

As for the R.H.S. in (C.10), we have by (C.8),

$$\begin{aligned}
\Big|\int_\Omega Pu_i h\Big| &= O\left(\|h\|_* \int_\Omega \Big(\sum_{j=1}^{m+k} \frac{\delta_j |x-q_j|^{2\alpha_j}}{(\delta_j^2 + |x-q_j|^{2(\alpha_j+1)})^{\frac{3}{2}}}\Big)|u_i|\right) + O(\|h\|_*) \\
&= O(p\|h\|_*)
\end{aligned} \tag{C.13}$$

since $|u_i| = O(|\log \delta_i|) = O(p)$ in Ω and

$$\int_{B(q_j,\varepsilon)} \frac{\delta_j |x-q_j|^{2\alpha_j}}{(\delta_j^2 + |x-q_j|^{2(\alpha_j+1)})^{\frac{3}{2}}} |u_i| \le Cp \int_{\mathbb{R}^2} \frac{|y|^{2\alpha_j}}{(1+|y|^{2(\alpha_j+1)})^{\frac{3}{2}}} = O(p).$$

Finally, by (C.8),

$$\text{(C.14)} \qquad \int_\Omega R_i \phi = O\left(\int_\Omega |x-q_i|^{2\alpha_i} e^{U_{\delta_i,q_i}} (|x-q_i| + e^{-\frac{p}{4}}) \right) = O(e^{-\frac{p}{4(\alpha_i+1)}}).$$

Hence, inserting (C.11)–(C.14) in (C.10), we obtain

$$\frac{16\pi}{3}(\alpha_i + 1) C_i = o(1)$$

for any $i = 1, \dots, m+k$. Thus $C_i = 0$, and the claim is proved.

5th Step. We establish the validity of the a priori estimate

$$\text{(C.15)} \qquad \|\phi\|_\infty \le Cp\|h\|_*$$

for solutions of problem (C.1)–(C.3) and $h \in C^{0,\alpha}(\bar\Omega)$. The previous step gives

$$\|\phi\|_\infty \le Cp\Big(\|h\|_* + \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} |c_{ij}| \Big)$$

since

$$\|e^{U_{\delta_i,q_i}} Z_{ij}\|_* \le 2\|e^{U_{\delta_i,q_i}}\|_* \le 16\,, \quad j = 1,2,\ i = m+1, \dots, m+k.$$

Hence, arguing by contradiction of (C.15), we can proceed as in Step 3 and suppose further that

$$p_n \|h_n\|_* \to 0\,,\ \ p_n \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} |c_{ij}^n| \ge \delta > 0 \quad \text{as } n \to +\infty.$$

We omit the dependence on n. It suffices to estimate the values of the constants c_{ij}. For $j = 1, 2$ and $i = m+1, \dots, m+k$, multiply (C.1) by PZ_{ij} and, integrating by parts, get

$$\text{(C.16)} \quad \sum_{h=1}^{2} \sum_{l=m+1}^{m+k} c_{lh} \big(PZ_{lh}, PZ_{ij}\big)_{\mathrm{H}_0^1} + \int_\Omega h PZ_{ij} = \int_\Omega a(x) W_\xi(x) \phi PZ_{ij} - \int_\Omega e^{U_{\delta_i,q_i}} Z_{ij} \phi,$$

since $\Delta PZ_{ij} = \Delta Z_{ij} = -e^{U_{\delta_i,q_i}} Z_{ij}$.

We now quote some well-known facts; see for example [13]. For $j = 1, 2$ and $i = m+1, \dots, m+k$, we have the expansions

$$PZ_{ij} = Z_{ij} - 8\pi\delta_i \frac{\partial H}{\partial (q_i)_j}(\cdot, q_i) + O\left(\delta_i^3\right)$$

$$PZ_{i0} = Z_{i0} - 1 + O\left(\delta_i^2\right) \tag{C.17}$$

in $C^1(\bar{\Omega})$ and

$$PZ_{ij} = -8\pi\delta_i \frac{\partial G}{\partial (q_i)_j}(\cdot, q_i) + O\left(\delta_i^3\right)$$

$$PZ_{i0} = O\left(\delta_i^2\right) \tag{C.18}$$

in $C^1_{\text{loc}}(\bar{\Omega} \setminus \{q_i\})$. By (C.17)–(C.18), we deduce the following "orthogonality" relations: for $j, h = 1, 2$ and $i, l = m+1, \dots, m+k$ with $i \neq l$,

$$\Big(PZ_{ij}, PZ_{ih}\Big)_{\mathrm{H}^1_0(\Omega)} = \left(64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}\right)\delta_{jh} + O(\delta_i^2)$$

$$\Big(PZ_{ij}, PZ_{lh}\Big)_{\mathrm{H}^1_0(\Omega)} = O(\delta_i\delta_l) \tag{C.19}$$

and

$$\Big(PZ_{i0}, PZ_{ij}\Big)_{\mathrm{H}^1_0(\Omega)} = O(\delta_i^2)$$

$$\Big(PZ_{i0}, PZ_{lh}\Big)_{\mathrm{H}^1_0(\Omega)} = O(\delta_i\delta_l) \tag{C.20}$$

uniformly on $\xi \in \mathcal{O}_\varepsilon$, where δ_{jh} denotes Kronecker's symbol.

Now, since

$$\Big|\int_\Omega hPZ_{ij}\Big| \le C' \int_\Omega |h| \le C\|h\|_*,$$

by (C.19) the L.H.S. of (C.16) can be estimated as

$$\text{L.H.S.} = Dc_{ij} + O\Big(e^{-\frac{p}{2}} \sum_{h=1}^{2} \sum_{l=m+1}^{m+k} |c_{lh}|\Big) + O(\|h\|_*), \tag{C.21}$$

where $D = 64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}$. Moreover, by Lemma C.1, the R.H.S. of (C.16) takes the form

$$\begin{aligned} \text{R.H.S.} &= \int_{B(q_i, \varepsilon\sqrt{\delta_i})} a(x)W_\xi(x)\phi PZ_{ij} - \int_\Omega e^{U_{\delta_i,q_i}}\phi Z_{ij} + O(e^{-\frac{p}{4}}\|\phi\|_\infty) \\ &= \int_{B(q_i, \varepsilon\sqrt{\delta_i})} \left(a(x)W_\xi(x) - e^{U_{\delta_i,q_i}}\right)\phi PZ_{ij} + \int_\Omega e^{U_{\delta_i,q_i}}\phi\left(PZ_{ij} - Z_{ij}\right) \\ &\quad + O(e^{-\frac{p}{4}}\|\phi\|_\infty) \\ &= \frac{1}{p}\int_{B\left(0, \frac{\varepsilon}{\sqrt{\delta_i}}\right)} \frac{32y_j}{(1+|y|^2)^3}\left(V^i - U^i - \frac{1}{2}(U^i)^2\right)\hat{\phi}_i + O(\frac{1}{p^2}\|\phi\|_\infty) \end{aligned} \tag{C.22}$$

in view of (C.17), where $\hat\phi_i(y) = \phi(\delta_i y + q_i)$. Inserting the estimates (C.21) and (C.22) into (C.16), we deduce that

$$Dc_{ij} + O\Big(e^{-\frac{p}{2}} \sum_{h=1}^{2} \sum_{l=m+1}^{m+k} |c_{lh}|\Big) = O\Big(\|h\|_* + \frac{1}{p}\|\phi\|_\infty\Big).$$

Hence, we obtain

$$\sum_{h=1}^{2} \sum_{l=m+1}^{m+k} |c_{lh}| = O\Big(\|h\|_* + \frac{1}{p}\|\phi\|_\infty\Big). \tag{C.23}$$

Since $\sum_{h=1}^{2} \sum_{l=m+1}^{m+k} |c_{lh}| = o(1)$, as in Step 4 we have

$$\hat\phi_i \to C_i \frac{|y|^2-1}{|y|^2+1} \text{ in } C^0_{\text{loc}}(\mathbb{R}^2)$$

for some constant C_i, $i = m+1, \dots, m+k$. Hence, in (C.22), we have a better estimate, since by Lebesgue's theorem, the term

$$\int_{B\left(0,\frac{\varepsilon}{\sqrt{\delta_i}}\right)} \frac{32 y_j}{(1+|y|^2)^3} \left(V^i - U^i - \frac{1}{2}(U^i)^2\right)(y)\hat\phi_i(y)$$

converges to

$$C_i \int_{\mathbb{R}^2} \frac{32 y_j (|y|^2-1)}{(1+|y|^2)^4} \left(V^i - U^i - \frac{1}{2}(U^i)^2\right)(y) = 0.$$

Therefore, the R.H.S. in (C.16) satisfies R.H.S. $= o(\frac{1}{p})$ and, in turn,

$$\sum_{h=1}^{2} \sum_{l=m+1}^{m+k} |c_{lh}| = O(\|h\|_*) + o(\frac{1}{p}).$$

This contradicts

$$p \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} |c_{ij}| \geq \delta > 0,$$

and the claim is established.

6th Step. We prove the solvability of (C.1)–(C.3). To this purpose, we consider the spaces

$$K_\xi = \Big\{ \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} c_{ij} P Z_{ij} : c_{ij} \in \mathbb{R} \text{ for } j = 1, 2,\ i = m+1, \dots, m+k \Big\}$$

and

$$K_\xi^\perp = \Big\{ \phi \in L^2(\Omega) : \int_\Omega e^{U_{\delta_i, q_i}} Z_{ij}\, \phi = 0 \text{ for } j = 1, 2, i = m+1, \dots, m+k \Big\}.$$

Define $\Pi_\xi : L^2(\Omega) \to K_\xi$ by

$$\Pi_\xi \phi = \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} c_{ij} P Z_{ij},$$

where c_{ij} are uniquely determined (as follows by (C.19)) by the system

$$\int_\Omega e^{U_{\delta_l, q_l}} Z_{lh} \left(\phi - \sum_{j=1}^{2} \sum_{i=m+1}^{m+k} c_{ij} P Z_{ij} \right) = 0 \quad \text{for any } h = 1, 2,\ l = m+1, \ldots, m+k.$$

Let $\Pi_\xi^\perp = \mathrm{Id} - \Pi_\xi : L^2(\Omega) \to K_\xi^\perp$. Problem (C.1)–(C.3), expressed in a weak form, is equivalent to find $\phi \in K_\xi^\perp \cap H_0^1(\Omega)$ such that

$$(\phi, \psi)_{\mathrm{H}_0^1(\Omega)} = \int_\Omega (a(x) W_\xi \phi - h)\ \psi\ dx, \qquad \text{for all } \psi \in K_\xi^\perp \cap H_0^1(\Omega).$$

With the aid of Riesz's representation theorem, this equation gets rewritten in $K_\xi^\perp \cap H_0^1(\Omega)$ in the operatorial form

$$(\mathrm{Id} - K)\phi = \tilde{h}, \tag{C.24}$$

where $\tilde{h} = \Pi_\xi^\perp \Delta^{-1} h$ and $K(\phi) = -\Pi_\xi^\perp \Delta^{-1} (a(x) W_\xi \phi)$ is a compact linear operator in $K_\xi^\perp \cap H_0^1(\Omega)$. The homogeneous equation $\phi = K(\phi)$ in $K_\xi^\perp \cap H_0^1(\Omega)$, which is equivalent to (C.1)–(C.3) with $h \equiv 0$, has only the trivial solution in view of the a priori estimate (C.15). Now, Fredholm's alternative guarantees unique solvability of (C.24) for any $\tilde{h} \in K_\xi^\perp$. Moreover, by elliptic regularity theory, this solution is in $W^{2,2}(\Omega)$.

At $p > p_0$ fixed, by the density of $C^{0,\alpha}(\bar{\Omega})$ in $(C(\bar{\Omega}), \|\cdot\|_\infty)$, we can approximate $h \in C(\bar{\Omega})$ by Hölderian functions; and, by (C.15) and elliptic regularity theory, we can show that the estimate $\|\phi\|_\infty \leq C\|h\|_*$ holds for any $h \in C(\bar{\Omega})$. The proof is complete. □

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