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Asymptotic Behaviour of a Thin Insulation Problem

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We consider a problem of thermal insulation and we study, by means of Γ-convergence, the best way to put the insulator when its mass tends to zero.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a $C^{1,1}$ bounded open set. For any point $\sigma \in \partial\Omega$, we denote by $n(\sigma)$ the unit outer normal to $\partial\Omega$. We cover externally the boundary of the region Ω by a thin layer Σ_ε of material of small conductivity $\delta > 0$ as follows. For any $h \in C^0(\partial\Omega, [0, +\infty[)$, we define

$$\Sigma_\varepsilon = \Sigma_\varepsilon(h) := \{x \in \mathbb{R}^n : x = \sigma + th(\sigma)n(\sigma),\ \sigma \in \partial\Omega,\ t \in (0, \varepsilon)\},$$

where $\varepsilon \in \mathbb{R}^+$ is small enough in a way to obtain $\overline{\Sigma}_\varepsilon \cap \overline{\Omega} = \partial\Omega$. Set

$$\Omega_\varepsilon := \overline{\Omega} \cup \Sigma_\varepsilon$$

and let $a_{\varepsilon,\delta} : \Omega_\varepsilon \to \mathbb{R}$ be the function representing the conductivity of Ω_ε, defined as

$$a_{\varepsilon,\delta}(x) := \begin{cases} 1 & \text{if } x \in \overline{\Omega} \\ \delta & \text{if } x \in \Sigma_\varepsilon . \end{cases}$$

Denoting by $u(x)$ the temperature at the point x (which is supposed to be identically equal to 0 out of Ω_ε), then $u(x)$ satisfies the nonhomogeneous equation

$$\begin{cases} u \in H^1_0(\Omega_\varepsilon) \\ -\operatorname{div}(a_{\varepsilon,\delta}\nabla u) = f \quad \text{in } \mathcal{D}'(\Omega_\varepsilon) \end{cases} \tag{1}$$

where $f \in L^p(\Omega)$, $p > \frac{2n}{n+2}$, represents a given heat source (f is extended to 0 out of Ω). Equivalently, u is a critical point of the functional $F_{\varepsilon,\delta} : L^1(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ defined by:

$$F_{\varepsilon,\delta}(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 + \dfrac{\delta}{2}\int_{\Sigma_\varepsilon} |\nabla u|^2 - \int_\Omega f\ u & \text{if } u \in H^1_0(\Omega_\varepsilon) \\ +\infty & \text{elsewhere.} \end{cases}$$

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Let us recall what it is known on the asymptotic behaviour of problem (1) as $(\varepsilon, \delta) \to (0,0)$. It is intuitively clear that, if $\varepsilon << \delta$, we may neglect the insulator and thus we expect to obtain in the limit the usual Dirichlet problem. If instead $\delta << \varepsilon$, we expect to have not heat transmission through $\partial\Omega$ and thus to obtain the Neumann problem.

The most interesting case is when $\varepsilon \approx \delta$, that is $\varepsilon = k\delta$. Up to a renormalization we can assume $k = 1$ and set $F_\varepsilon := F_{\varepsilon,\varepsilon}$. Results due to Acerbi and Buttazzo ([1],[7]) imply the following theorem (see also [3], [4], [6] and [8]), which is the starting point of our work.

Theorem 1.1. *For any $h \in C^0(\partial\Omega; [0,+\infty[)$, the sequence $\{F_\varepsilon\}$ Γ-converges with respect to the $L^1(\mathbb{R}^n)$-topology to the functional $F(h,\cdot) : L^1(\mathbb{R}^n) \to [0,+\infty]$ defined by:*

$$F(h,u) := \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_{\partial\Omega} \frac{u^2}{h} - \int_\Omega f\ u & \text{if } u \in H^1(\Omega) \\ +\infty & \text{elsewhere.} \end{cases} \tag{2}$$

Let $m > 0$; define Ξ_m as the set of all measurable functions $h : \partial\Omega \to [0,+\infty[$ such that $\int_{\partial\Omega} h = m$. In the sequel of the paper, the functional $F(h,u)$ will have the same expression also for functions $h \in \Xi_m$ which are not continuous in general.

Let $(\overline{h}_m, u_m)$ denote the pair which minimizes F in $\Xi_m \times H^1(\Omega)$ and set $h_m := \frac{\overline{h}_m}{m}$. The purpose of the present paper is to study, by means of $\Gamma-$convergence, the asympotic behaviour of h_m as m goes to 0.

Let us suppose from now on $p > n$. For fixed $h \in \Xi_m$, $F(h,u)$ is strictly convex in u and therefore it admits a unique minimizer $u^h \in H^1(\Omega)$, which satisfies the following equation:

$$\begin{cases} -\triangle u^h(x) = f(x) & x \in \Omega \\ h\frac{\partial u^h}{\partial n}(x) + u^h(x) = 0 & x \in \partial\Omega\,. \end{cases} \tag{3}$$

The aim now is to maximize the weighted average of the temperature $\int_\Omega f\ u$. If $f \geq 0$, this means to disperse heat as less as possible. This can be seen as a problem of optimal control (where h represents the control variable). A direct computation shows that

$$F(h,u^h) = -\frac{1}{2}\int_\Omega f\ u^h$$

and so we are led to consider the following problem:

$$\min_{h\in\Xi_m} \min_{u\in H^1(\Omega)} F(h,u)\,. \tag{4}$$

It can be seen (see Section 3) that u_m is the unique solution of the problem

$$\min_{u\in H^1(\Omega)} \left\{ \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2m}\left(\int_{\partial\Omega} |u|\right)^2 - \int_\Omega fu \right\}$$

and

$$h_m = \frac{|u_m|}{\int_{\partial\Omega} |u_m|}\,.$$

Before stating our main result, we first recall that, from classical elliptic theory and Sobolev embeddings (see for instance [5]), there exists an unique solution $\overline{u} \in C^{1,1-\frac{n}{p}}(\overline{\Omega}) \cap W^{2,p}(\Omega)$ of the following equation:

$$\begin{cases} -\triangle u(x) = f(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega\,. \end{cases} \tag{5}$$

We remark that the function $\overline{u}$ can be regarded as the temperature in case of no insulation. Let us denote

$$M = \max_{\partial\Omega} \left|\frac{\partial \overline{u}}{\partial n}\right|, \quad K^{\pm} := \{x \in \partial\Omega : \frac{\partial \overline{u}}{\partial n}(x) = \pm M\} \tag{6}$$

and define

$$v_m := \frac{u_m - \overline{u}}{m}\,.$$

Our main result is stated in the following

Theorem 1.2. *Up to a subsequence,* $\left\{\frac{v_m}{\int_{\partial\Omega}|v_m|}\right\}$ *weakly* converges in* $\mathcal{M}(\partial\Omega)$ *(see Definition 2.1) as* m *goes to* 0 *to a measure* λ *which satisfies*

$$spt\,\lambda \subseteq K^+ \cup K^- \tag{7}$$

$$\lambda \text{ is positive (resp. negative) in } K^- \text{ (resp. } K^+) \tag{8}$$

$$\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} d\lambda = -M. \tag{9}$$

Moreover

$$\lim_{m\to 0} \int_{\partial\Omega} |v_m| = |\lambda|(\partial\Omega) = M,$$

where $|\lambda|$ *denotes the total variation of* λ*.*

Theorem 1.2 and the fact that $h_m = \frac{|v_m|}{\int_{\partial\Omega}|v_m|}$ (see (11) below) imply the following result.

Theorem 1.3. *The sequence* $\{h_m\}$ *weakly* converges in* $\mathcal{M}(\partial\Omega)$ *as* m *goes to* 0 *to the total variation of* λ*.*

Theorem 1.3 states that $\{h_m\}$ converges to a measure concentrated on $K^+ \cup K^-$; physically this means that, when m is infinitesimal, the insulator has to be put in the points where the dispersion is maximal.

2. Notation and preliminaries

We first introduce some notation and briefly recall some basic results about Γ-convergence and Measure Theory.

Let Y be a metric space locally compact and separable. We denote by $\mathcal{M}(Y)$ the space of all finite and real Radon measures on Y and for any $\mu \in \mathcal{M}(Y)$ we denote by $|\mu|$ the total variation of μ.

Definition 2.1 (Weakly* convergence). Let $\mu \in \mathcal{M}(Y)$ and let $\{\mu_n\} \subset \mathcal{M}(Y)$. We say that $\{\mu_n\}$ weakly* converges to μ if

$$\lim_{n\to+\infty} \int_Y u\, d\mu_n = \int_Y u\, d\mu \qquad \text{for every } u \in C_0(Y).$$

Theorem 2.2. *Let $\{\mu_n\} \subset \mathcal{M}(Y)$ be a sequence such that $\sup_n |\mu_n|(Y) < +\infty$. Then $\{\mu_n\}$ has a weakly* converging subsequence. Moreover the map $\mu \mapsto |\mu|(Y)$ is lower semicontinuous with respect to the weak* convergence.*

Remark 2.3. The topology induced by the weak* convergence defined above is metrizable on the bounded sets of $\mathcal{M}(Y)$.

For more details about this topic see for instance [2].

Definition 2.4. Let X be a topological space and let $F_n, F : X \to \mathbb{R} \cup \{+\infty\}$. Define the Γ- lim inf and the Γ- lim sup of F_n as:

$$\Gamma\text{-}\liminf F_n(x) := \inf\left\{\liminf_{n\to\infty} F_n(x_n) : x_n \to x\right\}$$
$$\Gamma\text{-}\limsup F_n(x) := \inf\left\{\limsup_{n\to\infty} F_n(x_n) : x_n \to x\right\}.$$

We say that F_n Γ-converges to F if for all $x \in X$ we have:

$$\Gamma\text{-}\liminf_{n\to\infty} F_n(x) = \Gamma\text{-}\limsup_{n\to\infty} F_n(x) = F(x).$$

Definition 2.5. A sequence of functionals $F_n : X \to \mathbb{R} \cup \{+\infty\}$ is said to be sequentially equicoercive if, for any sequence $\{x_n\}$ such that $\sup_n F_n(x_n) < +\infty$, there exists a convergent subsequence.

Theorem 2.6. *Let $\{F_n\}$ be a sequence of sequentially equicoercive functionals defined on X and Γ-converging to F. Then there exists $\min_X F$ and $\min_X F = \lim_{n\to\infty} \inf_X F_n$. Moreover, if x_n is a minimizer of F_n, then every limit of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is a minimizer of F.*

For more details about this topic see for instance [9].

In the sequel we identify any $f \in L^1(\Omega)$ (resp. $f \in L^1(\partial\Omega)$) with the measure $f\,dx$ (resp. $f\,d\sigma$).

3. Main results

In order to study $\min_{h\in\Xi_m} \min_{u\in H^1(\Omega)} F(h,u)$, according to [7], we consider the equivalent problem:

$$\min_{u\in H^1(\Omega)} \min_{h\in\Xi_m} F(h,u). \tag{10}$$

Denoting by $\overline{h}_m(u) \in \Xi_m$ the unique solution of the problem

$$\min_{h\in\Xi_m} F(h,u),$$

it is possible to see that

$$\overline{h}_m(u) = m\frac{|u|}{\int_{\partial\Omega}|u|}\ . \tag{11}$$

Thus, setting

$$E_m(u) := F\left(\overline{h}_m(u), u\right) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2m}\left(\int_{\partial\Omega}|u|\right)^2 - \int_\Omega f\ u,$$

problem (10) becomes:

$$\min_{u\in H^1(\Omega)} E_m(u). \tag{12}$$

It is possible to prove (see [7]) that $E_m(u)$ is strictly convex, coercive and lower semicontinuous with respect to the weak topology of $H^1(\Omega)$.
Hence it immediately follows:

Theorem 3.1. *Problem* (12) *admits a unique solution* $u_m \in H^1(\Omega)$*, which satisfies*

$$\begin{cases} -\triangle u(x) = f(x) & x\in\Omega \\ \frac{\partial u}{\partial n}(x) + \frac{1}{m}sign(u(x))\displaystyle\int_{\partial\Omega}|u| \ni 0 & x\in\partial\Omega\,. \end{cases} \tag{13}$$

In (13) the map $sign$ is defined as $sign(t) := \pm 1$ if $t \gtrless 0$ and $sign(0) = [-1, 1]$.

Corollary 3.2. *Let* $\overline{h}_m(u)$ *be defined as in* (11) *and let* u_m *be given by Theorem 3.1. Then the pair* $(\overline{h}_m(u_m), u_m)$ *solves the minimization problem* (10) *or equivalently* (4)*.*

Defining $h_m := \dfrac{\overline{h}_m(u_m)}{m}$ and recalling (11), we have $\displaystyle\int_{\partial\Omega} h_m = 1$ and $h_m \geq 0$. Hence by Theorem 2.2 the sequence $\{h_m\}$ admits a subsequence weakly* converging in $\mathcal{M}(\partial\Omega)$ as $m \to 0$.

In the case of Ω an annular region in $\mathbb{R}^2$ and $f \equiv 1$, solving explicitly equation (13) in polar coordinates, it turns out that the optimal way to insulate Ω is achieved by putting the insulator on the inner circle, the set of boundary points with negative mean curvature, see [7] for details. This could suggest that h_m tends, as $m \to 0$, to a measure concentrated where the mean curvature of $\partial\Omega$ relative to Ω is minimal.
Nevertheless, this is false, as it follows from the following example.

Example 3.3. Let $f \equiv 1$ and $\Omega = \left(B_R(0)\setminus\overline{B_r(0)}\right)\cup B_\rho(p) \subset \mathbb{R}^2$ for $R > r > 0$, $\rho > 0$ and $|p| > R + \rho$. We define u_m as

$$u_m(x) = \begin{cases} -\frac{1}{4}\left(|x|^2 - r^2\right) + \frac{(R^2-r^2)}{4\log\frac{R}{r}}\log\frac{|x|}{r} & \text{if } x\in B_R(0)\setminus\overline{B_r(0)} \\ \frac{m}{4\pi} + \frac{\rho^2 - |x-p|^2}{4} & \text{if } x\in B_\rho(p). \end{cases}$$

It can be seen that $u_m |_{\partial B_r(0)} = u_m |_{\partial B_R(0)} \equiv 0$, $u_m |_{\partial B_\rho(p)} \equiv \frac{m}{4\pi}$ and

$$\frac{\partial u_m}{\partial n}(x) = \begin{cases} \frac{1}{2}r - \frac{R^2-r^2}{4r\log\frac{R}{r}} & \text{if } x \in \partial B_r(0) \\ -\frac{1}{2}R + \frac{R^2-r^2}{4R\log\frac{R}{r}} & \text{if } x \in \partial B_R(0) \\ -\frac{1}{2}\rho & \text{if } x \in \partial B_\rho(p). \end{cases}$$

Hence $\int_{\partial\Omega} |u_m| = \frac{m}{2}\rho$ and

$$0 \in \frac{\partial u_m}{\partial n}(x) + \frac{1}{m} sign(u(x)) \int_{\partial\Omega} |u_m|$$

if and only if $\rho \geq \rho_0 = \rho_0(R,r) = \max\left\{\left|R - \frac{R^2-r^2}{2R\log\frac{R}{r}}\right|, \left|r - \frac{R^2-r^2}{2r\log\frac{R}{r}}\right|\right\}$.
For fixed r, R, $\rho \geq \rho_0$, we can choose a point p so that $|p| > R + \rho$: so u_m turns out to be the unique solution of equation (13) and $h_m = \frac{|u_m|}{\int_{\partial\Omega}|u_m|} \in \Xi_m$ satisfies $\operatorname{spt} h_m = \partial B_\rho(p)$ for any $m > 0$.
The set of boundary points with minimal mean curvature is $B_r(0)$ and it is different from $\operatorname{spt} h_m$. According to Theorems 1.2 and 1.3, it is easy to check that $\partial B_\rho(p) \subseteq K^+ \cup K^-$ and the equality holds if $\rho > \rho_0$.

We observe that, if we write $u \in H^1(\Omega)$ in the form

$$u = \overline{u} + m\ v, \quad \overline{u} \text{ the solution of (5)}, \ v \in H^1(\Omega)$$

and if $I, \widetilde{J}_m : H^1(\Omega) \to [0, +\infty[$ are the functionals defined as

$$\begin{cases} I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f\ u \\ \widetilde{J}_m(v) := \frac{m}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\left(\int_{\partial\Omega}|v|\right)^2 + \int_{\partial\Omega}\frac{\partial\overline{u}}{\partial n}v, \end{cases} \tag{14}$$

then we have:

$$E_m(\overline{u} + m\ v) = I(\overline{u}) + m\tilde{J}_m(v). \tag{15}$$

In particular we can write the function u_m of Theorem 3.1 in the form

$$u_m = \overline{u} + m v_m,$$

where v_m is the unique minimizer of $\tilde{J}_m$ and satisfies:

$$\begin{cases} \triangle v_m(x) = 0 & x \in \Omega \\ sign(v_m(x))\int_{\partial\Omega}|v_m(x)| + m\frac{\partial v_m(x)}{\partial n} + \frac{\partial\overline{u}(x)}{\partial n} \ni 0 & x \in \partial\Omega\,. \end{cases} \tag{16}$$

Since $\overline{u}$ vanishes on $\partial\Omega$ we have

$$h_m = \frac{|u_m|}{\int_{\partial\Omega}|u_m|} = \frac{|v_m|}{\int_{\partial\Omega}|v_m|}\,.$$

We now focus the attention on the asymptotic behaviour of the functions v_m.

For any $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, there exists a unique function $v_\varphi \in H^1(\Omega)$ such that:

$$\int_\Omega |\nabla v_\varphi|^2 = \inf_{\{w \in H^1(\Omega), w|_{\partial\Omega} = \varphi\}} \int_\Omega |\nabla w|^2. \tag{17}$$

So we can define a new functional $J_m : \mathcal{M}(\partial\Omega) \to \mathbb{R}$ as:

$$J_m(\varphi) := \begin{cases} \dfrac{m}{2} \displaystyle\int_\Omega |\nabla v_\varphi|^2 + \frac{1}{2}\left(\int_{\partial\Omega} |\varphi|\right)^2 + \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n}\varphi & \text{if } \varphi \in H^{\frac{1}{2}}(\partial\Omega) \\ +\infty & \text{elsewhere.} \end{cases}$$

Notice that $v_m|_{\partial\Omega}$ is the unique minimizer of J_m. We now give a result of compactness and Γ-convergence for the functionals J_m.

Theorem 3.4 (Compactness for J_m). *Let $\{\varphi_m\} \subset H^{\frac{1}{2}}(\partial\Omega)$ be such that $\sup_m J_m(\varphi_m) < +\infty$. Then there exist $\mu \in \mathcal{M}(\partial\Omega)$ and a subsequence of $\{\varphi_m\}$ weakly* converging to μ in $\mathcal{M}(\partial\Omega)$ as $m \to 0$.*

Proof. Setting $M := \max_{x \in \partial\Omega} \left|\dfrac{\partial \overline{u}}{\partial n}(x)\right|$ and $T := \sup_m J_m(\varphi_m) < +\infty$, we have:

$$M \int_{\partial\Omega} |\varphi_m| - 2M^2 \leq \frac{1}{2}\left(\int_{\partial\Omega} |\varphi_m|\right)^2 - M \int_{\partial\Omega} |\varphi_m| \leq T$$

and hence $\displaystyle\int_{\partial\Omega} |\varphi_m| \leq \frac{T}{M} + 2M$. The thesis now follows from Theorem 2.2. □

Theorem 3.5 (Γ-convergence for J_m). *The sequence $\{J_m\}$ Γ-converges, with respect to the weak* convergence on $\mathcal{M}(\partial\Omega)$, as $m \to 0$, to the functional $J : \mathcal{M}(\partial\Omega) \to \mathbb{R}$ defined as*

$$J(\mu) := \frac{1}{2}\left(|\mu|(\partial\Omega)\right)^2 + \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n}\, d\mu.$$

Proof. The Γ-$\liminf$ inequality follows from the lower semicontinuity of the total variation with respect to the weak* convergence in $\mathcal{M}(\partial\Omega)$.
To prove the Γ-$\limsup$ inequality, it is enough to find, for any $\mu \in \mathcal{M}(\partial\Omega)$, a sequence $w_m \in H^1(\Omega)$ such that $\varphi_m := w_m|_{\partial\Omega}$ weakly* converges to μ in $\mathcal{M}(\partial\Omega)$ and $\limsup_{m \to 0} \widetilde{J}_m(w_m) \leq J(\mu)$, because, in view of (17),

$$J_m(\varphi_m) = \widetilde{J}_m(v_{\varphi_m}) \leq \widetilde{J}_m(w_m) \quad \forall\, m.$$

To this aim we make the following construction. Set $D := \{x \in \mathbb{R}^{n-1} : |x| < 1\}$, let $\{S_i, r_i : D \to S_i\}_{i=1,\dots,N}$ be a local coordinates system and let $\{\psi_i\}_{i=1,\dots,N}$ be a unit partition subordinate to the family $\{S_i\}$. Let $\mu_i \in \mathcal{M}(D)$ be defined as the push-forward of $\psi_i\mu$ through the map r_i^{-1}, $\mu_i := (r_i^{-1})_\#[\psi_i\mu]$.
Since $\operatorname{spt}\mu_i \subset D$, by Theorem 2.2 in [2] there exists a sequence $\{\widetilde{\mu}_i^\varepsilon\} \subset C_0^\infty(D)$ such that:

$$\widetilde{\mu}_i^\varepsilon \text{ weakly* converges to } \mu_i \tag{18}$$

$$|\widetilde{\mu}_i^\varepsilon|(D) \to |\mu_i|(D) \tag{19}$$

$$|\widetilde{\mu}_i^\varepsilon| = O\left(\frac{1}{\varepsilon^{n-1}}\right), \quad |\nabla \widetilde{\mu}_i^\varepsilon| = O\left(\frac{1}{\varepsilon^n}\right). \tag{20}$$

Let $\mu_i^\varepsilon \in \mathcal{M}(\partial\Omega)$ be defined as the push-forward of $\widetilde{\mu}_i^\varepsilon$ through the map r_i and let us identify the measure μ_i^ε with the function $\mu_i^\varepsilon := \widetilde{\mu}_i^\varepsilon \circ r_i^{-1} \in C_0^\infty(S_i)$. Set $\mu_\varepsilon := \sum_{i=1}^N \mu_i^\varepsilon$. By construction we get that, as $\varepsilon \to 0$,

$$\mu_\varepsilon \text{ weakly* converges to } \mu \text{ in } \mathcal{M}(\partial\Omega) \tag{21}$$

and

$$|\mu_\varepsilon|(\partial\Omega) \to |\mu|(\partial\Omega). \tag{22}$$

We now want to extend μ_ε in the interior of Ω. Set $d(x) := \operatorname{dist}(x, \partial\Omega)$ and for $\alpha > 0$ define $U := \{x \in \Omega : d(x) < \alpha\}$ and $\widetilde{U} := \{x \in U : d(x) \in (0, \frac{\alpha}{2})\}$. For α small enough, the orthogonal projection on $\partial\Omega$, $\pi : U \to \partial\Omega$ is well defined and let $\gamma_\varepsilon : [0,\alpha] \times \partial\Omega \to \mathbb{R}$ be the function defined as:

$$\gamma_\varepsilon(t,y) := \begin{cases} \mu_\varepsilon(y) & \text{if } t \in [0, \frac{\alpha}{2}] \\ \frac{2(\alpha-t)}{\alpha}\mu_\varepsilon(y) & \text{if } t \in \left(\frac{\alpha}{2}, \alpha\right]. \end{cases}$$

Setting

$$w_m(x) := \begin{cases} \gamma_{m^{1/(4n)}}(d(x), \pi(x)) & x \in U \\ 0 & x \in \Omega \backslash U, \end{cases}$$

it is easy to check that $w_m \in H^1(\Omega)$. Moreover, using (20), the lipschitz property of the map π and the fact that $|\nabla d(x)| = 1$ a.e., we infer the following estimate:

$$\begin{aligned} |\nabla w_{m^{1/(4n)}}(x)| &\le \left|\frac{\partial \gamma_{m^{1/(4n)}}}{\partial t}(d(x),\pi(x))\right| |\nabla d(x)| \\ &\quad + \left|\frac{\partial \gamma_{m^{1/(4n)}}}{\partial y}(d(x),\pi(x))\right| |\nabla \pi(x)| \\ &\le C\left[\left|\frac{\partial \gamma_{m^{1/(4n)}}}{\partial t}(d(x),\pi(x))\right| + \left|\frac{\partial \gamma_{m^{1/(4n)}}}{\partial y}(d(x),\pi(x))\right|\right] \\ &= \begin{cases} C\,|\nabla_\tau \mu_{m^{1/(4n)}}(\pi(x))| & x \in \widetilde{U} \\ C\left|-\frac{2}{\alpha}\mu_{m^{1/(4n)}}(\pi(x)) + \frac{2(\alpha-d(x))}{\alpha}\nabla_\tau \mu_{m^{1/(4n)}}(\pi(x))\right| & x \in U\backslash\widetilde{U} \end{cases} \\ &= O\left(\frac{1}{m^{1/4}}\right) \end{aligned}$$

for suitable $C > 0$. Therefore we get

$$m\int_\Omega |\nabla w_{m^{1/(4n)}}|^2 \le \widetilde{C}\,\sqrt{m} \to 0 \text{ as } m \to 0$$

for some $\widetilde{C} > 0$.
This conclude the proof of the Γ-limsup inequality. □

In order to characterize the properties of minimizers of J, we write it in a slightly different way. Set $\mathcal{M}_1(\partial\Omega) := \{\mu \in \mathcal{M}(\partial\Omega) : \ |\mu|(\partial\Omega) = 1\}$ and consider $\widetilde{J} : [0, +\infty) \times \mathcal{M}_1(\partial\Omega) \to \mathbb{R}$ defined as:

$$\widetilde{J}(t, \lambda) := J(t\lambda) = \frac{1}{2}t^2 + t \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} d\lambda. \tag{23}$$

We have:

$$\min_{\mu \in \mathcal{M}(\partial\Omega)} J(\mu) = \min_{\lambda \in \mathcal{M}_1(\partial\Omega)} \min_{t \geq 0} \tilde{J}(t, \lambda). \tag{24}$$

An easy computation shows that

$$\min_{t \geq 0} \widetilde{J}(t, \lambda) = \begin{cases} 0 & \text{if } \displaystyle\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} d\lambda \geq 0 \\ -\dfrac{1}{2}\left(\displaystyle\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} d\lambda\right)^2 & \text{if } \displaystyle\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} d\lambda < 0 \end{cases}$$

and hence, denoted by $(\bar{t}, \overline{\lambda})$ a pair which minimizes $\widetilde{J}$, we have that:

$$\bar{t} = -\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} d\overline{\lambda} > 0 \text{ and } \overline{\lambda} \text{ minimizes } \lambda \mapsto \int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} d\lambda. \tag{25}$$

We have that $\displaystyle\int_{\partial\Omega} \frac{\partial \overline{u}}{\partial n} d\lambda \geq -M$ and the equality holds if and only if

$$\operatorname{spt} \overline{\lambda} \subseteq K^+ \cup K^-, \qquad \overline{\lambda} \geq 0 \text{ (resp. } \leq 0) \text{ in } K^- \text{ (resp. } K^+), \qquad \bar{t} = M, \tag{26}$$

where M and $K^\pm$ are defined in (6). Finally by Theorems 2.6, 3.4, 3.5 we can deduce the proof of Theorem 1.2.

Proof of Theorem 1.2. Since $\sup_m J_m(v_m|_{\partial\Omega}) \leq \sup_m J_m(0) = 0 < +\infty$, up to a subsequence, $v_m|_{\partial\Omega}$ weakly* converges in $\mathcal{M}(\partial\Omega)$ to a minimum point μ of J which must have the form $\mu = t\lambda$ with (t, λ) satisfying (25).
Hence, since the total variation is lower semicontinuous with respect to the weak* convergence,

$$|\mu|(\partial\Omega) = M \leq \liminf_{m \to 0} \int_{\partial\Omega} |v_m|.$$

For any converging subsequence of $\int_{\partial\Omega} |v_m|$ with limit s we have $s \geq M$ and, along this subsequence,

$$-\frac{1}{2}M^2 = J(\mu) = \lim_{m \to 0} J_m(v_m|_{\partial\Omega}) \geq \frac{1}{2}s^2 - Ms.$$

Since $\frac{1}{2}t^2 - Mt > -\frac{1}{2}M^2$ for $t > M$, then $s = M$ for any converging subsequence of $\int_{\partial\Omega} |v_m|$ and this implies

$$\lim_{m \to 0} \int_{\partial\Omega} |v_m| = M.$$

Hence $\dfrac{v_m}{\int_{\partial\Omega} |v_m|}$ weakly* converges to the measure λ in $\mathcal{M}(\partial\Omega)$ and λ satisfies (26). □

Proof of Theorem 1.3. Up to a subsequence, $h_m \in \mathcal{M}_1(\partial\Omega)$ weakly* converges to some ν in $\mathcal{M}(\partial\Omega)$ with the properties $|\lambda| \leq \nu$ and $\nu(\partial\Omega) \leq 1 = |\lambda|(\partial\Omega)$. Hence $\nu = |\lambda|$ and the proof is completed. □

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