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Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent

Pierpaolo Esposito [a], Monica Musso [b,c,*], Angela Pistoia [d]

[a] Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S. Leonardo Murialdo, 1, 00146 Roma, Italy
[b] Departimento de Matematica, Pontificia Universidad Catolica de Chile, Avenida Vicuna Mackenna 4860, Macul, Santiago, Chile
[c] Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy
[d] Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", Via Scarpa, 16, 00166 Roma, Italy

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Abstract

We consider the boundary value problem $\Delta u + u^p = 0$ in a bounded, smooth domain Ω in $\mathbb{R}^2$ with homogeneous Dirichlet boundary condition and p a large exponent. We find topological conditions on Ω which ensure the existence of a positive solution u_p concentrating at exactly m points as $p \to \infty$. In particular, for a nonsimply connected domain such a solution exists for any given $m \geqslant 1$.
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1. Introduction and statement of main results

This paper is concerned with analysis of solutions to the boundary value problem:

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: esposito@mat.uniroma3.it (P. Esposito), mmusso@mat.puc.cl (M. Musso), pistoia@dmmm.uniroma1.it (A. Pistoia).

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where Ω is a smooth bounded domain in $\mathbb{R}^2$ and p is a large exponent. Let us consider the Rayleigh quotient

$$I_P(u) = \frac{\int_\Omega |\nabla u|^2}{(\int_\Omega |u|^{p+1})^{2/(p+1)}}, \quad u \in H_0^1(\Omega) \setminus \{0\},$$

and set

$$S_p = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} I_p(u).$$

Since $H_0^1(\Omega)$ is compactly embedded in $L^{p+1}(\Omega)$ for any $p > 0$, standard variational methods show that S_p is achieved by a positive function u_p which solves problem (1.1). The function u_p is known as least energy solution.

In [27,28] the authors show that the least energy solution has L^∞-norm bounded and bounded away from zero uniformly in p, for p large. Furthermore, up to subsequence, the renormalized energy density $p|\nabla u_p|^2$ concentrates as a Dirac delta around a critical point of the Robin function $H(x,x)$, where H is the regular part of Green function of the Laplacian in Ω with homogeneous Dirichlet boundary condition. Namely, the Green function $G(x,y)$ is the solution of the problem

$$\begin{cases} -\Delta_x G(x,y) = \delta_y(x) & x \in \Omega, \\ G(x,y) = 0 & x \in \partial\Omega, \end{cases}$$

and $H(x,y)$ is the regular part defined as

$$H(x,y) = G(x,y) - \frac{1}{2\pi} \log \frac{1}{|x-y|}.$$

In [1,16] the authors give a further description of the asymptotic behaviour of u_p, as $p \to \infty$, by identifying a limit profile problem of Liouville type:

$$\begin{cases} \Delta u + e^u = 0 & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < \infty, \end{cases} \tag{1.2}$$

and showing that $\|u_p\|_\infty \to \sqrt{e}$ as $p \to +\infty$.

Problem (1.2) possesses exactly a three-parameters family of solutions

$$U_{\delta,\xi} x = \log \frac{8\delta^2}{(\delta^2 + |x-\xi|^2)^2}, \tag{1.3}$$

where δ is a positive number and $\xi \in \mathbb{R}^2$ (see [6]).

The aim of this paper is to build solutions for problem (1.1) that, up to a suitable normalization, look like a sum of concentrated solutions for the limit profile problem (1.2) centered at several points $\xi_1, \ldots, \xi_m$, as $p \to \infty$. In this case, when m is possibly greater than 1, the function responsible to locate the concentration points $\xi_1, \ldots, \xi_m$ is more involved than the Robin function. In fact, location of such points is related to critical points of the function

$$\varphi_m(\xi_1, \ldots, \xi_m) = \sum_{j=1}^{m} H(\xi_j, \xi_j) + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} G(\xi_i, \xi_j).$$

Let us mention that the same function φ_m is responsible for the location of the points of concentration for solutions to the mean field equation in bounded domains $\Omega \subset \mathbb{R}^2$ (see [3,12,14]).

Our main result reads as follows.

Theorem 1.1. *Assume that Ω is not simply connected. Then given any $m \geqslant 1$ there exists $p_m > 0$ such that for any $p \geqslant p_m$ problem* (1.1) *has a solution u_p which concentrates at m different points in Ω, according to* (1.6), (1.7) *and* (1.8)*, as p goes to $+\infty$.*

This result is consequence of a more general theorem, which we state below, that ensures the existence of solutions to problem (1.1) which concentrate at m different points of Ω, under the assumption that the function φ_m has a *nontrivial critical value*.

Let Ω^m be $\Omega \times \Omega \times \cdots \times \Omega$ m times. We define

$$\varphi_m(x_1, \ldots, x_m) = +\infty \quad \text{if } x_i = x_j \text{ for some } i \neq j.$$

Let $\mathcal{D}$ be an open set compactly contained in Ω^m with smooth boundary. We recall that φ_m *links in $\mathcal{D}$ at critical level $\mathcal{C}$ relative to B and B_0* if B and B_0 are closed subsets of $\overline{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: Let us set Γ to be the class of all maps $\Phi \in C(B, \mathcal{D})$ with the property that there exists a function $\Psi \in C([0,1] \times B, \mathcal{D})$ such that:

$$\Psi(0, \cdot) = \mathrm{Id}_B, \qquad \Psi(1, \cdot) = \Phi, \qquad \Psi(T, \cdot)|_{B_0} = \mathrm{Id}_{B_0} \quad \text{for all } t \in [0,1].$$

We assume

$$\sup_{y \in B_0} \varphi_m(y) < \mathcal{C} \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} \varphi_m\big(\Phi(y)\big), \tag{1.4}$$

and for all $y \in \partial\mathcal{D}$ such that $\varphi_m(y) = \mathcal{C}$, there exists a vector τ_y tangent to $\partial\mathcal{D}$ at y such that

$$\nabla\varphi_m(y) \cdot \tau_y \neq 0. \tag{1.5}$$

Under these conditions a critical point $\bar{y} \in \mathcal{D}$ of φ_m with $\varphi_m(\bar{y}) = \mathcal{C}$ exists, as a standard deformation argument involving the negative gradient flow of φ_m shows. It is easy to check that the above conditions hold if

$$\inf_{x \in \mathcal{D}} \varphi_m(x) < \inf_{x \in \partial\mathcal{D}} \varphi_m(x), \quad \text{or} \quad \sup_{x \in \mathcal{D}} \varphi_m(x) > \sup_{x \in \partial\mathcal{D}} \varphi_m(x),$$

namely the case of (possibly degenerate) local minimum or maximum points of φ_m. We call $\mathcal{C}$ a nontrivial critical level of φ_m in $\mathcal{D}$.

Theorem 1.2. *Let $m \geqslant 1$ and assume that there is an open set $\mathcal{D}$ compactly contained in Ω^m, where φ_m has a nontrivial critical level $\mathcal{C}$. Then, there exists $p_m > 0$ such that for any $p \geqslant p_m$ problem* (1.1) *has a solution u_p which concentrates at m different points of Ω, i.e., as p goes to $+\infty$*

$$pu_p^{p+1} \rightharpoonup 8\pi e \sum_{i=1}^{m} \delta_{\xi_j} \quad \text{weakly in the sense of measure in } \overline{\Omega} \tag{1.6}$$

for some $\xi \in \mathcal{D}$ such that $\varphi_m(\xi_1, \dots, \xi_m) = \mathcal{C}$ and $\nabla\varphi_m(\xi_1, \dots, \xi_m) = 0$. More precisely, there is an m-tuple $\xi^p = (\xi_1^p, \dots, \xi_m^p) \in \mathcal{D}$ converging (*up to subsequence*) *to ξ such that, for any $\delta > 0$, as p goes to $+\infty$*

$$u_p \to 0 \quad \text{uniformly in } \Omega \setminus \bigcup_{j=1}^{m} B_\delta\big(\xi_i^p\big) \quad \text{and} \tag{1.7}$$

$$\sup_{x \in B_\delta(\xi_i^p)} u_p(x) \to \sqrt{e}. \tag{1.8}$$

The detailed proof of how Theorem 1.2 implies the result contained in Theorem 1.1 can be found in [12].

As already mentioned, the case of a (possibly degenerate) local maximum or minimum for φ_m is included. This simple fact allows us to obtain an existence result for solutions to problem (1.1) also when Ω is simply connected. Indeed, we can construct simply connected domains of dumbbell-type, where a large number of concentrating solutions can be found.

Let h be an integer. By h-dumbbell domain with thin handles we mean the following: let $\Omega_0 = \Omega_1 \cup \dots \cup \Omega_h$, with $\Omega_1, \dots, \Omega_h$ smooth bounded domains in $\mathbb{R}^2$ such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$. Assume that

$$\Omega_i \subset \big\{(x_1, x_2) \in \mathbb{R}^2\colon\ a_i \leqslant x_1 \leqslant b_i\big\}, \qquad \Omega_i \cap \{x_2 = 0\} \neq \emptyset,$$

for some $b_i < a_{i+1}$ and $i = 1, \dots, h$. Let

$$C_\varepsilon = \big\{(x_1, x_2) \in \mathbb{R}^2\colon\ |x_2| \leqslant \varepsilon,\ x_1 \in (a_1, b_h)\big\}, \quad \text{for some } \varepsilon > 0.$$

We say that Ω_ε is a h-dumbbell with thin handles if Ω_ε is a smooth simply connected domain such that $\Omega_0 \subset \Omega_\varepsilon \subset \Omega_0 \cup C_\varepsilon$, for some $\varepsilon > 0$.

The following result holds true.

Theorem 1.3. *There exist $\varepsilon_h > 0$ and $p_h > 0$ such that for any $\varepsilon \in (0, \varepsilon_h)$ and $p \geqslant p_h$ problem* (1.1) *in Ω_ε has at least $2^h - 1$ families of solutions which concentrate at different points of Ω_ε, according to* (1.6), (1.7) *and* (1.8), *as p goes to $+\infty$. More precisely, for any integer $1 \leqslant m \leqslant h$ there exist $\binom{h}{m}$ families of solutions of* (1.1) *which concentrate at m different points of Ω_ε.*

The detailed proof of how Theorem 1.2 implies the result contained in Theorem 1.3 can be found in [14]. We also refer the reader to [4,7], where domains like dumbbells with thin handles are considered.

The proof of all our results relies on a Lyapunov–Schmidt procedure, based on a proper choice of the ansatz for the solution we are looking for. Usually, in other related problems of asymptotic analysis, the ansatz for the solution is built as the sum of a main term, which is a solution (properly modified or projected) of the associated limit problem, and a lower order term, which can be determined by a fixed point argument. In our problem, this is not enough. Indeed, in order to perform the fix point argument to find the lower order term in the ansatz (see Lemma 4.1), we need to improve substantially the main term in the ansatz, adding two other terms in the expansion of the solution (see Section 2). This fact is basically due to estimate (4.7).

By performing a finite-dimensional reduction, we find an actual solution to our problem adjusting points ξ inside Ω to be critical points of a certain function $F(\xi)$ (see (5.2)). It is quite standard to show that this function $F(\xi)$ is a perturbation of $\varphi_m(\xi)$ in a C^0-sense. On the other hand, it is not at all trivial to show the C^1 closeness between F and φ_m. This difficulty is related to the difference between the exponential decay of the concentration parameters $\delta \sim e^{-p/4}$ (see (2.3)) and the polynomial decay $\frac{1}{p^4}$ of the error term $\|\Delta U_\xi + U_\xi^p\|_*$ of our approximating function U_ξ (see Proposition 2.1). We are able to overcome this difficulty (see Lemma 5.3) using a Pohozaev-type identity.

Now, we would like to compare problem (1.1) with some widely studied problems which have some analogies with it.

In higher dimension the problem equivalent to problem (1.1) is the slightly subcritical problem

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2}-\varepsilon} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, and ε is a positive parameter. Indeed, in dimension $N \geqslant 3$, the embedding of $H_0^1(\Omega)$ in $L^{p+1}(\Omega)$ is compact for every $p < \frac{N+2}{N-2}$. Hence the minimum of the Rayleigh quotient corresponding to problem (1.9) is achieved by a positive function u_ε, called least energy solution, which, after a multiplication by a suitable positive constant, is a solution to (1.9).

It is well known that, as ε goes to 0, the least energy solution u_ε concentrates around a point, which is a critical point of the Robin function of the corresponding Green function (see [2,17,19,25]). Also the converse is true: around any *stable* critical point of the Robin function one can build a family of solutions for (1.9) concentrating precisely there (see [23,25,26]).

In [2,21] the authors showed that also for problem (1.9) there exist solutions with concentration in multiple points and, as in the problem that we are considering in the present paper, the points of concentration are given by critical points of a certain function defined in terms of both the Green function and Robin function.

The analogies between problems (1.1) and (1.9) break down here. Indeed, while for (1.1) one can find solutions with an arbitrarily large number of condensation points in any given not simply connected domain Ω, in [2] the authors proved that solutions to (1.9) can have at most a finite number of peaks which depends on Ω (see, also, [15,21]).

The property of problem (1.1) to have a solution with an arbitrarily large number of points of condensation is what one expects to happen in the slightly supercritical version of problem (1.9), namely

$$\begin{cases} \Delta u + u^{\frac{N+2}{N-2}+\varepsilon} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

Indeed, a conjecture for (1.10) is that, given any domain Ω with a hole, one can see solutions with an arbitrarily large number of peaks (see [9,10,24]). For this fact, despite of being compact and hence subcritical in dimension 2, problem (1.1) shares patterns similar to the ones associated to slightly supercritical problem (1.10) in higher dimension.

However, again the analogies between (1.1) and (1.10) in higher dimension break down here. Indeed, the *dilation invariance*, which is crucial in the study of problem (1.10), does not play a role in finding solutions to problem (1.1), as already observed for a similar two-dimensional problem in [12,14] (see, also, [3]): only *translation invariance* is concerned in the study of (1.1).

The only *translation invariance* is the crucial key which allows to find solutions to the subcritical problem (see [11,18,20,22,29])

$$\begin{cases} -\varepsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where Ω is a bounded open domain in $\mathbb{R}^N$, $p > 1$ if $N = 2$ and $1 < p < \frac{N+2}{N-2}$ if $N \geqslant 3$, ε is a positive parameter.

Moreover, the property of problem (1.1) to have solution with an arbitrarily large number of points of condensation is what happens in the subcritical problem (1.11). In fact, in [8] the authors prove that if the reduced cohomology of Ω is not trivial, then for any integer k such a problem has at least one k-peaks solution, provided the parameter ε is small enough.

However, again the analogies between (1.1) and (1.11) break down here. Indeed, problem (1.1) is somehow almost critical in $\mathbb{R}^2$, since the limit problem as p goes to $+\infty$ is (1.2) which is critical in $\mathbb{R}^2$, while the limit problem of (1.11) as ε goes to 0 is the subcritical problem

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

The paper is organized as follows. In Section 2 we describe exactly the ansatz for the solution we are searching for. We rewrite the problem in term of a linear operator L for which a solvability theory is performed in Section 3. In Section 4 we solve an auxiliary nonlinear problem. We reduce (1.1) to solve a finite system $c_{ij} = 0$, as we will see in Section 5. Section 5 contains also the proof of Theorem 1.2.

2. A first approximation of the solution

In this section we will provide an ansatz for solutions of problem (1.1). A useful observation is that u satisfies Eq. (1.1) if and only if

$$v(y) = \delta^{\frac{2}{p-1}} u(\delta y + \xi), \quad y \in \Omega_{\xi,\delta}$$

satisfies

$$\begin{cases} \Delta v + v^p = 0 & \text{in } \Omega_{\xi,\delta}, \\ v \geqslant 0 \quad \text{in } \Omega_{\xi,\delta}, & v = 0 \quad \text{on } \partial\Omega_{\xi,\delta}, \end{cases} \tag{2.1}$$

where ξ is a given point in Ω, δ is a positive number with $\delta \to 0$, and $\Omega_{\xi,\delta}$ is the expanding domain defined by $\frac{\Omega-\xi}{\delta}$.

In this section we will show that the basic elements for the construction of an approximate solution to problem (1.1) which exhibits one point of concentration (or equivalently of problem (2.1)) are the radially symmetric solutions of problem (1.2) given by $U_{\delta,\xi}$ defined in (1.3).

For $U_{\delta,\xi}(x)$ defined in (1.3), we denote by $PU_{\delta,\xi}(x)$ its projection on the space $H_0^1(\Omega)$, namely $PU_{\delta,\xi}(x)$ is the unique solution of

$$\begin{cases} \Delta PU_{\delta,\xi} = \Delta U_{\delta,\xi} & \text{in } \Omega, \\ PU_{\delta,\xi}(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $PU_{\delta,\xi}(x) - U_{\delta,\xi}(x) + \log(8\delta^2) + 4\log\frac{1}{|x-\xi|} = O(\delta^2)$ uniformly on $x \in \partial\Omega$ as $\delta \to 0$ (together with any boundary derivatives), by harmonicity we get

$$\begin{aligned} PU_{\delta,\xi}(x) &= U_{\delta,\xi}(x) - \log(8\delta^2) + 8\pi H(x,\xi) + O(\delta^2) \quad \text{in } C^1(\overline{\Omega}), \\ PU_{\delta,\xi}(x) &= 8\pi G(x,\xi) + O(\delta^2) \quad \text{in } C^1_{\mathrm{loc}}(\overline{\Omega} \setminus \{\xi\}), \end{aligned} \tag{2.2}$$

provided ξ is bounded away from $\partial\Omega$.

Assume now that

$$\delta = \mu e^{-\frac{p}{4}}, \qquad \frac{1}{C} \leqslant \mu \leqslant C, \tag{2.3}$$

and define

$$u(x) = \frac{e^{\frac{p}{2(p-1)}}}{p^{p/(p-1)}\mu^{2/(p-1)}} PU_{\delta,\xi}(x), \quad x \in \Omega. \tag{2.4}$$

Observe that, as $p \to \infty$,

$$u(\xi) \to \sqrt{e} \quad \text{and} \quad u(x) = O\left(\frac{1}{p}\right) \quad \text{for } x \neq \xi.$$

Furthermore, under the extra assumption that the parameter μ is defined by the relation

$$\log(8\mu^4) = 8\pi H(\xi,\xi),$$

a direct computation shows that a good first approximation for a solution to problem (1.1) exhibiting only one point of concentration is given by a perturbation of the function u defined in (2.4).

Indeed, in the expanded variable $y = \frac{x-\xi}{\delta} \in \frac{\Omega-\xi}{\delta}$, if we define $v(y) = \delta^{2/(p-1)}u(\delta y + \xi)$, then our first approximation (2.4) looks like

$$\frac{1}{p^{p/(p-1)}}\big(p+U_{1,0}(y)+O\big(e^{-\frac{p}{4}}|y|+e^{-\frac{p}{4}}\big)\big) \tag{2.5}$$

and hence

$$\Delta v+v^p\sim\frac{1}{p^{p/(p-1)}}\bigg[-e^{U_{1,0}(y)}+\bigg(1+\frac{U_{1,0}(y)}{p}\bigg)^p\bigg],$$

which, roughly speaking, implies that the error for u to be a solution of (1.1) exhibiting one point of concentration, or equivalently for v to be a solution of (2.1), is of order $\frac{1}{p^2}$.

However, as we will see below, this is not enough to build an actual solution to (1.1) starting from $u(x)$. We need to refine this first approximation, or equivalently, according to (2.5), we need to go further in the expansion of $\bar{v}(y)=p+U_{1,0}(y)+o(1)$, by identifying first and second order terms in $\bar{v}-p-U_{1,0}$.

Let us call

$$v_\infty(y)=U_{1,0}(y)$$

and consider

$$\bar{v}(y)=v_\infty(y)+\frac{1}{p}w_0(y)+\frac{1}{p^2}w_1(y),$$

where w_0, w_1 solve

$$\Delta w_i+\frac{8}{(1+|y|^2)^2}w_i=\frac{1}{(1+|y|^2)^2}f_i(y)\quad\text{in }\mathbb{R}^2,\ i=1,2,$$

where

$$f_0=4v_\infty^2,\qquad f_1=8\bigg(w_0v_\infty-\frac{1}{3}v_\infty^3-\frac{1}{2}w_0^2-\frac{1}{8}v_\infty^4+\frac{1}{2}w_0v_\infty^2\bigg). \tag{2.6}$$

According to [5], for a radial function $f(y)=f(|y|)$ there exists a radial solution

$$w(r)=\frac{1-r^2}{1+r^2}\Bigg(\int_0^r\frac{\phi_f(s)-\phi_f(1)}{(s-1)^2}\,ds+\phi_f(1)\frac{r}{1-r}\Bigg)$$

for the equation

$$\Delta w+\frac{8}{(1+|y|^2)^2}w=f(y),$$

where

$$\phi_f(s)=\bigg(\frac{s^2+1}{s^2-1}\bigg)^2\frac{(s-1)^2}{s}\int_0^s t\frac{1-t^2}{1+t^2}f(t)\,dt\quad\text{for }s\neq 1$$

and $\phi_f(1) = \lim_{s\to 1}\phi_f(s)$. It is a straightforward computation to show that

$$w(r) = C_f \log r + D_f + O\left(\int_r^{+\infty} s|\log s||f|(s)\,ds + \frac{|\log r|}{r^2}\right) \quad \text{as } r \to +\infty,$$

where

$$C_f = \int_0^{+\infty} t\frac{t^2-1}{t^2+1} f(t)\,dt, \quad \text{provided} \int_0^{+\infty} t|f|(t)\,dt < +\infty.$$

Therefore, up to replacing $w(r)$ with $w(r) - D_f\frac{r^2-1}{r^2+1}$, we have shown:

Lemma 2.1. *Let $f \in C^1([0,+\infty))$ such that $\int_0^{+\infty} t|\log t||f|(t)\,dt < +\infty$. There exists a C^2 radial solution $w(r)$ of equation*

$$\Delta w + \frac{8}{(1+|y|^2)^2} w = f\big(|y|\big) \quad \textit{in } \mathbb{R}^2$$

such that as $r \to +\infty$

$$w(r) = \left(\int_0^{+\infty} t\frac{t^2-1}{t^2+1} f(t)\,dt\right)\log r + O\left(\int_r^{+\infty} s|\log s||f|(s)\,ds + \frac{|\log r|}{r^2}\right) \quad \textit{and}$$

$$\partial_r w(r) = \left(\int_0^{+\infty} t\frac{t^2-1}{t^2+1} f(t)\,dt\right)\frac{1}{t} + O\left(\frac{1}{r}\int_r^{+\infty} s|f|(s)\,ds + \frac{|\log r|}{r^3}\right).$$

By means of Lemma 2.1, since f_0 has at most logarithmic growth at infinity (see (2.6)), we can define $w_0(r)$ as a radial function satisfying

$$w_0(y) = C_0 \log|y| + O\left(\frac{1}{|y|}\right) \quad \text{as } |y| \to +\infty, \tag{2.7}$$

where

$$C_0 = 4\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3}\log^2\left(\frac{8}{(1+t^2)^2}\right) = 12 - 4\log 8.$$

More precisely, since we will need the exact expression of w_0, we have that

$$w_0(y) = \frac{1}{2}v_\infty^2(y) + 6\log\big(|y|^2+1\big) + \frac{2\log 8 - 10}{|y|^2+1}$$

$$+\frac{|y|^2-1}{|y|^2+1}\left(2\log^2\left(|y|^2+1\right)-\frac{1}{2}\log^2 8+4\int_{|y|^2}^{+\infty}\frac{ds}{s+1}\log\frac{s+1}{s}\right.$$
$$\left.-8\log|y|\log\left(|y|^2+1\right)\right), \tag{2.8}$$

as we can see by direct inspection. From (2.7) we get that also f_1 grows at most logarithmically at infinity and w_1 can be defined as a radial function satisfying

$$w_1(y)=C_1\log|y|+O\left(\frac{1}{|y|}\right)\quad \text{as } |y|\to+\infty, \tag{2.9}$$

for a suitable constant C_1.

We will see now that the profile $\frac{1}{p^{p/(p-1)}}(p+v_\infty(y)+\frac{1}{p}w_0(y)+\frac{1}{p^2}w_1(y))$ is a better approximation for a solution to the equation

$$\Delta v+v^p=0$$

in the region $|y|\leqslant Ce^{p/8}$. Indeed, by Taylor expansions of exponential and logarithmic function, we have that, for $|y|\leqslant Ce^{p/8}$,

$$\left(1+\frac{a}{p}+\frac{b}{p^2}+\frac{c}{p^3}\right)^p=e^a\left[1+\frac{1}{p}\left(b-\frac{a^2}{2}\right)+\frac{1}{p^2}\left(c-ab+\frac{a^3}{3}+\frac{b^2}{2}+\frac{a^4}{8}-\frac{a^2b}{2}\right)\right.$$
$$\left.+O\left(\frac{\log^6(|y|+2)}{p^3}\right)\right] \tag{2.10}$$

provided $-4\log(|y|+2)\leqslant a(y)\leqslant C$ and $|b(y)|+|c(y)|\leqslant C\log(|y|+2)$.

Observe that in our case $2C\geqslant a+\frac{b}{p}+\frac{c}{p^2}\geqslant-\frac{3}{4}p$ for $|y|\leqslant Ce^{p/8}$. Hence, by the choice of v_∞, w_0 and w_1 and expansion (2.10), we obtain that

$$\Delta v+v^p=O\left(\frac{1}{p^4}\frac{\log^6(|y|+2)}{(1+|y|^2)^2}\right)\quad \text{in } |y|\leqslant Ce^{\frac{p}{8}}.$$

As before, we will see now that a proper choice of the parameter μ will automatically imply that this approximation for v is also good for the boundary condition to be satisfied. Indeed, observe that by (2.7), (2.9) and Lemma 2.1 we get for $i=1,2$

$$P\left(w_i\left(\frac{x-\xi}{\delta}\right)\right)=w_i\left(\frac{x-\xi}{\delta}\right)-2\pi C_iH(x,\xi)+C_i\log\delta+O(\delta)\quad \text{in } C^1(\overline{\Omega})$$
$$P\left(w_i\left(\frac{x-\xi}{\delta}\right)\right)=-2\pi C_iG(x,\xi)+O(\delta)\quad \text{in } C^1_{\mathrm{loc}}\left(\overline{\Omega}\setminus\{\xi\}\right), \tag{2.11}$$

provided ξ is bounded away from $\partial\Omega$. If we take μ as a solution of

$$\log\left(8\mu^4\right)=8\pi H(\xi,\xi)\left(1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right)+\frac{\log\delta}{p}\left(C_0+\frac{C_1}{p}\right),$$

we get that

$$u(x)=\frac{e^{\frac{p}{2(p-1)}}}{p^{p/(p-1)}\mu^{2/(p-1)}}\left[PU_{\delta,\xi}(x)+\frac{1}{p}P\left(w_0\left(\frac{x-\xi}{\delta}\right)\right)+\frac{1}{p^2}P\left(w_1\left(\frac{x-\xi}{\delta}\right)\right)\right]$$

is a good first approximation in order to construct a solution for (1.1) with just one concentration point.

Let us remark that μ bifurcates, as p gets large, by $\bar{\mu}=e^{-3/4}e^{2\pi H(\xi,\xi)}$, solution of equation

$$\log\bigl(8\mu^4\bigr)=8\pi H(\xi,\xi)-\frac{C_0}{4}=8\pi H(\xi,\xi)-3+\log 8.$$

More precisely,

$$\mu=e^{-\frac{3}{4}}e^{2\pi H(\xi,\xi)}\left(1+O\left(\frac{1}{p}\right)\right).$$

Let us see now how things generalize if we want to construct a solution to problem (1.1) which exhibits m points of concentration. Let $\varepsilon>0$ fixed and take an m-tuple $\xi=(\xi_1,\dots,\xi_m)\in\mathcal{O}_\varepsilon$, where

$$\mathcal{O}_\varepsilon=\bigl\{\xi=(\xi_1,\dots,\xi_m)\in\Omega^m\colon\ \mathrm{dist}(\xi_i,\partial\Omega)\geqslant 2\varepsilon,\ |\xi_i-\xi_j|\geqslant 2\varepsilon,\ i\neq j\bigr\}.$$

Define

$$U_\xi(x)=\sum_{j=1}^m\frac{1}{\gamma\mu_j^{2/(p-1)}}\left[PU_{\delta_j,\xi_j}(x)+\frac{1}{p}P\left(w_0\left(\frac{x-\xi_j}{\delta_j}\right)\right)+\frac{1}{p^2}P\left(w_1\left(\frac{x-\xi_j}{\delta_j}\right)\right)\right],$$

where

$$\gamma=p^{\frac{p}{p-1}}e^{-\frac{p}{2(p-1)}}\quad\text{and}\quad\delta_j=\mu_je^{-\frac{p}{4}},\quad\frac{1}{C}\leqslant\mu_j\leqslant C. \tag{2.12}$$

The parameters μ_j will be chosen later. Observe that for any $j\neq i$ and $x=\delta_iy+\xi_i$

$$\begin{aligned}&\frac{1}{\gamma\mu_j^{2/(p-1)}}\left[PU_{\delta_j,\xi_j}(x)+\frac{1}{p}P\left(w_0\left(\frac{x-\xi_j}{\delta_j}\right)\right)+\frac{1}{p^2}P\left(w_1\left(\frac{x-\xi_j}{\delta_j}\right)\right)\right]\\&\quad=\frac{8\pi}{\gamma\mu_j^{2/(p-1)}}G(\xi_i,\xi_j)\left[1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right]+O\left(\frac{e^{-\frac{p}{4}}+|x-\xi_i|}{\gamma}\right).\end{aligned}$$

Hence, we get that the function $U_\xi(x)$ is a good approximation for a solution to problem (1.1) exhibiting m points of concentration provided

$$\begin{aligned}\log\bigl(8\mu_i^4\bigr)&=8\pi H(\xi_i,\xi_i)\left(1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right)+\frac{\log\delta_i}{p}\left(C_0+\frac{C_1}{p}\right)\\&\quad+8\pi\sum_{j\neq i}\frac{\mu_i^{2/(p-1)}}{\mu_j^{2/(p-1)}}G(\xi_j,\xi_i)\left[1-\frac{C_0}{4p}-\frac{C_1}{4p^2}\right].\end{aligned}$$

A direct computation shows that, for p large, μ satisfies

$$\mu_i = e^{-\frac{3}{4}} e^{2\pi H(\xi_i,\xi_i)+2\pi\sum_{j\neq i} G(\xi_j,\xi_i)}\left(1+O\left(\frac{1}{p}\right)\right). \tag{2.13}$$

Indeed, with this choice of the parameters μ_i, we have that

$$\sum_{j=1}^{m} \frac{1}{\gamma\mu_j^{2/(p-1)}}\left[PU_{\delta_j,\xi_j}(x) + \frac{1}{p}P\left(w_0\left(\frac{x-\xi_j}{\delta_j}\right)\right) + \frac{1}{p^2}P\left(w_1\left(\frac{x-\xi_j}{\delta_j}\right)\right)\right]$$
$$= \frac{1}{\gamma\mu_i^{2/(p-1)}}\left(p + v_\infty(y) + \frac{1}{p}w_0(y) + \frac{1}{p^2}w_1(y) + O\left(e^{-\frac{p}{4}}|y| + e^{-\frac{p}{4}}\right)\right) \tag{2.14}$$

for $x = \delta_i y + \xi_i$.

Remark 2.1. Let us remark that U_ξ is a positive function. Since $|v_\infty + \frac{1}{p}w_0 + \frac{1}{p^2}w_1| \leqslant C$ in $|y| \leqslant \frac{\epsilon}{\delta_i}$, by (2.14) we get that U_ξ is positive in $B(\xi_i,\epsilon)$ for any $i = 1,\dots,m$. Moreover, by (2.11) we get that

$$P\left(w_i\left(\frac{x-\xi_j}{\delta_j}\right)\right) \to -2\pi C_i G(\cdot,\xi_j)$$

in C^1-norm on $|x-\xi_j| \geqslant \epsilon$, $i = 0, 1$, and then

$$PU_{\delta_j,\xi_j} + \frac{1}{p}P\left(w_0\left(\frac{x-\xi_j}{\delta_j}\right)\right) + \frac{1}{p^2}P\left(w_1\left(\frac{x-\xi_j}{\delta_j}\right)\right) \to 8\pi G(\cdot,\xi_j)$$

in C^1-norm on $|x-\xi_j| \geqslant \epsilon$. Hence, since $\frac{\partial G}{\partial n}(\cdot,\xi_j) < 0$ on $\partial\Omega$, U_ξ is a positive function in Ω.

We will look for solutions u of problem (1.1) in the form $u = U_\xi + \phi$, where ϕ will represent an higher-order term in the expansion of u. Let us set

$$W_\xi(x) = pU_\xi^{p-1}(x).$$

In terms of ϕ, problem (1.1) becomes

$$\begin{cases} L(\phi) = -\left[R + N(\phi)\right] & \text{in } \Omega,\\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.15}$$

where

$$L(\phi) := \Delta\phi + W_\xi\phi \quad \text{and} \tag{2.16}$$

$$R_\xi := \Delta U_\xi + U_\xi^p, \qquad N(\phi) = \left[(U_\xi+\phi)^p - U_\xi^p - pU_\xi^{p-1}\phi\right]. \tag{2.17}$$

The main step in solving problem (2.15) for small ϕ, under a suitable choice of the points ξ_i, is that of a solvability theory for the linear operator L. In developing this theory, we will take into account the invariance, under translations and dilations, of the problem $\Delta v + e^v = 0$ in $\mathbb{R}^2$. We will perform the solvability theory for the linear operator L in weighted L^∞ spaces, following [12]. For any $h \in L^\infty(\Omega)$, define

$$\|h\|_* = \sup_{x\in\Omega} \left| \left(\sum_{j=1}^m \frac{\delta_j}{(\delta_j^2 + |x-\xi_j|^2)^{3/2}} \right)^{-1} h(x) \right|. \tag{2.18}$$

We conclude this section by proving an estimate of R in $\|\cdot\|_*$.

Proposition 2.1. *For fixed $\varepsilon > 0$, there exist $C > 0$ and $p_0 > 0$ such that for any $\xi \in \mathcal{O}_\varepsilon$ and $p \geqslant p_0$*

$$\left\| \Delta U_\xi + U_\xi^p \right\|_* \leqslant \frac{C}{p^4}. \tag{2.19}$$

Proof. Observe that

$$\begin{aligned} \Delta U_\xi &= \sum_{j=1}^m \frac{1}{\gamma \mu_j^{2/(p-1)}} \left(-e^{U_j} + \frac{1}{p\delta_j^2} \Delta w_0\left(\frac{x-\xi_j}{\delta_j}\right) + \frac{1}{p^2\delta_j^2} \Delta w_1\left(\frac{x-\xi_j}{\delta_j}\right)\right) \\ &= \sum_{j=1}^m \frac{1}{\gamma \mu_j^{2/(p-1)}} \left(-e^{U_j} + \frac{1}{p\delta_j^2} \tilde f_0\left(\frac{x-\xi_j}{\delta_j}\right) + \frac{1}{p^2\delta_j^2} \tilde f_1\left(\frac{x-\xi_j}{\delta_j}\right) \right. \\ &\quad \left. - \frac{1}{p} e^{U_j} w_0\left(\frac{x-\xi_j}{\delta_j}\right) - \frac{1}{p^2} e^{U_j} w_1\left(\frac{x-\xi_j}{\delta_j}\right)\right), \end{aligned} \tag{2.20}$$

where for $j = 1, \ldots, m$, $U_j = U_{\delta_j,\xi_j}$ and for $i = 0, 1$, $\tilde f_i(y) = \frac{1}{(1+|y|^2)^2} f_i(y)$ with f_0, f_1 given in (2.6). By (2.2) and (2.11), formula (2.20) gives that, if $|x - \xi_j| \geqslant \varepsilon$ for any $j = 1, \ldots, m$,

$$\left| \left(\sum_{j=1}^m \frac{\delta_j}{(\delta_j^2 + |x-\xi_j|^2)^{3/2}} \right)^{-1} \left(\Delta U_\xi + U_\xi^p\right)(x) \right| \leqslant C e^{\frac{p}{4}} \left(\left(\frac{C}{p}\right)^p + p e^{-\frac{p}{2}} \right) = O\left(p e^{-\frac{p}{4}}\right), \tag{2.21}$$

and, if $|x - \xi_i| \leqslant \varepsilon$ for some $i = 1, \ldots, m$,

$$\begin{aligned} \left|\Delta U_\xi + U_\xi^p\right| = \Bigg| & \frac{1}{\gamma \delta_i^2 \mu_i^{2/(p-1)}} \left(-\frac{8}{(1+|y|^2)^2} + \frac{1}{p} \tilde f_0(y) + \frac{1}{p^2} \tilde f_1(y) - \frac{1}{p} \frac{8}{(1+|y|^2)^2} w_0(y) \right. \\ & \left. - \frac{1}{p^2} \frac{8}{(1+|y|^2)^2} w_1(y) \right) + U_\xi^p(\delta_i y + \xi_i) + O\left(p e^{-\frac{p}{2}}\right) \Bigg|, \end{aligned} \tag{2.22}$$

where we denote $y = \frac{x-\xi_i}{\delta_i}$. By (2.14) we deduce that for $x = \delta_i y + \xi_i$,

$$U_\xi^p(x) = \left(\frac{p}{\gamma \mu_i^{2/(p-1)}} \right)^p \left(1 + \frac{1}{p} v_\infty(y) + \frac{1}{p^2} w_0(y) + \frac{1}{p^3} w_1(y) + O\left(\frac{e^{-\frac{p}{4}}}{p} |y| + \frac{e^{-\frac{p}{4}}}{p} \right) \right)^p.$$

Since $(\frac{p}{\gamma\mu_i^{2/(p-1)}})^p = \frac{1}{\gamma\delta_i^2\mu_i^{2/(p-1)}}$, by (2.10) we get for $|x-\xi_i| \leqslant \varepsilon\sqrt{\delta_i}$

$$
\begin{aligned}
U_\xi^p(x) = {} & \frac{1}{\gamma\delta_i^2\mu_i^{2/(p-1)}} \frac{8}{(1+|y|^2)^2}\left[1+\frac{1}{p}\left(w_0(y)-\frac{1}{2}\log^2\left(\frac{8}{(1+|y|^2)^2}\right)\right)\right. \\
& \frac{1}{p^2}\left(w_1 - \log\left(\frac{8}{(1+|y|^2)^2}\right)w_0 + \frac{1}{3}\log^3\left(\frac{8}{(1+|y|^2)^2}\right)\right. \\
& \left. + \frac{w_0^2}{2} + \frac{1}{8}\log^4\left(\frac{8}{(1+|y|^2)^2}\right) - \frac{w_0}{2}\log^2\left(\frac{8}{(1+|y|^2)^2}\right)\right) \\
& \left. + O\left(\frac{\log^6(|y|+2)}{p^3} + p^2 e^{-\frac{p}{4}} y + p^2 e^{-\frac{p}{4}}\right)\right], \quad y = \frac{x-\xi_i}{\delta_i}.
\end{aligned}
$$

Hence, in this region we obtain that

$$
\begin{aligned}
& \left|\left(\sum_{j=1}^m \frac{\delta_j}{(\delta_j^2+|x-\xi_j|^2)^{3/2}}\right)^{-1}\big(\Delta U_\xi + U_\xi^p\big)(x)\right| \\
& \leqslant \left|\frac{(\delta_i^2+|x-\xi_i|^2)^{\frac{3}{2}}}{\delta_i}\big(\Delta U_\xi + U_\xi^p\big)(x)\right| \\
& \leqslant \frac{C}{\gamma}\big(1+|y|^2\big)^{\frac{3}{2}} O\left(\frac{1}{p^3}\frac{\log^6(|y|+2)}{(1+|y|^2)^2}\right) \leqslant \frac{C}{p^4}, \quad y = \frac{x-\xi_i}{\delta_i}.
\end{aligned}
\tag{2.23}
$$

On the other hand, if $\varepsilon\sqrt{\delta_i} \leqslant |x-\xi_i| \leqslant \varepsilon$ we have that

$$
U_\xi^p(x) = O\left(\frac{e^{\frac{p}{2}}}{\gamma}\frac{1}{(1+|y|^2)^2}\right), \quad y = \frac{x-\xi_i}{\delta_i},
$$

since $(1+\frac{s}{p})^p \leqslant e^s$. Thus, in this region

$$
\begin{aligned}
& \left|\left(\sum_{j=1}^m \frac{\delta_j}{(\delta_j^2+|x-\xi_j|^2)^{3/2}}\right)^{-1}\big(\Delta U_\xi + U_\xi^p\big)(x)\right| \\
& = O\left(\frac{p}{(1+|y|^2)^{1/2}}\right) \leqslant Cpe^{-\frac{p}{8}}, \quad y = \frac{x-\xi_i}{\delta_i}.
\end{aligned}
\tag{2.24}
$$

By (2.21), (2.23) and (2.24) we obtain the desired result. □

3. Analysis of the linearized operator

In this section, we prove bounded invertibility of the operator L, uniformly on $\xi \in O_\varepsilon$, by using L^∞-norms introduced in (2.18). Let us recall that $L(\phi) = \Delta\phi + W_\xi\phi$, where $W_\xi(x) = pU_\xi^{p-1}(x)$. For simplicity of notation, we will omit the dependence of W_ξ on ξ.

As in Proposition 2.1, we have for the potential $W(x)$ the following expansions. If $|x-\xi_i|\leqslant\varepsilon$ for some $i=1,\dots,m$

$$\begin{aligned}W(x)&=p\left(\frac{p}{\gamma\mu_i^{2/(p-1)}}\right)^{p-1}\\&\quad\times\left(1+\frac{1}{p}v_\infty(y)+\frac{1}{p^2}w_0(y)+\frac{1}{p^3}w_1(y)+O\left(\frac{e^{-\frac{p}{4}}}{p}|y|+\frac{e^{-\frac{p}{4}}}{p}\right)\right)^{p-1}\\&=\delta_i^{-2}\left(1+\frac{1}{p}v_\infty(y)+\frac{1}{p^2}w_0(y)+\frac{1}{p^3}w_1(y)+O\left(\frac{e^{-\frac{p}{4}}}{p}|y|+\frac{e^{-\frac{p}{4}}}{p}\right)\right)^{p-1},\end{aligned}\tag{3.1}$$

where again we use the notation $y=\frac{x-\xi_i}{\delta_i}$. In this region, we have that

$$W(x)\leqslant\frac{C}{\delta_i^2}e^{v_\infty(y)}e^{-\frac{1}{p}v_\infty(y)}=O\big(e^{U_i(x)}\big),$$

since $v_\infty(y)\geqslant-2p$. Indeed, by Taylor expansions of exponential and logarithmic functions as in (2.10), we obtain that, if $|x-\xi_i|\leqslant\varepsilon\sqrt{\delta_i}$ (and $|y|\leqslant\frac{\varepsilon}{\sqrt{\delta_i}}$),

$$\begin{aligned}W(x)&=\delta_i^{-2}\left(1+\frac{1}{p}v_\infty(y)+\frac{1}{p^2}w_0(y)+\frac{1}{p^3}w_1(y)+O\left(\frac{e^{-\frac{p}{4}}}{p}|y|+\frac{e^{-\frac{p}{4}}}{p}\right)\right)^{p-1}\\&=\frac{8}{\delta_i^2(1+|y|^2)^2}\left(1+\frac{1}{p}\left(w_0-v_\infty-\frac{1}{2}v_\infty^2\right)+O\left(\frac{\log^4(|y|+2)}{p^2}\right)\right).\end{aligned}$$

If $|x-\xi_i|\geqslant\varepsilon$ for any $i=1,\dots,m$,

$$W(x)=O\left(p\left(\frac{C}{p}\right)^{p-1}\right).$$

Summing up, we have:

Lemma 3.1. *Let $\varepsilon>0$ be fixed. There exist $D_0>0$ and $p_0>0$ such that*

$$W(x)\leqslant D_0\sum_{j=1}^{m}e^{U_j(x)}$$

for any $\xi\in\mathcal{O}_\varepsilon$ and $p\geqslant p_0$. Furthermore,

$$W(x)=\frac{8}{\delta_i^2(1+|y|^2)^2}\left(1+\frac{1}{p}\left(w_0-v_\infty-\frac{1}{2}v_\infty^2\right)+O\left(\frac{\log^4(|y|+2)}{p^2}\right)\right)$$

for any $|x-\xi_i|\leqslant\varepsilon\sqrt{\delta_i}$, where $y=\frac{x-\xi_i}{\delta_i}$.

Remark 3.1. As for W, let us point out that, if $|x-\xi_i|\leqslant\varepsilon$ for some $i=1,\dots,m$, there holds

$$p\left(U_\xi+O\left(\frac{1}{p^3}\right)\right)^{p-2}\leqslant Cp\left(\frac{p}{\gamma\mu_i^{2/(p-1)}}\right)^{p-2}e^{\frac{p-2}{p}v_\infty(\frac{x-\xi_i}{\delta_i})}=O\big(e^{U_i(x)}\big).$$

Since this estimate is true if $|x-\xi_i|\geqslant\varepsilon$ for any $i=1,\dots,m$, we have that

$$p\left(U_\xi+O\left(\frac{1}{p^3}\right)\right)^{p-2}\leqslant C\sum_{j=1}^m e^{U_j(x)}. \tag{3.2}$$

In an heuristic way, the operator L is close to $\widetilde{L}$ defined by

$$\widetilde{L}(\phi)=\Delta\phi+\left(\sum_{i=1}^m e^{U_i}\right)\phi.$$

The operator $\widetilde{L}$ is "essentially" a superposition of linear operators which, after a dilation and translation, approach, as $p\to\infty$, the linear operator in $\mathbb{R}^2$,

$$\phi\to\Delta\phi+\frac{8}{(1+|y|^2)^2}\phi,$$

namely, equation $\Delta v+e^v=0$ linearized around the radial solution $\log\frac{8}{(1+|y|^2)^2}$. Set

$$z_0(y)=\frac{|y|^2-1}{|y|^2+1},\qquad z_i(y)=4\frac{y_i}{1+|y|^2},\quad i=1,2.$$

The first ingredient to develop the desired solvability theory for L is the well-known fact that any bounded solution of $L(\phi)=0$ in $\mathbb{R}^2$ is precisely a linear combination of the z_i, $i=0,1,2$, see [3] for a proof.

The second ingredient is a detailed analysis of $L-\widetilde{L}$. It has been proved in [12,14] that the operator $\widetilde{L}$ is invertible in the set of functions which, roughly speaking, are orthogonal to the functions z_i for $i=1,2$, and the operatorial norm of $\widetilde{L}^{-1}$ behaves like p as $p\to+\infty$. Since L is close to $\widetilde{L}$ up to terms of order at least $\frac{1}{p}$ (see Lemma 3.1), the invertibility of L becomes delicate and non trivial.

In [12,14] there were established a priori estimates respectively in weighted L^∞-norms and in $H_0^1(\Omega)$-norms. We will follow the approach in [12] since the estimates there are stronger and in this context very helpful.

Given $h\in C(\overline{\Omega})$, we consider the linear problem of finding a function $\phi\in W^{2,2}(\Omega)$ such that

$$L(\phi)=h+\sum_{i=1}^2\sum_{j=1}^m c_{ij}e^{U_j}Z_{ij}\quad\text{in }\Omega, \tag{3.3}$$

$$\phi=0\quad\text{on }\partial\Omega, \tag{3.4}$$

$$\int_\Omega e^{U_j}Z_{ij}\phi=0\quad\text{for all } i=1,2,\ j=1,\dots,m, \tag{3.5}$$

for some coefficients c_{ij}, $i=1,2$ and $j=1,\dots,m$. Here and in the sequel, for any $i=0,1,2$ and $j=1,\dots,m$ we denote

$$Z_{ij}(x) := z_i\left(\frac{x-\xi_j}{\delta_j}\right) = \begin{cases} \frac{|x-\xi_j|^2-\delta_j^2}{\delta_j^2+|x-\xi_j|^2} & \text{if } i=0,\\ \frac{4\delta_j(x-\xi_j)_i}{\delta_j^2+|x-\xi_j|^2} & \text{if } i=1,2. \end{cases}$$

The main result of this section is the following:

Proposition 3.1. *Let $\varepsilon>0$ be fixed. There exist $p_0>0$ and $C>0$ such that, for $h\in C(\overline{\Omega})$ there is a unique solution to problem* (3.3)–(3.5)*, for any $p>p_0$ and $\xi\in\mathcal{O}_\varepsilon$, which satisfies*

$$\|\phi\|_\infty \leqslant Cp\|h\|_*. \tag{3.6}$$

Proof. The proof of this result consists of six steps.

Step 1. The operator L satisfies the maximum principle in $\widetilde{\Omega} := \Omega\setminus\bigcup_{j=1}^m B(\xi_j,R\delta_j)$ for R large, independent on p. Namely,

$$\text{if } L(\psi)\leqslant 0 \text{ in } \widetilde{\Omega} \quad \text{and} \quad \psi\geqslant 0 \text{ on } \partial\widetilde{\Omega}, \qquad \text{then } \psi\geqslant 0 \quad \text{in } \widetilde{\Omega}.$$

In order to prove this fact, we show the existence of a positive function Z in $\widetilde{\Omega}$ satisfying $L(Z)<0$. We define Z to be

$$Z(x) = \sum_{j=1}^m z_0\left(\frac{a(x-\xi_j)}{\delta_j}\right), \quad a>0.$$

First, observe that, if $|x-\xi_j|\geqslant R\delta_j$ for $R>\frac{1}{a}$, then $Z(x)>0$. On the other hand, we have

$$W(x)Z(x) \leqslant D_0\left(\sum_{j=1}^m e^{U_j(x)}\right)Z(x) \leqslant D_0 Z(x)\sum_{j=1}^m \frac{8\delta_j^2}{|x-\xi_j|^4},$$

where D_0 is the constant in Lemma 3.1. Further, by definition of z_0,

$$-\Delta Z(x) = \sum_{j=1}^m a^2\frac{8\delta_j^2(a^2|x-\xi_j|^2-\delta_j^2)}{(a^2|x-\xi_j|^2+\delta_j^2)^3} \geqslant \frac{1}{3}\sum_{j=1}^m \frac{8a^2\delta_j^2}{(a^2|x-\xi_j|^2+\delta_j^2)^2} \geqslant \frac{4}{27}\sum_{j=1}^m \frac{8\delta_j^2}{a^2|x-\xi_j|^4}$$

provided $R>\frac{\sqrt{2}}{a}$. Hence

$$LZ(x) \leqslant \left(-\frac{4}{27}\frac{1}{a^2}+D_0\right)\sum_{j=1}^m \frac{8\delta_j^2}{|x-\xi_j|^4} < 0$$

provided that a is chosen sufficiently small, but independent of p. The function $Z(x)$ is what we are looking for.

Step 2. Let R be as before. Let us define the "inner norm" of ϕ in the following way

$$\|\phi\|_i = \sup_{x\in\bigcup_{j=1}^m B(\xi_j, R\delta_j)} |\phi|(x).$$

We claim that there is a constant $C > 0$ such that, if $L(\phi) = h$ in Ω, $h \in C^{0,\alpha}(\overline{\Omega})$, then

$$\|\phi\|_\infty \leqslant C\big[\|\phi\|_i + \|h\|_*\big],$$

for any $h \in C^{0,\alpha}(\overline{\Omega})$. We will establish this estimate with the use of suitable barriers. Let $M = 2\operatorname{diam}\Omega$. Consider the solution $\psi_j(x)$ of the problem:

$$\begin{cases} -\Delta\psi_j = \frac{2\delta_j}{|x-\xi_j|^3} & \text{in } R\delta_j < |x-\xi_j| < M, \\ \psi_j(x) = 0 & \text{on } |x-\xi_j| = R\delta_j \text{ and } |x-\xi_j| = M. \end{cases}$$

Namely, the function $\psi_j(x)$ is the positive function defined by

$$\psi_j(x) = -\frac{2\delta_j}{|x-\xi_j|} + A + B\log|x-\xi_j|,$$

where

$$B = 2\left(\frac{\delta_j}{M} - \frac{1}{R}\right)\frac{1}{\log\left(\frac{M}{R\delta_j}\right)} < 0$$

and

$$A = \frac{2\delta_j}{M} - B\log M.$$

Hence, the function $\psi - j(x)$ is uniformly bounded from above by a constant independent of p, since we have that, for $R\delta_j \leqslant |x-\xi_j| \leqslant M$,

$$\psi_j(x) \leqslant A + B\log(R\delta_j) = \frac{2\delta_j}{M} - B\log\frac{M}{R\delta_j} = \frac{2}{R}.$$

Define now the function

$$\tilde{\phi}(x) = 2\|\phi\|_i Z(x) + \|h\|_* \sum_{j=1}^m \psi_j(x),$$

where Z was defined in the previous step. First of all, observe that by the definition of Z, choosing R larger if necessary,

$$\tilde{\phi}(x) \geqslant 2\|\phi\|_i Z(x) \geqslant \|\phi\|_i \geqslant |\phi|(x) \quad \text{for } |x-\xi_j| = R\delta_j,\ j = 1, \dots, m,$$

and, by the positivity of $Z(x)$ and $\psi_j(x)$,

$$\tilde{\phi}(x) \geqslant 0 = |\phi|(x) \quad \text{for } x \in \partial\Omega.$$

Since by definition of $\|\cdot\|_*$ we have that

$$\left(\sum_{j=1}^{m} \frac{\delta_j}{(\delta_j^2 + |x-\xi_j|^2)^{3/2}}\right)\|h\|_* \geqslant \big|h(x)\big|, \tag{3.7}$$

finally, we obtain that

$$\begin{aligned} L\tilde{\phi} &\leqslant \|h\|_* \sum_{j=1}^{m} L\psi_j(x) = \|h\|_* \sum_{j=1}^{m}\left(-\frac{2\delta_j}{|x-\xi_j|^3} + W(x)\psi_j(x)\right) \\ &\leqslant \|h\|_* \sum_{j=1}^{m}\left(-\frac{2\delta_j}{|x-\xi_j|^3} + \frac{2mD_0}{R}e^{U_j(x)}\right) \\ &\leqslant -\|h\|_*\left(\sum_{j=1}^{m} \frac{\delta_j}{(\delta_j^2 + |x-\xi_j|^2)^{3/2}}\right) \leqslant -\big|h(x)\big| \leqslant |L\phi|(x) \end{aligned}$$

provided $R > 16mD_0$ and p large enough. Hence, by the maximum principle in step 1 we obtain that

$$|\phi|(x) \leqslant \tilde{\phi}(x) \quad \text{for } x \in \widetilde{\Omega},$$

and therefore, since $Z(x) \leqslant 1$ and $\psi_j(x) \leqslant \frac{2}{R}$,

$$\|\phi\|_\infty \leqslant C\big[\|\phi\|_i + \|h\|_*\big].$$

Step 3. We prove uniform a priori estimates for solutions ϕ of problem $L\phi = h$ in Ω, $\phi = 0$ on $\partial\Omega$, when $h \in C^{0,\alpha}(\overline{\Omega})$ and ϕ satisfies (3.5) and in addition the orthogonality conditions:

$$\int_\Omega e^{U_j} Z_{0j}\phi = 0 \quad \text{for } j = 1, \ldots, m. \tag{3.8}$$

Namely, we prove that there exists a positive constant C such that for any $\xi \in \mathcal{O}_\varepsilon$ and $h \in C^{0,\alpha}(\overline{\Omega})$

$$\|\phi\|_\infty \leqslant C\|h\|_*,$$

for p sufficiently large. By contradiction, assume the existence of sequences $p_n \to \infty$, points $\xi^n \in \mathcal{O}_\varepsilon$, functions h_n and associated solutions ϕ_n such that $\|h_n\|_* \to 0$ and $\|\phi_n\|_\infty = 1$.

Since $\|\phi_n\|_\infty = 1$, step 2 shows that $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$. Let us set $\hat\phi_j^n(y) = \phi_n(\delta_j^n y + \xi_j^n)$ for $j = 1, \dots, m$. By Lemma 3.1 and (3.7), elliptic estimates readily imply that $\hat\phi_j^n$ converges uniformly over compact sets to a bounded solution $\hat\phi_j^\infty$ of the equation in $\mathbb{R}^2$:

$$\Delta\phi + \frac{8}{(1+|y|^2)^2}\phi = 0.$$

This implies that $\hat\phi_j^\infty$ is a linear combination of the functions z_i, $i = 0, 1, 2$. Since $\|\hat\phi_j^n\|_\infty \leqslant 1$, by Lebesgue theorem the orthogonality conditions (3.5) and (3.8) on ϕ_n pass to the limit and give

$$\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_i(y)\hat\phi_j^\infty = 0 \quad \text{for any } i = 0, 1, 2.$$

Hence, $\hat\phi_j^\infty \equiv 0$ for any $j = 1, \dots, m$ contradicting $\liminf_{n\to+\infty} \|\phi_n\|_i > 0$.

Step 4. We prove that there exists a positive constant $C > 0$ such that any solution ϕ of equation $L\phi = h$ in Ω, $\phi = 0$ on $\partial\Omega$, satisfies

$$\|\phi\|_\infty \leqslant Cp\|h\|_*,$$

when $h \in C^{0,\alpha}(\overline{\Omega})$ and we assume on ϕ only the orthogonality conditions (3.5). Proceeding by contradiction as in step 3, we can suppose further that

$$p_n\|h_n\|_* \to 0 \quad \text{as } n \to +\infty, \tag{3.9}$$

but we loss in the limit the condition $\int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2} z_0(y)\hat\phi_j^\infty = 0$. Hence, we have that

$$\hat\phi_j^n \to C_j \frac{|y|^2-1}{|y|^2+1} \quad \text{in } C^0_{\text{loc}}(\mathbb{R}^2) \tag{3.10}$$

for some constants C_j. To reach a contradiction, we have to show that $C_j = 0$ for any $j = 1, \dots, m$. We will obtain it from the stronger condition (3.9) on h_n.

To this end, we perform the following construction. By Lemma 2.1, we find radial solutions w and t respectively of equations

$$\Delta w + \frac{8}{(1+|y|^2)^2} w = \frac{8}{(1+|y|^2)^2} z_0(y) \quad \text{and} \quad \Delta t + \frac{8}{(1+|y|^2)^2} t = \frac{8}{(1+|y|^2)^2} \quad \text{in } \mathbb{R}^2,$$

such that as $|y| \to +\infty$

$$w(y) = \frac{4}{3}\log|y| + O\left(\frac{1}{|y|}\right), \qquad t(y) = O\left(\frac{1}{|y|}\right),$$

since $8\int_0^{+\infty} t\frac{(t^2-1)^2}{(t^2+1)^4}\,dt = \frac{4}{3}$ and $8\int_0^{+\infty} t\frac{t^2-1}{(t^2+1)^3}\,dt = 0$.

For simplicity, from now on we will omit the dependence on n. For $j = 1, \dots, m$, define now

$$u_j(x) = w\left(\frac{x-\xi_j}{\delta_j}\right) + \frac{4}{3}(\log\delta_j)Z_{0j}(x) + \frac{8\pi}{3}H(\xi_j,\xi_j)t\left(\frac{x-\xi_j}{\delta_j}\right)$$

and denote by Pu_j the projection of u_j onto $H_0^j(\Omega)$. Since $u_j - Pu_j + \frac{4}{3}\log\frac{1}{|\cdot-\xi_j|} = O(\delta_j)$ on $\partial\Omega$ (together with boundary derivatives), by harmonicity we get

$$
\begin{aligned}
Pu_j &= u_j - \frac{8\pi}{3}H(\cdot,\xi_j) + O\big(e^{-\frac{p}{4}}\big) \quad \text{in } C^1(\overline{\Omega}),\\
Pu_j &= -\frac{8\pi}{3}G(\cdot,\xi_j) + O\big(e^{-\frac{p}{4}}\big) \quad \text{in } C^1_{\mathrm{loc}}\big(\overline{\Omega}\setminus\{\xi_j\}\big).
\end{aligned}
\tag{3.11}
$$

The function Pu_j solves

$$
\Delta Pu_j + W(x)Pu_j = e^{U_j}Z_{0j} + \big(W(x) - e^{U_j}\big)Pu_j + R_j, \tag{3.12}
$$

where

$$
R_j(x) = \left(Pu_j - u_j + \frac{8\pi}{3}H(\xi_j,\xi_j)\right)e^{U_j}.
$$

Multiply (3.12) by ϕ and integrate by parts to obtain

$$
\int_\Omega e^{U_j}Z_{0j}\phi + \int_\Omega \big(W(x) - e^{U_j}\big)Pu_j\phi = \int_\Omega Pu_j h - \int_\Omega R_j\phi. \tag{3.13}
$$

First of all, by Lebesgue theorem and (3.10) we get that

$$
\int_\Omega e^{U_j}Z_{0j}\phi \to C_j\int_{\mathbb{R}^2}\frac{8(|y|^2-1)^2}{(1+|y|^2)^4} = \frac{8\pi}{3}C_j. \tag{3.14}
$$

The more delicate term is $\int_\Omega (W(x) - e^{U_j})Pu_j\phi$. By Lemma 3.1 and (3.11) we have that

$$
\begin{aligned}
&\int_\Omega \big(W(x) - e^{U_j}\big)Pu_j\phi\\
&\quad= \int_{B(\xi_j,\varepsilon\sqrt{\delta_j})}\big(W(x) - e^{U_j}\big)Pu_j\phi - \frac{8\pi}{3}\sum_{k\neq j}G(\xi_k,\xi_j)\int_{B(\xi_k,\varepsilon\sqrt{\delta_k})}W(x)\phi - O\big(e^{-\frac{p}{8}}\big)\\
&\quad= \frac{4}{3}\frac{\log\delta_j}{p}\int_{B(0,\varepsilon/\sqrt{\delta_j})}\frac{8}{(1+|y|^2)^2}\left(w_0 - v_\infty - \frac{1}{2}v_\infty^2\right)z_0(y)\hat{\phi}_j\\
&\qquad - \frac{8\pi}{3}\sum_{k\neq j}G(\xi_k,\xi_j)\int_{B(0,\varepsilon/\sqrt{\delta_k})}\frac{8}{(1+|y|^2)^2}\hat{\phi}_k + O\left(\frac{1}{p}\right)\\
&= -\frac{C_j}{3}\int_{\mathbb{R}^2}\frac{8(|y|^2-1)^2}{(1+|y|^2)^4}\left(w_0 - v_\infty - \frac{1}{2}v_\infty^2\right)(y) - o(1)
\end{aligned}
$$

since Lebesgue theorem and (3.10) imply:

$$\int_{B(0,\varepsilon/\sqrt{\delta_j})} \frac{8}{(1+|y|^2)^2}\left(w_0 - v_\infty - \frac{1}{2}v_\infty^2\right) z_0(y)\hat{\phi}_j \to C_j \int_{\mathbb{R}^2} \frac{8(|y|^2-1)^2}{(1+|y|^2)^4}\left(w_0 - v_\infty - \frac{1}{2}v_\infty^2\right)$$

and

$$\int_{B(0,\varepsilon/\sqrt{\delta_k})} \frac{8}{(1+|y|^2)^2}\hat{\phi}_k \to C_k \int_{\mathbb{R}^2} \frac{8}{(1+|y|^2)^2}\frac{|y|^2-1}{|y|^2+1} = 0.$$

In a straightforward but tedious way, by (2.8) we can compute:

$$\int_{\mathbb{R}^2} \frac{8(|y|^2-1)^2}{(1+|y|^2)^4}\left(w_0 - v_\infty - \frac{1}{2}v_\infty^2\right)(y) = -8\pi,$$

so that we obtain

$$\int_{\Omega} \left(W(x) - e^{U_j}\right) P u_j \phi = \frac{8\pi}{3} C_j + o(1). \tag{3.15}$$

As far as the R.H.S. in (3.13), we have that by (3.11)

$$\left|\int_{\Omega} P u_j h\right| = O\left(\|h\|_* \int_{\Omega}\left(\sum_{k=1}^{m} \frac{\delta_k}{(\delta_k^2 + |x-\xi_k|^2)^{3/2}}\right)|P u_j|\right) = O\left(p\|h\|_*\right) \tag{3.16}$$

since $\int_{\Omega} |P u_j| = O(|\log \delta_j|) = O(p)$ and

$$\int_{B(\xi_j,\varepsilon)} \frac{\delta_j}{(\delta_j^2 + |x-\xi_j|^2)^{3/2}}|u_j| \leqslant \int_{\mathbb{R}^2} \frac{1}{(1+|y|^2)^{3/2}}|u_j|(\delta_j y + \xi_j) = O(p).$$

Finally, by (3.11)

$$\int_{\Omega} R_j \phi = O\left(\int_{\Omega} e^{U_j}\left(|x-\xi_j| + e^{-\frac{p}{4}}\right)\right) = O\left(e^{-\frac{p}{4}}\right). \tag{3.17}$$

Hence, inserting (3.14)–(3.17) in (3.13) we obtain that

$$\frac{16\pi}{3} C_j = o(1)$$

for any $j = 1, \ldots, m$. Necessarily, $C_j = 0$ and the claim is proved.

Step 5. We establish the validity of the a priori estimate:

$$\|\phi\|_\infty \leqslant Cp\|h\|_* \tag{3.18}$$

for solutions of problem (3.3)–(3.5) and $h \in C^{0,\alpha}(\overline{\Omega})$. The previous step gives

$$\|\phi\|_\infty \leqslant Cp\left(\|h\|_* + \sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}|\right)$$

$$\text{since} \quad \left\|e^{U_j} Z_{ij}\right\|_* \leqslant 2\left\|e^{U_j}\right\|_* \leqslant 16.$$

Hence, arguing by contradiction of (3.18), we can proceed as in step 3 and suppose further that

$$p_n\|h_n\|_* \to 0, \qquad p_n\sum_{i=1}^{2}\sum_{j=1}^{m}\left|c_{ij}^n\right| \geqslant \delta > 0 \quad \text{as } n \to +\infty.$$

We omit the dependence on n. It suffices to estimate the values of the constants c_{ij}. For $i = 1, 2$ and $j = 1, \ldots, m$, multiply (3.3) by PZ_{ij} and, integrating by parts, get:

$$\sum_{l=1}^{2}\sum_{h=1}^{m} c_{lh}(PZ_{lh}, PZ_{ij})_{H_0^1} + \int_\Omega hPZ_{ij} = \int_\Omega W(x)\phi PZ_{ij} - \int_\Omega e^{U_j} Z_{ij}\phi, \tag{3.19}$$

since $\Delta PZ_{ij} = \Delta Z_{ij} = -e^{U_j} Z_{ij}$. For $i = 1, 2$ and $j = 1, \ldots, m$ we have the following expansions:

$$PZ_{ij} = Z_{ij} - 8\pi\delta_j \frac{\partial H}{\partial(\xi_j)_i}(\cdot, \xi_j) + O\left(\delta_j^3\right), \qquad PZ_{0j} = Z_{0j} - 1 + O\left(\delta_j^2\right) \tag{3.20}$$

in $C^1(\overline{\Omega})$ and

$$PZ_{ij} = -8\pi\delta_j \frac{\partial G}{\partial(\xi_j)_i}(\cdot, \xi_j) + O\left(\delta_j^3\right), \qquad PZ_{0j} = O\left(\delta_j^2\right) \tag{3.21}$$

in $C^1_{\text{loc}}(\overline{\Omega} \setminus \{\xi_j\})$. By (3.20), (3.21) we deduce the following "orthogonality" relations: for $i, l = 1, 2$ and $j, h = 1, \ldots, m$ with $j \neq h$,

$$(PZ_{ij}, PZ_{lj})_{H_0^1(\Omega)} = \left(64\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}\right)\delta_{ij} + O\left(\delta_j^2\right), \qquad (PZ_{ij}, PZ_{lh})_{H_0^1(\Omega)} = O(\delta_j\delta_h) \tag{3.22}$$

and

$$(PZ_{0j}, PZ_{lj})_{H_0^1(\Omega)} = O\left(\delta_j^2\right), \qquad (PZ_{0j}, PZ_{lh})_{H_0^1(\Omega)} = O(\delta_j\delta_h) \tag{3.23}$$

uniformly on $\xi \in \mathcal{O}_\varepsilon$, where δ_{il} denotes the Kronecker's symbol. In fact, we have that

$$
\begin{aligned}
(PZ_{ij}, PZ_{lj})_{H_0^1(\Omega)} &= \int_\Omega e^{U_j} Z_{ij} PZ_{lj} \\
&= \int_{B(\xi_j,\varepsilon)} e^{U_j} Z_{ij}\left(Z_{ij} - 8\pi\delta_j \frac{\partial H}{\partial(\xi_j)_l}(\xi_j,\xi_j) + O\big(\delta_j|x-\xi_j| + \delta_j^3\big)\right) + O\big(\delta_j^4\big) \\
&= 128 \int_{\mathbb{R}^2} \frac{y_i y_l}{(1+|y|^2)^4} + O\big(\delta_j^2\big) = \left(64 \int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}\right)\delta_{ij} + O\big(\delta_j^2\big),
\end{aligned}
$$

$$
\begin{aligned}
(PZ_{ij}, PZ_{lh})_{H_0^1(\Omega)} &= \int_\Omega e^{U_j} Z_{ij} PZ_{lh} \\
&= \int_{B(\xi_j,\varepsilon)} e^{U_j} Z_{ij}\left(-8\pi\delta_h \frac{\partial G}{\partial(\xi_h)_l}(\xi_j,\xi_h) + O\big(\delta_h|x-\xi_j| + \delta_h^3\big)\right) + O\big(\delta_j^3\big) \\
&= O\big(\delta_j\delta_h\big),
\end{aligned}
$$

$$
\begin{aligned}
(PZ_{0j}, PZ_{lj})_{H_0^1(\Omega)} &= \int_\Omega e^{U_j} Z_{0j} PZ_{lj} \\
&= \int_{B(\xi_j,\varepsilon)} e^{U_j} Z_{0j}\left(Z_{lj} - 8\pi\delta_j \frac{\partial H}{\partial(\xi_j)_l}(\xi_j,\xi_j) + O\big(\delta_j|x-\xi_j| + \delta_j^3\big)\right) + O\big(\delta_j^3\big) \\
&= O\big(\delta_j^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
(PZ_{0j}, PZ_{lh})_{H_0^1(\Omega)} &= \int_\Omega e^{U_j} Z_{0j} PZ_{lh} \\
&= \int_{B(\xi_j,\varepsilon)} e^{U_j} Z_{0j}\left(-8\pi\delta_h \frac{\partial G}{\partial(\xi_h)_l}(\xi_j,\xi_h) + O\big(\delta_h|x-\xi_j| + \delta_h^3\big)\right) + O\big(\delta_j^2\big) \\
&= O\big(\delta_j\delta_h\big).
\end{aligned}
$$

Now, since

$$
\left|\int_\Omega hPZ_{ij}\right| \leqslant C' \int_\Omega |h| \leqslant C\|h\|_*,
$$

by (3.22) the L.H.S. of (3.19) is estimated as follows:

$$
\text{L.H.S.} = Dc_{ij} + O\left(e^{-\frac{p}{2}} \sum_{l=1}^{2}\sum_{h=1}^{m} |c_{lh}|\right) + O\big(\|h\|_*\big), \tag{3.24}
$$

where $D = 64\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4}$. Moreover, by Lemma 3.1 the R.H.S. of (3.19) takes the form:

$$\begin{aligned}
\text{L.H.S.} &= \int\limits_{B(\xi_j,\varepsilon\sqrt{\delta_j})} W(x)\phi PZ_{ij} - \int\limits_{\Omega} e^{U_j}\phi Z_{ij} + O\Big(\sqrt{\delta_j}\|\phi\|_\infty\Big) \\
&= \int\limits_{B(\xi_j,\varepsilon\sqrt{\delta_j})} \big(W(x) - e^{U_j}\big)\phi PZ_{ij} + \int\limits_{\Omega} e^{U_j}\phi (PZ_{ij} - Z_{ij}) + O\Big(\sqrt{\delta_j}\|\phi\|_\infty\Big) \\
&= \frac{1}{p} \int\limits_{B(\xi_j,\varepsilon/\sqrt{\delta_j})} \frac{32 y_i}{(1+|y|^2)^3}\bigg(w_0 - v_\infty - \frac{v_\infty^2}{2}\bigg)\hat{\phi}_j + O\bigg(\frac{1}{p^2}\|\phi\|_\infty\bigg) \qquad (3.25)
\end{aligned}$$

in view of (3.20), where $\hat{\phi}_j(y) = \phi(\delta_j y + \xi_j)$. Inserting the estimates (3.24) and (3.25) into (3.19), we deduce that

$$Dc_{ij} + O\Bigg(e^{-\frac{p}{2}}\sum_{l=1}^{2}\sum_{h=1}^{m}|c_{lh}|\Bigg) = O\bigg(\|h\|_* + \frac{1}{p}\|\phi\|_\infty\bigg).$$

Hence, we obtain that

$$\sum_{l=1}^{2}\sum_{h=1}^{m}|c_{lh}| = O\bigg(\|h\|_* + \frac{1}{p}\|\phi\|_\infty\bigg). \qquad (3.26)$$

Since $\sum_{l=1}^{2}\sum_{h=1}^{m}|c_{lh}| = o(1)$, as in step 4 we have that

$$\hat{\phi}_j \to C_j \frac{|y|^2 - 1}{|y|^2 + 1} \quad \text{in } C^0_{\mathrm{loc}}\big(\mathbb{R}^2\big)$$

for some constant Cj, $j = 1, \ldots, m$. Hence, in (3.25) we have a better estimate since by Lebesgue theorem the term

$$\int\limits_{B(0,\varepsilon/\sqrt{\delta_j})} \frac{32 y_i}{(1+|y|^2)^3}\bigg(w_0 - v_\infty - \frac{1}{2}v_\infty^2\bigg)(y)\hat{\phi}_j(y)$$

converges to

$$C_j \int\limits_{\mathbb{R}^2} \frac{32 y_i(|y|^2 - 1)}{(1+|y|^2)^4}\bigg(w_0 - v_\infty - \frac{1}{2}v_\infty^2\bigg)(y) = 0.$$

Therefore, we get that the R.H.S. in (3.19) satisfies: R.H.S. $= o(\frac{1}{p})$, and in turn, $\sum_{l=1}^{2}\sum_{h=1}^{m}|c_{lh}| = O(\|h\|_*) + o(\frac{1}{p})$. This contradicts

$$p\sum_{i=1}^{2}\sum_{j=1}^{m}|c_{ij}| \geqslant \delta > 0,$$

and the claim is established.

Step 6. We prove the solvability of (3.3)–(3.5). To this purpose, we consider the spaces:

$$K_\xi = \left\{\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} PZ_{ij}\colon\ c_{ij} \in \mathbb{R} \text{ for } i = 1, 2,\ j = 1, \dots, m\right\} \quad \text{and}$$

$$K_\xi^\perp = \left\{\phi \in L^2(\Omega)\colon \int_\Omega e^{U_j} Z_{ij}\phi = 0 \text{ for } i = 1, 2,\ j = 1, \dots, m\right\}.$$

Let $\Pi_\xi : L^2(\Omega) \to K_\xi$ defined as

$$\Pi_\xi \phi = \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} PZ_{ij},$$

where c_{ij} are uniquely determined (as it follows by (3.22), (3.23)) by the system:

$$\int_\Omega e^{U_h} Z_{lh}\left(\phi - \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} PZ_{ij}\right) = 0 \quad \text{for any } l = 1, 2,\ h = 1, \dots, m.$$

Let $\Pi_\xi^\perp = \mathrm{Id} - \Pi_\xi : L^2(\Omega) \to K_\xi^\perp$. Problem (3.3)–(3.5), expressed in a weak form, is equivalent to find $\phi \in K_\xi^\perp \cap H_0^1(\Omega)$ such that

$$(\phi, \psi)_{H_0^1(\Omega)} = \int_\Omega (W\phi - h)\psi\, dx \quad \text{for all } \psi \in K_\xi^\perp \cap H_0^1(\Omega).$$

With the aid of Riesz's representation theorem, this equation gets rewritten in $K_\xi^\perp \cap H_0^1(\Omega)$ in the operatorial form

$$(\mathrm{Id} - K)\phi = \tilde{h}, \tag{3.27}$$

where $\tilde{h} = \Pi_\xi^\perp \Delta^{-1} h$ and $K(\phi) = -\Pi_\xi^\perp \Delta^{-1}(W\phi)$ is a linear compact operator in $K_\xi^\perp \cap H_0^1(\Omega)$. The homogeneous equation $\phi = K(\phi)$ in $K_\xi^\perp \cap H_0^1(\Omega)$, which is equivalent to (3.3)–(3.5) with $h \equiv 0$, has only the trivial solution in view of the a priori estimate (3.18). Now, Fredholm's alternative guarantees unique solvability of (3.27) for any $\tilde{h} \in K_\xi^\perp$. Moreover, by elliptic regularity theory this solution is in $W^{2,2}(\Omega)$.

At $p > p_0$ fixed, by density of $C^{0,\alpha}(\overline{\Omega})$ in $(C(\overline{\Omega}), \|\cdot\|_\infty)$, we can approximate $h \in C(\overline{\Omega})$ by smooth functions and, by (3.18) and elliptic regularity theory, we can show that (3.6) holds for any $h \in C(\overline{\Omega})$. The proof is complete. □

Remark 3.2. Given $h \in C(\overline{\Omega})$, let $\bar{\phi}$ be the solution of (3.3)–(3.5) given by Proposition 3.1. Multiplying (3.3) by ϕ and integrating by parts, we get

$$\|\phi\|^2_{H_0^1(\Omega)} = \int_\Omega W\phi^2 - \int_\Omega h\phi.$$

By Lemma 3.1 we get

$$\|\phi\|_{H_0^1(\Omega)} \leqslant C\big(\|\phi\|_\infty + \|h\|_*\big).$$

4. The nonlinear problem

We want to solve the nonlinear auxiliary problem

$$\Delta(U_\xi + \phi) + (U_\xi + \phi)^p = \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} e^{U_j} Z_{ij} \quad \text{in } \Omega, \tag{4.1}$$

$$U_\xi + \phi > 0 \quad \text{in } \Omega, \tag{4.2}$$

$$\phi = 0 \quad \text{on } \partial\Omega, \tag{4.3}$$

$$\int_\Omega e^{U_j} Z_{ij}\phi = 0 \quad \text{for all } i = 1, 2,\ j = 1, \ldots, m, \tag{4.4}$$

for some coefficients c_{ij}, $i = 1, 2$ and $j = 1, \ldots, m$, which depend on ξ. Recalling that

$$N(\phi) = |U_\xi + \phi|^p - U_\xi^p - pU_\xi^{p-1}\phi, \qquad R = \Delta U_\xi + U_\xi^p,$$

we can rewrite (4.1) in the form

$$L(\phi) = -\big(R + N(\phi)\big) + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} e^{U_j} Z_{ij}.$$

Using the theory developed in the previous section for the linear operator L, we prove the following result:

Lemma 4.1. *Let $\varepsilon > 0$ be fixed. There exist $C > 0$ and $p_0 > 0$ such that, for any $p > p_0$ and $\xi \in \mathcal{O}_\varepsilon$, problem* (4.1)–(4.4) *has a unique solution ϕ_ξ which satisfies*

$$\|\phi_\xi\|_\infty \leqslant \frac{C}{p^3}. \tag{4.5}$$

Further, there holds:

$$\sum_{i=1}^{2}\sum_{j=1}^{m}\left|c_{ij}(\xi)\right| \leqslant \frac{C}{p^4}, \qquad \|\phi_\xi\|_{H_0^1(\Omega)} \leqslant \frac{C}{p^3}. \tag{4.6}$$

Proof. Let us denote by $\mathcal{C}_*$ the function space $C(\overline{\Omega})$ endowed with the norm $\|\cdot\|_*$. Proposition 3.1 implies that the unique solution $\phi = T(h)$ of (3.3)–(3.5) defines a continuous linear map from the Banach space $\mathcal{C}_*$ into $C_0(\overline{\Omega})$, with norm bounded by a multiple of p. Problem (4.1)–(4.4) becomes

$$\phi = A(\phi) := -T\bigl(R + N(\phi)\bigr).$$

For a given number $\gamma > 0$, let us consider the region

$$\mathcal{F}_\gamma := \left\{\phi \in C_0(\overline{\Omega}):\ \|\phi\|_\infty \leqslant \frac{\gamma}{p^3}\right\}.$$

We have the following estimates:

$$\bigl\|N(\phi)\bigr\|_* \leqslant Cp\|\phi\|_\infty^2, \qquad \bigl\|N(\phi_1) - N(\phi_2)\bigr\|_* \leqslant Cp\Bigl(\max_{i=1,2}\|\phi_i\|_\infty\Bigr)\|\phi_1 - \phi_2\|_\infty, \tag{4.7}$$

for any $\phi, \phi_1, \phi_2 \in \mathcal{F}_\gamma$. In fact, by Lagrange's theorem we have that

$$\bigl|N(\phi)\bigr| \leqslant p(p-1)\left(U_\xi + O\left(\frac{1}{p^3}\right)\right)^{p-2}\phi^2,$$

$$\bigl|N(\phi_1) - N(\phi_2)\bigr| \leqslant p(p-1)\left(U_\xi + O\left(\frac{1}{p^3}\right)\right)^{p-2}\Bigl(\max_{i=1,2}|\phi_i|\Bigr)|\phi_1 - \phi_2|$$

for any $x \in \Omega$, and hence, by (3.2) we get (4.7) since $\|\sum_{j=1}^m e^{U_j}\|_* = O(1)$. By (4.7), Propositions 2.1 and 3.1 imply that

$$\bigl\|A(\phi)\bigr\|_\infty \leqslant D'p\bigl(\bigl\|N(\phi)\bigr\|_* + \|R\|_*\bigr) \leqslant O\bigl(p^2\|\phi\|_\infty^2\bigr) + \frac{D}{p^3} \quad \text{and}$$

$$\bigl\|A(\phi_1) - A(\phi_2)\bigr\|_\infty \leqslant C'p\bigl\|N(\phi_1) - N(\phi_2)\bigr\|_* \leqslant Cp^2\Bigl(\max_{i=1,2}\|\phi_i\|_\infty\Bigr)\|\phi_1 - \phi_2\|_\infty$$

for any $\phi, \phi_1, \phi_2 \in \mathcal{F}_\gamma$, where D is independent of γ. Hence, if $\|\phi\|_\infty \leqslant \frac{2D}{p^3}$, we have that

$$\bigl\|A(\phi)\bigr\|_\infty = O\left(\frac{1}{p}\|\phi\|_\infty\right) + \frac{D}{p^3} \leqslant \frac{2D}{p^3}.$$

Choose $\gamma = 2D$. Then, A is a contraction mapping of $\mathcal{F}_\gamma$ since

$$\bigl\|A(\phi_1) - A(\phi_2)\bigr\|_\infty \leqslant \frac{1}{2}\|\phi_1 - \phi_2\|_\infty$$

for any $\phi_1, \phi_2 \in \mathcal{F}_\gamma$. Therefore, a unique fixed point ϕ_ξ of A exists in $\mathcal{F}_\gamma$. By (3.26), we get that

$$\sum_{i=1}^{2}\sum_{j=1}^{m}\left|c_{ij}(\xi)\right| = O\left(\left\|N(\phi_\xi)\right\|_* + \|R\|_* + \frac{1}{p}\|\phi_\xi\|_\infty\right) \leqslant \frac{C}{p^4}$$

and by Remark 3.2 we deduce that

$$\|\phi_\xi\|_{H_0^1(\Omega)} = O\left(\|\phi_\xi\|_\infty + N\left\|N(\phi_\xi)\right\|_* + \|R\|\right) \leqslant \frac{C}{p^3}.$$

The proof is now complete since ϕ_ξ solves (4.1)–(4.4): in order to show the validity of (4.2), let us remark that $p|\phi_\xi| \to 0$ in $C(\overline{\Omega})$ and by elliptic regularity theory $p|\phi_\xi| \to 0$ in $C^1(\overline{\Omega} \setminus \bigcup_{j=1}^m B(\xi_j, \epsilon))$ and so, we can proceed as in Remark 2.1 to show that $U_\xi + \phi_\xi > 0$ in Ω. □

Let $\xi_1, \xi_2 \in \mathcal{O}_\varepsilon$. Since

$$\begin{aligned}
&\Delta(\phi_{\xi_1} - \phi_{\xi_2}) + pU_{\xi_1}^{p-1}(\phi_{\xi_1} - \phi_{\xi_2}) \\
&\quad = \left((U_{\xi_2} + \phi_{\xi_2})^p - (U_{\xi_1} + \phi_{\xi_2})^p\right) + \left((U_{\xi_1} + \phi_{\xi_2})^p - (U_{\xi_1} + \phi_{\xi_2})^p - pU_{\xi_1}^{p-1}(\phi_{\xi_2} - \phi_{\xi_1})\right) \\
&\qquad + \Delta(U_{\xi_2} - U_{\xi_1}) + \sum_{i=1}^{2}\sum_{j=1}^{m}\left(c_{ij}(\xi_1) - c_{ij}(\xi_2)\right)e^{U_j}(\xi_1)Z_{ij}(\xi_1) \\
&\qquad + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}(\xi_2)\left(e^{U_j}(\xi_1)Z_{ij}(\xi_1) - e^{U_j}(\xi_2)Z_{ij}(\xi_2)\right)
\end{aligned}$$

and by (3.2)

$$\begin{aligned}
&\left\|\left((U_{\xi_1} + \phi_{\xi_2})^p - (U_{\xi_1} + \phi_{\xi_1})^p - pU_{\xi_1}^{p-1}(\phi_{\xi_2} - \phi_{\xi_1})\right)\right\|_* \\
&\quad \leqslant \frac{C}{p^2}\left\|\phi_{\xi_1} - \phi_{\xi_2}\right\|_\infty \left\|p\left(U_{\xi_1} + O\left(\frac{1}{p^3}\right)\right)^{p-2}\right\|_* = o\left(\frac{1}{p}\|\phi_{\xi_1} - \phi_{\xi_2}\|_\infty\right)
\end{aligned}$$

uniformly in $\mathcal{O}_\varepsilon$, by Proposition 3.1 and (4.6) we get

$$\begin{aligned}
\|\phi_{\xi_1} - \phi_{\xi_2}\|_\infty &\leqslant Cp\left\|(U_{\xi_2} + \phi_{\xi_2})^p - (U_{\xi_1} + \phi_{\xi_2})^p\right\|_* \\
&\quad + \frac{C}{p^3}\sum_{i=1}^{2}\sum_{j=1}^{m}\left\|e^{U_j}(\xi_1)Z_{ij}(\xi_1) - e^{U_j}(\xi_2)Z_{ij}(\xi_2)\right\|_* + Cp\left\|\Delta(U_{\xi_2} - U_{\xi_1})\right\|_*,
\end{aligned}$$

for any $p \geqslant p_0$ and $\xi_1, \xi_2 \in \mathcal{O}_\varepsilon$ (here, $\|\cdot\|_*$ is considered with respect to ξ_1). Hence, for fixed $p \geqslant p_0$, the map $\xi \to \phi_\xi$ is continuous in $C_0(\overline{\Omega})$ and, in view of Remark 3.2, in $H_0^1(\Omega)$. Further, this map is a C^1-function in $C_0(\overline{\Omega})$ as it follows by the Implicit Function Theorem applied to the equation:

$$G(\xi, \phi) := \Pi_\xi^\perp\left[U_\xi + \Pi_\xi^\perp\phi + \Delta^{-1}\left(U_\xi + \Pi_\xi^\perp\phi\right)^p\right] + \Pi_\xi\phi = 0, \quad \phi \in C_0(\overline{\Omega}),$$

where Π_ξ, $\Pi_\xi^\perp$ are the maps from $L^2(\Omega)$ respectively onto

$$K_\xi = \left\{\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} PZ_{ij}\colon\ c_{ij}\in\mathbb{R} \text{ for } i=1,2,\ j=1,\dots,m\right\} \quad \text{and}$$

$$K_\xi^\perp = \left\{\phi\in L^2(\Omega)\colon \int_\Omega e^{U_j} Z_{ij}\phi = 0 \text{ for } i=1,2,\ j=1,\dots,m\right\}$$

(see the notations in step 6 in the proof of Proposition 3.1). Let us remark that $\Pi_\xi\phi\in C_0^k(\overline{\Omega})$, for any $k\geqslant 0$. Indeed, $G(\xi,\phi_\xi)=0$ and the linearized operator:

$$\frac{\partial G}{\partial\phi}(\xi,\phi_\xi) = \Pi_\xi^\perp\left[\mathrm{id} + p\Delta^{-1}\left((U_\xi+\phi_\xi)^{p-1}\Pi_\xi^\perp\right)\right] + \Pi_\xi$$

is invertible for p large. In fact, easily we reduce the invertibility property to uniquely solve the equation $\frac{\partial G}{\partial\phi}(\xi,\phi_\xi)[\phi]=h$ in $K_\xi^\perp$ for any $h\in K_\xi^\perp\cap C_0(\overline{\Omega})$. By Fredholm's alternative, we need to show that in $K_\xi^\perp\cap C_0(\Omega)$ there is only the trivial solution for the equation $\frac{\partial G}{\partial\phi}(\xi,\phi_\xi)[\phi]=0$, or equivalently for

$$L\phi = p\left(U_\xi^{p-1} - (U_\xi+\phi_\xi)^{p-1}\right)\phi + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij} e^{U_j} Z_{ij},$$

for any choice of the coefficients c_{ij}, since by elliptic regularity theory $\phi\in C_0^2(\overline{\Omega})$. By Proposition 3.1 and (3.2), we derive that

$$\begin{aligned}\|\phi\|_\infty &\leqslant C' p\left\| p\left(U_\xi^{p-1} - (U_\xi+\phi_\xi)^{p-1}\right)\phi\right\|_* \\ &\leqslant C' p^2 \|\phi\|_\infty \|\phi_\xi\|_\infty \left\| p\left(U_\xi + O\left(\frac{1}{p^3}\right)\right)^{p-2}\right\|_* < \|\phi\|_*\end{aligned}$$

and hence, $\phi=0$. Similarly, we have also that $\xi\to\phi_\xi$ is a C^1-function in $H_0^1(\Omega)$.

5. Variational reduction

After problem (4.1)–(4.4) has been solved, we find a solution of (2.15) (and hence for (1.1)) if ξ is such that

$$c_{ij}(\xi)=0 \quad \text{for all } i=1,2,\ j=1,\dots,m, \tag{5.1}$$

where $c_{ij}(\xi)$ are the coefficients in (4.1). Problem (5.1) has a variational structure. Associated to (1.1), let us consider the energy functional J_p given by

$$J_P(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx, \quad u\in H_0^1(\Omega),$$

and the finite-dimensional restriction

$$F(\xi) := J_p(U_\xi + \phi_\xi), \tag{5.2}$$

where ϕ_ξ is the unique solution to problem (4.1)–(4.4) given by Lemma 4.1. Critical points of F correspond to solutions of (5.1) for large p, as the following result states:

Lemma 5.1. *The functional $F(\xi)$ is of class C^1. Moreover, for all p sufficiently large, if $D_\xi F(\xi) = 0$ then ξ satisfies* (5.1).

Proof. We have already shown that the map $\xi \to \phi_\xi$ is a C^1-map into $H_0^1(\Omega)$ and then, $F(\xi)$ is a C^1-function of ξ.

Since $D_\xi F(\xi) = 0$, we have that

$$\begin{aligned} 0 &= -\int_\Omega \big(\Delta(U_\xi + \phi_\xi) + (U_\xi + \phi_\xi)^p\big)(D_\xi U_\xi + D_\xi \phi_\xi) \\ &= -\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}(\xi) \int_\Omega e^{U_j} Z_{ij}(D_\xi U_\xi + D_\xi \phi_\xi) \\ &= -\sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}(\xi) \int_\Omega e^{U_j} Z_{ij} D_\xi U_\xi + \sum_{i=1}^{2}\sum_{j=1}^{m} c_{ij}(\xi) \int_\Omega D_\xi\big(e^{U_j} Z_{ij}\big)\phi_\xi \end{aligned}$$

since $\int_\Omega e^{U_j} Z_{ij}\phi_\xi = 0$. By the expression of U_ξ, we have that

$$\begin{aligned} \partial_{(\xi_j)_i} U_\xi &= -\sum_{s=1}^{m} \frac{1}{\gamma \mu_s^{2/(p-1)}} P\bigg[\frac{2}{p-1} U_{\delta_s,\xi_s} - 2Z_{0s} + \bigg(\frac{2}{p(p-1)} w_0 + \frac{2}{p^2(p-1)} w_1 \\ &\quad + \frac{1}{p}\nabla w_0 \cdot y + \frac{1}{p^2}\nabla w_1 \cdot y\bigg)\bigg|_{y=\frac{x-\xi_s}{\delta_s}}\bigg] \partial_{(\xi_j)_i} \log \mu_s \\ &\quad + \frac{1}{\gamma \delta_j \mu_j^{2/(p-1)}} P\bigg(Z_{ij} - \frac{1}{p}\partial_i w_0\bigg(\frac{x-\xi_j}{\delta_j}\bigg) - \frac{1}{p^2}\partial_i w_1\bigg(\frac{x-\xi_j}{\delta_j}\bigg)\bigg) \\ &= \frac{1}{\gamma \delta_j \mu_j^{2/(p-1)}} P\bigg(Z_{ij} - \frac{1}{p}\partial_i w_0\bigg(\frac{x-\xi_j}{\delta_j}\bigg) - \frac{1}{p^2}\partial_i w_1\bigg(\frac{x-\xi_j}{\delta_j}\bigg)\bigg) + O\bigg(\frac{1}{\gamma}\bigg), \end{aligned} \tag{5.3}$$

since $P : L^\infty(\Omega) \to L^\infty(\Omega)$ is a continuous operator (apply to $\frac{1}{p-1}U_{\delta_s,\xi_s}$, Z_{0s}, $\frac{1}{p}w_j(\frac{x-\xi_s}{\delta_s})$, $\nabla w_j(\frac{x-\xi_s}{\delta_s}) \cdot (\frac{x-\xi_s}{\delta_s})$ for $j = 0, 1$ which are bounded in Ω in view of Lemma 2.1), and

$$\begin{aligned} \partial_{(\xi_j)_i}\big(e^{U_h} Z_{lh}\big) &= -4\delta_h e^{U_h}\bigg(\frac{\delta_{il}}{\delta_h^2 + |x-\xi_h|^2} - 6\frac{(x-\xi_h)_l (x-\xi_j)_i}{(\delta_h^2 + |x-\xi_h|^2)^2}\bigg)\delta_{hj} \\ &\quad + 3e^{U_h} Z_{0h} Z_{lh} \partial_{(\xi_j)_i} \log \mu_h, \end{aligned} \tag{5.4}$$

where δ_{hj} denotes the Kronecker's symbol. Hence, by (5.3), (5.4) for $i=1,2$ and $j=1,\dots,m$, we get that

$$0=\partial_{(\xi_j)_i}F(\xi)=-\frac{1}{\gamma\delta_j\mu_j^{2/(p-1)}}\sum_{l=1}^{2}\sum_{h=1}^{m}c_{lh}(\xi)(PZ_{ij},PZ_{lh})_{H_0^1(\Omega)}$$
$$+O\bigg(\frac{1}{p\gamma\delta_j}+\|\phi_\xi\|_\infty\int_\Omega\big|\partial_{(\xi_j)_i}\big(e^{U_h}Z_{lh}\big)\big|\bigg)\sum_{l=1}^{2}\sum_{h=1}^{m}\big|c_{lh}(\xi)\big|$$

since $\partial_i w_j(\frac{x-\xi_s}{\delta_s})$ is bounded in Ω, $j=0,1$, in view of Lemma 2.1. Taking into account (3.22), (3.23), (4.5) and (5.4) we get

$$0=\frac{64}{\gamma\delta_j\mu_j^{2/(p-1)}}\bigg(\int_{\mathbb{R}^2}\frac{|y|^2}{(1+|y|^2)^4}\,dy\bigg)c_{ij}(\xi)+O\bigg(\frac{1}{p\gamma\delta_j}\sum_{l=1}^{2}\sum_{h,s=1}^{m}\big|c_{lh}(\xi)\big|\bigg)$$

which implies for p large (independent of $\xi\in\mathcal{O}_\varepsilon$) that $c_{ij}(\xi)=0$ for any $i=1,2$ and $j=1,\dots,m$. □

Next lemma shows that the leading term of the function $F(\xi)$ is given by $\varphi_m(\xi)$.

Lemma 5.2. *Let $\varepsilon>0$. The following expansion holds*:

$$F(\xi)=\frac{4\pi mp}{\gamma^2}-\frac{32\pi^2}{\gamma^2}\varphi_m(\xi_1,\dots,\xi_m)+\frac{4\pi m}{\gamma^2}+\frac{m}{2\gamma^2}\int_{\mathbb{R}^2}\bigg(\frac{8}{(1+|y|^2)^2}v_\infty-\Delta w_0\bigg)+O\bigg(\frac{1}{p^3}\bigg)$$

uniformly for $\xi\in\mathcal{O}_\varepsilon$.

Proof. First of all, multiply (4.1) by $U_\xi+\phi_\xi$ and integrate by parts to get:

$$\int_\Omega(U_\xi+\phi_\xi)^{p+1}=\int_\Omega\big|\nabla(U_\xi+\phi_\xi)\big|^2+\sum_{i=1}^{2}\sum_{j=1}^{m}c_{ij}\int_\Omega e^{U_j}Z_{ij}(U_\xi+\phi_\xi)$$
$$=\int_\Omega\big|\nabla(U_\xi+\phi_\xi)\big|^2+\sum_{i=1}^{2}\sum_{j=1}^{m}c_{ij}\int_\Omega e^{U_j}Z_{ij}U_\xi$$

in view of (4.4). Since U_ξ is a bounded function, by (4.6) we get that

$$\int_\Omega(U_\xi+\phi_\xi)^{p+1}=\int_\Omega\big|\nabla(U_\xi+\phi_\xi)\big|^2+O\bigg(\frac{1}{p^4}\bigg)$$

uniformly for $\xi\in\mathcal{O}_\varepsilon$. We can write

$$F(\xi)=\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega\left|\nabla(U_\xi+\phi_\xi)\right|^2+O\left(\frac{1}{p^4}\right)$$
$$=\left(\frac{1}{2}-\frac{1}{p+1}\right)\left(\int_\Omega|\nabla U_\xi|^2+2\int_\Omega\nabla U_\xi\nabla\phi_\xi+\int_\Omega|\nabla\phi_\xi|^2\right)+O\left(\frac{1}{p^4}\right). \tag{5.5}$$

We expand the term $\int_\Omega|\nabla U_\xi|^2$: in view of (2.14) and (2.20) we have that

$$\int_\Omega|\nabla U_\xi|^2=\sum_{j=1}^m\frac{1}{\gamma\mu_j^{2/(p-1)}}\int_{B(\xi_j,\varepsilon)}\left(e^{U_j}-\frac{1}{p\delta_j^2}\Delta w_0\left(\frac{x-\xi_j}{\delta_j}\right)\right.$$
$$\left.-\frac{1}{p^2\delta_j^2}\Delta w_1\left(\frac{x-\xi_j}{\delta_j}\right)+O\left(p^2e^{-\frac{p}{2}}\right)\right)U_\xi+O\left(e^{-\frac{p}{2}}\right)$$
$$=\sum_{j=1}^m\frac{1}{\gamma^2\mu_j^{4/(p-1)}}\int_{B(0,\varepsilon/\delta_j)}\left(\frac{8}{(1+|y|^2)^2}-\frac{1}{p}\Delta w_0-\frac{1}{p^2}\Delta w_1+O\left(p^2e^{-p}\right)\right)$$
$$\times\left(p+v_\infty+\frac{1}{p}w_0+\frac{1}{p^2}w_1+O\left(e^{-\frac{p}{4}}|y|+e^{-\frac{p}{4}}\right)\right)+O\left(e^{-\frac{p}{2}}\right)$$
$$=\sum_{j=1}^m\frac{1}{\gamma^2\mu_j^{4/(p-1)}}\left(8\pi p+\int_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty-\Delta w_0\right)+O\left(\frac{1}{p}\right)\right)$$
$$=\frac{8\pi mp}{\gamma^2}-\frac{32\pi}{\gamma^2}\sum_{j=1}^m\log\mu_j+\frac{m}{\gamma^2}\int_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty-\Delta w_0\right)+O\left(\frac{1}{p^3}\right)$$

since $\mu_j^{-\frac{4}{p-1}}=1-\frac{4}{p}\log\mu_j+O(\frac{1}{p^2})$. Recalling property (2.13) of μ_i, then we get that:

$$\int_\Omega|\nabla U_\xi|^2=\frac{8\pi mp}{\gamma^2}-\frac{64\pi^2}{\gamma^2}\varphi_m(\xi_1,\dots,\xi_m)+\frac{24\pi m}{\gamma^2}$$
$$+\frac{m}{\gamma^2}\int_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty-\Delta w_0\right)+O\left(\frac{1}{p^3}\right) \tag{5.6}$$

uniformly for $\xi\in\mathcal{O}_\varepsilon$. Hence, using (4.6) and (5.6) we get that

$$\int_\Omega\nabla U_\xi\nabla\phi_\xi+\frac{1}{2}\int_\Omega|\nabla\phi_\xi|^2=O\left(\frac{1}{p^{7/2}}\right). \tag{5.7}$$

Finally, inserting (5.6), (5.7) in (5.5), we get that

$$F(\xi)=\frac{4\pi mp}{\gamma^2}-\frac{32\pi^2}{\gamma^2}\varphi_m(\xi_1,\dots,\xi_m)+\frac{4\pi m}{\gamma^2}+\frac{m}{2\gamma^2}\int\limits_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty-\Delta w_0\right)+O\left(\frac{1}{p^3}\right)$$

uniformly for $\xi\in\mathcal{O}_\varepsilon$. □

Now, we want to show that the expansion of $F(\xi)$ in terms of $\varphi_m(\xi)$ holds in a C^1-sense.

Lemma 5.3. *Let* $\varepsilon>0$. *The following expansion holds*:

$$\nabla_{(\xi_j)_i}F(\xi)=-\frac{32\pi^2}{\gamma^2}\nabla_{(\xi_j)_i}\varphi_m(\xi_1,\dots,\xi_m)+o\left(\frac{1}{p^2}\right)$$

uniformly for $\xi\in\mathcal{O}_\varepsilon$, *for any* $j=1,\dots,m$ *and* $i=1,2$.

Proof. Let $j\in\{1,\dots,m\}$ and $i\in\{1,2\}$ be fixed. We want to expand the derivatives of $F(\xi)$ in ξ:

$$\partial_{(\xi_j)_i}F(\xi)=-\int\limits_\Omega\left(\Delta u_\xi+u_\xi^p\right)\partial_{(\xi_j)_i}u_\xi,$$

where $u_\xi=U_\xi+\phi_\xi$. Let us remark that it is very difficult to show directly that the expansion of $F(\xi)$ holds in a C^1-sense since there is a difference between the exponential decay of the concentration parameters $\delta_i=\mu_i e^{-\frac{p}{4}}$ and the polynomial decay $\frac{1}{p^4}$ of $\|R_\xi\|_*$ (see Proposition 2.1). As usual in similar contexts (also in higher dimensions), we should be able to show that $\|\partial_\xi\phi_\xi\|_\infty$ is of order $\frac{\|\phi_\xi\|_\infty}{\delta_i}$. Unfortunately, since $\|\phi_\xi\|_\infty$ is only of order $\frac{1}{p^3}$, $\partial_\xi\phi_\xi$ is not a small function and, at a first glance, there is no hope for a C^1-expansion of $F(\xi)$. To overcome the problem, the idea is the following: first, we replace the term $\partial_{(\xi_j)_i}u_\xi$ with $\partial_{x_i}u_\xi$ in the expression of $\partial_{(\xi_j)_i}F(\xi)$ in a neighborhood of ξ_j, up to higher order terms, and afterwards we use a Pohozaev-type identity based on integration by parts.

To this purpose, let η be a radial smooth cut-off function such that $0\leqslant\eta\leqslant 1$, $\eta\equiv 1$ for $|x|\leqslant\epsilon$ and $\eta\equiv 0$ for $|x|\geqslant 2\epsilon$. In view of (4.1) and (4.4), we can write

$$\begin{aligned}
&\int\limits_\Omega\left(\Delta u_\xi+u_\xi^p\right)\partial_{(\xi_j)_i}\phi_\xi\\
&=\sum_{k=1}^2\sum_{l=1}^m c_{kl}\int\limits_\Omega e^{U_l}Z_{kl}\partial_{(\xi_j)_i}\phi_\xi=-\sum_{k=1}^2\sum_{l=1}^m c_{kl}\int\limits_\Omega\partial_{(\xi_j)_i}\left(e^{U_l}Z_{kl}\right)\phi_\xi\\
&=\sum_{k=1}^2\sum_{l=1}^m c_{kl}\int\limits_\Omega\partial_{x_i}\left(e^{U_l}Z_{kl}\right)\eta(x-\xi_j)\phi_\xi\\
&\quad-\sum_{k=1}^2\sum_{l=1}^m c_{kl}\int\limits_\Omega\left[\partial_{(\xi_j)_i}\left(e^{U_l}Z_{kl}\right)+\eta(x-\xi_j)\partial_{x_i}\left(e^{U_l}Z_{kl}\right)\right]\phi_\xi
\end{aligned}$$

$$= -\sum_{k=1}^{2}\sum_{l=1}^{m} c_{kl}\int_{\Omega} e^{U_l} Z_{kl}\partial_{x_i}\big(\eta(x-\xi_j)\phi_\xi\big)$$
$$-\sum_{k=1}^{2}\sum_{l=1}^{m} c_{kl}\int_{\Omega}\big[\partial_{(\xi_j)_i}\big(e^{U_l}Z_{kl}\big)+\eta(x-\xi_j)\partial_{x_i}\big(e^{U_l}Z_{kl}\big)\big]\phi_\xi \tag{5.8}$$

by an integration by parts of the derivative in x_i. As for (5.4), we get that

$$\begin{aligned}
&\partial_{(\xi_j)_i}\big(e^{U_l}Z_{kl}\big)+\eta(x-\xi_j)\partial_{x_i}\big(e^{U_l}Z_{kl}\big)\\
&\quad = 4\delta_l e^{U_l}\frac{\eta(x-\xi_j)-\delta_{lj}}{\delta_l^2+|x-\xi_l|^2}\delta_{ik}+24\delta_l e^{U_l}\frac{(x-\xi_l)_k(x-\xi_l)_i}{(\delta_l^2+|x-\xi_l|^2)^2}\big(\delta_{lj}-\eta(x-\xi_j)\big)\\
&\qquad + 3e^{U_l}Z_{0l}Z_{kl}\partial_{(\xi_j)_i}\log\mu_l\\
&\quad = 3e^{U_l}Z_{0l}Z_{kl}\partial_{(\xi_j)_i}\log\mu_l + O\big(e^{-\frac{3}{4}p}\big),
\end{aligned}$$

where δ_{lj} denotes the Kronecker's symbol, and hence, we get that

$$\begin{aligned}
&\left|\sum_{k=1}^{2}\sum_{l=1}^{m} c_{kl}\int_{\Omega}\big[\partial_{(\xi_j)_i}\big(e^{U_l}Z_{kl}\big)+\eta(x-\xi_j)\partial_{x_i}\big(e^{U_l}Z_{kl}\big)\big]\phi_\xi\right|\\
&\quad \leqslant C\|\phi_\xi\|_\infty \max_{k,l}|c_{kl}|\int_{\Omega}\big(e^{U_l}+O\big(e^{-\frac{3}{4}p}\big)\big)=O\left(\frac{1}{p^7}\right)
\end{aligned}$$

in view of $|Z_{0l}Z_{kl}|\leqslant 2$, (4.5), (4.6). Inserting in (5.8), we get that

$$\int_{\Omega}\big(\Delta u_\xi+u_\xi^p\big)\partial_{(\xi_j)_i}\phi_\xi = -\sum_{k=1}^{2}\sum_{l=1}^{m} c_{kl}\int_{\Omega} e^{U_l}Z_{kl}\partial_{x_i}\big(\eta(x-\xi_j)\phi_\xi\big)+O\left(\frac{1}{p^7}\right). \tag{5.9}$$

Since $p\phi_\xi\to 0$ in C^1-norm away from $\xi_1,\dots,\xi_m$, (5.9) gives that

$$\begin{aligned}
\int_{\Omega}\big(\Delta u_\xi+u_\xi^p\big)\partial_{(\xi_j)_i}\phi_\xi &= -\sum_{k=1}^{2}\sum_{l=1}^{m} c_{kl}\int_{B(\xi_j,\epsilon)} e^{U_l}Z_{kl}\partial_{x_i}\phi_\xi + O\left(\frac{1}{p^7}\right)\\
&= -\int_{B(\xi_j,\epsilon)}\big(\Delta u_\xi+u_\xi^p\big)\partial_{x_i}\phi_\xi + O\left(\frac{1}{p^7}\right)
\end{aligned} \tag{5.10}$$

always in view of (4.1), (4.5) and (4.6).

Now, by Lemma 2.1 we get that $|\frac{1}{\delta_s}\partial_{x_i}w_l(\frac{x-\delta_s}{\delta_s})|\leqslant C$ for any $l=1,2$ and for x away from ξ_s. Hence, by the expression of U_ξ and (2.2), (2.11) we get that for $|x-\xi_j|\leqslant 2\epsilon$:

$$\begin{aligned}\partial_{x_i} U_\xi &= \sum_{s=1}^m \frac{1}{\gamma \mu_s^{2/(p-1)}} \left(\partial_{x_i} P U_{\delta_s,\xi_s} + \frac{1}{p} \partial_{x_i} P\left(w_0\left(\frac{x-\xi_s}{\delta_s}\right)\right) + \frac{1}{p^2} \partial_{x_i} P\left(w_1\left(\frac{x-\xi_s}{\delta_s}\right)\right)\right) \\ &= \sum_{s=1}^m \frac{1}{\gamma \mu_s^{2/(p-1)}} \left(\partial_{x_i} U_{\delta_s,\xi_s} + \frac{1}{p\delta_s} \partial_{x_i} w_0\left(\frac{x-\xi_s}{\delta_s}\right) + \frac{1}{p^2\delta_s} \partial_{x_i} w_1\left(\frac{x-\xi_s}{\delta_s}\right)\right) + O\left(\frac{1}{\gamma}\right) \\ &= -\frac{1}{\gamma \delta_j \mu_j^{2/(p-1)}} \left(Z_{ij} - \frac{1}{p} \partial_{x_i} w_0\left(\frac{x-\xi_j}{\delta_j}\right) - \frac{1}{p^2} \partial_{x_i} w_1\left(\frac{x-\xi_j}{\delta_j}\right)\right) + O\left(\frac{1}{\gamma}\right), \end{aligned} \tag{5.11}$$

since $\partial_{x_i} U_{\delta_s,\xi_s} = -\frac{1}{\delta_s} Z_{is}$. Since, as already observed, $\partial_{x_i} w_l(\frac{x-\xi_j}{\delta_j}) = O(\delta_j)$ uniformly away from ξ_j, $l = 1, 2$, by (5.11), in particular, we get

$$\partial_{x_i} U_\xi = O\left(\frac{1}{\gamma}\right), \tag{5.12}$$

for $\epsilon \leqslant |x-\xi_j| \leqslant 2\epsilon$. Moreover, the maximum principle and Lemma 2.1 imply that

$$P\left(\partial_{x_i} w_l\left(\frac{x-\xi_j}{\delta_j}\right)\right) - \partial_{x_i} w_l\left(\frac{x-\xi_j}{\delta_j}\right) = O(\delta_j)$$

in $C(\overline{\Omega})$ for any $l = 1, 2$, and hence, by (5.3) and (5.11) we get that

$$\partial_{(\xi_j)_i} U_\xi + \eta(x-\xi_j)\partial_{x_i} U_\xi = \frac{1}{\gamma \delta_j \mu_j^{2/(p-1)}} (PZ_{ij} - Z_{ij}) + O\left(\frac{1}{\gamma}\right) = O\left(\frac{1}{\gamma}\right) \tag{5.13}$$

in view of (3.20). Now, by (5.12), (5.13) we can write that:

$$\begin{aligned} &\int_\Omega \left(\Delta u_\xi + u_\xi^p\right) \partial_{(\xi_j)_i} U_\xi \\ &\quad = -\sum_{k=1}^2 \sum_{l=1}^m c_{kl} \int_\Omega e^{U_l} Z_{kl} \eta(x-\xi_j) \partial_{x_i} U_\xi \\ &\qquad + \sum_{k=1}^2 \sum_{l=1}^m c_{kl} \int_\Omega e^{U_l} Z_{kl} \left(\partial_{(\xi_j)_i} U_\xi + \eta(x-\xi_j)\partial_{x_i} U_\xi\right) \\ &\quad = -\sum_{k=1}^2 \sum_{l=1}^m c_{kl} \int_{B(\xi_j,\epsilon)} e^{U_l} Z_{kl} \partial_{x_i} U_\xi + O\left(\frac{1}{p^5}\right) = -\int_{B(\xi_j,\epsilon)} \left(\Delta u_\xi + u_\xi^p\right) \partial_{x_i} U_\xi + O\left(\frac{1}{p^5}\right) \end{aligned} \tag{5.14}$$

in view of (4.1) and (4.6). Resuming, by (5.10) and (5.14) we have:

$$\begin{aligned}\partial_{(\xi_j)_i} F(\xi) &= -\int_{\Omega} \left(\Delta u_\xi + u_\xi^p\right)(\partial_{(\xi_j)_i} U_\xi + \partial_{(\xi_j)_i}\phi_\xi) \\ &= \int_{B(\xi_j,\epsilon)} \left(\Delta u_\xi + u_\xi^p\right)(\partial_{x_i} U_\xi + \partial_{x_i}\phi_\xi) + O\left(\frac{1}{p^5}\right) \\ &= \int_{B(\xi_j,\epsilon)} \left(\Delta u_\xi + u_\xi^p\right)\partial_{x_i} u_\xi + O\left(\frac{1}{p^5}\right). \end{aligned} \tag{5.15}$$

Now, the role of $\nabla\varphi_m(\xi)$ becomes clear by means of the following Pohozaev-type identity (we follow some arguments of [13]): for any $B \subset \Omega$ and for any function u

$$\int_B \Delta u \nabla u = \int_{\partial B} \left(\partial_n u \nabla u - \frac{1}{2}|\nabla u|^2 n\right), \qquad \int_B u^p \nabla u = \frac{1}{1+p}\int_{\partial B} u^{p+1} n, \tag{5.16}$$

where $n(x)$ is the unit outer normal vector of ∂B at $x \in \partial B$. Let

$$\varphi_j(x) = H(x,\xi_j) + \sum_{l\neq j} G(x,\xi_l)$$

for any $j = 1,\dots,m$. Since, as already observed, $p\phi_\xi \to 0$ in C^1-norm away from $\xi_1,\dots,\xi_m$, for our function u_ξ by (2.2) and (2.11) we have the following asymptotic property:

$$pu_\xi(x) \to 8\pi\sqrt{e}\sum_{l=1}^m G(x,\xi_l) \quad \text{in } C^1_{\text{loc}}\left(\overline{\Omega}\setminus\{\xi_1,\dots,\xi_m\}\right). \tag{5.17}$$

Apply now (5.16) on $B = B(\xi_j,\varepsilon)$, $j = 1,\dots,m$, and use (5.17) to obtain as $p \to +\infty$

$$\begin{aligned}\int_B \left(\Delta u_\xi + u_\xi^p\right)\nabla u_\xi &= \int_{\partial B}\left(\partial_n u_\xi \nabla u_\xi - \frac{1}{2}|\nabla u_\xi|^2 n + \frac{1}{1+p}u_\xi^{p+1} n\right) \\ &= \frac{64\pi^2 e}{p^2}\int_{\partial B}\left[\left(-\frac{1}{2\pi\epsilon} + \partial_n\varphi_j\right)\left(-\frac{1}{2\pi}\frac{x-\xi_j}{|x-\xi_j|^2} + \nabla\varphi_j\right)\right. \\ &\qquad \left. - \frac{1}{2}\left|-\frac{1}{2\pi}\frac{x-\xi_j}{|x-\xi_j|^2} + \nabla\varphi_j\right|^2 n\right] + o\left(\frac{1}{p^2}\right) \\ &= -\frac{64\pi^2 e}{p^2}\nabla\varphi_j(\xi_j) + o\left(\frac{1}{p^2}\right), \end{aligned}$$

since we decompose $\sum_{l=1}^{m} G(x,\xi_l) = -\frac{1}{2\pi}\ln|x-\xi_j| + \varphi_j(x)$ with $\varphi_j(x)$ a harmonic function near ξ_j. In fact, we have used that

$$\frac{1}{2\pi\epsilon}\int\limits_{\partial B} \nabla\varphi_j = \nabla\varphi_j(\xi_j)$$

since $\nabla\varphi_j$ is harmonic near ξ_j, and by (5.16)

$$\int\limits_{\partial B}\left(\partial_n\varphi_j\nabla\varphi_j - \frac{1}{2}|\nabla\varphi_j|^2 n\right) = \int\limits_{B} \Delta\varphi_j\nabla\varphi_j = 0.$$

Combining with (5.15), finally, we get

$$\partial_{(\xi_j)_i}F(\xi) = -\frac{32\pi^2 e}{p^2}\partial_{(\xi_j)_i}\varphi_m(\xi) + o\left(\frac{1}{p^2}\right) = -\frac{32\pi^2}{\gamma^2}\partial_{(\xi_j)_i}\varphi_m(\xi) + o\left(\frac{1}{p^2}\right)$$

since $\nabla\varphi_j(\xi_j) = \frac{1}{2}\nabla_{\xi_j}\varphi_m(\xi)$. The proof is now complete. □

Finally, we carry out the proof of our main result.

Proof of Theorem 1.2. Let us consider the set $\mathcal{D}$ as in the statement of the theorem, $\mathcal{C}$ the associated critical value and $\xi \in \mathcal{D}$. According to Lemma 5.1, we have a solution of problem (1.1) if we adjust ξ so that it is a critical point of $F(\xi)$ defined by (5.2). This is equivalent to finding a critical point of

$$\widetilde{F}(\xi) = \frac{\gamma^2}{32\pi^2}\left[F(\xi) - \frac{4\pi mp}{\gamma^2} - \frac{4\pi m}{\gamma^2} - \frac{m}{2\gamma^2}\int\limits_{\mathbb{R}^2}\left(\frac{8}{(1+|y|^2)^2}v_\infty - \Delta w_0\right)\right].$$

On the other hand, from Lemmata 5.2 and 5.3, we have that for $\xi \in \mathcal{D}\cap\mathcal{O}_\varepsilon$,

$$\widetilde{F}(\xi) = \varphi_m(\xi) + o(1)\Theta_p(\xi),$$

where Θ_p and $\nabla_\xi\Theta_p$ are uniformly bounded in the considered region as $p \to \infty$.

Let us observe that if $M > \mathcal{C}$, then assumptions (1.4), (1.5) still hold for the function $\min\{M, \varphi_m(\xi)\}$ as well as for $\min\{M, \varphi_m(\xi) + o(1)\Theta_p(\xi)\}$. It follows that the function $\min\{M, \widetilde{F}(\xi)\}$ satisfies for all p large assumptions (1.4), (1.5) in $\mathcal{D}$ and therefore has a critical value $\mathcal{C}_p < M$ which is close to $\mathcal{C}$ in this region. If $\xi_p \in \mathcal{D}$ is a critical point at this level for $\widetilde{F}(\xi)$, then, since

$$\widetilde{F}(\xi_p) \leqslant \mathcal{C}_p < M,$$

we have that there exists $\varepsilon > 0$ such that $|\xi_{p,j} - \xi_{p,i}| > 2\varepsilon$, $\operatorname{dist}(\xi_{p,j}, \partial\Omega) > 2\varepsilon$. This implies C^1-closeness of $\widetilde{F}(\xi)$ and $\varphi_m(\xi)$ at this level, hence $\nabla\varphi_m(\xi_p) \to 0$. The function $u_p(x) = (U_{\xi_p}(x) + \phi_{\xi_p}(x))$ is therefore a solution with the qualitative properties predicted by the theorem. □

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