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Asymptotic behavior of radial solutions for a semilinear elliptic problem on an annulus through Morse index ☆

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Abstract

We study the asymptotic behavior of radial solutions for a singularly perturbed semilinear elliptic Dirichlet problem on an annulus. We show that Morse index informations on such solutions provide a complete description of the blow-up behavior. As a by-product, we exhibit some sufficient conditions to guarantee that radial ground state solutions blow-up and concentrate at the inner/outer boundary of the annulus.

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1. Introduction

In this paper, we study the asymptotic behavior as $\lambda \to +\infty$ of radial solutions to the problem:

$$\begin{cases} -\Delta u + \lambda V(x) u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

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where $p > 1$, $\Omega := \{x \in \mathbb{R}^N\colon\ 1 < |x| < 2\}$ is an annulus and $V : \bar{\Omega} \to \mathbb{R}$ is a radial smooth potential bounded away from zero:

$$\inf_{\Omega} V > 0. \tag{2}$$

The starting point of our analysis is the following, easy to prove, fact: since $H^1_{0,\mathrm{rad}}(\Omega)$, the space of $H^1_0(\Omega)$-radial functions, is compactly embedded into $L^{p+1}(\Omega)$ for any $p > 1$, radial solutions u_λ of (1) blow-up in $L^\infty(\Omega)$, i.e. $\max_\Omega u_\lambda \to +\infty$ as $\lambda \to +\infty$ (similar blow-up occurs in a general domain Ω as well, if $N = 2$ and $1 < p < +\infty$ or $N \geqslant 3$ and $1 < p \leqslant \frac{N+2}{N-2}$). It is then quite interesting, also in view of existence, to identify the limiting equation, to understand the nature of the blow-up set and to describe the asymptotic profile of u_λ: throughout the paper, $\lambda_n \to_n +\infty$ and then $\max_\Omega u_n \to_n +\infty$ (u_n corresponding solution of (1)).

Actually, we only know of a paper by Dancer [4] where some asymptotic analysis of (1) is carried over. It is limited to the case $V \equiv 1$ and p subcritical; by means of ODE techniques, Dancer shows that, for λ large, the only positive radial solution is the radial ground state, and it takes its unique maximum on a sphere whose radius goes to 1.

In some papers [1,2] by Ambrosetti, Malchiodi and Ni the knowledge of the limiting equation is used to obtain existence. Among other things, for potentials V satisfying (2) they found in [2] solutions u_λ blowing up as $\lambda \to +\infty$ on spheres of suitable radius. First, they introduce an auxiliary potential (see also [3])

$$M(r) := r^{n-1} V^\theta(r), \quad \theta = \frac{p+1}{p-1} - \frac{1}{2} \tag{3}$$

(here and in what follows we freely write x as $|x|$ and $V(x)$ as $V(|x|)$). Then, using constructive methods based on a nonlinear Lyapunov–Schmidt reduction, they build solutions u_λ which blow-up at the inner boundary (if $M'(1) > 0$) as well as solutions which blow-up at spheres whose radius is a strict local maximum (or minimum) of M. More in general, the Ambrosetti, Malchiodi and Ni work makes clear the crucial role of the "critical set":

$$\mathcal{M} = \left\{a \in [1,2]\colon\ (a-1)\dot{M}(a) \leqslant 0,\ (2-a)\dot{M}(a) \geqslant 0\right\}. \tag{4}$$

At least generically, any point $a \in \mathcal{M}$ should be a good candidate for being a blow-up radius, i.e. for the existence of (λ_n, u_n) solutions such that

$$\lambda_n \to +\infty, \qquad \max_{|r-a| \leqslant \delta} u_n(r) \to +\infty \quad \text{as } n \to +\infty,\ \forall \delta > 0.$$

One of our main results is that a blow-up radius has to belong to $\mathcal{M}$. Actually, the asymptotic analysis we develop in this paper relies on a Morse index assumption. Given solutions (λ_n, u_n) with $\lambda_n \to +\infty$ we will assume u_n have uniformly bounded Morse index, i.e.

$$\begin{cases} \exists \bar{k} \in \mathbb{N} \text{ such that, if } W \text{ is a linear subspace of } H^1_{0,\mathrm{rad}}(\Omega) \text{ and, for some } n \in \mathbb{N}, \\ \displaystyle\int_\Omega |\nabla v|^2 + \lambda_n V(x) v^2 - p u_n^{p-1} v^2 < 0,\ \forall v \in W \setminus \{0\},\ \text{then } \dim W \leqslant \bar{k}. \end{cases} \tag{5}$$

As a consequence of Theorem 3.1, of Corollary 3.2 and Theorem 4.2 we have the following:

Theorem 1.1. *Let $\lambda_n \to_n +\infty$, u_n be solutions to* (1) *satisfying* (5)*. Then, up to a subsequence, there are $k \leqslant \bar{k}$ and points $a_n^i \in (1,2)$, $i = 1, \dots, k$, with the following properties*: *a_n^i are the unique points of maximum of u_n, $u_n(a_n^i) \to +\infty$, a_n^i converge to points $a^i \in \mathcal{M}$, not necessarily distinct*; *furthermore, $u_n \to 0$ uniformly away from $\{a^1, \dots, a^k\}$.*

We recall that a radial ground state solution always satisfies (5): it has exactly Morse index one in $H^1_{0,\mathrm{rad}}(\Omega)$ (see [5]). Thus, as a by-product of Theorem 1.1, we obtain, generalizing [4], an explicit sequence of solutions blowing up on a sphere (compare with [2]):

Theorem 1.2. *Let u_λ be a radial ground state solution of* (1)*. For λ large, u_λ has a unique point of maximum a_λ and $u_\lambda(a_\lambda) \to +\infty$. Furthermore, if $a_{\lambda_j} \to a$, then*

$$\dot{M}(r) > 0\ \forall r \in (1,2] \quad \Rightarrow \quad a = 1 \quad \textit{while} \quad \dot{M}(r) < 0 \quad \forall r \in [1,2) \quad \Rightarrow \quad a = 2,$$
$$\dot{M}(1) < 0 < \dot{M}(2) \quad \Rightarrow \quad \dot{M}(a) = 0.$$

Thus, in any case, $a \in \mathcal{M}$. Finally, $u_n \to 0$ uniformly away from a.

The paper is organized as follows. In Section 2 we introduce a blow-up approach to identify the limit profile problem. In Section 3 we obtain the crucial global estimate (19) which will allow us in Section 4 to localize the blow-up set. In Appendix A, we briefly discuss the limiting problem and present a Pohozaev-type identity.

2. Local profile

In this section we give a complete identification of the limit profile problem and its spectral properties. Let U be the unique solution (see Appendix A) of the problem

$$\begin{cases} -\ddot{U} + \dfrac{2}{p+1} U = U^p & \text{in } \mathbb{R}, \\ 0 < U(r) \leqslant U(0) = 1 & \text{in } \mathbb{R}. \end{cases} \tag{6}$$

Proposition 2.1. *Let (λ_n, u_n) be solutions of* (1) *with u_n satisfying* (5)*. Let $a_n \in (1,2)$ be such that $u_n(a_n) \to +\infty$. Let $\varepsilon_n = u_n(a_n)^{-\frac{p-1}{2}}$ and $U_n(r) = \varepsilon_n^{\frac{2}{p-1}} u_n(\varepsilon_n r + a_n)$ for $r \in I_n$, where $I_n = (\frac{1-a_n}{\varepsilon_n}, \frac{2-a_n}{\varepsilon_n})$. Assume that*

$$\exists R_n \to +\infty: \quad u_n(a_n) = \max_{\{|r-a_n| \leqslant R_n \varepsilon_n\}} u_n. \tag{7}$$

Then, for a subsequence, we have that

$$\frac{1-a_n}{\varepsilon_n} \to_n -\infty, \qquad \frac{2-a_n}{\varepsilon_n} \to_n +\infty, \tag{8}$$

$$\lambda_n \varepsilon_n^2 V(a_n) \to_n \frac{2}{p+1} \tag{9}$$

and $U_n \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ as $n \to +\infty$, where U is the solution of (6)*. Moreover*

$$\exists R = R(U) > 0,\ \exists \psi_n \in C_0^\infty\big([a_n - R\varepsilon_n, a_n + R\varepsilon_n]\big):$$
$$\int_\Omega \big|\nabla\psi_n\big(|x|\big)\big|^2 + \big(\lambda_n V - p u_n^{p-1}\big)\psi_n\big(|x|\big)^2\,dx < 0 \quad \forall n \textit{ large}. \tag{10}$$

Proof. First, we rewrite (1) in polar coordinates:

$$\begin{cases} -\ddot{u}_n - \dfrac{N-1}{r}\dot{u}_n = u_n^p - \lambda_n V(r) u_n & \text{in } (1,2),\\ u_n > 0 & \text{in } (1,2),\\ u_n(1) = u_n(2) = 0. \end{cases}$$

Since a_n is a point of local maximum, we have $0 \leqslant -\ddot{u}_n(a_n) = u_n^p(a_n) - \lambda_n V(a_n) u_n(a_n)$, and hence, denoted $\omega(V) := [\max_{\bar{\Omega}} V][\min_{\bar{\Omega}} V]^{-1}$, it results

$$1 \geqslant \lambda_n V(a_n) u_n^{1-p}(a_n) = \lambda_n \varepsilon_n^2 V(a_n) \geqslant 0, \quad \lambda_n \varepsilon_n^2 V(r) \leqslant \omega(V). \tag{11}$$

Passing eventually to a subsequence, we can assume

$$\lambda_n \varepsilon_n^2 V(a_n) \to \mu, \quad \frac{a_n - 1}{\varepsilon_n} \to L_0, \quad \frac{2 - a_n}{\varepsilon_n} \to L_1 \quad \text{as } n \to +\infty, \tag{12}$$

for some $\mu \in [0,1]$, $L_0, L_1 \in [0, +\infty]$. Finally, notice that U_n satisfies the equation:

$$\begin{cases} -\ddot{U}_n - (N-1)\dfrac{\varepsilon_n}{\varepsilon_n r + a_n}\dot{U}_n = U_n^p - \lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) U_n, & r \in I_n,\\ U_n(0) = 1, \quad \dot{U}_n(0) = 0, \quad U_n(r) > 0, & r \in I_n,\\ U_n = 0, & r \in \partial I_n. \end{cases} \tag{13}$$

In the sequel, we will denote by $|A|$ the Lebesgue measure of a set A.

1st Step: For any closed bounded interval I with $0 \in I$, there exists $C = C(|I|) > 0$:

$$\|U_n\|_{C^{1,1}(I_n \cap I)} \leqslant C \quad \forall n \in \mathbb{N}. \tag{14}$$

Set $J_n = I_n \cap I$. Since I is bounded, (7) implies $U_n(r) \leqslant U_n(0) = 1$ for $n \geqslant n(|I|)$ and $r \in J_n$. Hence, by (11), (13):

$$\big|\dot{U}_n(r)\big| = \big|\dot{U}_n(r) - \dot{U}_n(0)\big| \leqslant |r| \int_0^1 \big|\ddot{U}_n(tr)\big|\,dt \leqslant (N-1)\big[1 + \omega(V)\big]\Big(\varepsilon_n \max_{s \in J_n}\big|\dot{U}_n(s)\big| + 1\Big)|r|$$
$$\leqslant \frac{1}{2}\max_{r \in J_n}\big|\dot{U}_n(r)\big| + (N-1)\big[1 + \omega(V)\big]|I|,$$

and then: $\max_{r \in J_n} |\dot{U}_n(r)| \leqslant 2(N-1)[1 + \omega(V)]|I|$ for $n \geqslant n(|I|)$. In turn, this implies

$$\begin{aligned}\left|\dot U_n(r)-\dot U_n(s)\right| &\leqslant |r-s|\int_0^1 \left|\ddot U_n\big(s+t(r-s)\big)\right|dt\\ &\leqslant (N-1)\big[1+\omega(V)\big]\Big(\varepsilon_n \max_{t\in J_n}\left|\dot U_n(t)\right|+1\Big)|r-s|\\ &\leqslant 2(N-1)\big[1+\omega(V)\big]|r-s| \quad \forall r,s\in J_n,\ n\geqslant n\big(|I|\big),\end{aligned}$$

i.e. (14) holds with $C=\max\{2(N-1)[1+\omega(V)][|I|+1]+1,\ \|U_n\|_{C^{1,1}(I_n\cap I)}\colon 1\leqslant n<n(|I|)\}$.

2nd Step: $L_0=L_1=+\infty$ and $U_n\to U$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ as $n\to+\infty$.

Assume that $L_0<+\infty$. Then, by (14), U_n is uniformly bounded in $C^{1,1}[-\frac{a_n-1}{\varepsilon_n},R]$, for any $R>0$. Since $L_0<+\infty$ implies $L_1=+\infty$, we can assume, up to a subsequence and a diagonal process, that $U_n\to U$ in $C^1_{\mathrm{loc}}[-L_0,+\infty)$ (and then $L_0>0$) where:

$$\begin{cases} -\ddot U+\mu U=U^p & \text{in } (-L_0,+\infty),\\ 0\leqslant U(r)\leqslant U(0)=1 & \text{in } (-L_0,+\infty),\\ U(-L_0)=0 \end{cases}$$

in view of (7), (12)–(13). Since U is even (see Appendix A), $U(L_0)=0$ and then $\dot U(L_0)=0$ because $U\geqslant 0$. Hence $U\equiv 0$, a contradiction. Thus $L_0=+\infty$. Similarly, $L_1=+\infty$.

3rd Step: $\mu=\frac{2}{p+1}$ and (10) holds.

As shown in Appendix A, U positive implies its energy is nonpositive:

$$\begin{aligned}0\geqslant H(U,\dot U) &:= \frac12\dot U^2-\frac12\mu U^2+\frac{1}{p+1}U^{p+1}\equiv \frac12\dot U^2(0)-\frac{\mu}{2}U^2(0)+\frac{1}{p+1}U^{p+1}(0)\\ &=\frac{1}{p+1}-\frac{\mu}{2}.\end{aligned}$$

Hence $\mu\geqslant\frac{2}{p+1}$. Now, $\mu>\frac{2}{p+1}$ implies (see Appendix A) U is a positive, possibly constant, periodic solution and there is a countable family of functions $\phi_j\in C_0^\infty(\mathbb{R})$ with mutually disjoint supports such that, for some $\delta>0$, it results

$$\int_{\mathbb{R}}\big(\dot\phi_j^2+\mu\phi_j^2-pU^{p-1}\phi_j^2\big)\,dr\leqslant-\delta<0.$$

Let $\phi_{j,n}(r)=\phi_j(\frac{r-a_n}{\varepsilon_n})$, so that $\operatorname{supp}\phi_{j,n}=a_n+\varepsilon_n\operatorname{supp}\phi_j$ are disjoint for different j's and contained in $\{a_n-R_j\varepsilon_n\leqslant|x|\leqslant a_n+R_j\varepsilon_n\}$, for some $R_j>0$. Moreover, if $a:=\lim_{n\to+\infty}a_n$ (along some subsequence), by Steps 1–2 we get:

$$\begin{aligned}&\varepsilon_n\int_\Omega\big(|\nabla\phi_{j,n}|^2+\big(\lambda_nV(r)-pu_n^{p-1}\big)\phi_{j,n}^2\big)\\ &\quad=\varepsilon_n\int_1^2 r^{N-1}\big((\dot\phi_{j,n})^2+\big(\lambda_nV(r)-pu_n^{p-1}\big)\phi_{j,n}^2\big)\end{aligned}$$

$$= \int\limits_{\operatorname{Supp}\phi_j} (\varepsilon_n r + a_n)^{N-1} \big[\dot{\phi}_j^2 + \big(\lambda_n \varepsilon_n^2 V(\varepsilon_n r + a_n) - pU_n^{p-1}\big)\phi_j^2\big]$$

$$\to_n a^{N-1} \int\limits_{\mathbb{R}} \big(\dot{\phi}_j^2 + (\mu - pU^{p-1})\phi_j^2\big) \leqslant -\delta < 0 \quad \forall n \geqslant n(j).$$

This contradicts (5) and hence $\mu = \frac{2}{p+1}$. As for (10), just notice that, by (6) we have

$$\int\limits_{\mathbb{R}} \left(\dot{U}^2 + \left(\frac{2}{p+1} - pU^{p-1}\right)U^2\right) = -(p-1)\int\limits_{\mathbb{R}} U^{p+1} < 0$$

(see (A.2) in Appendix A) and hence, by density, there exist $R = R(U)$ and $\psi \in C_0^\infty([-R, R])$ such that

$$\int\limits_{\mathbb{R}} \left(\dot{\psi}^2 + \left(\frac{2}{p+1} - pU^{p-1}\right)\psi^2\right) < 0.$$

As above, we see that $\psi_n(r) = \psi(\frac{r-a_n}{\varepsilon_n})$ satisfies the requirements in (10). This ends the proof of Proposition 2.1. □

3. Global behavior

Once the limit profile problem (6) has been identified and the local behavior around a blow-up sequence a_n has been described, our next task is to provide global estimates: we will show that the sequence u_n decays exponentially away from blow-up points and we will prove that the number of blow-up sequences cannot exceed $\bar{k}$, the upper bound for the Morse index of the (u_n)'s. We have the following global result:

Theorem 3.1. *Let $\lambda_n \to \infty$, u_n be solutions of* (1) *satisfying* (5)*. Up to a subsequence, there exist $a_n^1, \ldots, a_n^k$, $k \leqslant \bar{k}$ ($\bar{k}$ given in* (5)*), with $\varepsilon_n^i = u_n(a_n^i)^{-\frac{p-1}{2}} \to 0$ such that*

$$\lambda_n \big(\varepsilon_n^i\big)^2 V\big(a_n^i\big) \to \frac{2}{p+1} \quad \text{as } n \to +\infty \quad \forall i = 1, \ldots, k, \tag{15}$$

$$\varepsilon_n^1 \leqslant \varepsilon_n^i \leqslant C\varepsilon_n^1 \quad \forall i = 1, \ldots, k, \tag{16}$$

$$\frac{\varepsilon_n^i + \varepsilon_n^j}{|a_n^i - a_n^j|} \to 0 \quad \text{as } n \to +\infty \quad \forall i, j = 1, \ldots, k,\ i \neq j, \tag{17}$$

$$u_n\big(a_n^i\big) = \max_{\{|r-a_n^i| \leqslant R_n \varepsilon_n^i\}} u_n, \tag{18}$$

$$u_n(r) \leqslant C\big(\varepsilon_n^1\big)^{-\frac{2}{p-1}} \sum_{i=1}^k e^{-\gamma \frac{|r-a_n^i|}{\varepsilon_n^1}} \quad \forall r \in (1,2),\ \forall n \in \mathbb{N}, \tag{19}$$

for some $\gamma, C > 0$ and $R_n \to +\infty$ as $n \to +\infty$.

Proof. The proof is divided into two steps.

1st Step: There exist $k \leqslant \bar{k}$ sequences $a_n^1, \dots, a_n^k$ satisfying (15)–(18) such that:

$$\lim_{R\to+\infty}\Big(\limsup_{n\to+\infty}\Big[\big(\varepsilon_n^1\big)^{\frac{2}{p-1}}\max_{\{d_n(r)\geqslant R\varepsilon_n^1\}}u_n(r)\Big]\Big)=0, \tag{20}$$

where $d_n(r)=\min\{|r-a_n^i|\colon\ i=1,\dots,k\}$ is the distance function from $\{a_n^1,\dots,a_n^k\}$.

First of all, let a_n^1 be a point of global maximum of u_n: $u_n(a_n^1)=\max_{r\in(1,2)}u_n(r)$. Since (18) clearly holds for a_n^1, Proposition 2.1 applies, and (9) provides exactly (15). If (20) already holds for a_n^1, then we take $k=1$ and the claim is proved. If not (passing to a subsequence)

$$\exists\delta>0,\ \exists R_n\to+\infty\colon\quad \big(\varepsilon_n^1\big)^{\frac{2}{p-1}}\max_{\{|r-a_n^1|\geqslant R_n\varepsilon_n^1\}}u_n(r)\geqslant 2\delta>0. \tag{21}$$

Now, an application of Proposition 2.1 gives, eventually for a subsequence,

$$\big(\varepsilon_n^1\big)^{\frac{2}{p-1}}u_n\big(r\varepsilon_n^1+a_n^1\big)=U_n^1(r)\to_n U(r) \tag{22}$$

uniformly on bounded sets (U solution of (6)). By the decay of U (see (A.1)), there is $R_\delta>0$ such that $U(r)\leqslant\frac{\delta}{2}$ for $|r|\geqslant R_\delta$. Hence, using (22), we see that (R_j given in (21))

$$\forall j\ \exists n_j\colon\quad R_{n_j}\geqslant R_j\quad\text{and}\quad \big(\varepsilon_{n_j}^1\big)^{\frac{2}{p-1}}\max_{\{R_\delta\varepsilon_{n_j}^1\leqslant|r-a_{n_j}^1|\leqslant R_j\varepsilon_{n_j}^1\}}u_{n_j}(r)\leqslant\delta.$$

This, jointly with (21) gives

$$\begin{aligned}\big(\varepsilon_{n_j}^1\big)^{\frac{2}{p-1}}\max_{\{|r-a_{n_j}^1|\geqslant R_\delta\varepsilon_{n_j}^1\}}u_{n_j}(r)&=\big(\varepsilon_{n_j}^1\big)^{\frac{2}{p-1}}\max_{\{|r-a_{n_j}^1|\geqslant R_j\varepsilon_{n_j}^1\}}u_{n_j}(r)\geqslant 2\delta>\delta\\&\geqslant\big(\varepsilon_{n_j}^1\big)^{\frac{2}{p-1}}\max_{\{R_\delta\varepsilon_{n_j}^1\leqslant|r-a_{n_j}^1|\leqslant R_j\varepsilon_{n_j}^1\}}u_{n_j}(r)\quad\forall j.\end{aligned} \tag{23}$$

Hence, for any j:

$$\exists a_{n_j}^2\in\big\{\big|r-a_{n_j}^1\big|\geqslant R_j\varepsilon_{n_j}^1\big\}\colon\quad u_{n_j}\big(a_{n_j}^2\big)=\max_{\{|r-a_{n_j}^1|\geqslant R_\delta\varepsilon_{n_j}^1\}}u_{n_j}(r)\geqslant 2\delta\big(\varepsilon_{n_j}^1\big)^{-\frac{2}{p-1}}. \tag{24}$$

By (24) we get $\varepsilon_{n_j}^2:=u_{n_j}(a_{n_j}^2)^{-\frac{p-1}{2}}\leqslant\varepsilon_{n_j}^1(2\delta)^{-\frac{p-1}{2}}$, and since $\varepsilon_{n_j}^1\leqslant\varepsilon_{n_j}^2$ we see that (16) is fulfilled, as well as (17) because $|a_{n_j}^2-a_{n_j}^1|\geqslant R_j\varepsilon_{n_j}^1$. This inequality and (23) imply (18):

$$u_{n_j}\big(a_{n_j}^2\big)=\max_{\{|r-a_{n_j}^2|\leqslant[R_j-R_\delta](2\delta)^{\frac{2}{p-1}}\varepsilon_{n_j}^2\}}u_{n_j}(r).$$

In fact

$$\left|r-a^2_{n_j}\right| \leqslant [R_j-R_\delta](2\delta)^{\frac{p-1}{2}}\varepsilon^2_{n_j} \quad \Rightarrow$$
$$\left|r-a^1_{n_j}\right| \geqslant \left|a^2_{n_j}-a^1_{n_j}\right| - [R_j-R_\delta](2\delta)^{\frac{p-1}{2}}\varepsilon^2_{n_j} \geqslant R_j\varepsilon^1_{n_j} - [R_j-R_\delta]\varepsilon^1_{n_j} = R_\delta\varepsilon^1_{n_j}.$$

Up to the subsequence n_j, thus (16)–(18) hold true for $\{a^1_n, a^2_n\}$, and, if $\{a^1_n, a^2_n\}$ also satisfy (20), we are finished. Otherwise, we iterate the above argument: given s sequences $a^1_n, \dots, a^s_n$, let us denote $d_n(r) = \min\{|r-a^i_n|\colon\ i = 1, \dots, s\}$. If (15)–(18) are satisfied, but (20) is not, we have

$$\exists\delta > 0,\ \exists R_n \to +\infty\colon \quad \left(\varepsilon^1_n\right)^{\frac{2}{p-1}} \max_{\{d_n(r)\geqslant R_n\varepsilon^1_n\}} u_n(r) \geqslant 2\delta$$

and, by assumptions (16)–(18) and Proposition 2.1:

$$\exists\theta_i \in \left[\frac{1}{C}, 1\right]\colon \quad \frac{\varepsilon^1_n}{\varepsilon^i_n} \to \theta_i,$$
$$\left(\varepsilon^1_n\right)^{\frac{2}{p-1}} u_n\left(r\varepsilon^1_n + a^i_n\right) = \left(\frac{\varepsilon^1_n}{\varepsilon^i_n}\right)^{\frac{2}{p-1}} U^i_n\left(\frac{\varepsilon^1_n}{\varepsilon^i_n} r\right) \to \theta_i^{\frac{2}{p-1}} U(\theta_i r) \tag{25}$$

uniformly on bounded sets. By (A.1), $\theta_i^{\frac{2}{p-1}} U(\theta_i r) < \delta$ for $|r| \geqslant R_\delta$. Now things go as above, replacing $|r-a^1_n|$ with $d_n(r)$. Finally, the argument ends after at most $\bar{k}$ iteration, because Proposition 2.1 applies to any sequence a^i_n, $i = 1, \dots, k$, providing, for n large, radial functions $\psi^i_n \in C^\infty_0(\Omega)$ such that (10) holds with supp $\psi^i_n \subset \{a^i_n - R\varepsilon^i_n \leqslant |x| \leqslant a^i_n + R\varepsilon^i_n\}$, for some $R > 0$. By (17) we get that $\psi^1_n, \dots, \psi^k_n$ have disjoint compact supports for any n large and then $k \leqslant \bar{k}$.

2nd Step: Let $a^1_n, \dots, a^k_n$ be as in the first step. Then there are $\gamma, C > 0$ such that:

$$u_n(r) \leqslant C\left(\varepsilon^1_n\right)^{-\frac{2}{p-1}} \sum_{i=1}^k e^{-\gamma\frac{|r-a^i_n|}{\varepsilon^1_n}} \quad \forall r \in (1,2),\ \forall n \in \mathbb{N}.$$

By (20), for $R > 0$ large and $n \geqslant n(R)$, it results (recall that $\omega(V) := [\max_{\bar\Omega} V][\min_{\bar\Omega} V]^{-1}$)

$$\left(\varepsilon^1_n\right)^{\frac{2}{p-1}} \max_{\{d_n(r)\geqslant R\varepsilon^1_n\}} u_n(r) \leqslant \left(\frac{1}{(p+1)\omega(V)}\right)^{\frac{1}{p-1}},$$

and hence $(\varepsilon^1_n)^2 u_n^{p-1}(r) \leqslant \frac{1}{(p+1)\omega(V)}$ in $\{d_n(r) \geqslant R\varepsilon^1_n\}$. On the other hand, by (15) we get

$$\lambda_n\left(\varepsilon^1_n\right)^2 V(r) \geqslant \left[\omega(V)\right]^{-1}\lambda_n\left(\varepsilon^1_n\right)^2 V\left(a^1_n\right) \to_n \frac{2}{(p+1)\omega(V)}.$$

Hence, the following holds true: there are $R > 0$ and $n(R)$ such that, if $n \geqslant n(R)$, then

$$\left(\varepsilon^1_n\right)^2\left[\lambda_n V(r) - u_n^{p-1}(r)\right] \geqslant \frac{1}{2(p+1)\omega(V)} > 0 \quad \text{if } d_n(r) \geqslant R\varepsilon^1_n. \tag{26}$$

Now, consider the linear operator:

$$L_n\phi = -\Delta\phi + \big(\lambda_n V(r) - u_n^{p-1}(r)\big)\phi, \quad \phi \in C^2(\Omega).$$

Notice that $L_n u_n = 0$. Since $u_n > 0$ in Ω, L_n satisfies the minimum principle in any domain in Ω (see [6]). Let $\gamma > 0$ and $\phi_n^i(r) = e^{-\gamma(\varepsilon_n^1)^{-1}|r-a_n^i|}$. By (26), for R large it results

$$L_n\phi_n^i = \big(\varepsilon_n^1\big)^{-2}\phi_n^i\bigg[-\gamma^2 + (N-1)\frac{\varepsilon_n^1}{r}\gamma\frac{r-a_n^i}{|r-a_n^i|} + \big(\varepsilon_n^1\big)^2\big(\lambda_n V(r) - u_n^{p-1}(r)\big)\bigg] > 0$$

if $d_n(r) \geqslant R\varepsilon_n^1$, $\gamma^2 \leqslant \frac{1}{8(p+1)\omega(V)}$ and $n \geqslant n(R,\gamma)$. In addition, by (25) we have

$$\big(e^{\gamma R}\phi_n^i(r) - \big(\varepsilon_n^1\big)^{\frac{2}{p-1}}u_n(r)\big)\big|_{r=a_n^i \pm R\varepsilon_n^1} = 1 - \big(\varepsilon_n^1\big)^{\frac{2}{p-1}}u_n\big(a_n^i \pm R\varepsilon_n^1\big) \to 1 - \theta_i^{\frac{2}{p-1}}U(\pm\theta_i R) > 0.$$

Then $\Phi_n := e^{\gamma R}(\varepsilon_n^1)^{-\frac{2}{p-1}}\sum_{i=1}^k \phi_n^i$ satisfies

$$L_n(\Phi_n - u_n) > 0 \quad \text{in } \big\{d_n(r) > R\varepsilon_n^1\big\} \quad \text{and} \quad \Phi_n - u_n > 0 \quad \text{on } \big\{d_n(r) = R\varepsilon_n^1\big\} \cup \big\{|r| = 1, 2\big\}$$

(notice that, by (16)–(17) $\{d_n(r) > R\varepsilon_n^1\}$ are disjoint intervals for $n \geqslant n(R)$), and then, by minimum principle $u_n \leqslant \Phi_n$ in $\{d_n(r) > R\varepsilon_n^1\}$, if R is large and $n \geqslant n(R)$. That is

$$u_n(r) \leqslant e^{\gamma R}\big(\varepsilon_n^1\big)^{-\frac{2}{p-1}}\sum_{i=1}^k e^{-\gamma\frac{|r-a_n^i|}{\varepsilon_n^1}} \quad \text{if } d_n(r) \geqslant R\varepsilon_n^1 \text{ and } n \geqslant n(R). \tag{27}$$

Since

$$u_n(r) \leqslant \max_\Omega u_n = \big(\varepsilon_n^1\big)^{-\frac{2}{p-1}} \leqslant e^{\gamma R}\big(\varepsilon_n^1\big)^{-\frac{2}{p-1}}\sum_{i=1}^k e^{-\gamma\frac{|r-a_n^i|}{\varepsilon_n^1}} \quad \text{if } d_n(r) \leqslant R\varepsilon_n^1 \text{ and } n \geqslant n(R),$$

(27) holds for any $r \in (1,2)$ and $n \geqslant n(R)$. Thus, for some $C \geqslant e^{\gamma R}$ (19) holds true for any n and the proof is now complete. □

As a by-product, the number of points of local maximum is controlled by (5):

Corollary 3.2. *Let $\lambda_n \to \infty$, u_n be solutions of* (1) *satisfying* (5). *Up to a subsequence, u_n has, for n large, exactly k points of local maximum $a_n^1, \ldots, a_n^k$, $k \leqslant \bar{k}$, where $a_n^1, \ldots, a_n^k$ are given by Theorem* 3.1.

Proof. By (26) $u_n^p - \lambda_n V(r)u_n < 0$ $\forall r \in \{d_n(r) \geqslant R\varepsilon_n^1\}$, for R large and fixed and $n \geqslant n(R)$. Hence, by (1) all the points of local maximum of u_n stay, for n large, in the region $d_n(r) \leqslant R\varepsilon_n^1$. We are lead to show that $a_n^1, \ldots, a_n^k$ are, for n large, the only points of local maximum of u_n in $d_n(r) \leqslant R\varepsilon_n^1$.

By contradiction, let s_n be points of local maximum of u_n, with $0 < |s_n - a_n^i| \leqslant R\varepsilon_n^1$, for some $i \leqslant k$. Since 0 is the only critical point of the limit function U, by the $C^1_{\mathrm{loc}}(\mathbb{R})$ convergence of U_n^i to U we get $\tilde{s}_n := \frac{s_n - a_n^i}{\varepsilon_n^i} \to 0$ as $n \to +\infty$. By (13) and (15) we get:

$$-\ddot{U}_n^i(\tilde{s}_n) = \left(U_n^i\right)^p(\tilde{s}_n) - \lambda_n\left(\varepsilon_n^i\right)^2 V(\tilde{s}_n)U_n^i(\tilde{s}_n) \to_n 1 - \frac{2}{p+1} > 0.$$

Then, s_n is a strict local maximum and hence there is a local minimum at some t_n strictly in between s_n and a_n^i. However, as for s_n, it should be $\tilde{t}_n := \frac{t_n - a_n^i}{\varepsilon_n^i} \to 0$ as $n \to +\infty$ and $\ddot{U}_n^i(\tilde{t}_n) < 0$ for n large, a contradiction. □

4. Location of the blow-up set

In concentration phenomena, the role of the modified potential $M(r)$ given in (3) has been pointed out in papers of Ambrosetti, Malchiodi and Ni [1,2], when dealing with the same equation either in $\mathbb{R}^N$ or in a ball/annulus in $\mathbb{R}^N$ with homogeneous Dirichlet boundary condition. To show by an asymptotic approach the role of $M(r)$, we will combine the results in the previous section with a Pohozaev-type identity (see Appendix A).

Let us start with some asymptotic estimates for u_n, solutions of (1). By Corollary 3.2 u_n has, up to a subsequence, exactly k points of local maximum $a_n^1, \dots, a_n^k \in (1,2)$ with, say, $a_n^i \to a^i \in [1,2]$, $i = 1, \dots, k$. Let $J_i = \{j = 1, \dots, k\colon a_n^j \to_n a^i\}$. We have the following:

Lemma 4.1. *Let $g(r)$ be some smooth function on $[1,2]$. Let $q > 1$. Fix $i \in \{1, \dots, k\}$ and denote $I_\delta^i := [a^i - \delta, a^i + \delta] \cap (1,2)$ where $\delta > 0$ is so small that $I_\delta^i \cap \{a^1, \dots, a^k\} = \{a^i\}$. Then*

$$\int_{I_\delta^i} g(r)u_n^q = g\left(a^i\right)\left(\sum_{j\in J_i}\left(\varepsilon_n^j\right)^{\frac{p-1-2q}{p-1}}\right)\left(\int_{\mathbb{R}} U^q + o_n(1)\right) \tag{28}$$

where $o_n(1) \to 0$ as $n \to +\infty$. In particular, there holds:

$$\int_1^2 u_n^{p+1} = \left(\sum_{i=1}^k\left(\varepsilon_n^i\right)^{-\frac{p+3}{p-1}}\right)\left(\int_{\mathbb{R}} U^{p+1} + o_n(1)\right). \tag{29}$$

Proof. Let $d_n(r) := \min\{|r - a_n^i|\colon i = 1, \dots, k\}$. Given $R > 0$, (8), (16) and (17) imply that, for $n \geqslant n(R)$, $\{d_n(r) \leqslant R\varepsilon_n^1\}$ are mutually disjoint intervals and

$$\left\{d_n(r) \leqslant R\varepsilon_n^1\right\} \subset (1,2) \quad \text{and} \quad I_\delta^i \cap \left\{d_n(r) \leqslant R\varepsilon_n^1\right\} = \bigcup_{j\in J_i}\left\{\left|r - a_n^j\right| \leqslant R\varepsilon_n^1\right\}.$$

By (19) we know that $u_n^q \leqslant C(\varepsilon_n^1)^{-\frac{2q}{p-1}} \sum_{j=1}^k e^{-q\gamma\frac{|r-a_n^j|}{\varepsilon_n^1}}$. Thus

$$\int_{I_\delta^i} g(r)u_n^q = \int_{I_\delta^i \cap \{d_n(r) \leqslant R\varepsilon_n^1\}} g(r)u_n^q + \int_{I_\delta^i \cap \{d_n(r) \geqslant R\varepsilon_n^1\}} g(r)u_n^q$$

$$= \sum_{j \in J_i} \int_{\{|r-a_n^j| \leqslant R\varepsilon_n^1\}} g(r)u_n^q + O\left(\left(\varepsilon_n^1\right)^{-\frac{2q}{p-1}} \sum_{j=1}^k \int_{I_\delta^i \cap \{d_n(r) \geqslant R\varepsilon_n^1\}} e^{-q\gamma \frac{|r-a_n^j|}{\varepsilon_n^1}} \right)$$

$$= \sum_{j \in J_i} \left(\varepsilon_n^j\right)^{-\frac{2q-p+1}{p-1}} \int_{\{|r| \leqslant R \frac{\varepsilon_n^1}{\varepsilon_n^j}\}} g\left(\varepsilon_n^j r + a_n^j\right)\left(U_n^j\right)^q$$

$$+ O\left(\left(\varepsilon_n^1\right)^{-\frac{2q}{p-1}} \sum_{j=1}^k \varepsilon_n^j \int_{\{|r| \geqslant R \frac{\varepsilon_n^1}{\varepsilon_n^j}\}} e^{-q\gamma |r| \frac{\varepsilon_n^j}{\varepsilon_n^1}} \right).$$

Up to a subsequence, by (16) we can assume that $\varepsilon_n^1/\varepsilon_n^j \to_n \theta_j \in [\frac{1}{C}, 1]$ for any $j = 1, \dots, k$. Since $U_n^j \to_n U$ in $C^1_{\mathrm{loc}}(\mathbb{R})$ for any $j = 1, \dots, k$, we find, along some subsequence

$$\lim_{n \to +\infty} \left(\varepsilon_n^1\right)^{\frac{2q-p+1}{p-1}} \int_{I_\delta^i} u_n^q = g\left(a^i\right) \sum_{j \in J_i} \theta_j^{\frac{2q-p+1}{p-1}} \int_{\{|r| \leqslant R\theta_j\}} U^q + O\left(\sum_{j=1}^k \int_{\{|r| \geqslant R\theta_j\}} e^{-\frac{q\gamma|r|}{\theta_j}} \right).$$

Sending R to infinity, we get, along the same subsequence,

$$\lim_{n \to +\infty} \left(\varepsilon_n^1\right)^{\frac{2q-p+1}{p-1}} \int_{I_\delta^i} u_n^q = g\left(a^i\right) \left(\sum_{j \in J_i} \theta_j^{\frac{2q-p+1}{p-1}} \right) \int_{\mathbb{R}} U^q.$$

Since we found the same value along any convergent subsequence, and recalling the definition of θ_j, the proof of (28) is complete. Finally, since by (19) $u_n \to 0$ as $n \to +\infty$ uniformly far away from $\{a^1, \dots, a^k\}$, (28), with $q = p + 1$ and $g \equiv 1$, implies (29). □

The asymptotic expansions in Lemma 4.1, combined with the Pohozaev identity (A.3), leads to the identification of a^i, $i = 1, \dots, k$:

Theorem 4.2. *For any $i = 1, \dots, k$ $a^i \in \mathcal{M}$, where $\mathcal{M}$ is given in* (4).

Proof. Given $i = 1, \dots, k$, first consider the case $a^i \in (1, 2)$. Let I_δ^i be as in Lemma 4.1. By (15), (19) $\lambda_n u_n^2 \to_n 0$ uniformly away from the a^i's and elliptic regularity estimates imply the same for $\dot{u}_n$. Thus we see, plugging $a = a^i - \delta$, $b = a^i + \delta$ in (A.3), that:

$$\left(N - \frac{3}{2} - \frac{1}{p+1}\right) \int_{a^i-\delta}^{a^i+\delta} u_n^{p+1} + \lambda_n \int_{a^i-\delta}^{a^i+\delta} \left(\frac{r}{2}\dot{V} - (N-2)V\right) u_n^2 + \left(N - \frac{3}{2}\right) \int_{a^i-\delta}^{a^i+\delta} \frac{N-1}{2r^2} u_n^2 \to 0$$

as $n \to +\infty$. By (15) and (28) we get as $n \to +\infty$ (here $o_n(1) \to 0$ as $n \to +\infty$)

$$\int_{a^i-\delta}^{a^i+\delta} u_n^{p+1} = \Big(\sum_{j\in J_i} \big(\varepsilon_n^j\big)^{-\frac{p+3}{p-1}}\Big)\Big(\int_{\mathbb{R}} U^{p+1} + o_n(1)\Big), \qquad \int_{a^i-\delta}^{a^i+\delta} \frac{u_n^2}{r^2} = \frac{1}{\lambda_n} O\Big(\sum_{j\in J_i} \big(\varepsilon_n^j\big)^{-\frac{p+3}{p-1}}\Big),$$

$$\lambda_n \int_{a^i-\delta}^{a^i+\delta} \Big(\frac{r}{2}\dot V - (N-2)V\Big)u_n^2 = \frac{1}{p+1}\Big(a^i \frac{\dot V(a^i)}{V(a^i)} - 2(N-2)\Big)\Big(\sum_{j\in J_i} \big(\varepsilon_n^j\big)^{-\frac{p+3}{p-1}}\Big) \times \Big(\int_{\mathbb{R}} U^2 + o_n(1)\Big).$$

Hence, also making use of the relation $\int_{\mathbb{R}} U^{p+1} = \frac{4}{p+3}\int_{\mathbb{R}} U^2$ (see (A.1)), we get

$$\begin{aligned}
0 &= \Big(N - \frac{3}{2} - \frac{1}{p+1}\Big)\int_{\mathbb{R}} U^{p+1} - \frac{2(N-2)}{p+1}\int_{\mathbb{R}} U^2 + \frac{a^i}{p+1}\frac{\dot V(a^i)}{V(a^i)}\int_{\mathbb{R}} U^2 \\
&= \Big[N\Big(\frac{4}{p+3} - \frac{2}{p+1}\Big) - \Big(\frac{3}{2} + \frac{1}{p+1}\Big)\frac{4}{p+3} + \frac{4}{p+1} + \frac{a^i}{p+1}\frac{\dot V(a^i)}{V(a^i)}\Big]\int_{\mathbb{R}} U^2 \\
&= \Big[N - 1 + \frac{p+3}{2(p-1)}\frac{a^i \dot V(a^i)}{V(a^i)}\Big]\frac{2(p-1)}{(p+3)(p+1)}\int_{\mathbb{R}} U^2 \\
&= \frac{2(p-1)}{(p+3)(p+1)}\Big(\int_{\mathbb{R}} U^2\Big) V\big(a^i\big)^{-\theta}\big(a^i\big)^{2-N}\dot M\big(a^i\big).
\end{aligned}$$

Consider now the case $a^i = 1$. Let I_δ^i be as above. As before, $\lambda_n u_n^2 + \dot u_n^2 \to 0$ as $n \to +\infty$ at $1+\delta$. Taking $a = 1$, $b = 1+\delta$ in (A.3), we see that:

$$\begin{aligned}
&\Big(N - \frac{3}{2} - \frac{1}{p+1}\Big)\int_1^{1+\delta} u_n^{p+1} + \lambda_n \int_1^{1+\delta}\Big(\frac{r}{2}\dot V - (N-2)V\Big)u_n^2 + \Big(N - \frac{3}{2}\Big)\int_1^{1+\delta}\frac{N-1}{2r^2}u_n^2 \\
&\geqslant \Big(N - \frac{3}{2} - \frac{1}{p+1}\Big)\int_1^{1+\delta} u_n^{p+1} + \lambda_n \int_1^{1+\delta}\Big(\frac{r}{2}\dot V - (N-2)V\Big)u_n^2 \\
&\quad + \Big(N - \frac{3}{2}\Big)\int_1^{1+\delta}\frac{N-1}{2r^2}u_n^2 - \frac{1}{2}\dot u_n^2(1) \to 0
\end{aligned}$$

as $n \to +\infty$. Arguing as above, we get that

$$0 \leqslant \frac{2(p-1)}{(p+3)(p+1)}\left(N-1+\frac{p+3}{2(p-1)}\frac{\dot V(1)}{V(1)}\right)\int_{\mathbb{R}} U^2.$$

Hence, $\dot M(1) \geqslant 0$, and $a^i = 1 \in \mathcal{M}$ holds.

Case $a^i = 2$ can be dealt similarly, getting now $a^i = 2 \in \mathcal{M}$. Hence, the theorem is completely established. □

Appendix A

A.1. Phase plane analysis of the limiting equation

Let U be a C^2-solution of the equation

$$-\ddot U + \mu U = |U|^{p-1}U,$$

and $(U(r), \dot U(r))$ the corresponding (parametrized) orbit in the phase plane. Let

$$H(u,v) := \frac{1}{2}v^2 + G(u), \qquad G(u) := -\frac{\mu}{2}u^2 + \frac{1}{p+1}|u|^{p+1}$$

be the energy function; it is a conserved quantity: $h \equiv H(U(r), \dot U(r))$ is the energy of the orbit $(U, \dot U)$. Since level sets $\{H(u,v) = h\}$ are compact, U is globally defined. For simplicity, we will consider the case $\mu > 0$ (case $\mu = 0$ can be dealt in a similar and simpler way). Direct inspection on the level sets of H gives:

- $\{H(u,v) = h > 0\}$ is a closed orbit enclosing the unstable equilibrium $(0,0)$;
- $\{(u,v)\colon u > 0,\ H(u,v) = 0\}$ is an homoclinic orbit, asymptotic to $(0,0)$;
- $\{(u,v)\colon u > 0,\ H(u,v) < 0\}$ is a closed orbit enclosing the stable equilibrium $(\mu^{\frac{1}{p-1}}, 0)$.

As a consequence, U positive implies: $H(U, \dot U) \leqslant 0$.

From now on we will assume $U(0) = 1$, $\dot U(0) = 0$ (notice that U is even, because it satisfies the same Cauchy problem as $\tilde U(r) := U(-r)$). In this case, $H(U(r), \dot U(r)) \equiv \frac{1}{p+1} - \frac{\mu}{2} \leqslant 0$ iff $\mu \geqslant \frac{2}{p+1}$, so U positive implies $\mu \geqslant \frac{2}{p+1}$.

Case $\boldsymbol{\mu > \frac{2}{p+1}}$: $\boldsymbol{U}$ has infinite Morse index. From above: U is a positive periodic solution.

In case $U \equiv \mu^{\frac{1}{p-1}} = 1$ ($U(0) = 1$), the linearized equation at U is $\ddot v + (p-1)v = 0$. Let (a,b) be such that the first eigenvalue of the Dirichlet problem is smaller than $(p-1)$. Let φ be the corresponding positive eigenfunction. After setting $\varphi \equiv 0$ outside (a,b), we see that $\int_{\mathbb{R}} \dot\varphi^2 - (p-1)\varphi^2 < 0$.

Let $U \neq \mu^{\frac{1}{p-1}}$. Let $U(\bar r) = \min_{\mathbb{R}} U(r)$. By the above discussion, $0 < U(\bar r) < \mu^{\frac{1}{p-1}}$ and hence $G'(U(\bar r)) < 0$. If T is a period of U, $I_k := [\bar r + kT, \bar r + (k+1)T]$, $\varphi_k := [U - U(\bar r)]\chi_{I_k}$, then

$$\int_{I_k} \dot\varphi_k^2 + \mu\varphi_k^2 - pU^{p-1}\varphi_k^2 = \int_{I_k} \left[U^p - \mu U - \left(pU^{p-1} - \mu\right)\left(U - U(\bar r)\right)\right]\varphi_k \, dr.$$

But $U^p(r) - \mu U(r) - (pU^{p-1}(r) - \mu)(U(r) - U(\bar r)) = G'(U(r)) - G''(U(r))[U(r) - U(\bar r)] \leqslant G'(U(\bar r))$ because G' is convex on $(0, +\infty)$. Thus we have

$$\int_{\mathbb{R}} \dot\varphi_k^2 + \mu\varphi_k^2 - pU^{p-1}\varphi_k^2 \leqslant G'\big(U(\bar r)\big) \int_0^T \big[U - U(\bar r)\big] < 0.$$

By density, we can replace the φ_k with C_0^∞-functions with mutually disjoint supports.

Case $\mu = \frac{2}{p+1}$: exponential decay. Zero energy implies $(U, \dot U)$ is homoclinic to the zero equilibrium. Also, U is even and $\dot U(-r) > 0 > \dot U(r)$ $\forall r > 0$. We claim that

$$\exists C > 0\colon\ U(r) \leqslant Ce^{-\frac{|r|}{\sqrt{p+1}}} \quad \forall r \in \mathbb{R}, \qquad \frac{2}{p+1}\int_{\mathbb{R}} U^2 = \left(\frac{1}{2} + \frac{1}{p+1}\right)\int_{\mathbb{R}} U^{p+1}. \quad \text{(A.1)}$$

This follows from the conservation of energy: $\dot U^2 \equiv \frac{2}{p+1}(U^2 - U^{p+1})$. Since $\dot U < 0$ on $(0, +\infty)$ and $U(r) \to 0$ as $r \to +\infty$, we get that:

$$\frac{\dot U(r)}{U(r)} = \big(\ln U(r)\big)' = -\sqrt{\frac{2}{p+1}\big(1 - U^{p-1}(r)\big)} \to -\sqrt{\frac{2}{p+1}} \quad \text{as } r \to +\infty.$$

Hence, there exist $C > 0$ and $R > 0$ large so that $U(r) \leqslant Ce^{-\frac{r}{\sqrt{p+1}}}$ for $r \geqslant R$. In a similar way, we can get an exponential decay at $-\infty$. The conservation of energy gives an exponential decay for $\dot U$ as well, and by integration on $\mathbb{R}$ yields: $\frac{1}{2}\int_{\mathbb{R}} \dot U^2 = \frac{1}{p+1}(\int_{\mathbb{R}} U^2 - \int_{\mathbb{R}} U^{p+1})$.

Multiplying (6) by U and integrating on $\mathbb{R}$, we obtain that

$$\frac{1}{2}\int_{\mathbb{R}} \dot U^2 = -\frac{1}{p+1}\int_{\mathbb{R}} U^2 + \frac{1}{2}\int_{\mathbb{R}} U^{p+1}. \quad \text{(A.2)}$$

Taking the difference of these last two relations, (A.1) follows.

A.2. A Pohozaev-type identity

Lemma A.1. *Let u be a radial solution of* (1). *Let $1 \leqslant a < b \leqslant 2$. Then*

$$\begin{aligned}
\frac{a}{2}\dot u^2(a) &= \frac{b}{2}\dot u^2(b) + \left(r\frac{u^{p+1}}{p+1} - \frac{\lambda}{2}rVu^2 + \left(N - \frac{3}{2}\right)\dot u u + \left(N - \frac{3}{2}\right)\frac{N-1}{2r}u^2\right)\bigg|_a^b \\
&\quad + \left(N - \frac{3}{2} - \frac{1}{p+1}\right)\int_a^b u^{p+1} + \lambda\int_a^b \left(\frac{r}{2}\dot V - (N-2)V\right)u^2 \\
&\quad + \left(N - \frac{3}{2}\right)\int_a^b \frac{N-1}{2r^2}u^2. \qquad \text{(A.3)}
\end{aligned}$$

Proof. Multiply (1), written in polar coordinates, by $r\dot{u}$ and integrate on $[a,b]$:

$$\int_a^b \bigl(u^p - \lambda V u\bigr) r\dot{u} = \int_a^b \left(-\ddot{u} - \frac{N-1}{r}\dot{u}\right) r\dot{u} = -\frac{r}{2}\dot{u}^2\Big|_a^b - \left(N - \frac{3}{2}\right)\int_a^b \dot{u}^2.$$

An integration by parts gives

$$\int_a^b \bigl(u^p - \lambda V u\bigr) r\dot{u} = r\left(\frac{u^{p+1}}{p+1} - \frac{\lambda}{2} V u^2\right)\Big|_a^b - \frac{1}{p+1}\int_a^b u^{p+1} + \frac{\lambda}{2}\int_a^b (V + r\dot{V})u^2.$$

Hence, we obtain:

$$\frac{a}{2}\dot{u}^2(a) = \frac{b}{2}\dot{u}^2(b) + r\left(\frac{u^{p+1}}{p+1} - \frac{\lambda}{2} V u^2\right)\Big|_a^b + \left(N - \frac{3}{2}\right)\int_a^b \dot{u}^2 - \frac{1}{p+1}\int_a^b u^{p+1} + \frac{\lambda}{2}\int_a^b (V + r\dot{V})u^2. \tag{A.4}$$

Multiplying (1) by u and integrating on $[a,b]$, we get:

$$\int_a^b \bigl(u^{p+1} - \lambda V u^2\bigr) = \int_a^b \left(-\ddot{u} - \frac{N-1}{r}\dot{u}\right) u = -\dot{u}u\Big|_a^b + \int_a^b \dot{u}^2 - \frac{N-1}{2r}u^2\Big|_a^b - \int_a^b \frac{N-1}{2r^2}u^2$$

and so

$$\int_a^b \dot{u}^2 = \left(\dot{u}u + \frac{N-1}{2r}u^2\right)\Big|_a^b + \int_a^b \frac{N-1}{2r^2}u^2 + \int_a^b \bigl(u^{p+1} - \lambda V u^2\bigr). \tag{A.5}$$

Inserting (A.5) in (A.4), we finally get (A.3). □

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