A prescribed scalar curvature-type equation: almost critical manifolds and multiple solutions

Pierpaoalo Espisito\textsuperscript{a} and Gianni Mancini\textsuperscript{b,*}

\textsuperscript{a} Dipartimento di Matematica, Universit\`a degli Studi di Roma Tor Vergata, via della Ricerca Scientifica, Rome 00133, Italy
\textsuperscript{b} Dipartimento di Matematica, Universit\`a degli Studi Roma Tre, Largo San Leonardo Murialdo, 1, Rome 00146, Italy

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Abstract

We present an asymptotic analysis for a perturbed prescribed scalar curvature-type equation. A major consequence is a non-existence result in low dimension. Conversely, we prove an existence result in higher dimensions: to this aim we develop a general finite-dimensional reduction procedure for perturbed variational functionals. The general principle can be useful to discuss some other nonlinear elliptic PDE with Sobolev critical growth in bounded domains. © 2003 Elsevier Inc. All rights reserved.

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1. Introduction

Let $\Omega$ be a smooth bounded open set in $\mathbb{R}^N$, $N \geq 3$, and $f(x) \in C^\infty (\bar{\Omega})$ be a function positive somewhere. It is well known that the problem

\begin{equation}
\begin{aligned}
-\Delta u &= f(x)u^{N+2
\over
N-2} & \text{in } \Omega, \\
u > 0 & & \text{in } \Omega,
\end{aligned}
\end{equation}

(PSCE)

\begin{equation}
u = 0 & & \text{in } \partial \Omega
\end{equation}
has no solution, in general: by Pohozaev identity, and if \( \Omega \) is strictly star-shaped, a necessary condition is \( 0 < \sup_{x \in \Omega} \langle \nabla f(x), x \rangle \). Moreover, ground state solutions do never exist:

\[
\inf_{\{u \in H^1_0(\Omega) : \int_{\Omega} f |u|^{N/2-2} > 0\}} \frac{\int_{\Omega} |\nabla u|^2}{(\int_{\Omega} f |u|^{N-2})^{N/2}} = \frac{S}{(\max f)^{N/2}}
\]

(\( S \) = best Sobolev constant) is never attained.

We will discuss asymptotic behaviour and existence of multiple solutions for (PSCE) in the perturbative case: \( f = 1 + \delta a, a \in C^2(\bar{\Omega}) \) and \( \delta \to 0 \). We will refer to this perturbative problem as (PSCE)\(_\delta\).

In Section 2 we will perform a blow-up analysis for one-peak solutions of (PSCE)\(_\delta\), showing, in particular, that in quite general situations boundary concentration cannot occur. Another major outcome will be the non-existence, in low dimensions, of one-peak solutions (i.e. with energy close to \( S_N^N \)):

**Theorem 1.1.** Let \( N = 3, 4 \). If \( u_\delta \) are solutions of (PSCE)\(_\delta\) then

\[
\lim_{\delta \to 0} \inf \int_{\Omega} |\nabla u_\delta|^2 > S_N^N.
\]

As for existence, we state in Section 3 a variational principle for perturbative problems in presence of a manifold of “quasi critical points” for an unperturbed energy functional. Our principle extends to a more general setting, a nonlinear Lyapunov–Schmidt-type reduction introduced in [6] and recently improved by Ambrosetti and alias (see [5] and also the pioneering work of Rey [35]).

In Section 4 we will apply our reduction principle to (PSCE)\(_\delta\) to give some existence and multiplicity result (of one-peak solutions) in dimension \( N \geq 5 \):

**Theorem 1.2.** Let \( N \geq 5 \). Let \( x_0 \in \Omega \) be an isolated critical point of \( a \) with non-zero topological index and \( \Delta a(x_0) > 0 \). Then (PSCE)\(_\delta\) has solutions \( u_\delta \) which blow up, as \( \delta \) goes to zero, exactly at \( x_0 \).

On large balls, we obtain some new insight for (PSCE) giving an interpretation of the index counting condition introduced by Bahri and Coron (as for Ref. [10]); see Theorem 4.9 and related remarks.

In Section 5, we will discuss some other applications of the finite-dimensional reduction to the following class of problems:

\[
(P) \begin{cases}
-\Delta u = |u|^{p-1} u + g(\delta, x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where, here and elsewhere in the paper, \( p = \frac{N+2}{N-2} \). Here \( g(\delta, x, u) \) is a perturbation term, small if \( \delta \) is small, satisfying the growth condition
\[
\exists c > 0 : |g(\delta, x, u)| \leq c(1 + |u|^p).
\]

For \( g(\delta, x, u) = \delta u \) and \( 0 < \delta < \lambda_1(\Omega) \), precise existence results for (P) were established in [14] (see also [2] for sharp conditions in higher dimensions and general nonlinearities); existence of multiple solutions and asymptotic behaviour for \( \delta \to 0^+ \) were discussed in [26,35]. We generalize to a perturbation term \( g(\delta, x, u) = \delta a(x)|u|^{q-1}u, \ 1 \leq q < p, \ a(x) \in C^\infty(\bar{\Omega}) \). We cover also the case \( g(\delta, x, u) = |u + \delta a(x)|^{p-1}(u + \delta a(x)) - |u|^{p-1}u \), slightly improving existence results for non-homogeneous BVPs obtained in [37] (see also [16,17]).

2. Asymptotic analysis for (PSCE)\(_{\delta} \), boundary concentration and a non-existence result in low dimensions

Blow-up analysis for (PSCE) is a problem widely studied: see, to quote a few, [15,26,34,35] in case \( f \equiv 1 \), [27,28] in case \( f \) not constant and [23,39,40] for (PSCE) with an additional linear term (in [27,39,40] blow-up analysis of subcritical minimizers in a radial setting leads to an existence result). We will restrict our attention to “one-peak solutions” for

\[
(\text{PSCE})_{\delta} \quad \begin{cases} 
-\Delta u = (1 + \delta a(x))u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

I.e. we consider solutions \( u_\delta \) to (PSCE)\(_{\delta} \) such that, for some \( y_0 \in \bar{\Omega} \)
\[
|\nabla u_\delta|^2 \to S^2\delta_{y_0} \quad \text{as } \delta \to 0 \quad \text{in the sense of measures}.
\]

An important point here is to show that boundary concentration cannot occur if a non-degeneracy assumption on the critical points of \( a \) on \( \partial\Omega \) is fulfilled. Some non-degeneracy assumption seems to be in some sense necessary, since in general we cannot exclude such a phenomenon: in [18] it is exhibited a sequence of solutions for some perturbation of (PSCE) blowing up at a flat strict local maximum of \( f \) on the boundary.

As far as we know, the only known obstruction to boundary concentration is the following: \( \frac{\partial u}{\partial n} < 0 \) on \( \partial\Omega \), see [9] for a result in this direction for (PSCE) in the non-perturbative case. If \( \frac{\partial a}{\partial n} \leq 0 \) on \( \partial\Omega \), the method of [26], based on moving plane techniques as developed in [25], might exclude, in some cases, boundary concentration (one should ask, in addition, that \( a(x) \) increases in the inward normal direction in a neighbourhood of the boundary). Instead, we will use, for general \( a(x) \),
the method in \cite{35}: after improving some estimates and performing an accurate expansion of Pohozaev identities, it can be put at work to give the result.

Let us recall some well-known facts. For $\varepsilon > 0$ and $y \in \mathbb{R}^N$, let

\[ U_{\varepsilon,y}(x) = \varepsilon^{-\frac{N-2}{2}} U \left( \frac{x - y}{\varepsilon} \right), \quad U(x) = \frac{c_N}{(1 + |x|^2)^{\frac{N-2}{2}}}, \quad c_N = [N(N - 2)]^{-\frac{N-2}{4}}. \]

$U_{\varepsilon,y}$ are known to be the positive solutions in $\mathbb{R}^N$ of $-\Delta u = \frac{N+2}{N-2} u$. Denoted by $P: D^{1,2}(\mathbb{R}^N) \rightarrow H_0^1(\Omega)$ the orthogonal projection:

\[ \int_{\Omega} \nabla P\varphi \nabla \psi = \int_{\Omega} \nabla \varphi \nabla \psi \quad \forall \psi \in H_0^1(\Omega), \]

let

\[ T_{\varepsilon PU_{\varepsilon,y}} := \left\{ w \in H_0^1(\Omega) : \langle w, PU_{\varepsilon,y} \rangle = \langle w, \frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon} \rangle \right\} \]

\[ = \left\{ w, \frac{\partial PU_{\varepsilon,y}}{\partial y_i} \right\} = 0 \quad i = 1, \ldots, N \} \right\}. \]

The following facts are well known (see Proposition 2 in \cite{11} and \cite{35,38}):

**Proposition 2.1.** Let $\{u_\delta\}$ be as above. Then, for $\delta$ small,

\[ u_\delta = x_\delta PU_{\varepsilon_\delta,y_\delta} + w_\delta \quad (1) \]

with $x_\delta, \varepsilon_\delta \in (0, +\infty), y_\delta \in \Omega, w_\delta \in T_{\varepsilon_\delta PU_{\varepsilon_\delta,y_\delta}}$ and, as $\delta \to 0$,

\[ x_\delta \to 1, \quad y_\delta \to y_0, \quad \frac{\varepsilon_\delta}{\text{dist}(y_\delta, \partial \Omega)} \to 0, \quad w_\delta \to 0 \text{ in } H_0^1(\Omega) \]

Some notations are in order. Let $H(x, y)$ denote the regular part of the Green function of $\Omega$, i.e. for $x \in \Omega$

\[ \Delta_j H(x, y) = 0 \quad \text{in } \Omega, \]

\[ H(x, y) = |x - y|^{-(N-2)} \quad \text{on } \partial \Omega \]

and set $H(y) := H(y, y)$. Also, denote $D := \varepsilon_N^{p+1} \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{\frac{N+2}{2}}}.

The main result in this section is the following:

**Theorem 2.2.** Let $N \geq 3$, $a \in C^2(\bar{\Omega})$, $\text{Crit } a := \{ x \in \bar{\Omega} : \nabla a(x) = 0 \}$. Assume $\{u_\delta\}$ are solutions for \((\text{PSCE})_\delta\) such that, for some $y_0 \in \bar{\Omega}$

\[ |\nabla u_\delta|^2 \to S^N \delta_{y_0} \quad \text{as } \delta \to 0 \text{ in the sense of measures}. \quad (2) \]
Then \( N \geq 5, \nabla a(y_0) = 0 \) and \( \Delta a(y_0) \geq 0 \). Furthermore, \( y_0 \) cannot belong to \( \partial \Omega \), provided

\[
D^2 a \text{ is invertible } \forall x \in \text{Crit } \cap \partial \Omega.
\] (a)

In addition, if we write \( u_\delta \) as in (1), it results

\[
\varepsilon_\delta^{N-4} = \delta \frac{S^N \Delta a(y_0)}{N(N-2)DH(y_0)} + o(\delta) \text{ as } \delta \to 0.
\] (3)

We now derive Theorem 1.1 from the first statement in Theorem 2.2.

**Proof of Theorem 1.1.** First of all, let us remark that

\[
C_0 := \inf_{M} \int_{\Omega} |\nabla u|^2 > S^N_0,
\]

where \( M \) is the set of non-trivial solutions of \((\text{PSCE})_{\delta=0}\). Otherwise we could find a sequence \( \{u_n\}_{n \in \mathbb{N}} \) such that \( u_n \) solves \((\text{PSCE})_{\delta=0}\) and \( \int_{\Omega} |\nabla u_n|^2 \to S_\delta^N \) as \( n \to + \infty \).

Since \((\text{PSCE})_{\delta}\) has no ground-state solutions, \( u_n \to 0 \) weakly in \( H^1_0(\Omega) \) and \( |\nabla u_n|^2 \to S_\delta^N \) in the sense of measures, \( y_0 \in \tilde{\Omega} \) (see [38]). By (6), we have that

\[
z_n^2(N-2)\varepsilon_n^{N-2}H(y_n)D + \left( \frac{\varepsilon_n}{d_n} \right)^{N-2} = 0,
\]

where \( d_n := d(y_n, \partial \Omega) \) and \( z_n, \varepsilon_n, y_n \) are as in Proposition 2.1. A contradiction in view of \( d_N^{-2}H(y_n) = O(1) \) (see [35]).

Now, assume there are solutions \( u_\delta \) for \((\text{PSCE})_{\delta}\) with \( \delta \to 0 \) and \( \int_{\Omega} |\nabla u_\delta|^2 < \min C_0, dS_\delta^N \). From above, we derive that \( u_\delta \to 0 \) in \( H^1_0(\Omega) \) and hence Theorem 2.2 applies: \( N \geq 5 \). \( \Box \)

To prove Theorem 2.2, we will make use of Pohozaev identities (see [33]):

**Lemma 2.3.** Let \( u \) be a smooth solution of \((\text{PSCE})_{\delta}\), \( n(x) \) the unit outer normal to \( \partial \Omega \) in \( x \). Then, for any \( y \in \mathbb{R}^N \) and \( j = 1, \ldots, N \) we have

\[
\int_{\partial \Omega} \left( \frac{\partial u}{\partial n} \right)^2 \langle x - y, n(x) \rangle = \frac{N-2}{N} \delta \int_{\Omega} \langle x - y, \nabla a(x) \rangle u^{2N-2},
\] (4)

\[
\int_{\partial \Omega} \left( \frac{\partial u}{\partial n} \right)^2 n_j(x) = \frac{N-2}{N} \delta \int_{\Omega} \partial_j a(x)u^{2N-2}.
\] (5)
Proof of Theorem 2.2. We will plug $u_\delta$ (as given in (1)) in (4)–(5) and use several estimates from Appendix B. We will omit from now on the dependence on $\delta$. Inserting (B.3) and (B.5) into (4), we get

$$
\frac{\alpha^2(N-2)e^{N-2}}{\alpha^{p+1} S N^2 \delta e^2} \Delta a(y) = O \left( \frac{\epsilon^N}{d} \right)^{N-1} + \delta e \left( \frac{\epsilon^N}{d} \right)^{N-2} + \delta e^2 \ln \frac{1}{\epsilon} + \delta^2 e^2 + \left| x - y \right| \left[ \frac{\partial w}{\partial n} \right]_{L^2(\partial \Omega)} + d \left[ \frac{\partial w}{\partial n} \right]_{L^2(\partial \Omega)} + \int_{\partial \Omega} \left| x - y \right|^2 \left| \frac{\partial P U_{x,y}}{\partial n} \right| + d \int_{\partial \Omega} \left| \frac{\partial P U_{x,y}}{\partial n} \right| \left| \frac{\partial w}{\partial n} \right| .
$$

Here we used the following fact: if $\pi y$ denotes the projection of $y$ on $\partial \Omega$ and $d := \text{dist}(y, \partial \Omega) \leq d_0$ suitably small, then $\langle x - y, n(x) \rangle = \langle x - \pi y, n(x) \rangle + O(d) = O(\left| x - \pi y \right|^2 + d) = O(\left| x - y \right|^2 + d)$. Now, using (B.9)–(B.10) and (B.13)–(B.14) and

$$
\delta e \left( \frac{\epsilon^N}{d} \right)^{N-2} = O \left( \delta^2 e^2 + \delta^3 \left( \frac{\epsilon^N}{d} \right)^{N-2} \right) = o \left( \delta e^2 + \left( \frac{\epsilon}{d} \right)^{N-2} \right) \quad \text{as } \delta \to 0,
$$

we get

$$
\frac{\alpha^2(N-2)e^{N-2}}{\alpha^{p+1} S N^2 \delta e^2} H(y) D - \frac{1}{N} \frac{\alpha^{p+1} S N^2 \delta e^2}{\alpha^{N-1}} \Delta a(y) + o \left( \left( \frac{\epsilon^N}{d} \right)^{N-2} + \delta e^2 \right) = 0.
$$

Since $H(y)d^{N-2} \to C(y_0) > 0$ as $\delta \to 0$ (see [35]), we obtain $\Delta a(y_0) \geq 0$ and

$$
\frac{e^{N-4}}{d^{N-2}} = O(1).
$$

This implies, in particular, $N \geq 5$. Now, inserting (B.6) and (B.11) into (5), we obtain, for $j = 1, \ldots, N$,

$$
\delta \partial_j a(y) = O \left( \frac{\epsilon^{N-2}}{d^{N-1}} + \delta \left( \frac{\epsilon^N}{d} \right)^2 + \delta e^2 \right) + O \left( \left[ \frac{\partial w}{\partial n} \right]_{L^2(\partial \Omega)} ^2 + \int_{\partial \Omega} \left| \frac{\partial P U_{x,y}}{\partial n} \right| \left| \frac{\partial w}{\partial n} \right| \right).
$$

Hence, from (B.9) and (B.13) we get

$$
\nabla a(y) + O \left( \frac{\epsilon^{N-2}}{d^{N-1}} + \frac{\epsilon^2}{d^2} \right) = 0
$$

because

$$
\delta \left( \frac{\epsilon^N}{d} \right)^{N-2} = \frac{\delta \epsilon \epsilon^2}{d^2} \left( \frac{\epsilon^N}{d} \right)^{-N-1} = O \left( \delta^2 \frac{e^2}{d} + \frac{e^{N-2}}{d^{N-1}} \right).
$$
From (7) and (8), we get
\[ |\nabla a(y)| = O\left(\frac{\varepsilon^2}{d}\right) \]
and hence \( \nabla a(y_0) = 0 \). Also, assuming \( y_0 \in \partial \Omega \), (9) rewrites as
\[ \left| D^2 a(y_0) \left( \frac{y - y_0}{|y - y_0|} \right) + o\left(\frac{|y - y_0|}{|y - y_0|}\right) \right| = O\left(\frac{\varepsilon^2}{d^2}\right) \quad \text{as} \quad \delta \to 0 \]
for \( |y - y_0| \geq d \) and this implies \( D^2 a(y_0) \) is not invertible, contradicting (a). Hence \( y_0 \in \text{Crit} a \cap \Omega \). Finally, using \( \alpha \to 1 \), from (6) we get
\[ \varepsilon^{N-4} d^{-1} \to \frac{1}{N(N-2)D} \sum_{j} \frac{\Delta a(y_0)}{H(y_0)} \quad \text{as} \quad \delta \to 0. \quad \square \]

3. Almost critical manifolds and a reduction procedure: a general principle

We will develop in this section a perturbation theory for functionals of the form
\[ E_\delta(u) = E(u) - G(\delta, u), \quad u \in V, \quad \delta \in (-\delta_0, \delta_0), \]
where \( G \) is a “small” \( C^2 \) functional on the Hilbert space \( V \) and \( E \) has a “non-degenerate almost critical manifold”, that is:

There is a smooth immersion \( z : (0, +\infty) \times (0, +\infty) \times O \to V \), \( O \) smooth open set in \( \mathbb{R}^N \), parametrizing the smooth manifold \( \tilde{Z} = \{z(\alpha, \varepsilon, y) : \alpha > 0, \varepsilon > 0, y \in O\} \), such that

(A1) \( \tilde{Z} \) is bounded and \( \sup_{y \in O} ||\nabla E(z(\alpha, \varepsilon, y))|| = o(1) \) as \( (\alpha, \varepsilon) \to (1, 0) \),

(A2) there exists \( 0 < \varepsilon_0 < 1 \) such that \( L_z := \pi_\varepsilon^1 E''(z)|_{T^1_z} \in \text{Iso}(T^1_z, \tilde{T}^1_z) \forall z \in \tilde{Z} \) and \( \sup_{z \in \tilde{Z} \varepsilon_0} ||L_{z^{-1}}|| < \infty \),

where \( Z_\varepsilon := \{z(\alpha, \varepsilon, y) : 1 - s < \alpha < 1 + s, 0 < \varepsilon < s, y \in O\} \), \( 0 < s < 1 \), \( T_z \) is the tangent space at \( z \in Z \) and \( \pi_z : V \to T_z, \pi_{z^1} = \text{Id} - \pi_z \), are the orthogonal projections.

We will also require a good behaviour of \( E \) around points \( z \in \tilde{Z} \).

For \( R(z, w) := \nabla E(z + w) - [\nabla E(z) + E''(z)w] \), we will assume

(A3) \( \sup_{z \in \tilde{Z}} ||R(z, w)|| = o(||w||) \) and \( \sup_{z \in \tilde{Z}} ||D_w R(z, w)|| = o(1) \) as \( ||w|| \to 0 \).

As for the perturbation \( G \), we will assume

(A4) \( G(\delta, u), ||G'(\delta, u)||, ||G''(\delta, u)|| \to_{\delta \to 0} 0 \) uniformly on bounded sets.

We will perform, under these assumptions, a reduction procedure which follows the lines developed by Ambrosetti and collaborators; while they deal with perturbations of functionals which possess a non-degenerate manifold of critical points, we are perturbing a functional which, in general, has no critical points at all: the manifold of critical points is replaced here by a manifold of “quasi-critical points”. Actually, problems which fit into this framework have been widely
considered, starting from the pioneering work [35] (see also [1,3,10–13,36,37] to quote a few). So, this is an effort to give a general framework, in the spirit of the work of Ambrosetti, while borrowing basic analysis from Rey. First, we have:

**Lemma 3.1.** Let $E_\delta$ satisfy assumptions (A1)–(A4). Then there exist $0<\varepsilon_1<1$, $\delta_1>0$ and a smooth map $z\mapsto w(\delta, z)$, $z=z(\varepsilon, x, y)$, for $|\delta|<\delta_1$, $1-\varepsilon_1<\varepsilon<1+\varepsilon_1$, $0<\varepsilon<\varepsilon_1$ and $y\in O$, such that

\begin{enumerate}
  \item $\pi_z w(\delta, z) \equiv 0$
  \item $\pi_z^+ \nabla E_\delta(z + w(\delta, z)) \equiv 0$.
\end{enumerate}

Furthermore,

\[ ||w(\delta, z)|| = O(||\pi_z^+ \nabla E_\delta(z)||). \tag{10} \]

**Proof.** Set $L := \sup_{Z_{\varepsilon_0}} ||L_z^{-1}||$. Eqs. (i)–(ii) rewrite as a fixed point equation:

\[ w = -L_z^{-1} \pi_z^+ (\nabla E_\delta(z) - G''(\delta, z)w + R_\delta(z, w)), \quad w\in T_z^+ \tag{11} \]

where $L_z$ and $R_\delta$ are as above. For a given $\delta\in(-\delta_0, \delta_0)$ and $z\in Z_{\varepsilon_0}$, let us denote by $N_{\delta, z}$ the operator at the right-hand side in (11). We have

\[ ||N_{\delta, z}(w)|| \leq L(||\nabla E(z)|| + ||\nabla G(\delta, z)|| + ||G''(\delta, z)|| ||w|| + ||R_\delta(z, w)||). \]

By (A3) and (A4), we can find $\rho>0$, $0<\delta_1<\delta_0$ such that

\[ \sup_{z\in Z} ||R_\delta(z, w)|| + \sup_{z\in Z} ||D_w R_\delta(z, w)|| ||w|| \leq \frac{1}{4L} ||w||, \quad ||w|| \leq \rho, \quad |\delta| < \delta_1, \]

\[ \frac{1}{\rho} \left( ||\nabla G(\delta, z)|| + ||G''(\delta, z)|| \right) \leq \frac{1}{4L}, \quad |\delta| < \delta_1, \quad z\in Z. \]

By (A1) we can find $0<\varepsilon_1<\varepsilon_0$ such that $\sup_{z\in Z_{\varepsilon_0}} ||\nabla E(z)|| \leq \frac{1}{4\pi\rho}$. Hence,

\[ ||w|| \leq \rho \Rightarrow ||N_{\delta, z}(w)|| \leq \rho, \]

that is, $N_{\delta, z}$ maps $B_\rho := \{w\in T_z^+: ||w|| \leq \rho\}$ into itself for $z\in Z_{\varepsilon_1}$, $|\delta| < \delta_1$.

Since for $w_1, w_2 \in B_\rho$ we get

\[ ||N_{\delta, z}(w_1) - N_{\delta, z}(w_2)|| \leq L \left( \sup_{0\leq t\leq 1} ||D_w R_\delta(z, tw_1 + (1-t)w_2)|| + \frac{1}{4L} \right) ||w_1 - w_2|| \]

\[ \leq \frac{1}{2} ||w_1 - w_2||, \]
we see that \( N_{\delta,z} \) is a contraction on \( B_\rho \). Thus, \( N_{\delta,z}(\cdot) \) has a fixed point in \( B_\rho \), say \( w = w(\delta, z) \) for \( |\delta| < \delta_1 \) and \( z \in Z_{\eta} \). Now, from the fixed point equation,

\[
||w(\delta, z)|| = ||N_{\delta,z}(w(\delta, z))|| = O(||\pi_z^+ \nabla E_\delta(z)||) + o(1)||w(\delta, z)||,
\]

where \( o(1) \to 0 \) as \( \rho + \delta \to 0 \), and hence \( ||w(\delta, z)|| = O(||\pi_z^+ \nabla E_\delta(z)||) \).

Smoothness of \( z \to w(\delta, z) \) follows by the IFT applied to the equation

\[
\pi_z^+ \nabla E_\delta(z + \pi_z^+ u) + \pi_z u = 0, \quad u \in H_0^1(\Omega).
\]

In fact, the linearized operator at \( w = w(\delta, z) \), \( \pi_z^+ E''_\delta(z + w) \pi_z^+ + \pi_z \) is invertible, up to take \( \varepsilon_1, \delta_1 \) smaller, because sup \( z \in Z_{\eta} \) \( ||w(\delta, z)|| \to 0 \) as \( \varepsilon_1 + \delta_1 \to 0 \) and, at \( \delta = 0, w = 0 \), it is trivially invertible by (A.2). □

The final step in the reduction procedure is to prove that critical points of \( E_\delta \), close to \( Z \), correspond to critical points of

\[
E_\delta(x, \varepsilon, y) := E_\delta(z(x, \varepsilon, y) + w(\delta, z(x, \varepsilon, y)))).
\]

The proof relies on \( C^1 \) estimates of \( w(\delta, z) \) which involve the variation of \( T_z \). Let us first prove \( C^1 \) estimates under suitable assumptions.

**Lemma 3.2.** Assume (A1)–(A4) and let \( w(\delta, z) \) be given by Lemma 3.1. Then

\[
\left| \pi_z \frac{\partial w}{\partial z} \right| = O(||w||) \tag{12}
\]

provided the following assumption holds true:

\[
\exists c > 0 : \left| \pi_z \frac{\partial}{\partial z} (\pi_z^+ v) \right| \leq c ||\pi_z^+ v|| \quad \forall z \in Z, \quad \forall v \in V. \tag{A5}
\]

**Proof.** Let \( \bar{w} = w(\delta, \bar{z}) \) for some \( \bar{z} \in Z, \delta \) fixed. From \( \pi_z w(\delta, z) \equiv 0 \) it follows \( \pi_z \frac{\partial w}{\partial z} = -\frac{\partial}{\partial z} (\pi_z \bar{w}) \) at \( z = \bar{z} \). Since \( -\frac{\partial}{\partial z} (\pi_z \bar{w}) = \frac{\partial}{\partial z} (\pi_z^+ \bar{w}) \), we have, by (A5),

\[
\left| \pi_z \frac{\partial w}{\partial z} (\delta, \bar{z}) \right| \leq c ||\pi_z^+ \bar{w}||.
\]

This proves (12), because \( \pi_z^+ \bar{w} = \bar{w} \). □

**Theorem 3.3.** Assume (A1)–(A5) and let \( w(\delta, z) \) be given by Lemma 3.1. Then, for \( \varepsilon, \delta \) small, \( \nabla E_\delta(z_0 + w(\delta, z_0)) = 0 \) iff \( z_0 \) is a critical point of \( z \to E_\delta(z + w(\delta, z)) \).
Proof. Let \( z(t) \) be a smooth curve on \( Z \) with \( z(0) = z_0 \) and \( \dot{z}(0) = \pi_{z_0} \nabla E_\delta(z_0 + w(\delta, z_0)) \). By assumption,

\[
0 = \frac{d}{dt} E_\delta(z(t) + w(\delta, z(t)))|_{t=0} = \left\langle \nabla E_\delta(z_0 + w(\delta, z_0)), \dot{z}(0) + \frac{\partial w}{\partial z}(\delta, z_0)z(0) \right\rangle.
\]

Since \( \pi_{z_0}^{-1} \nabla E_\delta(z_0 + w(\delta, z_0)) = 0 \), using (10) and (12), we get

\[
||\dot{z}(0)||^2 \leq ||\dot{z}(0)||^2 \left|\pi_{z_0} \frac{\partial w}{\partial z}(\delta, z_0)\right| \leq c||w(\delta, z_0)|| ||\dot{z}(0)||^2 \leq \varepsilon ||\dot{z}(0)||^2 ||\nabla E_\delta(z_0)||
\]

and hence \( \dot{z}(0) = 0 \) because \( ||\nabla E_\delta(z)|| \leq 1 \) for \( z \in Z_{\varepsilon_1} \) if \( \varepsilon_1 + \delta_1 \) is small. \( \square \)

Remark 3.4 (The Melnikov function). Theorem 3.3 applies as follows: first, write

\[
z(\alpha, \varepsilon, y) = z(\tau), \quad \tau = (\alpha, \varepsilon, y)
\]

and

\[
E_\delta(z(\tau) + w(\delta, \tau)) = E(z(\tau)) - G(\delta, z(\tau)) + \int_0^1 \langle \nabla E_\delta(z(\tau) + tw(\delta, \tau)), w(\delta, \tau) \rangle dt.
\]

If we suppose \( E'' \) uniformly bounded on bounded sets, we have, by (10),

\[
E_\delta(z(\tau) + w(\delta, \tau)) = E(z(\tau)) - G(\delta, z(\tau)) + O(||\pi_{z(\tau)} \nabla E_\delta(z(\tau))||^2).
\]

In the applications, the remainder term will be “negligible” and one is led to look for critical points of the “Melnikov function”

\[
E_\delta(z(\tau)) = E(z(\tau)) - G(\delta, z(\tau)).
\]

4. Multiple solutions for (PSCE)_\delta

Here we complement the non-existence result contained in Theorem 1.1 by showing that for \( N \geq 5 \) there are branches of solutions for (PSCE)_\delta bifurcating from critical points of \( a(x) \) with positive laplacian, non-degenerate in some sense: this is the content of Theorem 1.2. To prove it, we will apply Theorem 3.3 to the functional

\[
E_\delta(u) = E(u) - G(\delta, u), \quad u \in H_0^1(\Omega),
\]

where

\[
E(u) = \frac{1}{2} \int_\Omega \nabla u^2 - \frac{1}{p+1} \int_\Omega |u|^{p+1}, \quad G(\delta, u) = \frac{\delta}{p+1} \int_\Omega a(x)|u|^{p+1}.
\]

The functional \( E(u) \) possesses a “non-degenerate almost critical manifold”

\[
Z := \{ \alpha \mathcal{P} U_{\varepsilon, \gamma} : \alpha > 0, \quad \varepsilon > 0, \quad y \in \Omega, \quad d(y, \partial \Omega) > \gamma \}, \quad \gamma > 0,
\]
where $PU_{e,y}$ are as in Section 2. In particular, $PU_{e,y}$ is the unique solution of
\[- \Delta PU_{e,y} = - \Delta U_{e,y} = U^p_{e,y} \quad \text{in } \Omega,
\]
\[PU_{e,y} = 0 \quad \text{on } \partial \Omega.
\]
We will omit, if not relevant, any reference to $\gamma$. We will use several facts stated in Appendix A.

Assumptions (A1) and (A2) are checked in Lemma A.6, while (A.3) follows from

**Lemma 4.1.** Let $\hat{p} = \min\{p, 2\}$. Then
\[\exists c > 0 : ||R(z, w)|| + ||D_uR(z, w)|| ||w|| \leq c ||w||^\hat{p} \quad \forall z \in \tilde{Z}.
\]

**Proof.** By direct computation, for any $\phi, \psi \in H^1_0(\Omega)$:
\[
\langle R(z, w), \phi \rangle = - \int_\Omega \left[ |z + w|^{p-1}(z + w) - zw^{p-1}w \right] \phi,
\]
\[
\langle D_uR(z, w)\phi, \psi \rangle = p \int_\Omega (z^{p-1} - |z + w|^{p-1}) \phi \psi.
\]

Using the elementary inequalities, for $a, b \in \mathbb{R}$,
\[
|a + b|a + b|^{p-1} - a|a|^{p-1} - p|a|^{p-1}b| \leq \begin{cases} c_p(|a|^{p-2}b^2 + |b|^p) & \text{if } p > 2, \\ c_p|b|^p & \text{if } p \leq 2, \end{cases}
\]
\[|a|^{p-1} - |a + b|^{p-1} \leq \begin{cases} c_p(|a|^{p-2}|b| + |b|^{p-1}) & \text{if } p > 2, \\ c_p|b|^{p-1} & \text{if } p \leq 2, \end{cases}
\]
and Hölder and Sobolev inequalities, the Lemma readily follows. $\square$

Assumption (A4) is easily checked and (A5) follows by Lemmas (A.4) and (A.5) and

**Remark 4.2.** Assumption (A5) involves the second derivatives of $z(x, e, y)$. Property (A5), and hence Lemma 3.2, Theorem 3.3, can be derived more directly by the following facts:
\[\exists c > 0 : \sum_{j,k} \frac{||\partial_{jk}z||^2}{||\partial_jz||^2||\partial_kz||^2} \leq c, \quad \langle \partial_tz, \partial_tz \rangle = o(||\partial_tz||||\partial_tz||) \quad \forall i \neq j. \quad (13)
\]

In fact, if $s = (x, e, y)$ and $z(s(t))$ is a curve in $Z$ such that $z(s(0)) = z$, property (A5) is equivalent to prove
\[
\left| \left| \pi_z \frac{d}{dt} (\pi_z v) \right| \right|_{t=0} \leq c \left| \left| \pi_z v \right| \right| \left| \left| \frac{dz}{dt} \right| \right|, \quad \forall v \in H^1_0(\Omega).
\]
If we write \( \pi_z \frac{d}{dt}(\pi_{z(t)}^+ v) \big|_{t=0} = \sum a_j \partial_z z \) and \( \frac{d}{dt} \big|_{t=0} = \sum_j \partial_z \frac{d s_j}{dt}(0) \), the second assumption in (13) implies that

\[
\left| \pi_z \frac{d}{dt}(\pi_{z(t)}^+ v) \right|_{t=0}^2 = (1 + o(1)) \sum_j a_j^2 ||\partial_z z||^2,
\]

\[
\left| \frac{d z}{dt} \right|_{t=0}^2 = (1 + o(1)) \sum_j \left( \frac{d s_j}{dt} (0) \right)^2 ||\partial_z z||^2.
\]

Since \( \langle \pi_{z(t)}^+ v, (\partial_z z)(s(t)) \rangle \equiv 0 \), we can get

\[
\left| \pi_z \frac{d}{dt}(\pi_{z(t)}^+ v) \right|_{t=0}^2 = \left\langle \pi_z \frac{d}{dt}(\pi_{z(t)}^+ v) , \sum_j a_j \partial_z z \right\rangle
\]

\[
= - \sum_j a_j \left( \frac{d s_j}{dt} (0) \right) \left( \sum_j a_j^2 ||\partial_z z||^2 \right)^{1/2} \left( \sum_j ||\partial_z z||^2 \right)^{1/2}
\]

\[
\leq c ||\pi_z^+ v|| \left| \pi_z \frac{d}{dt}(\pi_{z(t)}^+ v) \right|_{t=0} \left| \frac{d z}{dt} \right|_{t=0}.
\]

Hence (A5) follows. So, instead of (A5), one might more easily check (13).

Now, we are led to look for critical points of \( E_\delta(x, \varepsilon, y) := E_\delta(x PU_{e,y} + w(\delta, x, \varepsilon, y)) \). Accordingly with Remark 3.4, we need to estimate the remainder term. Since \( \psi_{e,y} := U_{e,y} - PU_{e,y} \) is an harmonic function, we get

\[
||\psi_{e,y}||_{\infty} \leq \max_{\partial \Omega} U_{e,y} = O(\varepsilon^{N/2}),
\]

If we write for any \( \Phi \in H_0^1(\Omega) \)

\[
\langle \nabla E(z), \Phi \rangle = \alpha \int_\Omega \nabla PU_{e,y} \nabla \Phi - \tilde{\alpha} \int_\Omega PU_{e,y} \Phi
\]

\[
= (\alpha - \tilde{\alpha}) \int_\Omega U_{e,y}^p \Phi + \tilde{\alpha} \int_\Omega (PU_{e,y} - PU_{e,y}) \Phi,
\]

\[
\langle \nabla G(\delta, z), \Phi \rangle = \delta \tilde{\alpha} \int_\Omega a(x) PU_{e,y}^p \Phi
\]

\[
= \delta \tilde{\alpha} \left[ a(x) \int_\Omega PU_{e,y}^p \Phi + \int_\Omega \langle \nabla a(y), x - y \rangle U_{e,y}^p \Phi \right]
\]

\[
+ \delta \tilde{\alpha} \left[ \int_\Omega (a(x) - a(y)) - \langle \nabla a(y), x - y \rangle U_{e,y}^p \Phi \right]
\]

\[
+ \int_\Omega a(x)(PU_{e,y}^p - U_{e,y}^p) \Phi,
\]
we can obtain, using Lemma A.1,

$$\|\pi_{\delta}^z \nabla E(z)\| = O(e^{N/2})$$,

$$\|\nabla E(z)\| = O(e^{N/2} + |z|),$$

$$\|\pi_{\delta}^z \nabla G(\delta, z)\| = O(\delta e|\nabla a(y)| + \delta^2 e^2),$$

$$\|\nabla G(\delta, z)\| = O(\delta),$$

because \(\int_\Omega U^p_{\Gamma, \gamma} \Phi = \int_\Omega \nabla P U_{\Gamma, \gamma} \nabla \Phi = 0\) for any \(\Phi \in T_z^{\perp}\). As for the remainder term,

$$\|\pi_{\delta}^z \nabla E_\delta(z)\|^2 = O(\delta^2 e^2 |\nabla a(y)|^2 + \delta^2 e^4),$$

$$\|\nabla E_\delta(z)\|^2 = O(\delta^2 + |z|^2).$$

According to Lemma A.5 in Appendix A, we have

$$E(\delta PU_{\Gamma, \gamma}) = \left(\frac{\alpha^2}{2} - \frac{\alpha^{p+1}}{p+1}\right) S^{N/2}_2 + D \left(-\frac{\alpha^2}{2} + \alpha^{p+1}\right) H(y)e^{N-2} + O(e^{N-1}),$$

where, as in Section 2, \(D = e^{N/2} \int_{\mathbb{R}^N} \frac{dy}{(1+|y|^2)^{N/2}}\). Finally, from Lemmas A.1 and A.2 we see that

$$G(\delta, \delta PU_{\Gamma, \gamma}) = -\frac{\delta}{p+1} \alpha^{p+1} \int_\Omega a(x) P U_{\Gamma, \gamma}^{p+1}$$

$$= -\frac{\delta}{p+1} \alpha^{p+1} \int_\Omega \left[a(y) + \sum_i \partial_i a(y)(x - y)_iight.\]$$

$$+ \frac{1}{2} \partial_i a(y)(x - y)_i(x - y)_j + O(|x - y|^3) \left] U_{\Gamma, \gamma}^{p+1} + O(\delta e^{N-2})$$

$$= -\alpha^{p+1} \frac{S^2}{p+1} \delta a(y) - \alpha^{p+1} \frac{S^2}{4N} \delta^2 e^2 \Delta a(y) + O(\delta^3)$$

because, by an integration by parts,

$$\int_{\mathbb{R}^N} \frac{|x|^2}{(1+|x|^2)^N} dx = \frac{N}{N-2} \int_{\mathbb{R}^N} \frac{dx}{(1+|x|^2)^N} = \frac{N}{N-2} S^2_2.$$ 

Summarizing, using (14), we get the following expansions for \(E_\delta(x, \varepsilon, y), z \in Z:\)
Lemma 4.3. Let $N \geq 5$. Then
\[
E_\delta(\varepsilon PU_{x,y} + w) = \left(\frac{x^2}{2} - \frac{x^p}{p+1}\right) S^N \frac{N}{2} + D\left(\frac{x^2}{2} + x^p + 1\right) H(y) \varepsilon^{N-2} - x^p+1 \frac{S^N}{p+1} \delta a(y) - x^p+1 \frac{S^N}{4N} \delta \varepsilon^2 \Delta a(y) + O(\varepsilon^{N-1} + \delta^2 \varepsilon^2 |\nabla a(y)|^2 + \delta \varepsilon^3).
\]

Next, we establish $C^1$ estimates.

Lemma 4.4. Let $N \geq 5$. Then
\[
\frac{\partial}{\partial y_i} E_\delta(\varepsilon PU_{x,y} + w) = -x^p+1 \frac{S^N}{p+1} \delta a(y) + O(\varepsilon^{N-2} + \delta \varepsilon + \delta^2 + \varepsilon^2 |1 - \varepsilon| + \delta |1 - \varepsilon|) (15)
\]
\[
\frac{\partial}{\partial \varepsilon} E_\delta(\varepsilon PU_{x,y} + w) = D(N-2) \left(\frac{x^2}{2} + x^p + 1\right) H(y) \varepsilon^{N-3} - x^p+1 \frac{S^N}{2N} \delta a(y) + O(\varepsilon^{N-2} + \delta^3 + \varepsilon^2 |\nabla a(y)| + \delta |1 - \varepsilon| + \delta |1 - \varepsilon||\nabla a(y)| + \delta \varepsilon |1 - \varepsilon|) (16)
\]
\[
\frac{\partial}{\partial x} E_\delta(\varepsilon PU_{x,y} + w) = S^N (x - x^p) - \delta a(y) x^p S^N + O(\delta \varepsilon + \varepsilon^2). (17)
\]

Proof. Since $\nabla E_\delta(\varepsilon PU_{x,y} + w) = O(||\nabla E_\delta(\varepsilon PU_{x,y})||)$, we have
\[
\frac{\partial}{\partial y_i} E_\delta(\varepsilon PU_{x,y} + w)
= \left(\nabla E_\delta(\varepsilon PU_{x,y} + w), \varepsilon \frac{\partial PU_{x,y}}{\partial y_i} + \pi_x \frac{\partial w}{\partial y_i}\right)
= x^2 \left(\frac{\partial PU_{x,y}}{\partial y_i}\right) - x \int_\Omega (1 + \delta a(x)) |\varepsilon PU_{x,y} + w|^{p-1} (\varepsilon PU_{x,y} + w) \frac{\partial PU_{x,y}}{\partial y_i}
+ O\left(||\nabla E_\delta(\varepsilon PU_{x,y})|| \left|\pi_x \frac{\partial w}{\partial y_i}\right|\right).
The first term is estimated in Lemma A.5:

\[
\left\langle PU_{e,y}, \frac{\partial PU_{e,y}}{\partial y_i} \right\rangle = -D \frac{\partial H}{\partial y_i}(y, y) e^{N-2} + O(e^{N-1}). \tag{18}
\]

As for the third term, we first derive from (12) and Lemma A.4:

\[
\left\| \pi_z \frac{\partial w}{\partial y_i} \right\| = \left\| \pi_z \frac{\partial w}{\partial z} \frac{\partial z}{\partial y_i} \right\| = O\left( \frac{1}{e} \|w\| \right)
\]

and hence

\[
\left\| \nabla E_{\delta}(z PU_{e,y}) \right\| \left\| \pi_z \frac{\partial w}{\partial y_i} \right\| = \frac{1}{e} O(\|\nabla E_{\delta}(z)\| \|w\|). \tag{19}
\]

It remains to estimate the second term. We claim that

\[
\int_{\Omega} (1 + \delta a(x)) |z PU_{e,y} + w|^{p-1} (z PU_{e,y} + w) \frac{\partial PU_{e,y}}{\partial y_i} \\
= \delta^{p} \frac{S_2}{p + 1} \delta a(y) + O \left( e^{N-2} + \frac{\delta}{e} \|w\| + e^{\frac{N-2}{2}} \|w\| + \frac{1}{e} \|\nabla E_{\delta}(z)\| \|w\| \right). \tag{20}
\]

Putting together estimates (18)–(20), we get

\[
\frac{\partial}{\partial y_i} E_{\delta}(PU_{e,y} + w) = -\delta^{p+1} \frac{S_2}{p + 1} \delta a(y) \\
+ O \left( e^{N-2} + \frac{\delta}{e} \|w\| + e^{\frac{N-2}{2}} \|w\| + \frac{1}{e} \|\nabla E_{\delta}(z)\| \|w\| \right)
\]

and hence (15) follows from (10) and (14). We now prove (20). We have

\[
\int_{\Omega} (1 + \delta a(x)) |z PU_{e,y} + w|^{p-1} (z PU_{e,y} + w) \frac{\partial PU_{e,y}}{\partial y_i} \\
= \delta^{p} \int_{\Omega} PU_{e,y}^{p-1} \frac{\partial PU_{e,y}}{\partial y_i} + \delta^{p} \int_{\Omega} a(x) PU_{e,y}^{p-1} \frac{\partial PU_{e,y}}{\partial y_i} \\
+ p \delta^{p-1} \int_{\Omega} PU_{e,y}^{p-1} \frac{\partial PU_{e,y}}{\partial y_i} w + h.o.t., \tag{21}
\]

where, by Taylor expansion,

\[
\text{h.o.t.} = O \left( \delta \int PU_{e,y}^{p-1} \left| \frac{\partial PU_{e,y}}{\partial y_i} \right| |w| + \int PU_{e,y}^{p-2} \left| \frac{\partial PU_{e,y}}{\partial y_i} \right| w^2 + \int \left| \frac{\partial PU_{e,y}}{\partial y_i} \right| |w|^p \right)
\]
if $N = 5$, while

$$
h.o.t. = O\left(\delta \int_{\Omega} U_{e,y}^{p-1} \left| \frac{\partial P U_{e,y}}{\partial y_i} \right| |w| + \int_{\Omega} P U_{e,y}^{p-2} \left| \frac{\partial P U_{e,y}}{\partial y_i} \right| w^2 \right)$$

if $N \geq 6$. The first term in (21) is estimated in Lemma A.5:

$$
\int_{\Omega} P U_{e,y}^{p} \frac{\partial P U_{e,y}}{\partial y_i} = -2D \frac{\partial H}{\partial y_i} (y, y) e^{N-2} + O\left(e^{N-1} \log \frac{1}{\varepsilon}\right).
$$

As for the second term in (21), we observe that, using Lemmas A.1 and A.2, we get

$$
\int_{\Omega} a(x) P U_{e,y}^{p} \frac{\partial P U_{e,y}}{\partial y_i}
= \int_{\Omega} \left[a(y) + \sum_j \partial_j a(y)(x - y)_j + O(|x - y|^2)\right] U_{e,y} \frac{\partial U_{e,y}}{\partial y_i} + O(\varepsilon^{N-3})
= \frac{N - 2}{N} \partial_a a(y) e_N^{p+1} \int_{\mathbb{R}^N} \frac{|x|^2}{\left(1 + |x|^2\right)^{N+1}} dx + O(\varepsilon)
= \frac{S^N}{p + 1} \partial_a a(y) + O(\varepsilon). \tag{23}
$$

As for the third term in (21), using $U_{e,y}^{p-1} - P U_{e,y}^{p-1} \ll c U_{e,y}^{p-2} \psi_{e,y}$, $\frac{\partial U_{e,y}}{\partial y_i} = O\left(\frac{U_{e,y}}{\varepsilon}\right)$ and Lemmas A.2 and A.1, we have that

$$
\int_{\Omega} P U_{e,y}^{p-1} \frac{\partial P U_{e,y}}{\partial y_i} w = \int_{\Omega} U_{e,y}^{p-1} \frac{\partial U_{e,y}}{\partial y_i} w + O\left(\varepsilon^{\frac{N-2}{2}} ||w||\right) = O\left(\varepsilon^{\frac{N-2}{2}} ||w||\right), \tag{24}
$$

because $p \int_{\Omega} U_{e,y}^{p-1} \frac{\partial U_{e,y}}{\partial y_i} w = \langle \frac{\partial P U_{e,y}}{\partial y_i}, w \rangle = 0$. Finally, using $U_{e,y}^{p-2} - P U_{e,y}^{p-2} \ll c U_{e,y}^{p-3} \psi_{e,y}$, $\frac{\partial U_{e,y}}{\partial y_i} = O\left(\frac{U_{e,y}}{\varepsilon}\right)$ and recalling also (see Lemma A.1) $\left(\int_{\Omega} U_{e,y}^{N(6-N)/2} - \frac{2}{N}\right) = O\left(\varepsilon^{4-N}\right)$, we estimate h.o.t. in case $N \geq 6:

$$
h.o.t. = O\left(\frac{\delta}{\varepsilon} \int_{\Omega} \frac{U_{e,y}^{p-1}}{\varepsilon} \left| \frac{\partial P U_{e,y}}{\partial y_i} \right| |w| + \int_{\Omega} \frac{P U_{e,y}^{p-2}}{\varepsilon} \left| \frac{\partial P U_{e,y}}{\partial y_i} - \psi_{e,y} \right| w^2 \right)
= O\left(\frac{\delta}{\varepsilon} ||w|| + \int_{\Omega} \left(\frac{U_{e,y}^{p-1}}{\varepsilon} + \frac{N-4}{\varepsilon^2} U_{e,y}^{p-2} + \varepsilon^{N-2} U_{e,y}^{p-3}\right) w^2 \right)
= O\left(\frac{\delta}{\varepsilon} ||w|| + \frac{||w||^2}{\varepsilon}\right). \tag{25}
$$
In case $N = 5$, we estimate the additional term using Lemma A.4:

$$
\int_{\Omega} \left| \frac{\partial PU_{e,y}}{\partial y_i} \right| |w|^p \leq c \left( \frac{||w||^p}{\varepsilon} \right).
$$

Estimates (22), (23) and (25)–(26) yield (20) and the claim is proved. \hfill \square

As for the $\varepsilon$-derivative, we can argue in a similar way:

- Eq. (18) is replaced (see Lemma A.5) by
  
  $$
  \left\langle PU_{e,y}, \frac{\partial PU_{e,y}}{\partial e} \right\rangle = -\frac{N-2}{2} DH(y)\varepsilon^{N-3} + O(\varepsilon^{N-2})
  $$

- Eq. (19) remains unchanged (see Lemma A.4)
- Eq. (20) is replaced by
  
  $$
  \int_{\Omega} (1 + \delta a(x))|zPU_{e,y} + w|^{p-1} (zPU_{e,y} + w) \frac{\partial PU_{e,y}}{\partial e}
  $$
  
  $$
  = -(N-2)Dx^p H(y)\varepsilon^{N-3} + \varepsilon^p \frac{S_N}{2N} \delta \varepsilon \Delta a(y)
  $$
  
  $$
  + O \left( \varepsilon^{N-2} + \delta \varepsilon^2 + \frac{||w||^2}{\varepsilon} + \varepsilon^{\frac{N-2}{2}} ||w|| + \frac{\delta}{\varepsilon} ||w|| \right).
  $$

Putting together (27), (19), (28) and using (14), we obtain (16).

Estimate (28) can be obtained as in (20):

Eq. (22) is replaced (see Lemma A.5) by

$$
\int_{\Omega} PU_{e,y}^p \frac{\partial PU_{e,y}}{\partial e} = -(N-2)DH(y)\varepsilon^{N-3} + O(\varepsilon^{N-2});
$$

Eq. (23) is replaced by

$$
\int_{\Omega} a(x)PU_{e,y}^p \frac{\partial PU_{e,y}}{\partial e} = \int_{\Omega} \left[ a(y) + \sum_j \partial_j a(y)(x - y)_j
  $$

$$
+ \frac{1}{2} \sum_{ij} \partial_{ij} a(y)(x - y)_i(x - y)_j + O(|x - y|^3) \right] \frac{\partial U_{e,y}^p}{\partial e} + O(\varepsilon^{N-3})
  $$

$$
= -\frac{N-2}{4N} \Delta a(y) c_N^{p+1} \varepsilon \int_{\mathbb{R}^N} \frac{|x|^2(1 - |x|^2)}{(1 + |x|^2)^{N+1}} dx + O(\varepsilon^2)
  $$

$$
= \frac{1}{2N} \frac{S_N}{\varepsilon} \Delta a(y) + O(\varepsilon^2);
$$

(30)
Eq. (24) is replaced by
\[
\int_{\Omega} \left[ P U_{e,y}^{p-1} \frac{\partial P U_{e,y}}{\partial \varepsilon} - w \right] + \int_{\Omega} U_{e,y}^{p-1} \frac{\partial U_{e,y}}{\partial \varepsilon} w + O\left( \frac{N-2}{2} ||w|| \right) = O\left( \frac{N-2}{2} ||w|| \right);
\]
(31)
as for the h.o.t., (25) and (26) become, respectively,
\[
\delta \int_{\Omega} U_{e,y}^{p-1} \frac{\partial P U_{e,y}}{\partial \varepsilon} |w| + \int_{\Omega} P U_{e,y}^{p-2} \frac{\partial U_{e,y}}{\partial \varepsilon} - \frac{\partial \psi_{e,y}}{\partial \varepsilon} w^2
\]
\[
= O\left( \frac{\delta}{\varepsilon} ||w|| + \int_{\Omega} \left( \frac{U_{e,y}^{p-1}}{\varepsilon} + \varepsilon \frac{N-4}{2} U_{e,y}^{p-2} + \varepsilon^{N-3} U_{e,y}^{p-3} \right) w^2 \right)
\]
\[
= O\left( \frac{\delta}{\varepsilon} ||w|| + \frac{||w||^2}{\varepsilon} \right)
\]
(32)
and
\[
\int_{\Omega} \left| \frac{\partial P U_{e,y}}{\partial \varepsilon} \right| w^p \leq c \left( \frac{||w||^p}{\varepsilon} \right).
\]
(33)
As for the \(\alpha\)-derivative, we can argue in a similar but more direct way. Using Lemma A.5, it is easy to see that
\[
\frac{\partial}{\partial \alpha} E_{\delta}(\alpha P U_{e,y} + w) = \left\langle \nabla E_{\delta}(\alpha P U_{e,y} + w), P U_{e,y} + \sqrt{\frac{\varepsilon}{\alpha}} \frac{\partial w}{\partial \alpha} \right\rangle
\]
\[
= \alpha ||P U_{e,y}||^2 - \int_{\Omega} (1 + \delta \alpha(x)) ||P U_{e,y} + w||^{p-1} (\alpha P U_{e,y} + w) P U_{e,y}
\]
\[
+ O(||\nabla E_{\delta}(z)|| ||w||)
\]
\[
= \alpha ||P U_{e,y}||^2 - \alpha^p \int_{\Omega} P U_{e,y}^{p+1} - \delta a(y) \alpha^p \int_{\Omega} P U_{e,y}^{p+1} + O(\delta \varepsilon + ||w||)
\]
\[
= S_{\alpha}^N (\alpha - \alpha^p) - \delta a(y) \alpha^p S_{\alpha}^N + O(\delta \varepsilon + \varepsilon^N),
\]
because \(||\sqrt{\alpha} \frac{\partial w}{\partial \alpha}|| = O(||w|| ||P U_{e,y}||) = O(||w||).
\]

**Remark 4.5.** In the expansion of the \(\varepsilon\)-derivative (16), we have a remainder term \(O(\delta^2 ||\nabla a(y)||)||. The presence of \(||\nabla a(y)||\) is needed only for \(N = 5\). In fact, in this case we will require \(\delta \sim \varepsilon\); then \(\delta^2\) is not small with respect to the second leading term in the \(\varepsilon\)-derivative which is of order \(\delta \varepsilon\).  

**Proof of Theorem 1.2.** Choose \(\frac{N-6}{2(N-4)} < s < \frac{N}{2(N-4)} < 1\) if \(N > 8\) and \(s = 1\) if \(5 \leq N \leq 8\). Introducing new variables \(\theta = \delta^{-\frac{1}{N-4}} \varepsilon\), \(\nu = \delta^{-s} (\alpha - 1)\), we are led to look for zeroes
for the vector field 
\[ \Phi(v, \theta, y) = (Y_\delta, \Theta_\delta, Y_\delta)(v, \theta, y), \]
where 
\[ Y_\delta(v, \theta, y) = v + \frac{a(y)}{p-1} \delta^{1-s} + o(1), \]
\[ \Theta_\delta(v, \theta, y) = DN(N-2)H(y)\theta^{N-4} - S^N \Delta a(y) + o(1) + O(\delta^{N-5} |\nabla a(y)|), \]
\[ Y_\delta(v, \theta, y) = \nabla a(y) + o(1), \]
where \( o(\cdot) \), \( O(\cdot) \) hold for \( \delta \to 0 \) uniformly in \( y \) and \( \theta, v \) bounded.

Now let \( y_0 \) be an (interior) isolated critical point of \( a(x) \) with \( \Delta a(y_0) > 0 \) and non-
zero topological index. Then \( \theta_0 := \left( \frac{S^N \Delta a(y_0)}{N(N-2)DH(y_0)} \right)^{\frac{1}{N-4}} \) is well defined and positive.

Let us set 
\[ v_0 = \begin{cases} \frac{a(y_0)}{p-1} & \text{if } 5 \leq N \leq 8, \\ 0 & \text{if } N > 8, \end{cases} \]
and 
\[ n = \begin{cases} 1 & \text{if } 5 \leq N \leq 8, \\ 0 & \text{if } N > 8. \end{cases} \]

We define the homotopy \( \Phi(t; v, \theta, y) \) by components as
\[ \Phi_1 = v + n \frac{a(y_0)}{p-1} + t \left( Y_\delta(v, \theta, y) - v - n \frac{a(y_0)}{p-1} \right), \]
\[ \Phi_2 = DN(N-2)H(y_0)\theta^{N-4} - S^N \Delta a(y_0) + t(\Theta_\delta(v, \theta, y) - DN(N-2)H(y)\theta^{N-4} - S^N \Delta a(y)), \]
\[ \Phi_3 = \nabla a(y) + t(Y_\delta(v, \theta, y) - \nabla a(y)). \]

Since \( |\nabla a(y)| = O(|y - y_0|) \), working on the first two components, it is possible to find \( r > 0 \) such that for \( \delta \) small 
\[ |\Phi(t; v_0 - 1, \theta, y)| + |\Phi(t; v_0 + 1, \theta, y)| + |\Phi(t; v, \frac{1}{2} \theta_0, y)| + |\Phi(t; v, \frac{3}{2} \theta_0, y)| > 0 \]
for \( t \in [0, 1], v \in [v_0 - 1, v_0 + 1], \theta \in [\frac{1}{2} \theta_0, \frac{3}{2} \theta_0] \) and \( y \in B_r(y_0) \). We fix such \( r > 0 \). Since 
\[ \inf_{y \in \partial B_r(y_0)} |\nabla a(y)| > 0 \]
by the third component, we have that for \( \delta \) small 
\[ \inf_{y \in \partial B_r(y_0)} |\Phi(t; v, \theta, y)| > 0 \quad \forall t \in [0, 1], v \in [v_0 - 1, v_0 + 1], \theta \in [\frac{1}{2} \theta_0, \frac{3}{2} \theta_0]. \]

So, for homotopic invariance, we can conclude that 
\[ \deg(\Phi(v, \theta, y), [v_0 - 1, v_0 + 1] \times [\frac{1}{2} \theta_0, \frac{3}{2} \theta_0] \times B_r(y_0), 0) \neq 0, \]
because

\[ \deg(\Phi(0; v, \theta, y), [v_0 - 1, v_0 + 1] \times \left[ \frac{1}{2} \theta_0, \frac{3}{2} \theta_0 \right] \times B_r(y_0), 0) = -\deg(\nabla a, B_r(y_0), 0). \]

So we find a free critical point \( u_\delta = z PU_{e,y} + w(\delta, \alpha, \epsilon, y) \) of \( E_\delta \) and we want to show that it is a positive function. Since for \( u_\delta \) there holds

\[ -\Delta u_\delta = (1 + \delta a(x))|u_\delta|^{p-1}u_\delta, \]

if we multiply and integrate for \( -u_\delta = -\max(-u_\delta, 0) \), we obtain

\[ \int_{\Omega} |\nabla u_\delta|^2 = \int_{\Omega} (1 + \delta a(x))(u_\delta)^{p+1}. \]

From the Sobolev embedding theorem and the above inequality, we get

\[ S\left( \int_{\Omega} (u_\delta^-)^{p+1} \right)^{\frac{2}{p+1}} \leq C \int_{\Omega} (u_\delta)^{p+1}. \quad (34) \]

Let us remark that, since \( PU_{e,y} > 0 \), we have \( u_\delta \leq |w(\delta, \alpha, \epsilon, y)| \). If, by contradiction, \( u_\delta \neq 0 \) for \( \delta \) small, we can simplify in (34) to obtain

\[ S \leq C \left( \int_{\Omega} (u_\delta^-)^{p+1} \right)^{\frac{p-1}{p+1}} \leq C_1(||w(\delta, \alpha, \epsilon, y)||)^{p-1} \rightarrow \delta \rightarrow 0. \]

Then, for \( \delta \) small, \( u_\delta \geq 0 \) and, by maximum principle, \( u_\delta > 0 \). This completes the proof of Theorem 1.2. \( \square \)

Because of geometric significance, \( (\text{PSCE}) \) has been widely studied in case \( \Omega = \mathbb{R}^N \) (see [4, 7, 8, 19, 20, 22, 29–32]). Regarding the problem on the whole space as a limiting problem, we will study now \( (\text{PSCE}) \) on large balls \( B_R \). Of course, the bifurcation result stated in Theorem 1.2 holds true. However, a more careful analysis brings to evidence a (possible) decay, as \( R \) goes to infinity, of the size of the perturbation insuring existence (and non-degeneracy) of bifurcating solutions.

From now on, \( \Omega = B_R \). For simplicity, we perform the finite-dimensional reduction and compute the “Melnikov function” with respect to

\[ Z := \{ P_R U_{e,y} : \epsilon > 0, \ |y| < r \}, \]

where \( P_R : D^{1,2}(\mathbb{R}^N) \rightarrow H^1_0(B_R) \) is the orthogonal projection. It is easy to see (see Lemma A.6) that \( Z \) is a “non-degenerate almost critical manifold”, in the sense that there holds

\[ (A1)' \]  
Z is bounded and \( \sup_{|y| < r} ||\nabla E(P_R U_{e,y})|| = o(1) \) as \( \frac{1}{R} \rightarrow 0 \),

\[ (A2)' \]  
there exists \( \epsilon_0 > 0 \) such that \( L_z := \pi_z^+ E''(z)|_{T_z^+} \in \text{Iso}(T_z^+, T_z^-) \) \( \forall z \in Z_{\epsilon_0} \) and 

\[ \sup_{z \in Z_{\epsilon_0}} ||L_z^{-1}|| < \infty, \] where \( Z_{\epsilon_0} := \{ P_R U_{e,y} : |y| < r, \ 0 < \epsilon < \epsilon_0 R \}. \)
From now on, we assume \( a \in C^3_b(\mathbb{R}^N) \), \( \Omega = B_R \), \( R \gg 1 \) and \( \text{Crit } a := \{ x \in \mathbb{R}^N : \nabla a(x) = 0 \} \subseteq B_r \). The finite-dimensional reduction can be performed, with a bound \( \delta \) on the size of the perturbation independent on \( R \). Similar computations as above can be carried over to obtain the estimate

\[
||\nabla E_\delta(PU_{\varepsilon,y})||^2 = O(\delta^2) + O\left( \frac{\varepsilon}{R} \right)^{N+2},
\]

as well as the following expansions for the functional \( E_\delta \) and its derivatives:

\[
E_\delta(PU_{\varepsilon,y} + w) = \frac{1}{N} S^N_N - \frac{S^N_N}{p + 1} a(y) \delta + \frac{Dd_N}{2(1 - R^{-2}|y|^2)^{N-2}} \left( \frac{\varepsilon}{R} \right)^{N-2}
- \frac{S^N_N}{4N} \Delta a(y) \delta \varepsilon^2 + O\left( \left( \frac{\varepsilon}{R} \right)^{N-1} + \delta \varepsilon^3 + \delta^2 \right),
\]

\[
\frac{\partial}{\partial \varepsilon} E_\delta(PU_{\varepsilon,y} + w) = (N - 2) \frac{Dd_N}{2(1 - R^{-2}|y|^2)^{N-2}} \frac{\varepsilon^{N-2}}{R^{N-2}} - \frac{S^N_N}{2N} \Delta a(y) \delta \varepsilon
+ O\left( \frac{\varepsilon^{N-2}}{R^{N-1}} + \delta \varepsilon^2 + \frac{\delta^2}{\varepsilon} \right),
\]

\[
\frac{\partial}{\partial y} E_\delta(PU_{\varepsilon,y} + w) = - \frac{S^N_N}{p + 1} \nabla a(y) \delta + O\left( \frac{\varepsilon^{N-2}}{R^{N-1}} + \delta \varepsilon^2 + \frac{\delta^2}{\varepsilon} \right),
\]

where \( d_N = \frac{1}{N(N - 2)} , D \) as above, \( w = w(\delta, R, \varepsilon, y) \) as in Lemma 3.1.

Now, after setting \( \theta = \tau^{\frac{1}{N-4\varepsilon}}, \tau := \delta R^{N-2} \), we are led to look for critical points of

\[
M_{\tau, R}(\theta, y) = \frac{S^N_N}{p + 1} a(y) - \frac{\tau^{N-4}}{4N} \left[ 2N - \frac{Dd_N}{(1 - R^{-2}|y|^2)^{N-2}} \theta^{N-2} - \frac{S^N_N}{4N} \Delta a(y) \theta^2 \right] + \tau^{N-4} o(1),
\]

where \( ||o(1)||_{C^1} \to 0 \) on compact subsets of \( \mathbb{R}^+ \times B_r \) as \( \tau \to 0 \). As above, isolated critical points of \( a \) with \( \Delta a > 0 \) and non-zero topological index generate critical points of \( M_{\tau, R}(\theta, y) \), provided \( \tau \ll 1 \). Hence, we get

**Theorem 4.6.** Let \( N \geq 6 \) and \( a \) as above. Then there exist \( \delta_0 \) small and \( R_0 \) such that, for any \( R \geq R_0 \) and \( \delta \leq \frac{\delta_0}{R^{N-2}} \), problem (PSCE) on \( B_R \) with \( f = 1 + \delta a \) has at least as many positive solutions as the number of non-degenerate critical points of \( a \) with positive laplacian.

**Remark 4.7.** The analysis in Theorem 4.6 is less accurate than in Theorem 1.2 because of the different choice of \( Z \). So we lose dimension \( N = 5 \).
Because of the decay $\delta \ll R^{2-N}$, we cannot obtain solutions on the whole space as limits of our bifurcating solutions: for this, we need solutions on large balls and uniform size of the perturbation. We first observe that our bifurcation result relies on the rather weak assumption “$a$ has non-degenerate critical points with positive laplacian”. Such an assumption should be compared with the much stronger “counting condition”

$$\sum_{\{x: \nabla a(x)=0, \Delta a(x)>0\}} i(\nabla a, x) \neq 0$$

discovered by Bahri and Coron, see Ref. [10], in their investigation of (PSCE) on the 3 sphere (see also [20]). A very nice interpretation of the “counting condition” is given, in term of degree theoretic arguments, in [4] (see also [24,31] for a Morse theory point of view).

We will show below that, while the bifurcating solutions might, for $R$ larger and larger, degenerate and cancel each other for $d$ smaller and smaller, the counting condition enters as an obstruction to a complete collapse of these solutions, insuring, via a continuation argument based on suitable a priori bounds, existence on large balls $B_R$ up to some $\tilde{\delta}$ independent on $R$. As noticed above, there is $\tilde{\delta}$ such that, for any given $\rho>0$, the reduced functional $E_\delta(\varepsilon, y) := E_\delta(z(\varepsilon, y) + w(\delta, R, \varepsilon, y))$ is defined on $D^+=\{(\varepsilon, y): \varepsilon^2 + |y|^2 < \rho^2, \varepsilon>0\}$ for $\delta \leq \tilde{\delta}$ and $R \geq \bar{R} = \bar{R}(\rho)$. We will assume, from now on,

$$D^2a(x) \in GL_N(R) \quad \text{and} \quad \Delta a(x) \neq 0 \quad \text{for any} \quad x \in \text{Crit } a. \quad (38)$$

Let $y_j, j = 1, \ldots, l$ be the critical points of $a$ with positive laplacian. The homotopy argument used in the proof of Theorem 4.6 gives, for $R$ given and $\delta \leq \delta(R)$, the existence of open neighbourhoods $V_j = (\tilde{\theta}_j, \tilde{\theta}_j) \times U_j$ of (\theta(y_j), y_j), where

$$\theta(y_j) = \theta(y_j, R) = \left(\frac{N}{N^2 \Delta a(y_j)(1-R^2|y_j|^2)^{N-2}}\right)^{\frac{1}{N-4}}$$

and $U_j$ are small neighbourhoods of $y_j$ with $||\nabla a||>0$ on $\partial U_j$, such that

$$\deg(\nabla M_{\tau,R}, V_j, 0) = -\deg(\nabla a, U_j, 0) = -i(\nabla a, y_j).$$

From Section 2, for $\delta \leq \delta(R)$ the critical points of $M_{\tau,R}$ in the $V_j$ are in one-to-one correspondence with the critical points of $E_\delta(\varepsilon, y)$ in $D^+_{\rho, \delta} = D^+_{\rho} \cap \{\varepsilon>\theta \delta^{\frac{1}{N-4}}\},$

$\theta = \min_j \{\tilde{\theta}_j\}, \rho>2r$, through the map $(\theta, y) \rightarrow ((\delta R^{N-2})^{\frac{1}{N-4}} \theta, y)$.

This readily implies
Lemma 4.8. There is $\tilde{R}$ and, for any $R \geq \tilde{R}$, there is $\delta = \delta(R)$, such that

$$\deg(-\nabla E_\delta(\varepsilon, y), D^+_\rho, 0) = -\sum_{j=1}^l i(\nabla a, x_j) \quad \forall \rho > 2r.$$ 

To continue this degree estimate up to some $\tilde{\delta}$ independent on $R$, we need suitable a priori bounds. First, we have

Claim 1. There is some $\tilde{R}$ such that, if $\delta \leq \tilde{\delta}$ and $R \geq \tilde{R}$, then $E_\delta(\varepsilon, y)$ has no critical points on $D^+_\rho \cap \{\varepsilon = \varepsilon_\delta\}$, $\varepsilon_\delta = \theta \delta^\frac{1}{N-4}$.

To have complete a priori bounds we will assume, following [4],

$$\exists \rho' > 0 : \langle \nabla a(x), x \rangle < 0 \quad \forall |x| > \rho',$$

$$\langle \nabla a(x), x \rangle \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \langle \nabla a(x), x \rangle < 0.$$ (39)

Claim 2. If (39) holds, there is some $\tilde{\delta}$ such that $E_\delta(\varepsilon, y)$ has no critical points on $\{\varepsilon^2 + |y|^2 = \rho^2, \varepsilon > \varepsilon_\delta\}$, for some $\rho > \max\{\rho', 2r\}$ and $\delta \leq \tilde{\delta}$.

By the above claims, we deduce that, for some $\tilde{R}$ large and $\tilde{\delta}$ small

$$\deg(-\nabla E_\delta(\varepsilon, y), D^+_\rho \cap \{\varepsilon > \varepsilon_\delta\}, 0) = -\sum_{j=1}^l i(\nabla a, x_j) \quad \forall \delta \leq \tilde{\delta}, \quad R \geq \tilde{R}$$

for some $\rho > 2r$ fixed. Hence, we have

Theorem 4.9. Let $N \geq 6$, $a \in C^3_0(\mathbb{R}^N)$, $\text{Crit } a \subset B_r$, a satisfying (38)–(39). Assume in addition

$$\sum_{\{x : \nabla a(x), \Delta a(x) > 0\}} i(\nabla a, x) \neq 0.$$ (40)

Then problem (PSCE) on $\Omega = B_R$ with $f = 1 + \delta a$ has a solution for $\delta \leq \tilde{\delta}$ and $R \geq \tilde{R}$, $\tilde{\delta}$ independent on $R$.

Proof. We have just to prove the claims. As for Claim 1, it follows from assumption (38) and expansions (36)–(37) of the derivatives of $E_\delta$.
on $D^+ \cap \{ \varepsilon = \varepsilon_\delta \}$

$$
\nabla_x E_\delta (PU_{x,y} + w) = - \frac{S^N}{2N} \Delta a(y) \delta^{N-4} + o(\delta^{N-4}),
$$

$$
\nabla_y E_\delta (PU_{x,y} + w) = - \frac{S^N}{p+1} \nabla a(y) \delta + o(\delta).
$$

Finally, we prove Claim 2. From (39), we can show that there exists $\rho \geq \rho'$ such that

$$
\langle \nabla_{(e,y)} \Gamma(e, y), (e, y) \rangle < 0 \quad \text{if} \quad \varepsilon^2 + |y|^2 \geq \rho^2,
$$

where $\Gamma$ is $\int_{\mathbb{R}^N} aU_{x,y}^{\rho+1} = \int_{\mathbb{R}^N} a(e + y)U_{x,y}^{\rho+1}$, extended as an even function in $\varepsilon$. Now, using previous computations, we get

$$
E_\delta (PU_{x,y} + w) = \frac{1}{N} S^N + \frac{Dd_N}{2(1 - R^{-2})} \left( \frac{\varepsilon}{R} \right)^{N-2}
$$

$$
- \frac{\delta}{p+1} \Gamma(e, y) + O \left( \left( \frac{\varepsilon}{R} \right)^{N-1} + \delta \left( \frac{\varepsilon}{R} \right)^{N-2} + \delta^2 \right),
$$

$$
\nabla_{(e,y)} E_\delta (PU_{x,y} + w) = \left( \frac{N-2}{2} Dd_N \frac{\varepsilon^{N-3}}{R^{N-2}}, (N-2) Dd_N \frac{\varepsilon^{N-2}}{R^{N-2}} \right) (1 + o(1))
$$

$$
- \frac{\delta}{p+1} \nabla_{(e,y)} \Gamma(e, y) + O \left( \delta \left( \frac{\varepsilon^{N-3}}{R^{N-2}} + \delta^2 \right) \right).
$$

In view of (41) and the positivity of the term

$$
\left( \left( \frac{N-2}{2} Dd_N \frac{\varepsilon^{N-3}}{R^{N-2}}, (N-2) Dd_N \frac{\varepsilon^{N-2}}{R^{N-2}} \right), (e, y) \right),
$$

we get that, for $\delta \leq 1$ and $R \gg 1$, on $\{ \varepsilon^2 + |y|^2 = \rho^2 \} \cap \{ \varepsilon > \varepsilon_\delta \}$ there holds

$$
\langle - \nabla_{(e,y)} E_\delta, (e, y) \rangle < 0.
$$

**Final remark.** A different situation occurs if we assume in (39) the reverse inequality. First, we observe that to compute $\text{deg}(-\nabla_{(e,y)} E_\delta, D^+_{\rho, \delta}, 0)$, we can also proceed as follows. From

$$
\nabla_{(e,y)} E_\delta = - \frac{\delta}{p+1} \nabla_{(e,y)} \Gamma + O \left( \frac{\varepsilon^{N-3}}{R^{N-2}} + \delta^2 \right),
$$

we see that for $\frac{M_1}{R^{N-2}} \leq \delta \leq \tilde{\delta}$, $M_1$ a large constant,

$$
\text{deg}(-\nabla_{(e,y)} E_\delta, D^+_{\rho, \delta}, 0) = \text{deg}(\nabla_{(e,y)} \Gamma, D^+_{\rho, \delta}, 0),
$$


whenever the r.h.s. is defined. This is the case if (39) holds, as well as if the reverse inequality is satisfied therein. Since, as can be easily seen,
\[
\Gamma(0, y) = \sum_{i=1}^{N} a(y), \quad \frac{\partial \Gamma}{\partial x}(0, y) = 0, \quad \frac{\partial^{2} \Gamma}{\partial y^{2}}(0, y) = 2 \Delta a(y)
\]
for some positive constant \(C\), we have, denoted \(D_{\rho} := \{e^{2} + |y|^{2} < \rho^{2}\}\),
\[
\deg(\nabla_{(x, y)} \Gamma, D_{\rho, \delta}, 0) = 2 \deg(\nabla_{(x, y)} \Gamma, D_{\rho, \delta}^{+}, 0) + \deg(\nabla_{(x, y)} \Gamma, D_{\rho} \cap \{|e| < \varepsilon_{\delta}\}, 0)
\]
\[
= 2 \deg(\nabla_{(x, y)} \Gamma, D_{\rho, \delta}^{+}, 0) + \sum_{\{x : \nabla a(x) = 0, \Delta a(x) > 0\}} i(\nabla a, x)
\]
\[
- \sum_{\{x : \nabla a(x) = 0, \Delta a(x) < 0\}} i(\nabla a, x).
\]
If the reverse inequality holds true in (39), we get the reverse inequality in (41), and then
\[
\sum_{\{x : \nabla a(x) = 0, \Delta a(x) > 0\}} i(\nabla a, x) + \sum_{\{x : \nabla a(x) = 0, \Delta a(x) < 0\}} i(\nabla a, x) = 1 = \deg(\nabla_{(x, y)} \Gamma, D_{\rho, \delta}, 0).
\]
Henceforth, for \(R^{2-N} \ll \delta \ll \delta\),
\[
\deg(-\nabla_{(x, y)} E_{\delta}, D_{\rho, \delta}^{+}, 0) = \deg(\nabla_{(x, y)} \Gamma, D_{\rho, \delta}^{+}, 0) = \sum_{\{x : \nabla a(x) = 0, \Delta a(x) < 0\}} i(\nabla a, x).
\]
On the other hand, Claims 1 and 2 still hold true and so we conclude that
\[
\deg(-\nabla_{(x, y)} E_{\delta}, D_{\rho, \delta}^{+}, 0)_{\delta \ll R^{-N}} = - \sum_{\{x : \nabla a(x) = 0, \Delta a(x) > 0\}} i(\nabla a, x)
\]
\[
\neq \sum_{\{x : \nabla a(x) = 0, \Delta a(x) < 0\}} i(\nabla a, x)
\]
\[
= \deg(-\nabla_{(x, y)} E_{\delta}, D_{\rho, \delta}^{+}, 0)_{\delta \gg R^{-N}}.
\]
In particular, no a priori bounds are available in this case.

5. Further applications of the reduction principle

We consider a generalization of [35]: given \(a(x)\) a smooth function in \(\hat{\Omega}\), \(\delta > 0\) a small parameter, \(1 \leq q < \frac{N+2}{N-2}\) and \(N \geq 3\), find \(u > 0\) such that
\[
(P)_{\delta} \quad \begin{cases} 
-\Delta u = \frac{N+2}{u^{N-2}} + \delta a(x) u^{q} & \text{in } \Omega, \\
u = 0 & \text{in } \partial \Omega.
\end{cases}
\]
In this case, the unperturbed functional is $E(u)$ and the finite-dimensional reduction is performed with respect to the “non-degenerate almost critical manifold”

$$Z := \{ PU_{e,y} : \varepsilon > 0, \ y \in \Omega, \ dist(y, \partial \Omega) > \gamma \}, \quad \gamma > 0,$$

in the sense that there holds

(A1)“ $Z$ is bounded and $\sup_{y \in \Omega, \ dist(y, \partial \Omega) > \gamma} ||\nabla E(\mathbf{PU}_{e,y})|| = o(1)$ as $\varepsilon \to 0$,

(A2)“ there exists $\varepsilon_0 > 0$ such that $L_z := \pi_z^* E''(z)|_{T_z} \in \text{Iso}(T_z^\perp, T_z^\perp)$ $\forall z \in Z_{\varepsilon_0}$ and $\sup_{z \in Z_{\varepsilon_0}} ||L_z^{-1}|| < \infty$,

where $Z_{\varepsilon_0} := \{ \mathbf{PU}_{e,y} : 0 < \varepsilon < \varepsilon_0, \ y \in \Omega, \ dist(y, \partial \Omega) > \gamma \}$ (see Lemma A.6). The perturbation is

$$G(\delta, u) = \frac{\delta}{q + 1} \int_{\Omega} a|u|^{q+1}.$$

Using Lemmas A.1 and A.2, one can get the following estimate for the remainder term:

$$||\nabla E_\delta(\mathbf{PU}_{e,y})||^2 = O(\varepsilon^{N-1} + \delta^2 A^2),$$

where

$$A = \begin{cases} \frac{N+2}{N-2} \left( -\frac{2}{q} \right)^{\frac{N-2}{N-2}} & \text{if } q > \frac{N+2}{2(N-2)}, \\ \frac{N+2}{N-2} (\log \frac{1}{\varepsilon})^\frac{N+2}{2N} & \text{if } q = \frac{N+2}{2(N-2)}, \\ \frac{N-2}{2} - \delta^q & \text{if } q < \frac{N+2}{2(N-2)}. \end{cases}$$

As for the “Melnikov function” (see Remark 3.4), if $q > \frac{2}{N-2}$, one gets

$$E_\delta(\mathbf{PU}_{e,y}) = E(\mathbf{PU}_{e,y}) - \frac{\delta}{q + 1} \int_{\Omega} a|\mathbf{PU}_{e,y}|^{q+1}$$

$$= \frac{1}{N} S_z^N + \frac{D}{2} H(y) e^{N-2} - \frac{F e^{\delta^q}}{q + 1} a(y) e^{N-2} (q + 1)$$

$$+ O(\varepsilon^{N-1}) + o(\delta e^{N-2} (q+1)), \quad (42)$$

where $F = \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{(q+1)/2}}$ and the expansion of $E_\delta(\mathbf{PU}_{e,y} + w)$ follows by

$$E_\delta(\mathbf{PU}_{e,y} + w) = E(\mathbf{PU}_{e,y}) + O(||\nabla E_\delta(\mathbf{PU}_{e,y})||^2),$$

where $w = w(\delta, e, y)$ is defined as in Lemma 3.1. After setting $\theta = \delta^q \frac{2}{(N-2)(q+1)-2} e$, if $q > \max\left\{ \frac{2}{N-2}, \frac{6-N}{2} \right\}$, the expansion of $E_\delta$ becomes

$$E_\delta(\mathbf{PU}_{e,y} + w) = \frac{1}{N} S_z^N + \frac{2(N-2)}{2(N-2)(q+1)-2} \left[ \frac{D}{2} H(y) \theta^{N-2} - \frac{F e^{\delta^q}}{q + 1} a(y) \theta^{N-2} (q + 1) + o(1) \right],$$

where $w = w(\delta, e, y)$ is defined as in Lemma 3.1.
where $o(1) \to 0$ as $\delta \to 0$ in $C^0$ norm for $\theta$ bounded and bounded away from zero. So we are led to study the “stable” critical points of

$$M(\theta, y) = DH(y)\theta^{N-2} - \frac{2}{q+1} c_N^{q+1} Fa(y)\theta^{N-2} \left(\frac{(N-2)(q+1)}{2}\right) \theta > 0, \quad y \in \Omega,$$

where $F, D$ and $c_N$ are as above. Since

$$\frac{\partial M}{\partial \theta} = 0 \Leftrightarrow \begin{cases} \theta = \theta(y) := \left(\frac{(2N-(N-2)(q+1))c_N^{q+1} Fa(y)}{(N-2)(q+1)DH(y)}\right)^{2(N-2)(q+1)-4}, \\ a(y) > 0 \end{cases}$$

and

$$M(\theta(y), y)) = D_{N,q} \left(\frac{a(y)^2}{2N-(N-2)(q+1)}\right)^{\frac{(N-2)}{(N-2)(q+1)-4}},$$

$$D_{N,q} = -\frac{(N-2)(q+1) - 4}{N-2} \left(\frac{2N-(N-2)(q+1)}{D(N-2)}\right)^{\frac{2N-(N-2)(q+1)}{(N-2)(q+1)-4}} \left(\frac{F c_N^{q+1}}{q+1}\right)^{\frac{2(N-2)}{(N-2)(q+1)-4}},$$

we can introduce

$$K(y) := \frac{a(y)^2}{2N-(N-2)(q+1)} \left(\frac{H(y)}{(N-2)}\right)^{\frac{(N-2)}{(N-2)(q+1)-4}}, \quad y \in \Omega$$

and the following result follows:

**Theorem 5.1.** Let $M, K$ be given as above and let $(\theta_j, y_j)$ be critical points of $M$. Let $1 < q < \frac{N+2}{N-2}$ if $N \geq 5$, $1 < q < 3$ if $N = 4$, $3 < q < 5$ if $N = 3$.

(i) If $(\theta_j, y_j)$ are $C^0$-stable, then there are $C_j$ disjoint compact neighbourhoods of $(\theta_j, y_j)$ and, for $\delta > 0$ small, there are $u_{\delta,j}$, solutions of $(P)_\delta$, such that

$$|\nabla u_{\delta,j}|^2 \to \sum_{j=1}^{N} \delta x_j \quad \text{as} \quad \delta \to 0 \quad \text{for some} \quad x_j \in C_j. \quad (43)$$

(ii) Let $C_j$ be disjoint compact subsets of $\Omega$ such that, for any $j$,

$$a(y) > 0 \quad \forall y \in C_j, \quad \max_{\partial C_j} K < \max_{C_j} K.$$

Then, for $\delta$ small, $(P)_\delta$ has solutions $u_{\delta,j}$ such that $(43)$ holds. Moreover, such solutions are positive.
Proof. We just derive (ii) from (i). For any given \( y \in \Omega \), let

\[
\theta(y) := \left( \frac{[2N-(N-2)(q+1)]e_N^{\frac{q+1}{2}} F_d(y)}{(N-2)(q+1)DH(y)} \right)^{\frac{2}{(N-2)(q+1)-4}}
\]

be the absolute minimizer of \( \theta \to M(\theta, y) \) and let

\[
0 < \theta < \min_{y \in C} \theta(y) \leq \max_{y \in C} \theta(y) < \bar{\theta}, \quad m := \min_{\partial([\theta, \bar{\theta}] \times C)} M, \quad m_b := \min_{\partial([\theta, \bar{\theta}] \times C)} M
\]

for \( C = C_j \) fixed. Since

\[
M(\theta(y), y)) = D_{N,q} K(y)^{\frac{(N-2)}{(N-2)(q+1)-4}}, \quad \forall y \in \Omega,
\]

\( D_{N,q} \) as above, one easily obtains \( \max_{\partial C} K < \max_{\partial C} K \Rightarrow m < m_b \) and then (ii) follows from (i).

The proof of the positivity for these solutions follows the same argument as in Theorem 1.2 because \( q \geq 1 \).

For the derivatives, similar computations as in Lemma 4.4 can be performed in case \( 1 \leq q < \frac{N+2}{N-2} \) if \( N \geq 5 \), \( \frac{5}{4} < q < 3 \) if \( N = 4 \). \( \square \)

**Theorem 5.2.** Let \( M, K \) be given as above and let \( (\theta_j, y_j) \) be critical points of \( M \). Let \( 1 \leq q < \frac{N+2}{N-2} \) if \( N \geq 5 \), \( \frac{5}{4} < q < 3 \) if \( N = 4 \).

(k) If \( (\theta_j, y_j) \) are \( C^1 \)-stable, then there are \( C_j \) disjoint compact neighbourhoods of \( (\theta_j, y_j) \) and, for \( \delta > 0 \) small, there are \( u_{\delta,j} \), solutions of \( (P)_\delta \), with property (43).

(kk) Let \( y_0 \) be a non-degenerate critical point of \( K \) with \( a(y_0) > 0 \). Then, for \( \delta \) small, \( (P)_\delta \) has a solution \( u_\delta \) satisfying (43) with limit Dirac mass in \( y_0 \).

Moreover, such solutions are positive.

**Proof.** By the assumptions \( \nabla K(y_0) = 0 \), \( D^2 K(y_0) \in GL_N(\mathbb{R}) \) and \( a(y_0) > 0 \), it follows that \( \nabla M(\theta(y_0), y_0) = 0 \) and \( D^2 M(\theta(y_0), y_0) \in GL_{N+1}(\mathbb{R}) \). The proof of this fact is a straightforward computation, we skip here the details. \( \square \)

**Remark 5.3.** (i) Non-degeneracy of critical points of \( K \) implies non-degeneracy of critical points of \( C^2 \)-perturbations of \( M \). This in turn would lead (see the proof of Theorem 5.1) to non-degeneracy and precise Morse index estimates of the corresponding variational functional associated to \( (P)_\delta \). However, we will not carry over \( C^2 \) estimates in this paper.

(ii) If \( a(x) \equiv 1 \), \( N > 4 \) and \( q = 1 \), then we find as many positive solutions as the number of non-degenerate critical points of \( H(y) \), which is exactly the famous result contained in [35].

Our approach applies as well to the non-homogeneous boundary value problem with small data. Let \( \Omega \subset \mathbb{R}^N \), \( N \geq 3 \), be a smooth open bounded domain and
Let us consider the following BVP:

\[
(BVP) \begin{cases} 
-\Delta u = |u|^{N-2}u & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}
\]

It can be seen (see [21] for a more general equation) that (BVP) has a “small” positive solution if \( \varphi \geq 0 \) is non-trivial and suitably small. We are interested for (BVP) with boundary data \( \delta \varphi, \delta > 0 \) small and \( \varphi \) positive somewhere, rewritten in the equivalent form:

\[
(BVP)_\delta \begin{cases} 
-\Delta u = |u + \delta a|^{N-2}(u + \delta a) & \text{in } \Omega, \\
u \in H^1_0(\Omega),
\end{cases}
\]

where \( a \) denotes the harmonic extension of \( \varphi \). Here the perturbation is

\[
G(\delta, u) = \frac{1}{p+1} \int_{\Omega} |u + \delta a|^{p+1} - |u|^{p+1},
\]

which is a \( C^2 \) functional converging to zero \( C^2 \)-uniformly on bounded sets. So we can find \( w \) according to Lemma 3.1 and the finite-dimensional reduction can be performed. Now we can expand \( G(\delta, u) \) in the form

\[
G(\delta, u) = G_1(\delta, u) + G_2(\delta, u)
\]

where

\[
|G_2(\delta, u)| = O \left( \delta^2 \int_{\Omega} |u|^{p-1} + \delta^{p+1} \right),
\]

\[
||\nabla G_2(\delta, u)|| = O \left( \delta^p + \delta^2 \left( \int_{\Omega} |u|^{(p-2)(p+1)} \right)^{\frac{p}{p+1}} \right) \text{ (if } p > 2 \).
\]

Let us stress that \( u \to \int_{\Omega} a(x)|u|^{p-1}u \) is not a \( C^2 \) functional for \( N > 6 \).

Some remarks are in order:

(a) the problem with a perturbation term \( G_1(\delta, u) = G(\delta, u) - G_2(\delta, u) = \delta \int_{\Omega} a(x)|u|^{p-1}u \) is exactly of the form \( (P)_\delta \) with \( q = p - 1 \), \( a(x) \) replaced by \( pa(x) \). So the expansion for \( E_1(\mathcal{P}a_\varepsilon, y) = E(\mathcal{P}a_\varepsilon, y) - G_1(\delta, \mathcal{P}a_\varepsilon, y) \) is given by (42) because \( q = p - 1 > \frac{2}{N-2} \);

(b) \( G_2(\delta, u) \) gives a contribution to the remainder term \( ||\nabla E_\delta||^2, E_\delta = E - G(\delta, \cdot) \), of order \( O(\delta^{2p} + \delta^4 \varepsilon^{N-2}) \) (if \( N < 6 \));

(c) \( G_2(\delta, \mathcal{P}a_\varepsilon, y) = O(\delta^{p+1} + \delta^2 \varepsilon) \);

(d) if \( \delta \sim \varepsilon^{\frac{N-2}{2}} \), there holds \( E_\delta(\mathcal{P}a_\varepsilon + w) = E_1(\mathcal{P}a_\varepsilon, y) + o(\delta \varepsilon^{N-2}) \). So it follows
Theorem 5.4. Let
\[ M(\theta, y) = c_N H(y) \theta^{N-2} - 2a(y) \theta^{N-2} \theta > 0, \quad y \in \Omega, \]
\[ K(y) = \frac{a(y)^2}{H(y)}, \quad y \in \Omega \]
and let \((\theta_j, y_j)\) be critical points of \(M\).

(i) If \((\theta_j, y_j)\) are \(C^0\)-stable, then there are \(C_j\) disjoint compact neighbourhoods of \((\theta_j, y_j)\) and, for \(\delta > 0\) small, there are \(u_{\delta,j}\), solutions of \((\text{BVP})_{\delta}\), such that
\[ |\nabla u_{\delta,j}|^2 \to \sum_{x_j}^N \delta_{x_j} \quad \text{as} \quad \delta \to 0 \quad \text{for some} \quad x_j \in C_j. \tag{45} \]

(ii) Let \(C_j\) be disjoint compact subsets of \(\Omega\) such that, for any \(j\),
\[ a(y) > 0 \quad \forall y \in C_j, \quad \max_{\partial C_j} K < \max_{C_j} K. \]

Then, for \(\delta\) small, \((\text{BVP})_{\delta}\) has solutions \(u_{\delta,j}\) such that
\[ |\nabla u_{\delta,j}|^2 \to \sum_{x_j}^N \delta_{x_j} \quad \text{as} \quad \delta \to 0 \quad \text{for some} \quad x_j \in C_j. \tag{46} \]
Moreover, if \(\varphi \geq 0\), such solutions are positive.

Proof. We need only to prove that the solutions are positive if \(\varphi \geq 0\). If this case, we define \(v_{\delta}\) as the “small” positive solution of \((\text{BVP})_{\delta}\), \(\delta > 0\) small, whose existence is ensured by [21]. We verify that \(u = u_{\delta} - v_{\delta}\) is positive (for simplicity, we will omit the dependence on \(\delta\)). Since for \(u\) there holds
\[ -\Delta u = |u + \delta a + v_{\delta}|^{p-1}(u + \delta a + v_{\delta}) - (\delta a + v_{\delta})^p, \]
we have that, for any \(\phi \in H_0^1(\Omega)\),
\[ \int_{\Omega} \nabla u \nabla \phi = p \int_{\Omega} u \phi \int_0^1 |su + \delta a + v_{\delta}|^{p-1} ds. \]
By choosing \(\phi = -u^- = -\max(-u, 0)\), we obtain
\[ \int_{\Omega} |\nabla u^-|^2 = p \int_{\Omega} (u^-)^2 \int_0^1 |su^- + \delta a + v_{\delta}|^{p-1} ds \]
\[ \leq o(1) \left( \int_{\Omega} (u^-)^{p+1} \right)^{\frac{2}{p+1}} + C_2 \int_{\Omega} (u^-)^{p+1}. \]
From the Sobolev embedding theorem and the above inequality we get

$$
S \left( \int_\Omega (u^-)^{p+1} \right)^{\frac{2}{p+1}} \leq o(1) \left( \int_\Omega (u^-)^{p+1} \right)^{\frac{2}{p+1}} + C_2 \int_\Omega (u^-)^{p+1}. \tag{47}
$$

Let us remark that since $PU_{e,y} > 0$, we have $u^- \leq \|w(\delta, e, y)\| + v_\delta$. If, by contradiction, $u^- \neq 0$ for $\delta$ small, we can simplify in (47) to obtain

$$
S \leq o(1) + C_2 \left( \int_\Omega (u^-)^{p+1} \right)^{\frac{p-1}{p+1}} \leq o(1) + C_3(||w(\delta, e, y)||^{p-1} + ||v_\delta||^{p-1}) \to \delta \to 0.
$$

Then, for $\delta$ small, $u_\delta \geq v_\delta > 0$. This completes the proof of Theorem 5.4. \qed

Similar computations can be performed for the derivatives leading to the counterpart of Theorem 5.2. Essentially, if $j \leq 0$ and $\delta > 0$ is small, problem (BVP)$_{\delta}$ has as many positive solutions as the non-degenerate critical points of $K$ with $a > 0$. This is almost the same result for this problem contained in [37]. However, Theorem 5.4 represents a slight improvement because it permits to handle dimension $N = 3$ and it provides an existence result (in any dimension) corresponding to the strict relative maxima of $K$.

With the aid of Theorem 5.4, we can provide an example where some highly oscillating boundary data produce a large number of solutions:

**An example.** Let $\Omega = B_1(0)$ be the unit open ball, $n$ any positive integer. Let $y_j \in \partial B_1$, $j = 1, \ldots, n$ and $t > 1$. We want to show that

$$
-\Delta u = u^{\frac{N+2}{N-2}} \quad \text{in} \quad B_1,
$$

$$
u = \delta \sum_{j=1}^n \frac{1}{|y - ty_j|^{N-2}} \quad \text{on} \quad \partial B_1
$$

has at least $n$ positive solutions if $t < t_{p,n} := 1 + \frac{\rho^2}{4\rho^4}$, $\rho \leq \min_i \frac{|y_i - y_j|}{\sqrt{2}}$ and $\delta$ smaller than some $\delta_t$.

Denoted $d'(y) := \sum_{j=1}^n \frac{1}{|y - ty_j|^{N-2}}$ and $K'(y) := \frac{d'(y)^2}{H(y)}$, it is easy to check, to apply Theorem 5.4, that

$$
m_t := \max \{ K'(y) : y \in B_1(0), |y - y_j| \geq \rho \ \forall j \} 
$$

$$< \max \{ K'(y) : y \in B_1(0), |y - y_i| \leq \rho \} \ \forall i = 1, \ldots, n \ \text{provided} \ t < t_{p,n}. $$
Appendix A

Here, we recall several kinds of estimates for

$$U_{e,y}(x) = c_N \frac{N-2}{e^2 + |x-y|^2}^{N-2}, \quad c_N = [N(N-2)]^{\frac{N-2}{4}}, \quad \epsilon > 0, \quad y \in \mathbb{R}^N.$$ 

Also, \( \int_{\mathbb{R}^N} |\nabla U_{e,y}|^2 = \int_{\mathbb{R}^N} U_{e,y}^{p+1} = S_N^2 \) and

$$\frac{\partial U_{e,y}}{\partial x_i}(x) = -c_N (N-2) \frac{N-2}{e^2 + |x-y|^2} x_i - y_i \left| \frac{\partial U_{e,y}}{\partial x_i}(x) \right| \leq \frac{N-2}{2e} U_{e,y}(x), \quad (A.1)$$

$$\frac{\partial U_{e,y}}{\partial \epsilon}(x) = -c_N \left( \frac{N-2}{2} \frac{N-4}{e^2 + |x-y|^2} \right) e^2 - |x-y|^2 \left| \frac{\partial U_{e,y}}{\partial \epsilon}(x) \right| \leq \frac{N-2}{2e} U_{e,y}(x). \quad (A.2)$$

Direct computations give the following estimates.

**Lemma A.1.**

$$\int_{\Omega} U_{e,y}^q = \begin{cases} 
O(e^{N-\frac{N-2}{2}q}) & \text{if } q > \frac{N}{N-2}, \\
O \left( \frac{N}{e^2} \log \frac{\text{diam} \Omega}{\epsilon} \right) & \text{if } q = \frac{N}{N-2}, \\
O \left( \frac{N-2}{q} (\text{diam} \Omega)^{N-(N-2)q} \right) & \text{if } q < \frac{N}{N-2},
\end{cases}$$

$$\int_{B_r(y)} |x-y|^s U_{e,y}^q = O \left( \frac{N-2}{e^{\frac{N-2}{q}} (N-2q-N-s)} \right) \quad \text{if } q > \frac{N+s}{N-2},$$

where \( r > 0 \).

Now to get estimates for \( PU_{e,y} \) (recall that \( \Delta P U_{e,y} = \Delta U_{e,y}, \ P U_{e,y} \equiv 0 \) on \( \partial \Omega \)), let us introduce

$$\psi_{e,y} := U_{e,y} - PU_{e,y}, \quad f_{e,y} := \psi_{e,y} - c_N H(y, \cdot) \epsilon^{N-2},$$

where \( H(y, x) \) denotes the regular part of the Green’s function, i.e., \( \forall y \in \Omega \), \( \Delta_x H(y, x) = 0 \) in \( \Omega \) and \( H(y, x)|_{x \in \partial \Omega} = |x-y|^{-(N-2)} \). For any given \( y \in \Omega \) we will denote \( d := \text{dist}(y, \partial \Omega) \) and \( H(y) = H(y, y) \). By the maximum principle:

$$0 \leq \psi_{e,y} \leq U_{e,y}, \quad ||\psi_{e,y}||_{\infty} \leq \max_{x \in \partial \Omega} U_{e,y}(x) \leq c_N \frac{N-2}{d^{N-2}}.$$
In particular, \(0 \leq U_{e,y}^p - PU_{e,y}^p \leq c \frac{\varepsilon}{d^{N-2}} U_{e,y}^{p-1}\). We also have \(f_{e,y} = O\left(\frac{\varepsilon^2}{d^N}\right)\) because \(f_{e,y}\) is harmonic in \(\Omega\) with boundary data

\[
f_{e,y}(x) = c_{N} \frac{N-2}{\varepsilon^2} \left[ \frac{1}{(\theta^2 + |x-y|^2)^{\frac{N-2}{2}}} - \frac{1}{|x-y|^{N-2}} \right] = O\left(\frac{N^2}{\varepsilon^2} \frac{1}{d^N}\right).
\]

Similarly, one gets estimates for the derivatives of \(\psi_{e,y}\) and \(f_{e,y}\). Summarizing (see also \([35]\) for more details)

**Lemma A.2.** Given \(\varepsilon > 0\), \(\psi_{e,y}, f_{e,y}, d\) as above, then

\[
\psi_{e,y} = O\left(\frac{N-2}{\varepsilon^2} \frac{1}{d^{N-2}} \right) \frac{\partial \psi_{e,y}}{\partial y_j} = O\left(\frac{N-2}{\varepsilon^2} \frac{1}{d^{N-1}} \right) \frac{\partial \psi_{e,y}}{\partial \varepsilon} = O\left(\frac{N-4}{\varepsilon^2} \frac{1}{d^{N-2}} \right), \quad (A.3)
\]

\[
\frac{\partial^2 \psi_{e,y}}{\partial y_j \partial y_j} = O\left(\frac{N-2}{\varepsilon^2} \frac{1}{d^N} \right) \frac{\partial^2 \psi_{e,y}}{\partial y_j \partial \varepsilon} = O\left(\frac{N-4}{\varepsilon^2} \frac{1}{d^{N-1}} \right) \frac{\partial^2 \psi_{e,y}}{\partial \varepsilon^2} = O\left(\frac{N-6}{\varepsilon^2} \frac{1}{d^{N-2}} \right), \quad (A.4)
\]

\[
f_{e,y} = O\left(\frac{N+2}{\varepsilon^2} \frac{1}{d^N} \right) \frac{\partial f_{e,y}}{\partial y_j} = O\left(\frac{N+2}{\varepsilon^2} \frac{1}{d^{N+1}} \right) \frac{\partial f_{e,y}}{\partial \varepsilon} = O\left(\frac{N}{\varepsilon^2} \frac{1}{d^N} \right). \quad (A.5)
\]

We are now interested in some estimate for the \(L^{p+1}\)-norm of \(\tilde{\psi}_{e,y}\). Let us define

\[
\tilde{\psi}_{e,y}(x) := \begin{cases} \psi_{e,y}(x) & \text{if } x \in \Omega, \\ U_{e,y}(x) & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}
\]

We have that \(\tilde{\psi}_{e,y} \in D^{1,2}(\mathbb{R}^N), D^{1,2}(\mathbb{R}^N)\) being the completion of \(C_0^\infty(\mathbb{R}^N)\) with respect to the \(L^2\)-norm of the gradient, and, by Sobolev inequality,

\[
\left( \int_{\mathbb{R}^N} \tilde{\psi}_{e,y}^{p+1} \right)^{\frac{p+1}{2}} \leq \frac{1}{S} \int_{\mathbb{R}^N} |\nabla \tilde{\psi}_{e,y}|^2,
\]

where \(S\) is the Sobolev constant. For the r.h.s. we can obtain

\[
\int_{\mathbb{R}^N} |\nabla \tilde{\psi}_{e,y}|^2 = \int_{\mathbb{R}^N} |\nabla U_{e,y}|^2 - \int_{\Omega} |\nabla PU_{e,y}|^2 \]

\[
= S^2 - \int_{\Omega} U_{e,y}^{p+1} + \int_{\Omega} U_{e,y}^{p} \psi_{e,y} = O\left(\frac{\varepsilon}{d^N}\right),
\]
because \( \int_{\Omega} \nabla U_{e,y} \nabla P U_{e,y} = \int_{\Omega} |\nabla P U_{e,y}|^2 \). Hence,

\[
\int_{\mathbb{R}^N} \psi_{e,y}^{p+1} = \int_{\Omega} \psi_{e,y}^{p+1} + O \left( \left( \frac{\varepsilon}{d} \right)^N \right) = O \left( \left( \frac{\varepsilon}{d} \right)^N \right)
\]

which proves

**Lemma A.3.**

\[
|\psi_{e,y}|_{L^p(\Omega)} = O \left( \left( \frac{\varepsilon}{d} \right)^{N-\frac{2}{p}} \right).
\]  

(A.6)

Now, using estimates on \( \psi_{e,y} \) and its derivatives, we can get for the first and second derivatives of \( P U_{e,y} \):

**Lemma A.4.** Let \( \gamma > 0 \). Then, for all \( i \neq j \), we have

\[
\left\| \frac{\partial P U_{e,y}}{\partial y_i} \right\|^2 = \frac{c_1}{\varepsilon^2} + O(\varepsilon^{N-3}), \quad \left\| \frac{\partial P U_{e,y}}{\partial \varepsilon} \right\|^2 = \frac{c_2}{\varepsilon^2} + O(\varepsilon^{N-4}),
\]

\[
\left\langle \frac{\partial P U_{e,y}}{\partial y_i}, \frac{\partial P U_{e,y}}{\partial y_j} \right\rangle = O(\varepsilon^{N-3}), \quad \left\langle \frac{\partial P U_{e,y}}{\partial y_i}, \frac{\partial P U_{e,y}}{\partial \varepsilon} \right\rangle = O(\varepsilon^{N-3}),
\]

\[
\left\| \frac{\partial^2 P U_{e,y}}{\partial y_i \partial y_j} \right\| = O \left( \frac{1}{\varepsilon^2} \right), \quad \left\| \frac{\partial^2 P U_{e,y}}{\partial y_i \partial \varepsilon} \right\| = O \left( \frac{1}{\varepsilon^2} \right), \quad \left\| \frac{\partial^2 P U_{e,y}}{\partial \varepsilon^2} \right\| = O \left( \frac{1}{\varepsilon^2} \right)
\]

uniformly for \( y \in \Omega \) with \( d(y, \partial \Omega) > \gamma \).

**Proof.** For the norm and scalar product of first derivatives, by Lemma A.1, Lemma A.2 and \( \frac{d U_{e,y}}{d y_i} = O \left( \frac{U_{e,y}}{\varepsilon^2} \right) \), we get, for \( i \neq j \),

\[
\left\| \frac{\partial P U_{e,y}}{\partial y_i} \right\|^2 = p \int_{\mathbb{R}^N} U_{e,y}^{p-1} \left( \frac{\partial U_{e,y}}{\partial y_i} \right)^2 + O(\varepsilon^{N-3}) = \frac{c_1}{\varepsilon^2} + O(\varepsilon^{N-3}),
\]

\[
\left\| \frac{\partial P U_{e,y}}{\partial \varepsilon} \right\|^2 = p \int_{\mathbb{R}^N} U_{e,y}^{p-1} \left( \frac{\partial U_{e,y}}{\partial \varepsilon} \right)^2 + O(\varepsilon^{N-4}) = \frac{c_2}{\varepsilon^2} + O(\varepsilon^{N-4}),
\]

\[
\left\langle \frac{\partial P U_{e,y}}{\partial y_i}, \frac{\partial P U_{e,y}}{\partial y_j} \right\rangle = O \left( \int_{\Omega \setminus B_{\varepsilon}(y)} U_{e,y}^{p-1} \left| \frac{\partial U_{e,y}}{\partial y_i} \right| \left| \frac{\partial U_{e,y}}{\partial y_j} \right| + \varepsilon^{N-3} \right) = O(\varepsilon^{N-3}),
\]

\[
\left\langle \frac{\partial P U_{e,y}}{\partial y_i}, \frac{\partial P U_{e,y}}{\partial \varepsilon} \right\rangle = O \left( \int_{\Omega \setminus B_{\varepsilon}(y)} U_{e,y}^{p-1} \left| \frac{\partial U_{e,y}}{\partial y_i} \right| \left| \frac{\partial U_{e,y}}{\partial \varepsilon} \right| + \varepsilon^{N-3} \right) = O(\varepsilon^{N-3}).
\]
For the second derivatives, by Lemma A.2, we get for the first relation

$$
\int_\Omega \left[ \nabla^2 P_{e,y} \right]^2 = O\left( \int_\Omega \left[ \frac{2N}{N+2} \right] \frac{dx}{(1+|x|^2)} \right)
$$

because

$$
\frac{\partial^2 U_{e,y}}{\partial y_i \partial y_j} = O\left( \frac{U_{e,y}}{e^2} \right)
$$

We proceed in an analogous way for the remaining relations.

Now, we carry out a more subtle analysis with the aid of the expansion of $\psi_{e,y}$ in term of the regular part of Green’s function.

**Lemma A.5.** Let $D = \frac{2N}{N+2} \int_{\mathbb{R}^N} \frac{dx}{(1+|x|^2)}$ and $\gamma > 0$. Then

$$
||P_{e,y}||^2 = \int_\Omega \left[ \nabla^2 P_{e,y} \right] = S^2 - Dh(y)e^{N-2} + O(e^{N-1}),
$$

$$
\int_\Omega P_{e,y}^{p+1} = S^2 - (p+1)DH(y)e^{N-2} + O(e^{N-1}),
$$

$$
\left\langle P_{e,y}, \frac{\partial P_{e,y}}{\partial y_i} \right\rangle = -D \frac{\partial H}{\partial y_i}(y,y)e^{N-2} + O(e^{N-1}),
$$

$$
\left\langle P_{e,y}, \frac{\partial P_{e,y}}{\partial \epsilon} \right\rangle = -\frac{N-2}{2} DH(y)e^{N-3} + O(e^{N-2}),
$$

$$
\int_\Omega P_{e,y}^p \frac{\partial P_{e,y}}{\partial y_i} = -2D \frac{\partial H}{\partial y_i}(y,y)e^{N-2} + O\left( e^{N-1} \log \frac{1}{\epsilon} \right)
$$

$$
\int_\Omega P_{e,y}^p \frac{\partial P_{e,y}}{\partial \epsilon} = -(N-2)DH(y)e^{N-3} + O(e^{N-2}),
$$

uniformly for $y \in \Omega$ with $d(y, \partial \Omega) > \gamma$.

**Proof.** Let us recall that

$$
\int_{\mathbb{R}^N} |\nabla U_{e,y}|^2 = \int_{\mathbb{R}^N} U_{e,y}^{p+1} = S^2.
$$

Now, for the first relation, by Lemma A.1, Lemma A.2 and using Taylor expansion for $H(y,x)$, we get

$$
\int_\Omega \left[ \nabla P_{e,y} \right] = \int_\Omega \left[ U_{e,y}^{p+1} - \int_\Omega U_{e,y}^p \psi_{e,y} \right]
$$

$$
= \int_{\mathbb{R}^N} U_{e,y}^{p+1} - c_N e^{N-2} \int_\Omega U_{e,y}^p [H(y) + O(\|x - y\|)] + O(e^N)
$$

$$
= S^2 - DH(y)e^{N-2} + O(e^{N-1}),
$$

where $\psi_{e,y}$ is the Green’s function.
because

$$\int_{\Omega} U^p_{e,y} |x - y| = O(\varepsilon^{N/2}).$$

Similarly, for the second one we have

$$\int_{\Omega} P U^{p+1}_{e,y} = \int_{\Omega} U^{p+1}_{e,y} - (p+1) \int_{\Omega} U^p_{e,y} \psi_{e,y} + O(\varepsilon^{N-1})
= S^{N/2} - (p+1) DH(y)\varepsilon^{N-2} + O(\varepsilon^{N-1}).$$

Next, by Lemma A.1, Lemma A.2 and Taylor expansion for $\partial H/\partial y_i (y, x)$:

$$\left\langle PU_{e,y}, \partial PU_{e,y} / \partial y_i \right\rangle = - c_N \varepsilon^{N-2} \int_{\Omega} U^p_{e,y} \left[ \partial H / \partial y_i (y, y) + O(|x - y|) \right] + O(\varepsilon^{N-1})
= - D \partial H / \partial y_i (y, y)\varepsilon^{N-2} + O(\varepsilon^{N-1}),$$

because

$$\frac{1}{p+1} \int_{\mathbb{R}^N} U^{p+1}_{e,y} = \text{cost.} \Rightarrow \int_{\Omega} U^p_{e,y} \partial U_{e,y} / \partial y_i = - \int_{\mathbb{R}^N} U^p_{e,y} \partial U_{e,y} / \partial y_i = O(\varepsilon^{N-1}).$$

Similarly,

$$\left\langle PU_{e,y}, \partial PU_{e,y} / \partial \varepsilon \right\rangle = - \frac{N-2}{2} c_N \varepsilon^{N-4} \int_{\Omega} U^p_{e,y} [H(y) + O(|x - y|)] + O(\varepsilon^{N-1})
= - \frac{N-2}{2} DH(y)\varepsilon^{N-3} + O(\varepsilon^{N-2}),$$

because, as above,

$$\int_{\Omega} U^p_{e,y} \partial U_{e,y} / \partial \varepsilon = - \int_{\mathbb{R}^N,\Omega} U^p_{e,y} \partial U_{e,y} / \partial \varepsilon = O(\varepsilon^{N-1}).$$

For the last but one relation, we get

$$\int_{\Omega} P U_{e,y} \partial PU_{e,y} / \partial y_i = \int_{\Omega} U^p_{e,y} \partial PU_{e,y} / \partial y_i - p \int_{\Omega} U^{p-1}_{e,y} \partial PU_{e,y} / \partial y_i \psi_{e,y}
+ O\left( \int_{\Omega} U^{p-2}_{e,y} \frac{\partial PU_{e,y}}{\partial y_i} \psi_{e,y}^2 \right).$$
Now, oddness implies \( \int_{B_r(y)} U_{e,y}^{p-1} \frac{\partial U_{e,y}}{\partial y_i} \psi_{e,y} = 0 \), and hence, using Lemmas A.1 and A.2,

\[
p \int_{\Omega} U_{e,y}^{p-1} \frac{\partial P U_{e,y}}{\partial y_i} \psi_{e,y} = p \int_{B_r(y)} U_{e,y}^{p-1} \frac{\partial U_{e,y}}{\partial y_i} \psi_{e,y} + O(\varepsilon^{N-1})
\]

\[
= pc_N \varepsilon^{N-2} \int_{B_r(y)} U_{e,y}^{p-1} \frac{\partial U_{e,y}}{\partial y_i} \left[ \sum_j \frac{\partial H(y,y)(x_j - y_j)}{\partial y_j} + O(\|x - y\|^2) \right] + O(\varepsilon^{N-1})
\]

\[
= D \frac{\partial H}{\partial y_i}(y,y)\varepsilon^{N-2} + O\left(\varepsilon^{N-1} \log \frac{1}{\varepsilon}\right),
\]

because

\[
p c_N \int_{\mathbb{R}^N} U_{e,y}^{p-1} \frac{\partial U_{e,y}}{\partial y_i} (x_j - y_j) = - c_N \int_{\mathbb{R}^N} \frac{\partial}{\partial x_i} (U_{e,y}^{p})(x_j - y_j)
\]

\[
= c_N \int_{\mathbb{R}^N} U_{e,y}^{p} \delta_{ij} = D \varepsilon^{N-2} \delta_{ij}.
\]

For the remainder term, by Lemmas A.1 and A.2, we get

\[
\int_{\Omega} U_{e,y}^{p-2} \left| \frac{\partial P U_{e,y}}{\partial y_i} \right| \psi_{e,y}^2 = O\left(\varepsilon^{N-2} \int_{\Omega} U_{e,y}^{p-2} \left( \left| \frac{\partial U_{e,y}}{\partial y_i} \right| + \varepsilon \frac{N-2}{2} \right) \right)
\]

\[
= O\left(\varepsilon^{2N-5} \int_{0}^{\text{diam } \Omega} \rho^N \frac{1}{(1 + \rho^2)^3 \varepsilon^N} + \varepsilon \log \frac{1}{\varepsilon} \right) = O\left(\varepsilon^{N} \log \frac{1}{\varepsilon}\right).
\]

Thus, from the third relation of this Lemma A.5, we obtain the requested expansion.

Finally, we have

\[
\int_{\Omega} P U_{e,y}^{p} \frac{\partial P U_{e,y}}{\partial \varepsilon} = \int_{\Omega} U_{e,y}^{p} \frac{\partial P U_{e,y}}{\partial \varepsilon} - p \int_{B_r(y)} U_{e,y}^{p-1} \frac{\partial U_{e,y}}{\partial \varepsilon} \psi_{e,y} + O\left(\varepsilon^{N-1} \log \frac{1}{\varepsilon}\right),
\]

because, as above,

\[
\int_{\Omega} U_{e,y}^{p-2} \left| \frac{\partial P U_{e,y}}{\partial \varepsilon} \right| \psi_{e,y}^2 = O\left(\varepsilon^{N-2} \int_{\Omega} U_{e,y}^{p-2} \left( \left| \frac{\partial U_{e,y}}{\partial \varepsilon} \right| + \varepsilon \frac{N-4}{2} \right) \right)
\]

\[
= O\left(\varepsilon^{2N-5} \int_{0}^{\text{diam } \Omega} \rho^{N-1} \frac{1}{(1 + \rho^2)^2} + \varepsilon \log \frac{1}{\varepsilon} \right) = O\left(\varepsilon^{N-1} \log \frac{1}{\varepsilon}\right).
\]
Once again, we need to estimate the different terms.

\[
p \int_{B_r(y)} U_{x,y}^{p-1} \frac{\partial U_{x,y}}{\partial x} \psi_{x,y} = pcN^\varepsilon \int_{B_r(y)} U_{x,y}^{p-1} \frac{\partial U_{x,y}}{\partial x} \left[ H(y) + O(|x-y|) \right] + O(\varepsilon^{N-1})
\]
\[
= \frac{N-2}{2} DH(y)e^{N-3} + O(\varepsilon^{N-2}).
\]

Finally, from the fourth relation in this Lemma A.5, we obtain

\[
\int_\Omega PU_{x,y}^p \frac{\partial PU_{x,y}}{\partial x} = -(N-2)DH(y)e^{N-3} + O(\varepsilon^{N-2}).
\]

We conclude this appendix by showing that all the manifolds \( Z \) considered in the paper are “non-degenerate almost critical manifold” for the functional \( E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega |u|^{p+1}, u \in H^1_0(\Omega) \).

**Lemma A.6.** Let \( d := \text{dist}(y, \partial \Omega), \text{diam} \Omega \leq R \). Then

(i) \( \exists \alpha_N : ||\nabla E(\alpha PU_{x,y})|| \leq N+2 \varepsilon R^{N-3} + |1 - \alpha| \) if \( N > 6 \),

\[
\alpha_6 \left( \frac{1}{d^{3-\varepsilon}} \log \frac{1}{\varepsilon} \right) + |1 - \alpha| \) if \( N = 6 \),

\[
\alpha_6 \left( \left( \frac{1}{d} \right)^{N-2} + |1 - \alpha| \right) \) if \( 3 \leq N < 6 \).

for \( \alpha \) bounded. Furthermore, \( \exists 0 < \varepsilon_0 < 1, c > 0 \):

(ii) \( ||\pi_z E''(z)w|| \geq c ||w||, z = \alpha PU_{x,y}, \) for any \( w \in T_1 := \{ w \in H^1_0(\Omega) : \langle w, PU_{x,y} \rangle = \langle w, \frac{\partial PU_{x,y}}{\partial y} \rangle = 0 \ \forall i = 1, \ldots, N \} \) and for \( 0 < \varepsilon < \varepsilon_0 d, 1 - \varepsilon \leq \alpha < 1 + \varepsilon_0 \);

(iii) \( ||\pi_z E''(z)w|| \geq c ||w||, z = PU_{x,y}, \) for any \( w \in T_2 := \{ w \in H^1_0(\Omega) : \langle w, \frac{\partial PU_{x,y}}{\partial y} \rangle = \langle w, \frac{\partial PU_{x,y}}{\partial y} \rangle = 0 \ \forall i = 1, \ldots, N \} \) and for \( 0 < \varepsilon < \varepsilon_0 d \).

**Proof.** (i) Since \( \int_\Omega \nabla PU_{x,y} \nabla \varphi = \int_\Omega U_{x,y}^p \varphi \ \forall \varphi \in H^1_0(\Omega) \) and \( \int_\Omega U_{x,y}^{p+1} = S^N_2 \), we have

\[
\langle \nabla E(\alpha PU_{x,y}), \varphi \rangle \leq \alpha \int_\Omega \nabla PU_{x,y} \nabla \varphi - \varphi \alpha \int_\Omega PU_{x,y}^p \varphi \leq \alpha S^{-\frac{1}{2}} ||\varphi|| \left( \int_\Omega \left( U_{x,y}^p - PU_{x,y}^{p+1} \right)^{\frac{p}{p+1}} \right)^{\frac{p}{p+1}} + |\alpha - \alpha\varphi| S^{N+2} \frac{1}{4} ||\varphi||
\]
\[
\leq p \alpha S^{-\frac{1}{2}} ||\varphi|| \left( \int_\Omega \left( U_{x,y}^{p+1} \right)^{\frac{(p-1)(p+1)}{p}} \right)^{\frac{p}{p+1}} + |\alpha - \alpha\varphi| S^{N+2} \frac{1}{4} ||\varphi||.
\]

By Lemma A.1 and (A.3), estimate (i) follows.
It is well known that (see Appendix D in [35])
\[
\int_\Omega |\nabla w|^2 - p \int_\Omega U_{e,y}^{p-1} w^2 \geq \frac{4}{N+4} \int_\Omega |\nabla w|^2
\]
(A.7)
for any \( w \in T_1 \). Hence, we get
\[
||\pi_{T_1} E''(xPU_{e,y})w)|| \geq \frac{1}{||w||} \langle E''(xPU_{e,y})w, w \rangle
\]
\[
= \frac{1}{||w||} \left[ \int_\Omega |\nabla w|^2 - p x^{p-1} \int_\Omega PU_{e,y}^{p-1} w^2 \right] \geq \frac{2}{N+4} ||w||
\]
for any \( w \in T_1 \) and for \( 0 < \varepsilon < \varepsilon_0 d, 1 - \varepsilon_0 < \varepsilon < 1 + \varepsilon_0 \). Hence (ii) holds.

We can write any \( w \in T_2 \) in the form \( w = \lambda \pi_{T_2} PU_{e,y} + v, v \in T_1 \), \( \lambda = \langle w, PU_{e,y} \rangle / \langle PU_{e,y}, PU_{e,y} \rangle \).
Since \( \pi_{T_2} PU_{e,y} = PU_{e,y} + o(1) \) as \( \frac{\varepsilon}{d} \to 0 \) in view of Lemma A.4, setting \( w_1 = -\lambda \pi_{T_2} PU_{e,y} + v \), we can get
\[
\int_\Omega \nabla w \nabla w_1 - \int_\Omega PU_{e,y}^{p-1} w w_1 = \lambda^2 \left[ p \int_\Omega PU_{e,y}^{p+1} - \int_\Omega |\nabla PU_{e,y}|^2 \right] + \int_\Omega |\nabla v|^2 - p \int_\Omega PU_{e,y}^{p-1} v^2 + o(||w||^2)
\]
\[\geq (p - 1) S \frac{N}{2} \lambda^2 + \frac{4}{N+4} \int_\Omega |\nabla v|^2 + o(||w||^2)
\]
\[\geq c ||w|| |w||w_1||
\]
for \( \frac{\varepsilon}{d} \) small, \( c \) a positive constant. Finally, we can conclude that
\[
||\pi_{T_2} E''(PU_{e,y})w)|| \geq \frac{1}{||w||} \left[ \int_\Omega \nabla w \nabla w_1 - \int_\Omega PU_{e,y}^{p-1} w w_1 \right] \geq c ||w||
\]
for \( \frac{\varepsilon}{d} \) small, \( w \in T_2 \), and then (iii).

\( \square \)

Appendix B

In this appendix, we give the proofs of all facts needed in the expansion of Pohozaev identities.

Proposition 2.1 gives a decomposition of \( u_\delta \) in the form \( u_\delta = x_\delta PU_{\bar{e},y} + w_\delta, w_\delta \in T_{x_\delta PU_{\bar{e},y}}, w_\delta \to 0 \) as \( \delta \to 0 \) (from now on, we will omit for simplicity the dependence on \( \delta \)), but it does not give any information about the rate of convergence of \( w \). However, assuming \( w \to 0, x \to 1 \) and using the equation for \( w \), we can gain something more:
Lemma B.1. Let \( \hat{q} = \min\{ \frac{N}{2}, N - 2 \} \). Then

\[
||w|| = O\left( \left( \frac{e}{d} \right)^{\hat{q}} + \delta \varepsilon \right). \tag{B.1}
\]

Proof. In fact, the function \( w \) solves

\[
-\Delta w = [(\alpha PU_{x,y} + w)^p - \alpha U_{x,y}^p] + \delta a(x)(\alpha PU_{x,y} + w)^p \quad \text{in } \Omega,
\]

\[
w = 0 \quad \text{on } \partial \Omega. \tag{B.2}
\]

Using

\[
(a + b)^p - a^p = O(a^{p-1}|b| + |b|^p),
\]

\[
(a + b)^p - a^p - p a^{p-1} b = O(|b|^p + d^{p-2}|b|^2 (\text{if } p > 2))
\]

for \( a \geq 0, a + b \geq 0 \), we can get, by multiplying (B.2) for \( w \) and integrating,

\[
\int_{\Omega} |\nabla w|^2 = (\alpha^p - \alpha) \int_{\Omega} U_{x,y}^p w + p \alpha^{p-1} \int_{\Omega} U_{x,y}^{p-1} w^2
\]

\[
+ \delta \alpha^{p-1} a(y) \int_{\Omega} U_{x,y}^p w + O\left( \int_{\Omega} U_{x,y}^{p-1} |w| \psi_{x,y} + |w|(|w|^p + |\psi_{x,y}|^p)\right)
\]

\[
+ \int_{\Omega} U_{x,y}^{p-2} |w|(w^2 + \psi_{x,y}^2) (\text{if } p > 2)
\]

\[
+ \delta \int_{\Omega} |x - y| U_{x,y}^p |w| + \delta \int_{\Omega} U_{x,y}^{p-1} w^2.
\]

By Lemma A.1 and (A.3), (A.6), for the term \( \int_{\Omega} U_{x,y}^{p-1} |w| \psi_{x,y} \) we can get

\[
\int_{\Omega} U_{x,y}^{p-1} |w| \psi_{x,y} = \int_{B_d(y)} U_{x,y}^{p-1} |w| \psi_{x,y} + \int_{\Omega \setminus B_d(y)} U_{x,y}^{p-1} |w| \psi_{x,y}
\]

\[
= O\left( \frac{N-2}{d^N-2} \left( \frac{p (p-1) (p+1)}{p+1} \right) \int_{B_d(y)} U_{x,y}^{p-1} \right) ||w||
\]

\[
= O\left( \left( \frac{e}{d} \right)^{\hat{q}} ||w|| \right).
\]

Hence, from \( \int_{\Omega} U_{x,y}^p w = \int_{\Omega} \nabla PU_{x,y} \nabla w = 0, \alpha \to 1 \) and (A.6) we derive

\[
(1 + o(1)) \int_{\Omega} |\nabla w|^2 - p \int_{\Omega} U_{x,y}^{p-1} w^2 = O\left( \left( \frac{e}{d} \right)^{\hat{q}} + \delta \varepsilon \right) ||w||.
\]
In view of (A.7) we get the estimate

$$||w|| = O\left(\frac{e}{d} + \delta e\right).$$

Now we give crucial estimates for expanding the Pohozaev identities for $u_\phi$.

**Lemma B.2.** Let $n(x)$ be the unit outer normal to $\partial \Omega$ in $x$, $D$ as in Section 2. Then

$$\int_{\partial \Omega} \left(\frac{\partial PU_{e,y}}{\partial n}\right)^2 \langle x - y, n(x) \rangle = (N - 2)e^{N-2} H(y) D + O\left(\left(\frac{e}{d}\right)^{N-1}\right), \quad (B.3)$$

$$\int_{\partial \Omega} \left(\frac{\partial PU_{e,y}}{\partial n}\right)^2 n_j(x) = 2e^{N-2} D \partial_j H(y) + O\left(\frac{e^{N-1}}{d^N}\right), \quad j = 1, \ldots, N. \quad (B.4)$$

**Proof.** Multiplying

$$-\Delta PU_{e,y} = U_{e,y}^p \quad \text{in } \Omega,$$

$$PU_{e,y} = 0 \quad \text{on } \partial \Omega$$

for $\langle x - y, \nabla PU_{e,y} \rangle$ and $\partial_{x_j} PU_{e,y}$, we can get by some integration by parts

$$\frac{N - 2}{2} \int_{\Omega} U_{e,y}^p PU_{e,y} + \frac{1}{2} \int_{\partial \Omega} \left(\frac{\partial PU_{e,y}}{\partial n}\right)^2 \langle x - y, n(x) \rangle$$

$$= \int_{\Omega} \Delta PU_{e,y} \langle x - y, \nabla PU_{e,y} \rangle$$

$$= -\int_{\Omega} U_{e,y}^p \langle x - y, \nabla PU_{e,y} \rangle$$

$$= \frac{N - 2}{2} \int_{\Omega} U_{e,y}^p PU_{e,y} - p e \int_{\Omega} U_{e,y}^{p-1} PU_{e,y} \partial_{x_\ell} U_{e,y},$$

because $\langle x - y, \nabla_y U_{e,y} \rangle = \frac{N-2}{2} U_{e,y} + e \partial_{x_\ell} U_{e,y}$, and

$$-\frac{1}{2} \int_{\partial \Omega} \left(\frac{\partial PU_{e,y}}{\partial n}\right)^2 n_j(x) = \int_{\Omega} -\Delta PU_{e,y} \partial_{x_j} PU_{e,y}$$

$$= \int_{\Omega} U_{e,y}^p \partial_{x_j} PU_{e,y} = p \int_{\Omega} U_{e,y}^{p-1} PU_{e,y} \partial_{x_j} U_{e,y},$$
respectively. So, by the first equality we get
\[
\int_{\partial \Omega} \left( \frac{\partial PU_{t,y}}{\partial n} \right)^2 \langle x - y, n(x) \rangle = -2p e \int_{\Omega} U_{t,y}^{p-1} P U_{t,y} \partial_t U_{t,y}
\]
\[
= 2e^{\frac{N}{2}} c_N H(y) \left[ \partial_t \left( \int_{\mathbb{R}^N} U_{t,y}^p \right) + \frac{1}{e} \int_{\mathbb{R}^N \setminus \Omega} U_{t,y}^p \right]
\]
\[
+ O \left( \int_{\Omega} \left( |f_{t,y}| + e^{\frac{N-2}{2}} \frac{|x - y|}{d^{N-1}} \right) U_{t,y}^p \right)
\]
\[
= O \left( \int_{\mathbb{R}^N \setminus \Omega} U_{t,y}^{p+1} \right)
\]
\[
= (N - 2)e^{N-2} H(y) D + O \left( \left( \frac{e}{d} \right)^{N-1} \right),
\]
where we have used Lemma A.1 and the estimates \( H(y) + d|\nabla H(y)| = O \left( \frac{1}{d^{N-1}} \right), \partial_x U_{t,y} = O \left( \frac{U_{t,y}}{e} \right) \) and (A.5). Hence (B.3) holds. Finally, by the second equality we derive
\[
\int_{\partial \Omega} \left( \frac{\partial PU_{t,y}}{\partial n} \right)^2 n_j(x) = -2p \int_{\Omega} U_{t,y}^{p-1} P U_{t,y} \partial_{y_j} U_{t,y}
\]
\[
= 2e^{\frac{N-2}{2}} c_N \int_{\Omega} \left[ H(y) + \langle \nabla H(y), x - y \rangle \right]
\]
\[
+ O \left( \frac{|x - y|^2}{d^N} \right) U_{t,y}^{p-1} \partial_{y_j} U_{t,y} + \frac{1}{e} O \left( \int_{\Omega} |f_{t,y}| U_{t,y}^p + \int_{\mathbb{R}^N \setminus \Omega} U_{t,y}^{p+1} \right)
\]
\[
= 2 \frac{N + 2}{N} e^{N-2} c_N^{p+1} \partial_j H(y, y) \int_{\mathbb{R}^N} \frac{|x|^2}{(1 + |x|^2)^{N+2}} dx + O \left( \frac{e^{N-1}}{d^{N-1}} \right),
\]
where we have used Lemma A.1 and the estimates \( H(y) + d|\nabla H(y)| + d^2 |D_j H(y)| = O \left( \frac{1}{d^{N-1}} \right), \partial_{y_j} U_{t,y} = O \left( \frac{U_{t,y}}{e} \right), \) (A.5) and
\[
\int_{\Omega} |x - y|^2 U_{t,y}^{p-1} |\partial_{y_j} U_{t,y}| = O \left( \frac{N}{e^2} \right).
\]
Hence (B.4) follows because, by an integration by parts,
\[
\int_{\mathbb{R}^N} \frac{|x|^2}{(1 + |x|^2)^{N+2}} dx = \frac{N}{N + 2} \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{N+2}}.
\]
\[\square\]
Lemma B.3. There holds
\[
\frac{N - 2}{N} \delta \int_{\Omega} \langle x - y, \nabla a(x) \rangle (xPU_{e,y} + w)^{p+1}
= \frac{1}{N} \varepsilon^{p+1} S^{N} \delta \varepsilon^{2} \Delta a(y) + O\left(\frac{\varepsilon}{d} N, \frac{\varepsilon^{N - 2}}{d} + \varepsilon^{3} \ln \frac{1}{\varepsilon} + \varepsilon^{2} \Delta^{2}\right), \quad (B.5)
\]
\[
\frac{N - 2}{N} \delta \int_{\Omega} \partial_{i} a(x)(xPU_{e,y} + w)^{p+1}
= \frac{N - 2}{N} \varepsilon^{p+1} S^{N} \delta \partial_{i} a(y) + O\left(\frac{\varepsilon}{d} N, \frac{\varepsilon^{N - 2}}{d} + \varepsilon^{2} \Delta^{2}\right), \quad j = 1, \ldots, N. \quad (B.6)
\]

Proof. Using
\[
(a + b)^{p+1} - a^{p+1} = O(a^{p}|b| + |b|^{p+1}),
\]
\[
(a + b)^{p+1} - a^{p+1} - pd^{p}b = O(a^{p-1}b^{2} + |b|^{p+1}),
\]
for \(a \geq 0, a + b \geq 0\), we can get by Lemma A.1
\[
\int_{\Omega} \langle x - y, \nabla a(x) \rangle (xPU_{e,y} + w)^{p+1}
= \varepsilon^{p+1} \int_{\Omega} \langle x - y, \nabla a(x) \rangle U_{e,y}^{p+1}
+ O\left(\int_{\Omega} |x - y|U_{e,y}^{p,1}(|w| + |\psi_{e,y}|) + ||w||^{p+1} + \int_{\Omega} \psi_{e,y}^{p+1}\right)
= \frac{1}{N} \varepsilon^{p+1} \varepsilon^{p+1} \Delta a(y) \int_{\mathbb{R}^{N}} \frac{|x|^{2}}{(1 + |x|^{2})^{N}} dx
+ O\left(\frac{\varepsilon}{d} N, \frac{\varepsilon^{N - 2}}{d} + \varepsilon^{3} \ln \frac{1}{\varepsilon} + \varepsilon|w|| + \varepsilon|\psi_{e,y}|_{L^{p+1}(\Omega)} + ||w||^{p+1} + |\psi_{e,y}|_{L^{p+1}(\Omega)}^{p+1}\right)
\]
because
\[
\langle x - y, \nabla a(x) \rangle = \langle x - y, \nabla a(y) \rangle + \langle D^{2} a(y)(x - y), x - y \rangle + O(|x - y|^{3}),
\]
and

\[
\int_{\Omega} \partial_j a(x) (x P U_{\varepsilon,y} + w)^{p+1}
\]

\[
= \alpha^{p+1} \int_{\Omega} [\partial_j a(y) + \langle \nabla \partial_j a(y), x - y \rangle + O(|x - y|^2)] U_{\varepsilon,y}^{p+1}
\]

\[
+ px^p \int_{\Omega} [\partial_j a(y) + O(|x - y|)] U_{\varepsilon,y}^p (w - \varphi_{\varepsilon,y}) + O \left( \frac{||w||^2 + \left( \int_{\Omega} \psi_{\varepsilon,y}^{p+1} \right)^{p+1}}{\partial_j a(y)} \right)
\]

\[
= \alpha^{p+1} \partial_j a(y) S^N + O \left( \left( \frac{\varepsilon}{a} \right)^N + \varepsilon^2 + \varepsilon ||w|| + ||\varphi_{\varepsilon,y}||_\infty \int_{\Omega} U_{\varepsilon,y}^p + ||w||^2 + ||\varphi_{\varepsilon,y}||_2^{p+1} \Omega \right),
\]

because \( \int_{\Omega} U_{\varepsilon,y}^p w = 0 \).

Using now (A.3), (A.6) and (B.1) in the above expansions, we conclude the proof. \( \square \)

Let us remark that, by an integration by parts,

\[
\int_{\mathbb{R}^N} \frac{|x|^2}{(1 + |x|^2)^N} dx = \frac{N}{2(N-1)} \int_{\mathbb{R}^N} \frac{dx}{(1 + |x|^2)^{N-1}}
\]

\[
= \frac{N}{2(N-2)} \left[ \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^N} dx + \int_{\mathbb{R}^N} \frac{|x|^2}{(1 + |x|^2)^N} dx \right].
\]

Let us introduce a smooth cut-off function \( \xi \) on \( \mathbb{R}^N \) such that \( 0 \leq \xi \leq 1 \), \( \xi = 0 \) on \( B_1(0) \) and \( \xi = 1 \) on \( B_1(0)^c \). Set \( \eta(x) := \xi \left( \frac{\varepsilon - \xi^2}{d^2} \right) \).

For \( \gamma \in \{0, 1\} \), we consider the function \( z(x) := \eta(x)|x - y|^\gamma w(x) \) which solves

\[
- \Delta z = g(x) \text{ in } \Omega,
\]

\[
z = 0 \text{ on } \partial \Omega,
\]

with

\[
g(x) := -\eta(x)|x - y|^\gamma \Delta w(x) - \Delta \eta(x)|x - y|^\gamma w(x) - \gamma (N + \gamma - 2)|x \\
- y|^{\gamma-2} \eta(x)w(x) - 2\gamma \langle \nabla \eta(x), x - y \rangle |x - y|^{\gamma-2} w(x) \\
- 2|y - y|^\gamma \langle \nabla \eta(x), \nabla w(x) \rangle - 2\gamma \eta(x)|x - y|^\gamma |\nabla w(x), x - y \rangle.
\]

Similarly, we define \( v(x) := \eta(x)|x - y|^\gamma P U_{\varepsilon,y}(x) \) which solves

\[
- \Delta v = h(x) \text{ in } \Omega,
\]

\[
v = 0 \text{ on } \partial \Omega,
\]
with
\[
\begin{align*}
    h(x) &:= \eta(x)|x - y|^\gamma U_{e,y}^x - \Delta \eta(x)|x - y|^\gamma PU_{e,y}(x) \\
    &- \gamma(N + \gamma - 2)|x - y|^{\gamma-2} \eta(x) PU_{e,y}(x) \\
    &- 2\gamma \langle \nabla \eta(x), x - y \rangle |x - y|^{\gamma-2} PU_{e,y}(x) \\
    &- 2|x - y|^{\gamma} \langle \nabla \eta(x), \nabla PU_{e,y}(x) \rangle \\
    &- 2\gamma \eta(x)|x - y|^{\gamma-2} \langle \nabla PU_{e,y}(x), x - y \rangle.
\end{align*}
\]

By elliptic regularity theory and the theory of traces, we have the inequalities
\[
\begin{align*}
    |x - y|^{\gamma} \frac{\partial w}{\partial n} \bigg|_{L^2(\partial \Omega)}^2 &= \frac{\partial}{\partial n}(\eta|x - y|^{\gamma}w) \bigg|_{L^2(\partial \Omega)}^2 \leq C|g|_{L^q(\Omega)}^2 \\
    |x - y|^{\gamma} \frac{\partial PU_{e,y}}{\partial n} \bigg|_{L^2(\partial \Omega)}^2 &= \frac{\partial}{\partial n}(\eta|x - y|^{\gamma}PU_{e,y}) \bigg|_{L^2(\partial \Omega)}^2 \leq C|h|_{L^q(\Omega)}^2
\end{align*}
\]
for some constant \( C > 0 \) and \( q := \frac{2N}{N+1} \).

**Remark B.4.** With the function \( z \), we are cutting \(|x - y|^\gamma w\) to be zero in a small neighbourhood \( B_d(y) \) of the concentration point \( y \). In this way, we will expect that the estimate for \(|x - y|^\gamma \frac{\partial w}{\partial n} \big|_{L^2(\partial \Omega)} \) becomes sharper. This idea is already present in [35] where an estimate for \(|\frac{\partial w}{\partial n} \big|_{L^2(\partial \Omega)} \) is obtained: it corresponds to the choice \( \gamma = 0 \) but this estimate is not enough for our purposes.

Multiplying \( \eta(x)w \) also for \(|x - y|\), we can expect to gain in the estimate some power of \( d \) as a multiplying factor. It is just what happens and it will be crucial in the proof of Theorem 2.2. We apply the same method also to obtain some estimate for \(|x - y|^\gamma \frac{\partial PU_{e,y}}{\partial n} \|_{L^2(\partial \Omega)} \).

We are now in position to prove

**Lemma B.5.** There holds
\[
\begin{align*}
    \left| \frac{\partial w}{\partial n} \right|_{L^2(\partial \Omega)}^2 &= o \left( \frac{\varepsilon^{N-2}}{d^{N-1}} + \delta \varepsilon^2 \right), \quad (B.9) \\
    \left| x - y \right| \left| \frac{\partial w}{\partial n} \right|_{L^2(\partial \Omega)}^2 &= o \left( \left( \frac{\varepsilon}{d} \right)^{N-2} + \delta \varepsilon^2 \right). \quad (B.10)
\end{align*}
\]
Proof. It is enough to estimate each term of \( g(x) \) in \( L^q \)-norm, \( q = \frac{2N}{N+1} \). Taking into account that \( |\Delta w| = O(U_{e^{-\eta}}^p + |w|^p) \), it is easy to see that

\[
\left( \int_\Omega (\eta |x - y|^7 U_{e^{-\eta}}^p)^q \right)^{\frac{2}{q}} = O \left( e^{2^{\gamma-1}} \left( \int_\Omega^{\frac{d}{N+1}} \frac{r^q + N-1}{(1 + r^2)^{(N+2)q}} \, dr \right) \frac{N+1}{N} \right),
\]

\[
= O \left( e^{2^{\gamma-1}} \left( \frac{N}{d} \right)^{N+3-2\gamma} \right),
\]

\[
\left( \int_\Omega (|\Delta \eta||x - y|^7 |w|)^q \right)^{\frac{2}{q}} = O \left( d^{2^{\gamma-4}} |B_{d}(y)|^3 \left( \int_\Omega |w|^{p+1} \right)^{\frac{2}{p+1}} \right) = O(d^{2^{\gamma-1}} |w|^2),
\]

\[
|\gamma| \left( \int_\Omega (\eta |x - y|^7 |w|)^q \right)^{\frac{2}{q}} = |\gamma| O \left( \left( \int_\Omega |x - y|^{2(N+2) - 2\gamma} \right)^{\frac{3}{N}} \left( \int_\Omega |w|^{p+1} \right)^{\frac{2}{p+1}} \right) = O(|w|^2),
\]

\[
\left( \int_\Omega (|\nabla \eta||x - y|^7 |\nabla w|)^q \right)^{\frac{2}{q}} = O(d^{2^{\gamma-2}} |B_{d}(y)|^\frac{1}{N} |w|^2) = O(d^{2^{\gamma-1}} |w|^2),
\]

\[
|\gamma| \left( \int_\Omega (\eta |x - y|^7 |\nabla w|)^q \right)^{\frac{2}{q}} = |\gamma| O \left( \left( \int_\Omega |x - y|^{2(N+2) - 2\gamma} \right)^{\frac{1}{N}} |w|^2 \right) = O(|w|^2).
\]

It remains to estimate \( \left( \int_\Omega (\eta |x - y|^7 |w|^p)^q \right)^{\frac{2}{q}} \), the most difficult because \( pq > p + 1 \). We multiply \(-\Delta w \) for \( \eta^\frac{2(N-2)}{N+1} |x - y|^\frac{2(N-2)}{N+1} |w|^\frac{2}{N+1} \) and, integrating by parts, with some manipulations, we can get

\[
\int_\Omega -\Delta w \eta^\frac{2(N-2)}{N+1} |x - y|^\frac{2(N-2)}{N+1} |w|^\frac{2}{N+1}
\]

\[
= \frac{(N+1)(N+3)}{(N+2)^2} \int_\Omega |\nabla (\eta^{\frac{N-2}{N+1}} |x - y|^{\frac{2(N-2)}{N+1} |w|^\frac{N+2}{N+1}})|^2
\]

\[
+ O \left( \int_\Omega |\nabla w| |\nabla \eta| |x - y|^\frac{2(N-2)}{N+1} |w|^\frac{N+3}{N+1} + |\gamma| \int_\Omega |\nabla w| |x - y|^\frac{2(N-2)}{N+1} |w|^\frac{N+3}{N+1} \right).
\]
Since
\[
\int_\Omega |\nabla w||\nabla \eta||x-y|^{2/(N-1)}|w|^{N+3} = O\left(d^{(2\gamma-1)(N-2)/N+1}|w|^{2/(N+1)}\right),
\]
\[
|\gamma\int_\Omega |\nabla w||x-y|^{2/(N-1)}|w|^{N+3} = |\gamma|O\left(|w|^{2/(N+1)}\right),
\]
\[
\int_\Omega |\nabla \eta|^2|x-y|^{2/(N-1)}|w|^{2/(N+1)} = O\left(d^{(2\gamma-1)(N-2)/N+1}|w|^{2/(N+1)}\right),
\]
\[
|\gamma|\int_\Omega |x-y|^{2/(N-1)-2}|w|^{2/(N+1)} = |\gamma|O\left(|w|^{2/(N+1)}\right),
\]
and using $|\Delta w| = O(U_{\varepsilon,y}^p + |w|^p)$, by the Sobolev inequality we get
\[
S\left(\int_\Omega (\eta|x-y|^2|w|^p)^q\right)^{2/p+1} + O\left(d^{(2\gamma-1)(N-2)/N+1}|w|^{2/(N+1)}\right) + |\gamma|O\left(|w|^{2/(N+1)}\right)
\leq \int_\Omega |\Delta w|^q \eta^{2/(N-1)}|x-y|^{2/(N-1)}|w|^{N+3} \leq O\left(\int_\Omega |w|^{p-1} \left(\eta^{2/(N-1)}|x-y|^{2/(N-1)}|w|^{2/(N+1)}\right) + \int_\Omega U_{\varepsilon,y}^p \eta^{2/(N-1)}|x-y|^{2/(N-1)}|w|^{N+1}\right)
= O\left(|w|^{p-1} \left(\int_\Omega (\eta|x-y|^2|w|^p)^q\right)^{2/p+1} \right)
+ O\left(d^{(2\gamma-1)(N-2)/N+1}|w|^{N+3} \left(\int_\Omega \frac{r^{2N(N-2)+N-1}}{N+3} \frac{N^2 + N^2 + 2N^2 + N^2}{N^2 + N^2 + 2N^2 + N^2} dr \right)\right)
= O\left(\left(\int_\Omega (\eta|x-y|^2|w|^p)^q\right)^{2/p+1} \right) + O\left(|w|^{N+3} \left(\frac{2\gamma-1)(N-2)}{N+1} + e^{2\gamma-1} \left(\frac{e^{N^2 + 5N^2 + 2N^2} - 2N^2}{2(N-1)^2} \right)^2 \right)\right).
\]

It follows that
\[
\left(\int_\Omega (\eta|x-y|^2|w|^p)^q\right)^{2/q} = O\left(d^{2\gamma-1}|w|^{2/(N-2)} + |\gamma||w|^{2/(N-2)} + e^{2\gamma-1} \left(\frac{e^{N^2 + 5N^2 + 2N^2} - 2N^2}{2(N-1)^2} \right)^2 \right).
\]

Resuming all this estimates, we get that for $\gamma = 0$
\[
\left|\frac{\partial w}{\partial n}\right|_{L^2(\partial \Omega)}^2 = O\left(\left(\frac{e^{N^2 + 2}}{d^N} + \frac{|w|}{d^N} + \frac{|w|^{N+3}}{d^{N+2}} + \frac{N^2 + 5N^2}{d^{N+2}} \right)\right).
\]
and for $\gamma = 1$

$$
|x - y| \left| \frac{\partial w}{\partial n} \right|_{L^2(\partial \Omega)}^2 = O \left( \frac{\varepsilon}{d} \frac{N + 1}{d^2} + ||w||^2 + ||w|| \frac{N + 3}{N - 2} \left( \frac{\varepsilon}{d} \frac{N^2 + N + \delta}{2(N - 2)} \right) \right).
$$

Inserting (B.1), using $\frac{N^2 + 5N - 2}{2(N - 2)} = \frac{N^2 + N + 6}{2(N - 2)} + 2 \geq \frac{N}{2}$ and

$$
\frac{(\delta \varepsilon)^{N - 2}}{\varepsilon} \left( \frac{\varepsilon}{d} \right)^{N^2 + 5N - 2} = O \left( \delta \left( \frac{\varepsilon}{d} \right)^{\frac{N}{2}} \right) = O \left( \delta^4 \frac{\varepsilon}{d^2} \frac{1}{d} \frac{N - 2}{d^2} \frac{N}{d^2} \right) = O \left( \delta^3 \frac{\varepsilon^2}{d} + d^2 \frac{\varepsilon^{N - 2}}{d^{N - 1}} \right),
$$

we can obtain the required estimates. \( \square \)

Similarly, we can proceed to prove

**Lemma B.6.** There holds

$$
\left| x - y \right| \left( \frac{\partial P U_{x,y}}{\partial n} \right)_{L^2(\partial \Omega)}^2 = O \left( \frac{\varepsilon^{N - 2}}{d^{N - 1}} \right), \quad (B.11)
$$

$$
\left| x - y \right| \left( \frac{\partial P U_{x,y}}{\partial n} \right)_{L^2(\partial \Omega)}^2 = O \left( \left( \frac{\varepsilon}{d} \right)^{N - 2} \right). \quad (B.12)
$$

**Proof.** We need to estimate $h$ in $L^q$-norm, $q = \frac{2N}{N + 1}$. By Lemma A.1, we have that

$$
\left( \int_{\Omega} \left( \eta |x - y|^q U_{x,y}^q \right)^{\frac{2}{q}} \right)^{\frac{2}{q}} = O \left( \varepsilon^{2q - 1} \left( \int_{d}^{+ \infty} \frac{2N}{r^{N + 1} + N - 1} \left( \frac{N}{N + 1} \right) \right) \right)
$$

$$
= O \left( \varepsilon^{2q - 1} \left( \frac{\varepsilon}{d} \right)^{N + 3 - 2q} \right),
$$

$$
\left( \int_{\Omega} \left( |\Delta \eta||x - y|^q P U_{x,y}^q \right)^{\frac{2}{q}} \right)^{\frac{2}{q}} = O \left( \varepsilon^{2q - 1} \left( \int_{\Omega, B_{\frac{d}{2}}} U_{x,y}^{p + 1} \right)^{\frac{2}{p + 1}} \right) = O \left( \varepsilon^{2q - 1} \left( \frac{\varepsilon}{d} \right)^{N - 2} \right),
$$

$$
|\gamma| \left( \int_{\Omega} \left( |x - y|^\gamma - \eta P U_{x,y}^q \right)^{\frac{2}{q}} \right)^{\frac{2}{q}} = |\gamma| O \left( \left( \int_{\Omega, B_{\frac{d}{2}}} U_{x,y}^{p + 1} \right)^{\frac{2}{p + 1}} \right) = |\gamma| O \left( \left( \frac{\varepsilon}{d} \right)^{N - 2} \right),
$$
\[
|\gamma| \left( \int_{\Omega} (|\nabla \eta| |x - y|^{\gamma-1} PU_{e,y})^q \right)^{\frac{2}{q}} = |\gamma| O \left( d^{2\gamma-1} \left( \int_{\Omega \setminus B_d(y)} |\nabla PU_{e,y}|^2 \right)^{\frac{2}{q}} \right) = |\gamma| O \left( \frac{\varepsilon}{d} N^{-2} \right),
\]

\[
\left( \int_{\Omega} (|x - y|^q |\nabla \eta(x)||\nabla PU_{e,y})^q \right)^{\frac{2}{q}} = O \left( d^{2\gamma-1} \left( \int_{\Omega \setminus B_d(y)} |\nabla PU_{e,y}|^2 \right) \right) = O \left( d^{2\gamma-1} \left( \frac{\varepsilon}{d} N^{-2} \right) \right),
\]

\[
|\gamma| \left( \int_{\Omega} (|x - y|^{\gamma-1} |\nabla PU_{e,y}|^q \right)^{\frac{2}{q}} = |\gamma| O \left( \int_{\Omega \setminus B_d(y)} |\nabla PU_{e,y}|^2 \right) = |\gamma| O \left( \left( \frac{\varepsilon}{d} \right)^{N-2} \right),
\]

where we have used

\[
\int_{\Omega \setminus B_d(y)} |\nabla PU_{e,y}|^2 \leq 2 \left( \int_{\Omega \setminus B_d(y)} |\nabla U_{e,y}|^2 + \int_{\Omega \setminus B_d(y)} |\nabla \psi_{e,y}|^2 \right) = O \left( \frac{\varepsilon}{d} N^{-2} \right),
\]

in view of

\[
\int_{\Omega} |\nabla \psi_{e,y}|^2 = \int_{\Omega} |\nabla U_{e,y}|^2 = \int_{\Omega} U_{e,y}^p PU_{e,y} = O \left( \left( \frac{\varepsilon}{d} \right)^{N-2} \right).
\]

It follows that

\[
\left| \frac{\partial PU_{e,y}}{\partial n} \right|^2_{L^2(\partial \Omega)} = O \left( \frac{\varepsilon^{N-2}}{d^{N-1}} \right), \quad \left| x - y \right|^2 \left| \frac{\partial PU_{e,y}}{\partial n} \right|^2_{L^2(\partial \Omega)} = O \left( \left( \frac{\varepsilon}{d} \right)^{N-2} \right).
\]

By Lemmas (B.5) and (B.6), it can be easily deduced that

**Lemma B.7.** There holds

\[
\int_{\partial \Omega} \left| \frac{\partial PU_{e,y}}{\partial n} \right| \left| \frac{\partial w}{\partial n} \right| = o \left( \frac{\varepsilon^{N-2}}{d^{N-1} + \delta \frac{\varepsilon^2}{d}} \right), \quad (B.13)
\]
\[
\int_{\partial \Omega} |x - y|^2 \left| \frac{\partial P \mathbf{U}_{\varepsilon, \lambda}}{\partial \mathbf{n}} \right| \left| \frac{\partial \mathbf{w}}{\partial \mathbf{n}} \right| = o \left( \left( \frac{\varepsilon}{d} \right)^{N-2} + \delta \varepsilon^2 \right). 
\]

(B.14)

References


[33] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965) 1408–1411.