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A prescribed scalar curvature-type equation: almost critical manifolds and multiple solutions ☆

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Abstract

We present an asymptotic analysis for a perturbed prescribed scalar curvature-type equation. A major consequence is a non-existence result in low dimension. Conversely, we prove an existence result in higher dimensions: to this aim we develop a general finite-dimensional reduction procedure for perturbed variational functionals. The general principle can be useful to discuss some other nonlinear elliptic PDE with Sobolev critical growth in bounded domains.

Keywords: Nonlinear; Elliptic; Critical growth; Finite-dimensional reduction; Melnikov function; Scalar curvature; Asymptotic analysis

1. Introduction

Let Ω be a smooth bounded open set in $\mathbf{R}^N$, $N \geqslant 3$, and $f(x) \in C^\infty(\bar{\Omega})$ be a function positive somewhere. It is well known that the problem

$$\text{(PSCE)} \quad \begin{cases} -\Delta u = f(x) u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

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has no solution, in general: by Pohozaev identity, and if Ω is strictly star-shaped, a necessary condition is $0 < \sup_{x\in\Omega} \langle \nabla f(x), x\rangle$. Moreover, ground state solutions do never exist:

$$\inf_{\{u\in H_0^1(\Omega)\,:\,\int_\Omega f|u|^{\frac{2N}{N-2}}>0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega f|u|^{\frac{2N}{N-2}}\right)^{\frac{N-2}{N}}} = \frac{S}{(\max f)^{\frac{N-2}{N}}}$$

($S =$ best Sobolev constant) is never attained.

We will discuss asymptotic behaviour and existence of multiple solutions for (PSCE) in the perturbative case: $f = 1 + \delta a$, $a \in C^2(\bar{\Omega})$ and $\delta \to 0$. We will refer to this perturbative problem as $(\text{PSCE})_\delta$.

In Section 2 we will perform a blow-up analysis for one-peak solutions of $(\text{PSCE})_\delta$, showing, in particular, that in quite general situations boundary concentration cannot occur. Another major outcome will be the non-existence, in low dimensions, of one-peak solutions (i.e. with energy close to $S^{\frac{N}{2}}$):

Theorem 1.1. *Let $N = 3, 4$. If u_δ are solutions of* $(\text{PSCE})_\delta$ *then*

$$\liminf_{\delta\to 0} \int_\Omega |\nabla u_\delta|^2 > S^{\frac{N}{2}}.$$

As for existence, we state in Section 3 a variational principle for perturbative problems in presence of a manifold of "quasi critical points" for an unperturbed energy functional. Our principle extends to a more general setting, a nonlinear Lyapunov–Schmidt-type reduction introduced in [6] and recently improved by Ambrosetti and alias (see [5] and also the pioneering work of Rey [35]).

In Section 4 we will apply our reduction principle to $(\text{PSCE})_\delta$ to give some existence and multiplicity result (of one-peak solutions) in dimension $N \geqslant 5$:

Theorem 1.2. *Let $N \geqslant 5$. Let $x_0 \in \Omega$ be an isolated critical point of a with non-zero topological index and $\Delta a(x_0) > 0$. Then* $(\text{PSCE})_\delta$ *has solutions u_δ which blow up, as δ goes to zero, exactly at x_0.*

On large balls, we obtain some new insight for (PSCE) giving an interpretation of the index counting condition introduced by Bahri and Coron (as for Ref. [10]); see Theorem 4.9 and related remarks.

In Section 5, we will discuss some other applications of the finite-dimensional reduction to the following class of problems:

$$(\text{P}) \quad \begin{cases} -\Delta u = |u|^{p-1}u + g(\delta, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where, here and elsewhere in the paper, $p = \frac{N+2}{N-2}$. Here $g(\delta, x, u)$ is a perturbation term, small if δ is small, satisfying the growth condition

$$\exists c > 0 : |g(\delta, x, u)| \leqslant c(1 + |u|^p).$$

For $g(\delta, x, u) = \delta u$ and $0 < \delta < \lambda_1(\Omega)$, precise existence results for (P) were established in [14] (see also [2] for sharp conditions in higher dimensions and general nonlinearities); existence of multiple solutions and asymptotic behaviour for $\delta \to 0^+$ were discussed in [26,35]. We generalize to a perturbation term $g(\delta, x, u) = \delta a(x)|u|^{q-1}u$, $1 \leqslant q < p$, $a(x) \in C^\infty(\bar{\Omega})$. We cover also the case $g(\delta, x, u) = |u + \delta a(x)|^{p-1}(u + \delta a(x)) - |u|^{p-1}u$, slightly improving existence results for non-homogeneous BVPs obtained in [37] (see also [16,17]).

2. Asymptotic analysis for $(\mathrm{PSCE})_\delta$, boundary concentration and a non-existence result in low dimensions

Blow-up analysis for (PSCE) is a problem widely studied: see, to quote a few, [15,26,34,35] in case $f \equiv 1$, [27,28] in case f not constant and [23,39,40] for (PSCE) with an additional linear term (in [27,39,40] blow-up analysis of subcritical minimizers in a radial setting leads to an existence result). We will restrict our attention to "one-peak solutions" for

$$(\mathrm{PSCE})_\delta \quad \begin{cases} -\Delta u = (1 + \delta a(x))u^{\frac{N+2}{N-2}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

I.e. we consider solutions u_δ to $(\mathrm{PSCE})_\delta$ such that, for some $y_0 \in \bar{\Omega}$

$$|\nabla u_\delta|^2 \rightharpoonup S^{\frac{N}{2}} \delta_{y_0} \quad \text{as } \delta \to 0 \text{ in the sense of measures.}$$

An important point here is to show that boundary concentration cannot occur if a non-degeneracy assumption on the critical points of a on $\partial\Omega$ is fulfilled. Some non-degeneracy assumption seems to be in some sense necessary, since in general we cannot exclude such a phenomenon: in [18] it is exhibited a sequence of solutions for some perturbation of (PSCE) blowing up at a flat strict local maximum of f on the boundary.

As far as we know, the only known obstruction to boundary concentration is the following: $\frac{\partial a}{\partial n} < 0$ on $\partial\Omega$, see [9] for a result in this direction for (PSCE) in the non-perturbative case. If $\frac{\partial a}{\partial n} \leqslant 0$ on $\partial\Omega$, the method of [26], based on moving plane techniques as developed in [25], might exclude, in some cases, boundary concentration (one should ask, in addition, that $a(x)$ increases in the inward normal direction in a neighbourhood of the boundary). Instead, we will use, for general $a(x)$, the method in [35]: after improving some estimates and performing an accurate expansion of Pohozaev identities, it can be put at work to give the result.

Let us recall some well-known facts. For $\varepsilon > 0$ and $y \in \mathbf{R}^N$, let

$$U_{\varepsilon,y}(x) = \varepsilon^{-\frac{N-2}{2}} U\left(\frac{x-y}{\varepsilon}\right), \quad U(x) = \frac{c_N}{(1+|x|^2)^{\frac{N-2}{2}}}, \quad c_N = [N(N-2)]^{\frac{N-2}{4}}.$$

$U_{\varepsilon,y}$ are known to be the positive solutions in $\mathbf{R}^N$ of $-\Delta u = u^{\frac{N+2}{N-2}}$. Denoted by P: $D^{1,2}(\mathbf{R}^N) \to H^1_0(\Omega)$ the orthogonal projection:

$$\int_\Omega \nabla P\varphi \nabla \psi = \int_\Omega \nabla \varphi \nabla \psi \quad \forall \psi \in H^1_0(\Omega),$$

let

$$T_{\alpha PU_{\varepsilon,y}} := \left\{ w \in H^1_0(\Omega) : \langle w, PU_{\varepsilon,y} \rangle = \left\langle w, \frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon} \right\rangle = \left\langle w, \frac{\partial PU_{\varepsilon,y}}{\partial y_i} \right\rangle = 0 \quad i = 1, \dots, N \right\}.$$

The following facts are well known (see Proposition 2 in [11] and [35,38]):

Proposition 2.1. *Let $\{u_\delta\}$ be as above. Then, for δ small,*

$$u_\delta = \alpha_\delta PU_{\varepsilon_\delta, y_\delta} + w_\delta \tag{1}$$

with α_δ, $\varepsilon_\delta \in (0, +\infty)$, $y_\delta \in \Omega$, $w_\delta \in T_{\alpha_\delta PU_{\varepsilon_\delta, y_\delta}}$ and, as $\delta \to 0$,

$$\alpha_\delta \to 1, \quad y_\delta \to y_0, \quad \frac{\varepsilon_\delta}{dist(y_\delta, \partial\Omega)} \to 0, \quad w_\delta \to 0 \text{ in } H^1_0(\Omega)$$

Some notations are in order. Let $H(x,y)$ denote the regular part of the Green function of Ω, i.e. for $x \in \Omega$

$$\Delta_y H(x,y) = 0 \quad \text{in } \Omega,$$

$$H(x,y) = |x-y|^{-(N-2)} \quad \text{on } \partial\Omega$$

and set $H(y) := H(y,y)$. Also, denote $D := c_N^{p+1} \int_{\mathbf{R}^N} \frac{dx}{(1+|x|^2)^{\frac{N+2}{2}}}$.

The main result in this section is the following:

Theorem 2.2. *Let $N \geq 3$, $a \in C^2(\bar{\Omega})$, $Crit\, a := \{x \in \bar{\Omega} : \nabla a(x) = 0\}$. Assume $\{u_\delta\}$ are solutions for* $(\text{PSCE})_\delta$ *such that, for some $y_0 \in \bar{\Omega}$*

$$|\nabla u_\delta|^2 \rightharpoonup S^{\frac{N}{2}} \delta_{y_0} \quad \text{as } \delta \to 0 \text{ in the sense of measures.} \tag{2}$$

Then $N \geq 5$, $\nabla a(y_0) = 0$ *and* $\Delta a(y_0) \geq 0$. *Furthermore*, y_0 *cannot belong to* $\partial\Omega$, *provided*

$$D^2 a \text{ is invertible } \quad \forall x \in Crit\, a \cap \partial\Omega. \tag{a}$$

In addition, *if we write* u_δ *as in* (1), *it results*

$$\varepsilon_\delta^{N-4} = \delta \frac{S^{\frac{N}{2}} \Delta a(y_0)}{N(N-2)DH(y_0)} + o(\delta) \quad \text{as } \delta \to 0. \tag{3}$$

We now derive Theorem 1.1 from the first statement in Theorem 2.2.

Proof of Theorem 1.1. First of all, let us remark that

$$C_0 := \inf_M \int_\Omega |\nabla u|^2 > S^{\frac{N}{2}},$$

where M is the set of non-trivial solutions of $(\text{PSCE})_{\delta=0}$. Otherwise we could find a sequence $\{u_n\}_{n \in \mathbf{N}}$ such that u_n solves $(\text{PSCE})_{\delta=0}$ and $\int_\Omega |\nabla u_n|^2 \to S^{\frac{N}{2}}$ as $n \to +\infty$. Since $(\text{PSCE})_\delta$ has no ground-state solutions, $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $|\nabla u_n|^2 \rightharpoonup S^{\frac{N}{2}} \delta_{y_0}$ in the sense of measures, $y_0 \in \bar{\Omega}$ (see [38]). By (6), we have that

$$\alpha_n^2 (N-2) \varepsilon_n^{N-2} H(y_n) D + o\left(\left(\frac{\varepsilon_n}{d_n}\right)^{N-2}\right) = 0,$$

where $d_n := d(y_n, \partial\Omega)$ and $\alpha_n, \varepsilon_n, y_n$ are as in Proposition 2.1. A contradiction in view of $d_n^{N-2} H(y_n) = O(1)$ (see [35]).

Now, assume there are solutions u_δ for $(\text{PSCE})_\delta$ with $\delta \to 0$ and $\int_\Omega |\nabla u_\delta|^2 < \min C_0, dS^{\frac{N}{2}}$. From above, we derive that $u_\delta \rightharpoonup 0$ in $H_0^1(\Omega)$ and hence Theorem 2.2 applies: $N \geq 5$. $\square$

To prove Theorem 2.2, we will make use of Pohozaev identities (see [33]):

Lemma 2.3. *Let* u *be a smooth solution of* $(\text{PSCE})_\delta$, $n(x)$ *the unit outer normal to* $\partial\Omega$ *in* x. *Then*, *for any* $y \in \mathbf{R}^N$ *and* $j = 1, \ldots, N$ *we have*

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 \langle x - y, n(x) \rangle = \frac{N-2}{N} \delta \int_\Omega \langle x - y, \nabla a(x) \rangle u^{\frac{2N}{N-2}}, \tag{4}$$

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 n_j(x) = \frac{N-2}{N} \delta \int_\Omega \partial_j a(x) u^{\frac{2N}{N-2}}. \tag{5}$$

Proof of Theorem 2.2. We will plug u_δ (as given in (1)) in (4)–(5) and use several estimates from Appendix B. We will omit from now on the dependence on δ. Inserting (B.3) and (B.5) into (4), we get

$$\begin{aligned}
&\alpha^2(N-2)\varepsilon^{N-2}H(y)D-\frac{1}{N}\alpha^{p+1}S^{\frac{N}{2}}\delta\varepsilon^2\Delta a(y)\\
&\quad=O\left(\left(\frac{\varepsilon}{d}\right)^{N-1}+\delta\varepsilon\left(\frac{\varepsilon}{d}\right)^{\frac{N-2}{2}}+\delta\varepsilon^3\ln\frac{1}{\varepsilon}+\delta^2\varepsilon^2+\left\||x-y|\frac{\partial w}{\partial n}\right\|^2_{L^2(\partial\Omega)}+d\left\|\frac{\partial w}{\partial n}\right\|^2_{L^2(\partial\Omega)}\right.\\
&\qquad\left.+\int_{\partial\Omega}|x-y|^2\left|\frac{\partial PU_{\varepsilon,y}}{\partial n}\right|\left|\frac{\partial w}{\partial n}\right|+d\int_{\partial\Omega}\left|\frac{\partial PU_{\varepsilon,y}}{\partial n}\right|\left|\frac{\partial w}{\partial n}\right|\right).
\end{aligned}$$

Here we used the following fact: if πy denotes the projection of y on $\partial\Omega$ and $d\coloneqq dist(y,\partial\Omega)\leqslant d_0$ suitably small, then $\langle x-y,n(x)\rangle=\langle x-\pi y,n(x)\rangle+O(d)=O(|x-\pi y|^2+d)=O(|x-y|^2+d)$. Now, using (B.9)–(B.10) and (B.13)–(B.14) and

$$\delta\varepsilon\left(\frac{\varepsilon}{d}\right)^{\frac{N-2}{2}}=O\left(\delta^{\frac{3}{2}}\varepsilon^2+\delta^{\frac{1}{2}}\left(\frac{\varepsilon}{d}\right)^{N-2}\right)=o\left(\delta\varepsilon^2+\left(\frac{\varepsilon}{d}\right)^{N-2}\right)\quad\text{as }\delta\to0,$$

we get

$$\alpha^2(N-2)\varepsilon^{N-2}H(y)D-\frac{1}{N}\alpha^{p+1}S^{\frac{N}{2}}\delta\varepsilon^2\Delta a(y)+o\left(\left(\frac{\varepsilon}{d}\right)^{N-2}+\delta\varepsilon^2\right)=0. \tag{6}$$

Since $H(y)d^{N-2}\to C(y_0)>0$ as $\delta\to0$ (see [35]), we obtain $\Delta a(y_0)\geqslant0$ and

$$\frac{\varepsilon^{N-4}}{\delta d^{N-2}}=O(1). \tag{7}$$

This implies, in particular, $N\geqslant5$. Now, inserting (B.6) and (B.11) into (5), we obtain, for $j=1,\dots,N$,

$$\delta\partial_j a(y)=O\left(\frac{\varepsilon^{N-2}}{d^{N-1}}+\delta\left(\frac{\varepsilon}{d}\right)^{\frac{N}{2}}+\delta\varepsilon^2\right)+O\left(\left\|\frac{\partial w}{\partial n}\right\|^2_{L^2(\partial\Omega)}+\int_{\partial\Omega}\left|\frac{\partial PU_{\varepsilon,y}}{\partial n}\right|\left|\frac{\partial w}{\partial n}\right|\right).$$

Hence, from (B.9) and (B.13) we get

$$\nabla a(y)+O\left(\frac{\varepsilon^{N-2}}{\delta d^{N-1}}+\frac{\varepsilon^2}{d}\right)=0 \tag{8}$$

because

$$\delta\left(\frac{\varepsilon}{d}\right)^{\frac{N}{2}}=\frac{\delta\varepsilon}{d^{\frac{1}{2}}}\frac{\varepsilon^{\frac{N-2}{2}}}{d^{\frac{N-1}{2}}}=O\left(\delta^2\frac{\varepsilon^2}{d}+\frac{\varepsilon^{N-2}}{d^{N-1}}\right).$$

From (7) and (8), we get

$$|\nabla a(y)| = O\left(\frac{\varepsilon^2}{d}\right) \tag{9}$$

and hence $\nabla a(y_0) = 0$. Also, assuming $y_0 \in \partial\Omega$, (9) rewrites as

$$\left| D^2 a(y_0)\left(\frac{y - y_0}{|y - y_0|}\right) + \frac{o(|y - y_0|)}{|y - y_0|}\right| = O\left(\frac{\varepsilon^2}{d^2}\right) \quad \text{as } \delta \to 0$$

because $|y - y_0| \geqslant d$ and this implies $D^2 a(y_0)$ is not invertible, contradicting (a). Hence $y_0 \in Crit\, a \cap \Omega$. Finally, using $\alpha \to 1$, from (6) we get

$$\varepsilon^{N-4}\delta^{-1} \to \frac{1}{N(N-2)D} S^{\frac{N}{2}} \frac{\Delta a(y_0)}{H(y_0)} \quad \text{as } \delta \to 0. \qquad \square$$

3. Almost critical manifolds and a reduction procedure: a general principle

We will develop in this section a perturbation theory for functionals of the form

$$E_\delta(u) = E(u) - G(\delta, u), \quad u \in V, \quad \delta \in (-\delta_0, \delta_0),$$

where G is a "small" C^2 functional on the Hilbert space V and E has a "non-degenerate almost critical manifold", that is:

There is a smooth immersion $z : (0, +\infty) \times (0, +\infty) \times O \to V$, O smooth open set in $\mathbf{R}^N$, parametrizing the smooth manifold $Z = \{z(\alpha, \varepsilon, y) : \alpha > 0,\ \varepsilon > 0,\ y \in O\}$, such that

(A1) Z is bounded and $\sup_{y \in O} ||\nabla E(z(\alpha, \varepsilon, y))|| = o(1)$ as $(\alpha, \varepsilon) \to (1, 0)$,

(A2) there exists $0 < \varepsilon_0 < 1$ such that $L_z := \pi_z^\perp E''(z)|_{T_z^\perp} \in Iso(T_z^\perp, T_z^\perp)\, \forall z \in Z_{\varepsilon_0}$ and $\sup_{z \in Z_{\varepsilon_0}} ||L_z^{-1}|| < \infty$,

where $Z_s := \{z(\alpha, \varepsilon, y) : 1 - s < \alpha < 1 + s,\ 0 < \varepsilon < s,\ y \in O\}$, $0 < s < 1$, T_z is the tangent space at $z \in Z$ and $\pi_z : V \to T_z$, $\pi_z^\perp = Id - \pi_z$, are the orthogonal projections. We will also require a good behaviour of E around points $z \in Z$.

For $R(z, w) := \nabla E(z + w) - [\nabla E(z) + E''(z)w]$, we will assume

(A3) $\sup_{z \in Z} ||R(z, w)|| = o(||w||)$ and $\sup_{z \in Z} ||D_w R(z, w)|| = o(1)$ as $||w|| \to 0$.

As for the perturbation G, we will assume

(A4) $G(\delta, u), ||G'(\delta, u)||, ||G''(\delta, u)|| \to_{\delta \to 0} 0$ uniformly on bounded sets.

We will perform, under these assumptions, a reduction procedure which follows the lines developed by Ambrosetti and collaborators; while they deal with perturbations of functionals which possess a non-degenerate manifold of critical points, we are perturbing a functional which, in general, has no critical points at all: the manifold of critical points is replaced here by a manifold of "quasi-critical points". Actually, problems which fit into this framework have been widely considered, starting from the pioneering work [35] (see also [1,3,10–13,36,37] to quote a few). So, this is an effort to give a general framework, in the spirit of the work of Ambrosetti, while borrowing basic analysis from Rey. First, we have:

Lemma 3.1. *Let E_δ satisfy assumptions* (A1)–(A4). *Then there exist* $0<\varepsilon_1<1$, $\delta_1>0$ *and a smooth map* $z\to w(\delta,z)$, $z=z(\alpha,\varepsilon,y)$, for $|\delta|<\delta_1$, $1-\varepsilon_1<\alpha<1+\varepsilon_1$, $0<\varepsilon<\varepsilon_1$ *and* $y\in O$, *such that*

$$\text{(i)}\ \ \pi_z w(\delta,z)\equiv 0$$

and

$$\text{(ii)}\ \ \pi_z^\perp \nabla E_\delta(z+w(\delta,z))\equiv 0.$$

Furthermore,

$$||w(\delta,z)||=O(||\pi_z^\perp \nabla E_\delta(z)||). \tag{10}$$

Proof. Set $L:=\sup_{Z_{\varepsilon_0}}||L_z^{-1}||$. Eqs. (i)–(ii) rewrite as a fixed point equation:

$$w=-L_z^{-1}\pi_z^\perp(\nabla E_\delta(z)-G''(\delta,z)w+R_\delta(z,w)),\quad w\in T_z^\perp, \tag{11}$$

where L_z and R_δ are as above. For a given $\delta\in(-\delta_0,\delta_0)$ and $z\in Z_{\varepsilon_0}$, let us denote by $N_{\delta,z}$ the operator at the right-hand side in (11). We have

$$||N_{\delta,z}(w)||\leqslant L(||\nabla E(z)||+||\nabla G(\delta,z)||+||G''(\delta,z)||||w||+||R_\delta(z,w)||).$$

By (A3) and (A4), we can find $\rho>0$, $0<\delta_1<\delta_0$ such that

$$\sup_{z\in Z}||R_\delta(z,w)||+\sup_{z\in Z}||D_wR_\delta(z,w)||||w||\leqslant\frac{1}{4L}||w||,\quad ||w||\leqslant\rho,\quad |\delta|<\delta_1,$$
$$\frac{1}{\rho}||\nabla G(\delta,z)||+||G''(\delta,z)||\leqslant\frac{1}{4L},\quad |\delta|<\delta_1,\quad z\in Z.$$

By (A1) we can find $0<\varepsilon_1<\varepsilon_0$ such that $\sup_{z\in Z_{\varepsilon_1}}||\nabla E(z)||\leqslant\frac{1}{4L}\rho$. Hence,

$$||w||\leqslant\rho\Rightarrow||N_{\delta,z}(w)||\leqslant\rho,$$

that is, $N_{\delta,z}$ maps $B_\rho:=\{w\in T_z^\perp:||w||\leqslant\rho\}$ into itself for $z\in Z_{\varepsilon_1}$, $|\delta|<\delta_1$.

Since for $w_1,w_2\in B_\rho$ we get

$$\begin{aligned}||N_{\delta,z}(w_1)-N_{\delta,z}(w_2)||&\leqslant L\left(\sup_{0\leqslant t\leqslant 1}||D_wR_\delta(z,tw_1+(1-t)w_2)||+\frac{1}{4L}\right)||w_1-w_2||\\&\leqslant\tfrac{1}{2}||w_1-w_2||,\end{aligned}$$

we see that $N_{\delta,z}$ is a contraction on B_ρ. Thus, $N_{\delta,z}(\cdot)$ has a fixed point in B_ρ, say $w = w(\delta, z)$ for $|\delta| < \delta_1$ and $z \in Z_{\varepsilon_1}$. Now, from the fixed point equation,

$$||w(\delta, z)|| = ||N_{\delta,z}(w(\delta, z))|| = O(||\pi_z^\perp \nabla E_\delta(z)||) + o(1)||w(\delta, z)||,$$

where $o(1) \to 0$ as $\rho + \delta \to 0$, and hence $||w(\delta, z)|| = O(||\pi_z^\perp \nabla E_\delta(z)||)$.

Smoothness of $z \to w(\delta, z)$ follows by the IFT applied to the equation

$$\pi_z^\perp \nabla E_\delta(z + \pi_z^\perp u) + \pi_z u = 0, \quad u \in H_0^1(\Omega).$$

In fact, the linearized operator at $w = w(\delta, z)$, $\pi_z^\perp E_\delta''(z + w)\pi_z^\perp + \pi_z$ is invertible, up to take ε_1, δ_1 smaller, because $\sup_{z \in Z_{\varepsilon_1}} ||w(\delta, z)|| \to 0$ as $\varepsilon_1 + \delta_1 \to 0$ and, at $\delta = 0$, $w = 0$, it is trivially invertible by (A.2). $\square$

The final step in the reduction procedure is to prove that critical points of E_δ, close to Z, correspond to critical points of

$$E_\delta(\alpha, \varepsilon, y) := E_\delta(z(\alpha, \varepsilon, y) + w(\delta, z(\alpha, \varepsilon, y))).$$

The proof relies on C^1 estimates of $w(\delta, z)$ which involve the variation of T_z. Let us first prove C^1 estimates under suitable assumptions.

Lemma 3.2. *Assume* (A1)–(A4) *and let* $w(\delta, z)$ *be given by Lemma* 3.1. *Then*

$$\left|\left|\pi_z \frac{\partial w}{\partial z}\right|\right| = O(||w||) \tag{12}$$

provided the following assumption holds true:

$$\exists c > 0 : \left|\left|\pi_z \frac{\partial}{\partial z}(\pi_z^\perp v)\right|\right| \leqslant c||\pi_z^\perp v|| \ \forall z \in Z, \quad \forall v \in V. \tag{A5}$$

Proof. Let $\bar{w} = w(\delta, \bar{z})$ for some $\bar{z} \in Z$, δ fixed. From $\pi_z\, w(\delta, z) \equiv 0$ it follows $\pi_z \frac{\partial w}{\partial z} = -\frac{\partial}{\partial z}(\pi_z \bar{w})$ at $z = \bar{z}$. Since $-\frac{\partial}{\partial z}(\pi_z \bar{w}) = \frac{\partial}{\partial z}(\pi_z^\perp \bar{w})$, we have, by (A5),

$$\left|\left|\pi_{\bar{z}} \frac{\partial w}{\partial z}(\delta, \bar{z})\right|\right| \leqslant c||\pi_{\bar{z}}^\perp \bar{w}||.$$

This proves (12), because $\pi_{\bar{z}}^\perp \bar{w} = \bar{w}$. $\square$

Theorem 3.3. *Assume* (A1)–(A5) *and let* $w(\delta, z)$ *be given by Lemma* 3.1. *Then, for* ε, δ *small,* $\nabla E_\delta(z_0 + w(\delta, z_0)) = 0$ *iff* z_0 *is a critical point of* $z \to E_\delta(z + w(\delta, z))$.

Proof. Let $z(t)$ be a smooth curve on Z with $z(0) = z_0$ and $\dot{z}(0) = \pi_{z_0} \nabla E_\delta(z_0 + w(\delta, z_0))$. By assumption,

$$0 = \frac{d}{dt} E_\delta(z(t) + w(\delta, z(t)))|_{t=0} = \left\langle \nabla E_\delta(z_0 + w(\delta, z_0)), \dot{z}(0) + \frac{\partial w}{\partial z}(\delta, z_0)\dot{z}(0) \right\rangle.$$

Since $\pi_{z_0}^\perp \nabla E_\delta(z_0 + w(\delta, z_0)) = 0$, using (10) and (12), we get

$$||\dot{z}(0)||^2 \leqslant ||\dot{z}(0)||^2 \left|\left| \pi_{z_0} \frac{\partial w}{\partial z}(\delta, z_0) \right|\right| \leqslant c||w(\delta, z_0)||||\dot{z}(0)||^2 \leqslant \tilde{c}||\dot{z}(0)||^2 ||\nabla E_\delta(z_0)||$$

and hence $\dot{z}(0) = 0$ because $||\nabla E_\delta(z)|| \ll 1$ for $z \in Z_{\varepsilon_1}$ if $\varepsilon_1 + \delta_1$ is small. □

Remark 3.4 (The Melnikov function). Theorem 3.3 applies as follows: first, write $z(\alpha, \varepsilon, y) = z(\tau)$, $\tau = (\alpha, \varepsilon, y)$ and

$$\begin{aligned} E_\delta(z(\tau) + w(\delta, \tau)) &= E(z(\tau)) - G(\delta, z(\tau)) \\ &\quad + \int_0^1 \langle \nabla E_\delta(z(\tau) + tw(\delta, \tau)), w(\delta, \tau) \rangle \, dt. \end{aligned}$$

If we suppose E'' uniformly bounded on bounded sets, we have, by (10),

$$E_\delta(z(\tau) + w(\delta, \tau)) = E(z(\tau)) - G(\delta, z(\tau)) + O(||\pi_{z(\tau)}^\perp \nabla E_\delta(z(\tau))||^2).$$

In the applications, the remainder term will be "negligible" and one is led to look for critical points of the "Melnikov function"

$$E_\delta(z(\tau)) = E(z(\tau)) - G(\delta, z(\tau)).$$

4. Multiple solutions for $(\mathrm{PSCE})_\delta$

Here we complement the non-existence result contained in Theorem 1.1 by showing that for $N \geqslant 5$ there are branches of solutions for $(\mathrm{PSCE})_\delta$ bifurcating from critical points of $a(x)$ with positive laplacian, non-degenerate in some sense: this is the content of Theorem 1.2. To prove it, we will apply Theorem 3.3 to the functional $E_\delta(u) = E(u) - G(\delta, u)$, $u \in H_0^1(\Omega)$, where

$$E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p+1} \int_\Omega |u|^{p+1}, \quad G(\delta, u) = \frac{\delta}{p+1} \int_\Omega a(x)|u|^{p+1}.$$

The functional $E(u)$ possesses a "non-degenerate almost critical manifold"

$$Z := \{\alpha P U_{\varepsilon,y} : \alpha > 0,\ \varepsilon > 0,\ y \in \Omega,\ d(y, \partial\Omega) > \gamma\}, \quad \gamma > 0,$$

where $PU_{\varepsilon,y}$ are as in Section 2. In particular, $PU_{\varepsilon,y}$ is the unique solution of

$$-\Delta PU_{\varepsilon,y} = -\Delta U_{\varepsilon,y} = U^p_{\varepsilon,y} \quad \text{in } \Omega,$$

$$PU_{\varepsilon,y} = 0 \quad \text{on } \partial\Omega.$$

We will omit, if not relevant, any reference to γ. We will use several facts stated in Appendix A.

Assumptions (A1) and (A2) are checked in Lemma A.6, while (A.3) follows from

Lemma 4.1. *Let* $\hat{p} = \min\{p, 2\}$. *Then*

$$\exists c > 0 : \|R(z,w)\| + \|D_w R(z,w)\| \|w\| \leqslant c \|w\|^{\hat{p}} \quad \forall z \in \tilde{Z}.$$

Proof. By direct computation, for any $\phi, \psi \in H^1_0(\Omega)$:

$$\langle R(z,w), \phi \rangle = -\int_\Omega [|z+w|^{p-1}(z+w) - z^p - pz^{p-1}w]\phi,$$

$$\langle D_w R(z,w)\phi, \psi \rangle = p\int_\Omega (z^{p-1} - |z+w|^{p-1})\phi\psi.$$

Using the elementary inequalities, for $a, b \in \mathbf{R}$,

$$|(a+b)|a+b|^{p-1} - a|a|^{p-1} - p|a|^{p-1}b| \leqslant \begin{cases} c_p(|a|^{p-2}b^2 + |b|^p) & \text{if } p > 2, \\ c_p|b|^p & \text{if } p \leqslant 2, \end{cases}$$

$$||a|^{p-1} - |a+b|^{p-1}| \leqslant \begin{cases} c_p(|a|^{p-2}|b| + |b|^{p-1}) & \text{if } p > 2, \\ c_p|b|^{p-1} & \text{if } p \leqslant 2, \end{cases}$$

and Hölder and Sobolev inequalities, the Lemma readily follows. □

Assumption (A4) is easily checked and (A5) follows by Lemmas (A.4) and (A.5) and

Remark 4.2. Assumption (A5) involves the second derivatives of $z(\alpha, \varepsilon, y)$. Property (A5), and hence Lemma 3.2, Theorem 3.3, can be derived more directly by the following facts:

$$\exists c > 0 : \sum_{j,k} \frac{\|\partial_{jk} z\|^2}{\|\partial_j z\|^2 \|\partial_k z\|^2} \leqslant c, \quad \langle \partial_i z, \partial_j z \rangle = o(\|\partial_i z\| \|\partial_j z\|) \quad \forall i \neq j. \tag{13}$$

In fact, if $s = (\alpha, \varepsilon, y)$ and $z(s(t))$ is a curve in Z such that $z(s(0)) = z$, property (A5) is equivalent to prove

$$\left\| \pi_z \frac{d}{dt} (\pi^\perp_{z(t)} v) \Big|_{t=0} \right\| \leqslant c \|\pi^\perp_z v\| \left\| \frac{dz}{dt} \Big|_{t=0} \right\|, \quad \forall v \in H^1_0(\Omega).$$

If we write $\pi_z \frac{d}{dt}(\pi^{\perp}_{z(t)} v)|_{t=0} = \sum a_j \partial_j z$ and $\frac{dz}{dt}|_{t=0} = \sum_j \partial_j z \frac{ds_j}{dt}(0)$, the second assumption in (13) implies that

$$\left|\left|\pi_z \frac{d}{dt}(\pi^{\perp}_{z(t)} v)\Big|_{t=0}\right|\right|^2 = (1+o(1)) \sum_j a_j^2 ||\partial_j z||^2,$$

$$\left|\left|\frac{dz}{dt}\Big|_{t=0}\right|\right|^2 = (1+o(1)) \sum_j \left(\frac{ds_j}{dt}(0)\right)^2 ||\partial_j z||^2.$$

Since $\langle \pi^{\perp}_{z(t)} v, (\partial_j z)(s(t)) \rangle \equiv 0$, we can get

$$\begin{aligned}
\left|\left|\pi_z \frac{d}{dt}(\pi^{\perp}_{z(t)} v)\Big|_{t=0}\right|\right|^2 &= \left\langle \pi_z \frac{d}{dt}(\pi^{\perp}_{z(t)} v)\Big|_{t=0}, \sum_j a_j \partial_j z \right\rangle \\
&= -\sum_{j,k} a_j \left\langle \pi_z^{\perp} v, \partial_{jk} z \frac{ds_k}{dt}(0) \right\rangle \\
&\leqslant ||\pi_z^{\perp} v|| \left(\sum_k \frac{ds_k}{dt}(0)^2 ||\partial_k z||^2 \right)^{\frac{1}{2}} \left(\sum_j a_j^2 ||\partial_j z||^2 \right)^{\frac{1}{2}} \left(\sum_{j,k} \frac{||\partial_{jk} z||^2}{||\partial_j z||^2 ||\partial_k z||^2} \right)^{\frac{1}{2}} \\
&\leqslant c ||\pi_z^{\perp} v|| \left|\left|\pi_z \frac{d}{dt}(\pi^{\perp}_{z(t)} v)\Big|_{t=0}\right|\right| \left|\left|\frac{dz}{dt}\Big|_{t=0}\right|\right|.
\end{aligned}$$

Hence (A5) follows. So, instead of (A5), one might more easily check (13).

Now, we are led to look for critical points of $E_\delta(\alpha, \varepsilon, y) := E_\delta(\alpha PU_{\varepsilon,y} + w(\delta, \alpha, \varepsilon, y))$. Accordingly with Remark 3.4, we need to estimate the remainder term. Since $\psi_{\varepsilon,y} := U_{\varepsilon,y} - PU_{\varepsilon,y}$ is an harmonic function, we get

$$||\psi_{\varepsilon,y}||_\infty \leqslant \max_{\partial\Omega} U_{\varepsilon,y} = O(\varepsilon^{\frac{N-2}{2}}).$$

If we write for any $\Phi \in H_0^1(\Omega)$

$$\begin{aligned}
\langle \nabla E(z), \Phi \rangle &= \alpha \int_\Omega \nabla PU_{\varepsilon,y} \nabla \Phi - \alpha^p \int_\Omega PU^p_{\varepsilon,y} \Phi \\
&= (\alpha - \alpha^p) \int_\Omega U^p_{\varepsilon,y} \Phi + \alpha^p \int_\Omega (U^p_{\varepsilon,y} - PU^p_{\varepsilon,y}) \Phi, \\
\langle \nabla G(\delta, z), \Phi \rangle &= \delta \alpha^p \int_\Omega a(x) PU^p_{\varepsilon,y} \Phi \\
&= \delta \alpha^p \left[a(y) \int_\Omega U^p_{\varepsilon,y} \Phi + \int_\Omega \langle \nabla a(y), x - y \rangle U^p_{\varepsilon,y} \Phi \right] \\
&\quad + \delta \alpha^p \left[\int_\Omega (a(x) - a(y) - \langle \nabla a(y), x - y \rangle) U^p_{\varepsilon,y} \Phi \right. \\
&\quad \left. + \int_\Omega a(x) (PU^p_{\varepsilon,y} - U^p_{\varepsilon,y}) \Phi \right],
\end{aligned}$$

we can obtain, using Lemma A.1,

$$||\pi_z^\perp \nabla E(z)|| = O(\varepsilon^{\frac{N}{2}}), \quad ||\nabla E(z)|| = O(\varepsilon^{\frac{N}{2}} + |1-\alpha|),$$
$$||\pi_z^\perp \nabla G(\delta, z)|| = O(\delta\varepsilon|\nabla a(y)| + \delta\varepsilon^2), \quad ||\nabla G(\delta, z)|| = O(\delta),$$

because $\int_\Omega U_{\varepsilon,y}^p \Phi = \int_\Omega \nabla PU_{\varepsilon,y} \nabla \Phi = 0$ for any $\Phi \in T_z^\perp$. As for the remainder term,

$$||\pi_z^\perp \nabla E_\delta(z)||^2 = O(\varepsilon^N + \delta^2\varepsilon^2|\nabla a(y)|^2 + \delta^2\varepsilon^4),$$
$$||\nabla E_\delta(z)||^2 = O(\varepsilon^N + \delta^2 + |1-\alpha|^2). \tag{14}$$

According to Lemma A.5 in Appendix A, we have

$$E(\alpha PU_{\varepsilon,y}) = \left(\frac{\alpha^2}{2} - \frac{\alpha^{p+1}}{p+1}\right) S^{\frac{N}{2}} + D\left(-\frac{\alpha^2}{2} + \alpha^{p+1}\right) H(y)\varepsilon^{N-2} + O(\varepsilon^{N-1}),$$

where, as in Section 2, $D = c_N^{\frac{2N}{N-2}} \int_{\mathbf{R}^N} \frac{dy}{(1+|y|^2)^{\frac{N+2}{2}}}$. Finally, from Lemmas A.1 and A.2 we see that

$$\begin{aligned} G(\delta, \alpha PU_{\varepsilon,y}) &= -\frac{\delta}{p+1}\alpha^{p+1} \int_\Omega a(x) PU_{\varepsilon,y}^{p+1} \\ &= -\frac{\delta}{p+1}\alpha^{p+1} \int_\Omega \Bigg[a(y) + \sum_i \partial_i a(y)(x-y)_i \\ &\quad + \frac{1}{2}\partial_{ij} a(y)(x-y)_i(x-y)_j + O(|x-y|^3)\Bigg] U_{\varepsilon,y}^{p+1} + O(\delta\varepsilon^{N-2}) \\ &= -\alpha^{p+1} \frac{S^{\frac{N}{2}}}{p+1}\delta a(y) - \alpha^{p+1}\frac{S^{\frac{N}{2}}}{4N}\delta\varepsilon^2 \Delta a(y) + O(\delta\varepsilon^3) \end{aligned}$$

because, by an integration by parts,

$$\int_{\mathbf{R}^N} \frac{|x|^2}{(1+|x|^2)^N} dx = \frac{N}{N-2}\int_{\mathbf{R}^N} \frac{dx}{(1+|x|^2)^N} = \frac{N}{N-2} S^{\frac{N}{2}}.$$

Summarizing, using (14), we get the following expansions for $E_\delta(\alpha, \varepsilon, y)$, $z \in Z$:

Lemma 4.3. *Let* $N \geqslant 5$. *Then*

$$E_\delta(\alpha PU_{\varepsilon,y}+w)=\left(\frac{\alpha^2}{2}-\frac{\alpha^{p+1}}{p+1}\right)S^{\frac{N}{2}}+D\left(-\frac{\alpha^2}{2}+\alpha^{p+1}\right)H(y)\varepsilon^{N-2}$$
$$-\ \alpha^{p+1}\frac{S^{\frac{N}{2}}}{p+1}\delta a(y)-\alpha^{p+1}\frac{S^{\frac{N}{2}}}{4N}\delta\varepsilon^2\Delta a(y)$$
$$+\ O(\varepsilon^{N-1}+\delta^2\varepsilon^2|\nabla a(y)|^2+\delta\varepsilon^3).$$

Next, we establish C^1 estimates.

Lemma 4.4. *Let* $N \geqslant 5$. *Then*

$$\frac{\partial}{\partial y_i}E_\delta(\alpha PU_{\varepsilon,y}+w)=-\alpha^{p+1}\frac{S^{\frac{N}{2}}}{p+1}\delta\partial_i a(y)+O(\varepsilon^{N-2}+\delta\varepsilon+\delta^2$$
$$+\ \varepsilon^{\frac{N-2}{2}}|1-\alpha|+\delta|1-\alpha|) \tag{15}$$

$$\frac{\partial}{\partial\varepsilon}E_\delta(\alpha PU_{\varepsilon,y}+w)=D(N-2)\left(-\frac{\alpha^2}{2}+\alpha^{p+1}\right)H(y)\varepsilon^{N-3}$$
$$-\ \alpha^{p+1}\frac{S^{\frac{N}{2}}}{2N}\delta\varepsilon\Delta a(y)+O(\varepsilon^{N-2}+\delta\varepsilon^{\frac{3}{2}}+\delta^2|\nabla a(y)|$$
$$+\ \delta^2\varepsilon+\varepsilon^{\frac{N-2}{2}}|1-\alpha|+\delta|1-\alpha||\nabla a(y)|+\delta\varepsilon|1-\alpha|) \tag{16}$$

$$\frac{\partial}{\partial\alpha}E_\delta(\alpha PU_{\varepsilon,y}+w)=S^{\frac{N}{2}}(\alpha-\alpha^p)-\delta a(y)\alpha^p S^{\frac{N}{2}}+O(\delta\varepsilon+\varepsilon^{\frac{N}{2}}). \tag{17}$$

Proof. Since $\nabla E_\delta(\alpha PU_{\varepsilon,y}+w)=O(||\nabla E_\delta(\alpha PU_{\varepsilon,y})||)$, we have

$$\frac{\partial}{\partial y_i}E_\delta(\alpha PU_{\varepsilon,y}+w)$$
$$=\left\langle \nabla E_\delta(\alpha PU_{\varepsilon,y}+w),\alpha\frac{\partial PU_{\varepsilon,y}}{\partial y_i}+\pi_z\frac{\partial w}{\partial y_i}\right\rangle$$
$$=\alpha^2\left\langle PU_{\varepsilon,y},\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right\rangle-\alpha\int_\Omega(1+\delta a(x))|\alpha PU_{\varepsilon,y}+w|^{p-1}(\alpha PU_{\varepsilon,y}+w)\frac{\partial PU_{\varepsilon,y}}{\partial y_i}$$
$$+\ O\left(||\nabla E_\delta(\alpha PU_{\varepsilon,y})||\left|\left|\pi_z\frac{\partial w}{\partial y_i}\right|\right|\right).$$

The first term is estimated in Lemma A.5:

$$\left\langle PU_{\varepsilon,y}, \frac{\partial PU_{\varepsilon,y}}{\partial y_i} \right\rangle = -D\frac{\partial H}{\partial y_i}(y,y)\varepsilon^{N-2} + O(\varepsilon^{N-1}). \tag{18}$$

As for the third term, we first derive from (12) and Lemma A.4:

$$\left|\left|\pi_z \frac{\partial w}{\partial y_i}\right|\right| = \left|\left|\pi_z \frac{\partial w}{\partial z}\frac{\partial z}{\partial y_i}\right|\right| = O\left(\frac{1}{\varepsilon}||w||\right)$$

and hence

$$||\nabla E_\delta(\alpha PU_{\varepsilon,y})||\left|\left|\pi_z \frac{\partial w}{\partial y_i}\right|\right| = \frac{1}{\varepsilon}O(||\nabla E_\delta(z)||||w||). \tag{19}$$

It remains to estimate the second term. We claim that

$$\begin{aligned}&\int_\Omega (1+\delta a(x))|\alpha PU_{\varepsilon,y}+w|^{p-1}(\alpha PU_{\varepsilon,y}+w)\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\\ &\quad = \delta\alpha^p\frac{S^{\frac{N}{2}}}{p+1}\partial_i a(y) + O\left(\varepsilon^{N-2}+\frac{\delta}{\varepsilon}||w||+\varepsilon^{\frac{N-2}{2}}||w||+\frac{||w||^2}{\varepsilon}\right).\end{aligned} \tag{20}$$

Putting together estimates (18)–(20), we get

$$\begin{aligned}\frac{\partial}{\partial y_i}E_\delta(PU_{\varepsilon,y}+w) = &-\delta\alpha^{p+1}\frac{S^{\frac{N}{2}}}{p+1}\partial_i a(y)\\ &+O\left(\varepsilon^{N-2}+\frac{\delta}{\varepsilon}||w||+\varepsilon^{\frac{N-2}{2}}||w||+\frac{1}{\varepsilon}||\nabla E_\delta(z)||w||\right)\end{aligned}$$

and hence (15) follows from (10) and (14). We now prove (20). We have

$$\begin{aligned}&\int_\Omega (1+\delta a(x))|\alpha PU_{\varepsilon,y}+w|^{p-1}(\alpha PU_{\varepsilon,y}+w)\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\\ &\quad = \alpha^p\int_\Omega PU^p_{\varepsilon,y}\frac{\partial PU_{\varepsilon,y}}{\partial y_i} + \delta\alpha^p\int_\Omega a(x)PU^p_{\varepsilon,y}\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\\ &\qquad + p\alpha^{p-1}\int_\Omega PU^{p-1}_{\varepsilon,y}\frac{\partial PU_{\varepsilon,y}}{\partial y_i}w + h.o.t.,\end{aligned} \tag{21}$$

where, by Taylor expansion,

$$\text{h.o.t.} = O\left(\delta\int_\Omega U^{p-1}_{\varepsilon,y}\left|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right||w| + \int_\Omega PU^{p-2}_{\varepsilon,y}\left|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right|w^2 + \int_\Omega\left|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right||w|^p\right)$$

if $N=5$, while

$$\text{h.o.t.} = O\left(\delta \int_\Omega U_{\varepsilon,y}^{p-1}\left|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right||w| + \int_\Omega PU_{\varepsilon,y}^{p-2}\left|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right|w^2\right)$$

if $N\geqslant 6$. The first term in (21) is estimated in Lemma A.5:

$$\int_\Omega PU_{\varepsilon,y}^p \frac{\partial PU_{\varepsilon,y}}{\partial y_i} = -2D\frac{\partial H}{\partial y_i}(y,y)\varepsilon^{N-2} + O\left(\varepsilon^{N-1}\log\frac{1}{\varepsilon}\right). \tag{22}$$

As for the second term in (21), we observe that, using Lemmas A.1 and A.2, we get

$$\begin{aligned}
&\int_\Omega a(x)PU_{\varepsilon,y}^p \frac{\partial PU_{\varepsilon,y}}{\partial y_i}\\
&= \int_\Omega \left[a(y) + \sum_j \partial_j a(y)(x-y)_j + O(|x-y|^2)\right] U_{\varepsilon,y}^p \frac{\partial U_{\varepsilon,y}}{\partial y_i} + O(\varepsilon^{N-3})\\
&= \frac{N-2}{N}\partial_i a(y) c_N^{p+1} \int_{\mathbf{R}^N} \frac{|x|^2}{(1+|x|^2)^{N+1}}\,dx + O(\varepsilon)\\
&= \frac{S^{\frac{N}{2}}}{p+1}\partial_i a(y) + O(\varepsilon).
\end{aligned} \tag{23}$$

As for the third term in (21), using $U_{\varepsilon,y}^{p-1} - PU_{\varepsilon,y}^{p-1} \leqslant cU_{\varepsilon,y}^{p-2}\psi_{\varepsilon,y}$, $\frac{\partial U_{\varepsilon,y}}{\partial y_i} = O\left(\frac{U_{\varepsilon,y}}{\varepsilon}\right)$ and Lemmas A.2 and A.1, we have that

$$\int_\Omega PU_{\varepsilon,y}^{p-1}\frac{\partial PU_{\varepsilon,y}}{\partial y_i} w = \int_\Omega U_{\varepsilon,y}^{p-1}\frac{\partial U_{\varepsilon,y}}{\partial y_i} w + O(\varepsilon^{\frac{N-2}{2}}||w||) = O(\varepsilon^{\frac{N-2}{2}}||w||), \tag{24}$$

because $p\int_\Omega U_{\varepsilon,y}^{p-1}\frac{\partial U_{\varepsilon,y}}{\partial y_i}w = \langle \frac{\partial PU_{\varepsilon,y}}{\partial y_i}, w\rangle = 0$. Finally, using $U_{\varepsilon,y}^{p-2} - PU_{\varepsilon,y}^{p-2} \leqslant cU_{\varepsilon,y}^{p-3}\psi_{\varepsilon,y}$, $\frac{\partial U_{\varepsilon,y}}{\partial y_i} = O\left(\frac{U_{\varepsilon,y}}{\varepsilon}\right)$ and recalling also (see Lemma A.1) $\left(\int_\Omega U_{\varepsilon,y}^{\frac{N(6-N)}{2(N-2)}}\right)^{\frac{2}{N}} = O\left(\varepsilon^{\frac{6-N}{2}}\log\frac{1}{\varepsilon}\right)$, $\left(\int_\Omega U_{\varepsilon,y}^{\frac{N(4-N)}{N-2}}\right)^{\frac{2}{N}} = O(\varepsilon^{4-N})$, we estimate h.o.t. in case $N\geqslant 6$:

$$\begin{aligned}
\text{h.o.t} &= O\left(\delta\int_\Omega U_{\varepsilon,y}^{p-1}\left|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right||w| + \int_\Omega PU_{\varepsilon,y}^{p-2}\left|\frac{\partial U_{\varepsilon,y}}{\partial y_i} - \frac{\partial \psi_{\varepsilon,y}}{\partial y_i}\right|w^2\right)\\
&= O\left(\frac{\delta}{\varepsilon}||w|| + \int_\Omega\left(\frac{U_{\varepsilon,y}^{p-1}}{\varepsilon} + \varepsilon^{\frac{N-4}{2}}U_{\varepsilon,y}^{p-2} + \varepsilon^{N-2}U_{\varepsilon,y}^{p-3}\right)w^2\right)\\
&= O\left(\frac{\delta}{\varepsilon}||w|| + \frac{||w||^2}{\varepsilon}\right).
\end{aligned} \tag{25}$$

In case $N = 5$, we estimate the additional term using Lemma A.4:

$$\int_\Omega \left|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right| |w|^p \leqslant c\left(\frac{||w||^p}{\varepsilon}\right). \tag{26}$$

Estimates (22), (23) and (25)–(26) yield (20) and the claim is proved. □

As for the ε-derivative, we can argue in a similar way:

- Eq. (18) is replaced (see Lemma A.5) by

$$\left\langle PU_{\varepsilon,y}, \frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon}\right\rangle = -\frac{N-2}{2} DH(y)\varepsilon^{N-3} + O(\varepsilon^{N-2}) \tag{27}$$

- Eq. (19) remains unchanged (see Lemma A.4)
- Eq. (20) is replaced by

$$\begin{aligned}\int_\Omega &(1+\delta a(x))|\alpha PU_{\varepsilon,y}+w|^{p-1}(\alpha PU_{\varepsilon,y}+w)\frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon}\\ &= -(N-2)D\alpha^p H(y)\varepsilon^{N-3} + \alpha^p \frac{S^{\frac{N}{2}}}{2N}\delta\varepsilon\Delta a(y)\\ &\quad + O\left(\varepsilon^{N-2} + \delta\varepsilon^2 + \frac{||w||^2}{\varepsilon} + \varepsilon^{\frac{N-2}{2}}||w|| + \frac{\delta}{\varepsilon}||w||\right).\end{aligned} \tag{28}$$

Putting together (27), (19), (28) and using (14), we obtain (16).

Estimate (28) can be obtained as in (20):
Eq. (22) is replaced (see Lemma A.5) by

$$\int_\Omega PU^p_{\varepsilon,y}\frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon} = -(N-2)DH(y)\varepsilon^{N-3} + O(\varepsilon^{N-2}); \tag{29}$$

Eq. (23) is replaced by

$$\begin{aligned}\int_\Omega a(x)PU^p_{\varepsilon,y}\frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon} &= \int_\Omega\Bigg[a(y) + \sum_j \partial_j a(y)(x-y)_j\\ &\quad + \frac{1}{2}\sum_{i,j}\partial_{ij}a(y)(x-y)_i(x-y)_j + O(|x-y|^3)\Bigg]U^p_{\varepsilon,y}\frac{\partial U_{\varepsilon,y}}{\partial \varepsilon} + O(\varepsilon^{N-3})\\ &= -\frac{N-2}{4N}\Delta a(y)c_N^{p+1}\varepsilon\int_{\mathbf{R}^N}\frac{|x|^2(1-|x|^2)}{(1+|x|^2)^{N+1}}dx + O(\varepsilon^2)\\ &= \frac{1}{2N}S^{\frac{N}{2}}\varepsilon\Delta a(y) + O(\varepsilon^2);\end{aligned} \tag{30}$$

Eq. (24) is replaced by

$$\int_\Omega PU_{\varepsilon,y}^{p-1}\frac{\partial PU_{\varepsilon,y}}{\partial\varepsilon}w=\int_\Omega U_{\varepsilon,y}^{p-1}\frac{\partial U_{\varepsilon,y}}{\partial\varepsilon}w+O(\varepsilon^{\frac{N-2}{2}}||w||)=O(\varepsilon^{\frac{N-2}{2}}||w||);\quad (31)$$

as for the h.o.t., (25) and (26) become, respectively,

$$\begin{aligned}\delta\int_\Omega U_{\varepsilon,y}^{p-1}\left|\frac{\partial PU_{\varepsilon,y}}{\partial\varepsilon}\right||w|&+\int_\Omega PU_{\varepsilon,y}^{p-2}\left|\frac{\partial U_{\varepsilon,y}}{\partial\varepsilon}-\frac{\partial\psi_{\varepsilon,y}}{\partial\varepsilon}\right|w^2\\&=O\left(\frac{\delta}{\varepsilon}||w||+\int_\Omega\left(\frac{U_{\varepsilon,y}^{p-1}}{\varepsilon}+\varepsilon^{\frac{N-4}{2}}U_{\varepsilon,y}^{p-2}+\varepsilon^{N-3}U_{\varepsilon,y}^{p-3}\right)w^2\right)\\&=O\left(\frac{\delta}{\varepsilon}||w||+\frac{||w||^2}{\varepsilon}\right)\end{aligned}\quad (32)$$

and

$$\int_\Omega\left|\frac{\partial PU_{\varepsilon,y}}{\partial\varepsilon}\right||w|^p\leqslant c\left(\frac{||w||^p}{\varepsilon}\right).\quad (33)$$

As for the α-derivative, we can argue in a similar but more direct way. Using Lemma A.5, it is easy to see that

$$\begin{aligned}\frac{\partial}{\partial\alpha}E_\delta(\alpha PU_{\varepsilon,y}+w)&=\left\langle\nabla E_\delta(\alpha PU_{\varepsilon,y}+w),PU_{\varepsilon,y}+\pi_z\frac{\partial w}{\partial\alpha}\right\rangle\\&=\alpha||PU_{\varepsilon,y}||^2-\int_\Omega(1+\delta a(x))|\alpha PU_{\varepsilon,y}+w|^{p-1}(\alpha PU_{\varepsilon,y}+w)PU_{\varepsilon,y}\\&\quad+O(||\nabla E_\delta(z)||\,||w||)\\&=\alpha||PU_{\varepsilon,y}||^2-\alpha^p\int_\Omega PU_{\varepsilon,y}^{p+1}-\delta a(y)\alpha^p\int_\Omega PU_{\varepsilon,y}^{p+1}+O(\delta\varepsilon+||w||)\\&=S^{\frac{N}{2}}(\alpha-\alpha^p)-\delta a(y)\alpha^pS^{\frac{N}{2}}+O(\delta\varepsilon+\varepsilon^{\frac{N}{2}}),\end{aligned}$$

because $||\pi_z^\perp\frac{\partial w}{\partial\alpha}||=O(||w||\,||PU_{\varepsilon,y}||)=O(||w||)$.

Remark 4.5. In the expansion of the ε-derivative (16), we have a remainder term $O(\delta^2|\nabla a(y)|)$. The presence of $|\nabla a(y)|$ is needed only for $N=5$. In fact, in this case we will require $\delta\sim\varepsilon$: then δ^2 is not small with respect to the second leading term in the ε-derivative which is of order $\delta\varepsilon$.

Proof of Theorem 1.2. Choose $\frac{N-6}{2(N-4)}<s<\frac{N}{2(N-4)}<1$ if $N>8$ and $s=1$ if $5\leqslant N\leqslant 8$. Introducing new variables $\theta=\delta^{-\frac{1}{N-4}}\varepsilon$, $\upsilon=\delta^{-s}(\alpha-1)$, we are led to look for zeroes for the vector field

$$\Phi(\upsilon,\theta,y) = (\Upsilon_\delta, \Theta_\delta, Y_\delta)(\upsilon,\theta,y),$$

where

$$\begin{aligned}
\Upsilon_\delta(\upsilon,\theta,y) &= \upsilon + \frac{a(y)}{p-1}\delta^{1-s} + o(1),\\
\Theta_\delta(\upsilon,\theta,y) &= DN(N-2)H(y)\theta^{N-4} - S^{\frac{N}{2}}\Delta a(y) + o(1) + O(\delta^{\frac{N-5}{N-4}}\theta^{-1}|\nabla a(y)|),\\
Y_\delta(\upsilon,\theta,y) &= \nabla a(y) + o(1),
\end{aligned}$$

where $o(\cdot)$, $O(\cdot)$ hold for $\delta \to 0$ uniformly in y and θ, υ bounded.

Now let y_0 be an (interior) isolated critical point of $a(x)$ with $\Delta a(y_0) > 0$ and non-zero topological index. Then $\theta_0 := \left(\frac{S^{\frac{N}{2}}\Delta a(y_0)}{N(N-2)DH(y_0)}\right)^{\frac{1}{N-4}}$ is well defined and positive. Let us set

$$\upsilon_0 = \begin{cases} -\frac{a(y_0)}{p-1} & \text{if } 5 \leqslant N \leqslant 8,\\ 0 & \text{if } N > 8,\end{cases} \quad \text{and} \quad n = \begin{cases} 1 & \text{if } 5 \leqslant N \leqslant 8,\\ 0 & \text{if } N > 8.\end{cases}$$

We define the homotopy $\Phi(t;\upsilon,\theta,y)$ by components as

$$\begin{aligned}
\Phi_1 &= \upsilon + n\frac{a(y_0)}{p-1} + t\left(\Upsilon_\delta(\upsilon,\theta,y) - \upsilon - n\frac{a(y_0)}{p-1}\right),\\
\Phi_2 &= DN(N-2)H(y_0)\theta^{N-4} - S^{\frac{N}{2}}\Delta a(y_0) + t(\Theta_\delta(\upsilon,\theta,y)\\
&\quad - DN(N-2)H(y)\theta^{N-4} - S^{\frac{N}{2}}\Delta a(y)),\\
\Phi_3 &= \nabla a(y) + t(Y_\delta(\upsilon,\theta,y) - \nabla a(y)).
\end{aligned}$$

Since $|\nabla a(y)| = O(|y-y_0|)$, working on the first two components, it is possible to find $r > 0$ such that for δ small

$$|\Phi(t;\upsilon_0 - 1,\theta,y)| + |\Phi(t;\upsilon_0 + 1,\theta,y)| + |\Phi(t;\upsilon,\tfrac{1}{2}\theta_0,y)| + |\Phi(t;\upsilon,\tfrac{3}{2}\theta_0,y)| > 0$$

for $t \in [0,1]$, $\upsilon \in [\upsilon_0 - 1, \upsilon_0 + 1]$, $\theta \in [\frac{1}{2}\theta_0, \frac{3}{2}\theta_0]$ and $y \in B_r(y_0)$. We fix such $r > 0$. Since $\inf_{y \in \partial B_r(y_0)} |\nabla a(y)| > 0$ by the third component, we have that for δ small

$$\inf_{y \in \partial B_r(y_0)} |\Phi(t;\upsilon,\theta,y)| > 0 \quad \forall t \in [0,1],\ \upsilon \in [\upsilon_0 - 1, \upsilon_0 + 1],\ \theta \in \left[\tfrac{1}{2}\theta_0, \tfrac{3}{2}\theta_0\right].$$

So, for homotopic invariance, we can conclude that

$$deg(\Phi(\upsilon,\theta,y), [\upsilon_0 - 1, \upsilon_0 + 1] \times \left[\tfrac{1}{2}\theta_0, \tfrac{3}{2}\theta_0\right] \times B_r(y_0), 0) \neq 0,$$

because

$$deg(\Phi(0;\upsilon,\theta,y),[\upsilon_0-1,\upsilon_0+1]\times\left[\tfrac{1}{2}\theta_0,\tfrac{3}{2}\theta_0\right]\times B_r(y_0),0)=-deg(\nabla a,B_r(y_0),0).$$

So we find a free critical point $u_\delta=\alpha PU_{\varepsilon,y}+w(\delta,\alpha,\varepsilon,y)$ of E_δ and we want to show that it is a positive function. Since for u_δ there holds

$$-\Delta u_\delta=(1+\delta a(x))|u_\delta|^{p-1}u_\delta,$$

if we multiply and integrate for $-u_\delta^-=-\max(-u_\delta,0)$, we obtain

$$\int_\Omega|\nabla u_\delta^-|^2=\int_\Omega(1+\delta a(x))(u_\delta^-)^{p+1}.$$

From the Sobolev embedding theorem and the above inequality, we get

$$S\left(\int_\Omega(u_\delta^-)^{p+1}\right)^{\frac{2}{p+1}}\leqslant C\int_\Omega(u_\delta^-)^{p+1}. \tag{34}$$

Let us remark that, since $PU_{\varepsilon,y}>0$, we have $u_\delta^-\leqslant|w(\delta,\alpha,\varepsilon,y)|$. If, by contradiction, $u_\delta^-\neq0$ for δ small, we can simplify in (34) to obtain

$$S\leqslant C\left(\int_\Omega(u_\delta^-)^{p+1}\right)^{\frac{p-1}{p+1}}\leqslant C_1(\|w(\delta,\alpha,\varepsilon,y)\|^{p-1})\to_{\delta\to0}0.$$

Then, for δ small, $u_\delta\geqslant0$ and, by maximum principle, $u_\delta>0$. This completes the proof of Theorem 1.2. □

Because of geometric significance, (PSCE) has been widely studied in case $\Omega=\mathbf{R}^N$ (see [4,7,8,19,20,22,29–32]). Regarding the problem on the whole space as a limiting problem, we will study now (PSCE) on large balls B_R. Of course, the bifurcation result stated in Theorem 1.2 holds true. However, a more careful analysis brings to evidence a (possible) decay, as R goes to infinity, of the size of the perturbation insuring existence (and non-degeneracy) of bifurcating solutions.

From now on, $\Omega=B_R$. For simplicity, we perform the finite-dimensional reduction and compute the "Melnikov function" with respect to

$$Z:=\{P_RU_{\varepsilon,y}:\varepsilon>0,\ |y|<r\},$$

where $P_R:D^{1,2}(\mathbf{R}^N)\to H_0^1(B_R)$ is the orthogonal projection. It is easy to see (see Lemma A.6) that Z is a "non-degenerate almost critical manifold", in the sense that there holds

(A1)′ Z is bounded and $\sup_{|y|<r}\|\nabla E(P_RU_{\varepsilon,y})\|=o(1)$ as $\frac{\varepsilon}{R}\to0$,

(A2)′ there exists $\varepsilon_0>0$ such that $L_z:=\pi_z^\perp E''(z)|_{T_z^\perp}\in Iso(T_z^\perp,T_z^\perp)$ $\forall z\in Z_{\varepsilon_0}$ and $\sup_{z\in Z_{\varepsilon_0}}\|L_z^{-1}\|<\infty$, where $Z_{\varepsilon_0}:=\{P_RU_{\varepsilon,y}:|y|<r,\ 0<\varepsilon<\varepsilon_0R\}$.

From now on, we assume $a\in C_b^3(\mathbf{R}^N)$, $\Omega = B_R$, $R\gg 1$ and $Crit\, a := \{x\in\mathbf{R}^N:\ \nabla a(x) = 0\}\subset B_r$. The finite-dimensional reduction can be performed, with a bound $\bar\delta$ on the size of the perturbation independent on R. Similar computations as above can be carried over to obtain the estimate

$$||\nabla E_\delta(PU_{\varepsilon,y})||^2 = O(\delta^2) + O\left(\left(\frac{\varepsilon}{R}\right)^{N+2}\right),$$

as well as the following expansions for the functional E_δ and its derivatives:

$$E_\delta(PU_{\varepsilon,y}+w) = \frac{1}{N}S^{\frac{N}{2}} - \frac{S^{\frac{N}{2}}}{p+1}a(y)\delta + \frac{Dd_N}{2(1-R^{-2}|y|^2)^{N-2}}\left(\frac{\varepsilon}{R}\right)^{N-2} - \frac{S^{\frac{N}{2}}}{4N}\Delta a(y)\delta\varepsilon^2 + O\left(\left(\frac{\varepsilon}{R}\right)^{N-1} + \delta\varepsilon^3 + \delta^2\right), \tag{35}$$

$$\frac{\partial}{\partial\varepsilon}E_\delta(PU_{\varepsilon,y}+w) = (N-2)\frac{Dd_N}{2(1-R^{-2}|y|^2)^{N-2}}\frac{\varepsilon^{N-3}}{R^{N-2}} - \frac{S^{\frac{N}{2}}}{2N}\Delta a(y)\delta\varepsilon + O\left(\frac{\varepsilon^{N-2}}{R^{N-1}} + \delta\varepsilon^2 + \frac{\delta^2}{\varepsilon}\right), \tag{36}$$

$$\frac{\partial}{\partial y}E_\delta(PU_{\varepsilon,y}+w) = -\frac{S^{\frac{N}{2}}}{p+1}\nabla a(y)\delta + O\left(\frac{\varepsilon^{N-2}}{R^{N-1}} + \delta\varepsilon^2 + \frac{\delta^2}{\varepsilon}\right), \tag{37}$$

where $d_N = \frac{1}{N(N-2)\omega_N}$, D as above, $w = w(\delta, R, \varepsilon, y)$ as in Lemma 3.1.

Now, after setting $\theta = \tau^{-\frac{1}{N-4}}\varepsilon$, $\tau := \delta R^{N-2}$, we are led to look for critical points of

$$M_{\tau,R}(\theta,y) = \frac{S^{\frac{N}{2}}}{p+1}a(y) - \frac{\tau^{\frac{2}{N-4}}}{4N}\left[2N\frac{Dd_N}{(1-R^{-2}|y|^2)^{N-2}}\theta^{N-2} - S^{\frac{N}{2}}\Delta a(y)\theta^2\right] + \tau^{\frac{2}{N-4}}o(1),$$

where $||o(1)||_{C^1}\to 0$ on compact subsets of $\mathbf{R}^+\times B_r$ as $\tau\to 0$. As above, isolated critical points of a with $\Delta a>0$ and non-zero topological index generate critical points of $M_{\tau,R}(\theta,y)$, provided $\tau\ll 1$. Hence, we get

Theorem 4.6. *Let $N\geqslant 6$ and a as above. Then there exist δ_0 small and R_0 such that, for any $R\geqslant R_0$ and $\delta\leqslant\frac{\delta_0}{R^{N-2}}$, problem* (PSCE) *on B_R with $f = 1+\delta a$ has at least as many positive solutions as the number of non-degenerate critical points of a with positive laplacian.*

Remark 4.7. The analysis in Theorem 4.6 is less accurate than in Theorem 1.2 because of the different choice of Z. So we lose dimension $N = 5$.

Because of the decay $\delta \ll R^{2-N}$, we cannot obtain solutions on the whole space as limits of our bifurcating solutions: for this, we need solutions on large balls and uniform size of the perturbation. We first observe that our bifurcation result relies on the rather weak assumption "a has non-degenerate critical points with positive laplacian". Such an assumption should be compared with the much stronger "counting condition"

$$\sum_{\{x\,:\,\nabla a(x)=0,\ \Delta a(x)>0\}} i(\nabla a, x) \neq 0$$

discovered by Bahri and Coron, see Ref. [10], in their investigation of (PSCE) on the 3 sphere (see also [20]). A very nice interpretation of the "counting condition" is given, in term of degree theoretic arguments, in [4] (see also [24,31] for a Morse theory point of view).

We will show below that, while the bifurcating solutions might, for R larger and larger, degenerate and cancel each other for δ smaller and smaller, the counting condition enters as an obstruction to a complete collapse of these solutions, insuring, via a continuation argument based on suitable a priori bounds, existence on large balls B_R up to some $\bar{\delta}$ independent on R. As noticed above, there is $\bar{\delta}$ such that, for any given $\rho > 0$, the reduced functional $E_\delta(\varepsilon, y) \coloneqq E_\delta(z(\varepsilon, y) + w(\delta, R, \varepsilon, y))$ is defined on $D^+_\rho = \{(\varepsilon, y) : \varepsilon^2 + |y|^2 < \rho^2, \varepsilon > 0\}$ for $\delta \leqslant \bar{\delta}$ and $R \geqslant \bar{R} = \bar{R}(\rho)$. We will assume, from now on,

$$D^2 a(x) \in Gl_N(\mathbf{R}) \text{ and } \Delta a(x) \neq 0 \quad \text{for any } x \in Crit\, a. \tag{38}$$

Let y_j, $j = 1, \dots, l$ be the critical points of a with positive laplacian. The homotopy argument used in the proof of Theorem 4.6 gives, for R given and $\delta \leqslant \delta(R)$, the existence of open neighbourhoods $V_j = (\underline{\theta}_j, \bar{\theta}_j) \times U_j$ of $(\theta(y_j), y_j)$, where $\theta(y_j) = \theta(y_j, R) = \left(\frac{S^{\frac{N}{2}} \Delta a(y_j)(1 - R^{-2}|y_j|^2)^{N-2}}{N(N-2)Dd_N} \right)^{\frac{1}{N-4}}$ and U_j are small neighbourhoods of y_j with $||\nabla a|| > 0$ on ∂U_j, such that

$$deg(\nabla M_{\tau,R}, V_j, 0) = -deg(\nabla a, U_j, 0) = -i(\nabla a, y_j).$$

From Section 2, for $\delta \leqslant \delta(R)$ the critical points of $M_{\tau,R}$ in the V_j are in one-to-one correspondence with the critical points of $E_\delta(\varepsilon, y)$ in $D^+_{\rho,\delta} \coloneqq D^+_\rho \cap \{\varepsilon > \underline{\theta}\delta^{\frac{1}{N-4}}\}$, $\underline{\theta} = \min_j \{\underline{\theta}_j\}$, $\rho > 2r$, through the map $(\theta, y) \to ((\delta R^{N-2})^{\frac{1}{N-4}}\theta, y)$.

This readily implies

Lemma 4.8. *There is $\bar{R}$ and, for any $R \geqslant \bar{R}$, there is $\delta = \delta(R)$, such that*

$$deg(-\nabla E_\delta(\varepsilon, y), D^+_{\rho,\delta}, 0) = -\sum_{j=1}^{l} i(\nabla a, x_j) \quad \forall \rho > 2r.$$

To continue this degree estimate up to some $\bar{\delta}$ independent on R, we need suitable a priori bounds. First, we have

Claim 1. *There is some $\bar{R}$ such that, if $\delta \leqslant \bar{\delta}$ and $R \geqslant \bar{R}$, then $E_\delta(\varepsilon, y)$ has no critical points on $D^+_\rho \cap \{\varepsilon = \varepsilon_\delta\}$, $\varepsilon_\delta = \underline{\theta}\delta^{\frac{1}{N-4}}$.*

To have complete a priori bounds we will assume, following [4],

$$\exists \rho' > 0 : \langle \nabla a(x), x \rangle < 0 \quad \forall |x| > \rho',$$
$$\langle \nabla a(x), x \rangle \in L^1(\mathbf{R}^N), \quad \int_{\mathbf{R}^N} \langle \nabla a(x), x) \rangle < 0. \tag{39}$$

Claim 2. *If* (39) *holds, there is some $\bar{\delta}$ such that $E_\delta(\varepsilon, y)$ has no critical points on $\{\varepsilon^2 + |y|^2 = \rho^2, \ \varepsilon > \varepsilon_\delta\}$, for some $\rho > \max\{\rho', 2r\}$ and $\delta \leqslant \bar{\delta}$.*

By the above claims, we deduce that, for some $\bar{R}$ large and $\bar{\delta}$ small

$$deg(-\nabla E_\delta(\varepsilon, y), D^+_\rho \cap \{\varepsilon > \varepsilon_\delta\}, 0) = -\sum_{j=1}^{l} i(\nabla a, x_j) \quad \forall \delta \leqslant \bar{\delta}, \ \ R \geqslant \bar{R}$$

for some $\rho > 2r$ fixed. Hence, we have

Theorem 4.9. *Let $N \geqslant 6$, $a \in C^3_b(\mathbf{R}^N)$, $Crit\, a \subset B_r$, a satisfying* (38)–(39). *Assume in addition*

$$\sum_{\{x\,:\,\nabla a(x),\ \Delta a(x) > 0\}} i(\nabla a, x) \neq 0. \tag{40}$$

Then problem (PSCE) *on $\Omega = B_R$ with $f = 1 + \delta a$ has a solution for $\delta \leqslant \bar{\delta}$ and $R \geqslant \bar{R}$, $\bar{\delta}$ independent on R.*

Proof. We have just to prove the claims. As for Claim 1, it follows from assumption (38) and expansions (36)–(37) of the derivatives of E_δ

on $D^+_\rho \cap \{\varepsilon = \varepsilon_\delta\}$

$$\nabla_\varepsilon E_\delta(PU_{\varepsilon,y} + w) = -\frac{S^{\frac{N}{2}}}{2N}\Delta a(y)\delta^{\frac{N-3}{N-4}} + o(\delta^{\frac{N-3}{N-4}}),$$

$$\nabla_y E_\delta(PU_{\varepsilon,y} + w) = -\frac{S^{\frac{N}{2}}}{p+1}\nabla a(y)\delta + o(\delta).$$

Finally, we prove Claim 2. From (39), we can show that there exists $\rho \geqslant \rho'$ such that

$$\langle \nabla_{(\varepsilon,y)}\Gamma(\varepsilon, y), (\varepsilon, y)\rangle < 0 \quad \text{if } \varepsilon^2 + |y|^2 \geqslant \rho^2, \tag{41}$$

where Γ is $\int_{\mathbf{R}^N} aU^{p+1}_{\varepsilon,y} = \int_{\mathbf{R}^N} a(\varepsilon x + y)U^{p+1}_{1,0}$, extended as an even function in ε. Now, using previous computations, we get

$$\begin{aligned} E_\delta(PU_{\varepsilon,y} + w) = &\frac{1}{N}S^{\frac{N}{2}} + \frac{Dd_N}{2(1 - R^{-2}|y|^2)^{N-2}}\left(\frac{\varepsilon}{R}\right)^{N-2} \\ &- \frac{\delta}{p+1}\Gamma(\varepsilon, y) + O\left(\left(\frac{\varepsilon}{R}\right)^{N-1} + \delta\left(\frac{\varepsilon}{R}\right)^{N-2} + \delta^2\right), \end{aligned}$$

$$\begin{aligned} \nabla_{(\varepsilon,y)}E_\delta(PU_{\varepsilon,y} + w) = &\left(\frac{N-2}{2}Dd_N\frac{\varepsilon^{N-3}}{R^{N-2}}, (N-2)Dd_N\frac{\varepsilon^{N-2}}{\mathbf{R}^N}y\right)(1 + o(1)) \\ &- \frac{\delta}{p+1}\nabla_{(\varepsilon,y)}\Gamma(\varepsilon, y) + O\left(\delta\frac{\varepsilon^{N-3}}{R^{N-2}} + \frac{\delta^2}{\varepsilon}\right). \end{aligned}$$

In view of (41) and the positivity of the term

$$\left\langle\left(\frac{N-2}{2}Dd_N\frac{\varepsilon^{N-3}}{R^{N-2}}, (N-2)Dd_N\frac{\varepsilon^{N-2}}{\mathbf{R}^N}y\right), (\varepsilon, y)\right\rangle,$$

we get that, for $\delta \ll 1$ and $R \gg 1$, on $\{\varepsilon^2 + |y|^2 = \rho^2\} \cap \{\varepsilon > \varepsilon_\delta\}$ there holds $\langle -\nabla_{(\varepsilon,y)}E_\delta, (\varepsilon, y)\rangle < 0$.

Final remark. A different situation occurs if we assume in (39) the reverse inequality. First, we observe that to compute $deg(-\nabla_{(\varepsilon,y)}E_\delta, D^+_{\rho,\delta}, 0)$, we can also proceed as follows. From

$$\nabla_{(\varepsilon,y)}E_\delta = -\frac{\delta}{p+1}\nabla_{(\varepsilon,y)}\Gamma + O\left(\frac{\varepsilon^{N-3}}{R^{N-2}} + \frac{\delta^2}{\varepsilon}\right),$$

we see that for $\frac{M_1}{R^{N-2}} \leqslant \delta \leqslant \bar{\delta}$, M_1 a large constant,

$$deg(-\nabla_{(\varepsilon,y)}E_\delta, D^+_{\rho,\delta}, 0) = deg(\nabla_{(\varepsilon,y)}\Gamma, D^+_{\rho,\delta}, 0),$$

whenever the r.h.s. is defined. This is the case if (39) holds, as well as if the reverse inequality is satisfied therein. Since, as can be easily seen,

$$\Gamma(0,y) = S^{\frac{N}{2}}a(y), \quad \frac{\partial \Gamma}{\partial \varepsilon}(0,y) = 0, \quad \frac{\partial^2 \Gamma}{\partial \varepsilon^2}(0,y) = C\Delta a(y)$$

for some positive constant C, we have, denoted $D_\rho \coloneqq \{\varepsilon^2 + |y|^2 < \rho^2\}$,

$$\begin{aligned} deg(\nabla_{(\varepsilon,y)}\Gamma, D_{\rho,\delta}, 0) &= 2\, deg(\nabla_{(\varepsilon,y)}\Gamma, D^+_{\rho,\delta}, 0) + deg(\nabla_{(\varepsilon,y)}\Gamma, D_\rho \cap \{|\varepsilon| < \varepsilon_\delta\}, 0) \\ &= 2\, deg(\nabla_{(\varepsilon,y)}\Gamma, D^+_{\rho,\delta}, 0) + \sum_{\{x\,:\,\nabla a(x)=0,\ \Delta a(x)>0\}} i(\nabla a, x) \\ &\quad - \sum_{\{x\,:\,\nabla a(x)=0,\ \Delta a(x)<0\}} i(\nabla a, x). \end{aligned}$$

If the reverse inequality holds true in (39), we get the reverse inequality in (41), and then

$$\sum_{\{x\,:\,\nabla a(x)=0,\ \Delta a(x)>0\}} i(\nabla a, x) + \sum_{\{x\,:\,\nabla a(x)=0,\ \Delta a(x)<0\}} i(\nabla a, x) = 1 = deg(\nabla_{(\varepsilon,y)}\Gamma, D_{\rho,\delta}, 0).$$

Henceforth, for $R^{2-N} \ll \delta \leqslant \bar{\delta}$,

$$deg(-\nabla_{(\varepsilon,y)}E_\delta, D^+_{\rho,\delta}, 0) = deg(\nabla_{(\varepsilon,y)}\Gamma, D^+_{\rho,\delta}, 0) = \sum_{\{x\,:\,\nabla a(x)=0,\ \Delta a(x)<0\}} i(\nabla a, x).$$

On the other hand, Claims 1 and 2 still hold true and so we conclude that

$$\begin{aligned} deg(-\nabla_{(\varepsilon,y)}E_\delta, D^+_{\rho,\delta}, 0)|_{\delta \ll R^{2-N}} &= - \sum_{\{x\,:\,\nabla a(x)=0,\ \Delta a(x)>0\}} i(\nabla a, x) \\ &\neq \sum_{\{x\,:\,\nabla a(x)=0,\ \Delta a(x)<0\}} i(\nabla a, x) \\ &= deg(-\nabla_{(\varepsilon,y)}E_\delta, D^+_{\rho,\delta}, 0)|_{\delta \gg R^{2-N}}. \end{aligned}$$

In particular, no a priori bounds are available in this case.

5. Further applications of the reduction principle

We consider a generalization of [35]: given $a(x)$ a smooth function in $\bar{\Omega}$, $\delta > 0$ a small parameter, $1 \leqslant q < \frac{N+2}{N-2}$ and $N \geqslant 3$, find $u > 0$ such that

$$(\mathrm{P})_\delta \quad \begin{cases} -\Delta u = u^{\frac{N+2}{N-2}} + \delta a(x) u^q & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

In this case, the unperturbed functional is $E(u)$ and the finite-dimensional reduction is performed with respect to the "non-degenerate almost critical manifold"

$$Z := \{PU_{\varepsilon,y} : \varepsilon > 0,\ y \in \Omega,\ dist(y, \partial\Omega) > \gamma\}, \quad \gamma > 0,$$

in the sense that there holds

(A1)$''$ Z is bounded and $\sup_{y\in\Omega,\, dist(y,\partial\Omega)>\gamma} ||\nabla E(PU_{\varepsilon,y})|| = o(1)$ as $\varepsilon \to 0$,

(A2)$''$ there exists $\varepsilon_0 > 0$ such that $L_z := \pi_z^{\perp} E''(z)|_{T_z^{\perp}} \in Iso(T_z^{\perp}, T_z^{\perp})$ $\forall z \in Z_{\varepsilon_0}$ and $\sup_{z\in Z_{\varepsilon_0}} ||L_z^{-1}|| < \infty$,

where $Z_{\varepsilon_0} := \{PU_{\varepsilon,y} : 0 < \varepsilon < \varepsilon_0,\ y \in \Omega,\ dist(y, \partial\Omega) > \gamma\}$ (see Lemma A.6). The perturbation is

$$G(\delta, u) = \frac{\delta}{q+1} \int_\Omega a|u|^{q+1}.$$

Using Lemmas A.1 and A.2, one can get the following estimate for the remainder term:

$$||\nabla E_\delta(PU_{\varepsilon,y})||^2 = O(\varepsilon^{N-1} + \delta^2 \Lambda^2),$$

where

$$\Lambda = \begin{cases} \varepsilon^{\frac{N+2}{2} - \frac{N-2}{2}q} & \text{if } q > \frac{N+2}{2(N-2)}, \\ \varepsilon^{\frac{N+2}{4}} (\log \frac{1}{\varepsilon})^{\frac{N+2}{2N}} & \text{if } q = \frac{N+2}{2(N-2)}, \\ \varepsilon^{\frac{N-2}{2}q} & \text{if } q < \frac{N+2}{2(N-2)}. \end{cases}$$

As for the "Melnikov function" (see Remark 3.4), if $q > \frac{2}{N-2}$, one gets

$$\begin{aligned} E_\delta(PU_{\varepsilon,y}) &= E(PU_{\varepsilon,y}) - \frac{\delta}{q+1} \int_\Omega a PU_{\varepsilon,y}^{q+1} \\ &= \frac{1}{N} S^{\frac{N}{2}} + \frac{D}{2} H(y) \varepsilon^{N-2} - \frac{F c_N^{q+1}}{q+1} a(y) \delta \varepsilon^{N - \frac{N-2}{2}(q+1)} \\ &\quad + O(\varepsilon^{N-1}) + o(\delta \varepsilon^{N-\frac{N-2}{2}(q+1)}), \end{aligned} \tag{42}$$

where $F = \int_{\mathbf{R}^N} \frac{dx}{(1+|x|^2)^{\frac{(N-2)(q+1)}{2}}}$ and the expansion of $E_\delta(PU_{\varepsilon,y} + w)$ follows by

$$E_\delta(PU_{\varepsilon,y} + w) = E(PU_{\varepsilon,y}) + O(||\nabla E_\delta(PU_{\varepsilon,y})||^2),$$

where $w = w(\delta, \varepsilon, y)$ is defined as in Lemma 3.1. After setting $\theta = \delta^{-\frac{2}{(N-2)(q+1)-4}} \varepsilon$, if $q > \max\{\frac{2}{N-2}, \frac{6-N}{N-2}\}$, the expansion of E_δ becomes

$$E_\delta(PU_{\varepsilon,y} + w) = \frac{1}{N} S^{\frac{N}{2}} + \delta^{\frac{2(N-2)}{(N-2)(q+1)-4}} \left[\frac{D}{2} H(y) \theta^{N-2} - \frac{F c_N^{q+1}}{q+1} a(y) \theta^{N-\frac{N-2}{2}(q+1)} + o(1) \right],$$

where $o(1)\to 0$ as $\delta\to 0$ in C^0 norm for θ bounded and bounded away from zero. So we are led to study the "stable" critical points of

$$M(\theta,y)=DH(y)\theta^{N-2}-\frac{2}{q+1}c_N^{q+1}Fa(y)\theta^{N-\frac{(N-2)(q+1)}{2}}\theta>0,\quad y\in\Omega,$$

where F, D and c_N are as above. Since

$$\frac{\partial M}{\partial\theta}=0\ \Leftrightarrow\ \begin{cases}\theta=\theta(y):=\left(\frac{[2N-(N-2)(q+1)]c_N^{q+1}Fa(y)}{(N-2)(q+1)DH(y)}\right)^{\frac{2}{(N-2)(q+1)-4}}\\ a(y)>0\end{cases}$$

and

$$M(\theta(y),y))=D_{N,q}\left(\frac{a(y)^2}{H(y)^{\frac{2N-(N-2)(q+1)}{(N-2)}}}\right)^{\frac{(N-2)}{(N-2)(q+1)-4}},$$

$$D_{N,q}=-\frac{(N-2)(q+1)-4}{N-2}\left(\frac{2N-(N-2)(q+1)}{D(N-2)}\right)^{\frac{2N-(N-2)(q+1)}{(N-2)(q+1)-4}}\left(\frac{Fc_N^{q+1}}{q+1}\right)^{\frac{2(N-2)}{(N-2)(q+1)-4}},$$

we can introduce

$$K(y):=\frac{a(y)^2}{H(y)^{\frac{2N-(N-2)(q+1)}{(N-2)}}},\quad y\in\Omega$$

and the following result follows:

Theorem 5.1. *Let M, K be given as above and let (θ_j,y_j) be critical points of M. Let $1\leqslant q<\frac{N+2}{N-2}$ if $N\geqslant 5$, $1<q<3$ if $N=4$, $3<q<5$ if $N=3$.*

(i) *If (θ_j,y_j) are C^0-stable, then there are C_j disjoint compact neighbourhoods of (θ_j,y_j) and, for $\delta>0$ small, there are $u_{\delta,j}$, solutions of $(\mathrm{P})_\delta$, such that*

$$|\nabla u_{\delta,j}|^2\rightharpoonup S^{\frac{N}{2}}\delta_{x_j}\quad \text{as } \delta\to 0 \text{ for some } x_j\in C_j. \tag{43}$$

(ii) *Let C_j be disjoint compact subsets of Ω such that, for any j,*

$$a(y)>0\quad\forall y\in C_j,\quad \max_{\partial C_j}K<\max_{C_j}K.$$

Then, for δ small, $(\mathrm{P})_\delta$ has solutions $u_{\delta,j}$ such that (43) holds.
Moreover, such solutions are positive.

Proof. We just derive (ii) from (i). For any given $y \in \Omega$, let

$$\theta(y) := \left(\frac{[2N-(N-2)(q+1)]c_N^{q+1} Fa(y)}{(N-2)(q+1)DH(y)} \right)^{\frac{2}{(N-2)(q+1)-4}}$$

be the absolute minimizer of $\theta \to M(\theta, y)$ and let

$$0 < \underline{\theta} < \min_{y \in C} \theta(y) \leqslant \max_{y \in C} \theta(y) < \bar{\theta}, \quad m := \min_{[\underline{\theta}, \bar{\theta}] \times C} M, \quad m_b := \min_{\partial([\underline{\theta}, \bar{\theta}] \times C)} M$$

for $C = C_j$ fixed. Since

$$M(\theta(y), y)) = D_{N,q} K(y)^{\frac{(N-2)}{(N-2)(q+1)-4}}, \quad \forall y \in \Omega,$$

$D_{N,q}$ as above, one easily obtains $\max_{\partial C} K < \max_C K \Rightarrow m < m_b$ and then (ii) follows from (i).

The proof of the positivity for these solutions follows the same argument as in Theorem 1.2 because $q \geqslant 1$.

For the derivatives, similar computations as in Lemma 4.4 can be performed in case $1 \leqslant q < \frac{N+2}{N-2}$ if $N \geqslant 5$, $\frac{5}{4} < q < 3$ if $N = 4$. □

Theorem 5.2. *Let M, K be given as above and let (θ_j, y_j) be critical points of M. Let $1 \leqslant q < \frac{N+2}{N-2}$ if $N \geqslant 5$, $\frac{5}{4} < q < 3$ if $N = 4$.*

(k) *If (θ_j, y_j) are C^1-stable, then there are C_j disjoint compact neighbourhoods of (θ_j, y_j) and, for $\delta > 0$ small, there are $u_{\delta,j}$, solutions of* $(\mathrm{P})_\delta$, *with property* (43).

(kk) *Let y_0 be a non-degenerate critical point of K with $a(y_0) > 0$. Then, for δ small,* $(\mathrm{P})_\delta$ *has a solution u_δ satisfying* (43) *with limit Dirac mass in y_0.*

Moreover, such solutions are positive.

Proof. By the assumptions $\nabla K(y_0) = 0$, $D^2 K(y_0) \in Gl_N(\mathbf{R})$ and $a(y_0) > 0$, it follows that $\nabla M(\theta(y_0), y_0) = 0$ and $D^2 M(\theta(y_0), y_0) \in Gl_{N+1}(\mathbf{R})$. The proof of this fact is a straightforward computation, we skip here the details. □

Remark 5.3. (i) Non-degeneracy of critical points of K implies non-degeneracy of critical points of C^2-perturbations of M. This in turn would lead (see the proof of Theorem 5.1) to non-degeneracy and precise Morse index estimates of the corresponding variational functional associated to $(\mathrm{P})_\delta$. However, we will not carry over C^2 estimates in this paper.

(ii) If $a(x) \equiv 1$, $N > 4$ and $q = 1$, then we find as many positive solutions as the number of non-degenerate critical points of $H(y)$, which is exactly the famous result contained in [35].

Our approach applies as well to the non-homogeneous boundary value problem with small data. Let $\Omega \subset \mathbf{R}^N$, $N \geqslant 3$, be a smooth open bounded domain and

$\varphi \in C^{\alpha}(\partial\Omega)$, $\alpha \in (0,1)$. Let us consider the following BVP:

$$(\mathrm{BVP})\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}}u & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

It can be seen (see [21] for a more general equation) that (BVP) has a "small" positive solution if $\varphi \geqslant 0$ is non-trivial and suitably small. We are interested for (BVP) with boundary data $\delta\varphi$, $\delta > 0$ small and φ positive somewhere, rewritten in the equivalent form:

$$(\mathrm{BVP})_{\delta}\begin{cases} -\Delta u = |u+\delta a|^{\frac{4}{N-2}}(u+\delta a) & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases}$$

where a denotes the harmonic extension of φ. Here the perturbation is

$$G(\delta,u) = \frac{1}{p+1}\int_{\Omega} |u+\delta a|^{p+1} - |u|^{p+1},$$

which is a C^2 functional converging to zero C^2-uniformly on bounded sets. So we can find w according to Lemma 3.1 and the finite-dimensional reduction can be performed. Now we can expand $G(\delta,u)$ in the form $G(\delta,u) = \delta\int_{\Omega} a(x)|u|^{p-1}u + G_2(\delta,u)$ where

$$|G_2(\delta,u)| = O\left(\delta^2 \int_{\Omega} |u|^{p-1} + \delta^{p+1}\right),$$
$$\|\nabla G_2(\delta,u)\| = O\left(\delta^p + \delta^2\left(\int_{\Omega} |u|^{\frac{(p-2)(p+1)}{p}}\right)^{\frac{p}{p+1}} (\text{if } p>2)\right).$$

Let us stress that $u \to \int_{\Omega} a(x)|u|^{p-1}u$ is not a C^2 functional for $N>6$.

Some remarks are in order:

(a) the problem with a perturbation term $G_1(\delta,u) = G(\delta,u) - G_2(\delta,u) = \delta\int_{\Omega} a(x)|u|^{p-1}u$ is exactly of the form $(\mathrm{P})_{\delta}$ with $q = p-1$, $a(x)$ replaced by $pa(x)$. So the expansion for $E_1(PU_{\varepsilon,y}) = E(PU_{\varepsilon,y}) - G_1(\delta, PU_{\varepsilon,y})$ is given by (42) because $q = p-1 > \frac{2}{N-2}$;

(b) $G_2(\delta,u)$ gives a contribution to the remainder term $\|\nabla E_{\delta}\|^2$, $E_{\delta} = E - G(\delta,\cdot)$, of order $O(\delta^{2p} + \delta^4\varepsilon^{\frac{6-N}{2}}(\text{if } N<6))$;

(c) $G_2(\delta, PU_{\varepsilon,y}) = O(\delta^{p+1} + \delta^2\varepsilon)$;

(d) if $\delta \sim \varepsilon^{\frac{N-2}{2}}$, there holds $E_{\delta}(PU_{\varepsilon} + w) = E_1(PU_{\varepsilon,y}) + o(\delta\varepsilon^{\frac{N-2}{2}})$. So it follows

Theorem 5.4. *Let*

$$M(\theta, y) = c_N H(y)\theta^{N-2} - 2a(y)\theta^{\frac{N-2}{2}} \theta > 0, \quad y \in \Omega,$$

$$K(y) = \frac{a(y)^2}{H(y)}, \quad y \in \Omega$$

and let (θ_j, y_j) *be critical points of* M.

(i) *If* (θ_j, y_j) *are* C^0*-stable, then there are* C_j *disjoint compact neighbourhoods of* (θ_j, y_j) *and, for* $\delta > 0$ *small, there are* $u_{\delta,j}$, *solutions of* $(\mathrm{BVP})_\delta$, *such that*

$$|\nabla u_{\delta,j}|^2 \rightharpoonup S^{\frac{N}{2}} \delta_{x_j} \quad \text{as } \delta \to 0 \text{ for some } x_j \in C_j. \tag{45}$$

(ii) *Let* C_j *be disjoint compact subsets of* Ω *such that, for any* j,

$$a(y) > 0 \quad \forall y \in C_j, \qquad \max_{\partial C_j} K < \max_{C_j} K.$$

Then, for δ *small,* $(\mathrm{BVP})_\delta$ *has solutions* $u_{\delta,j}$ *such that*

$$|\nabla u_{\delta,j}|^2 \rightharpoonup S^{\frac{N}{2}} \delta_{x_j} \quad \text{as } \delta \to 0 \text{ for some } x_j \in C_j. \tag{46}$$

Moreover, if $\varphi \geqslant 0$, *such solutions are positive.*

Proof. We need only to prove that the solutions are positive if $\varphi \geqslant 0$. If this case, we define v_δ as the "small" positive solution of $(\mathrm{BVP})_\delta$, $\delta > 0$ small, whose existence is ensured by [21]. We verify that $u = u_\delta - v_\delta$ is positive (for simplicity, we will omit the dependence on δ). Since for u there holds

$$-\Delta u = |u + \delta a + v_\delta|^{p-1}(u + \delta a + v_\delta) - (\delta a + v_\delta)^p,$$

we have that, for any $\phi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \nabla \phi = p \int_\Omega u\phi \int_0^1 |su + \delta a + v_\delta|^{p-1} \, ds.$$

By choosing $\phi = -u^- = -\max(-u, 0)$, we obtain

$$\int_\Omega |\nabla u^-|^2 = p \int_\Omega (u^-)^2 \int_0^1 |-su^- + \delta a + v_\delta|^{p-1} \, ds$$
$$\leqslant o(1)\left(\int_\Omega (u^-)^{p+1} \right)^{\frac{2}{p+1}} + C_2 \int_\Omega (u^-)^{p+1}.$$

From the Sobolev embedding theorem and the above inequality we get

$$S\left(\int_\Omega (u^-)^{p+1}\right)^{\frac{2}{p+1}} \leqslant o(1)\left(\int_\Omega (u^-)^{p+1}\right)^{\frac{2}{p+1}} + C_2\int_\Omega (u^-)^{p+1}. \quad (47)$$

Let us remark that since $PU_{\varepsilon,y}>0$, we have $u^- \leqslant |w(\delta,\varepsilon,y)| + v_\delta$. If, by contradiction, $u^- \neq 0$ for δ small, we can simplify in (47) to obtain

$$S \leqslant o(1) + C_2\left(\int_\Omega (u^-)^{p+1}\right)^{\frac{p-1}{p+1}} \leqslant o(1) + C_3(||w(\delta,\varepsilon,y)||^{p-1} + ||v_\delta||^{p-1}) \to_{\delta\to 0} 0.$$

Then, for δ small, $u_\delta \geqslant v_\delta > 0$. This completes the proof of Theorem 5.4. □

Similar computations can be performed for the derivatives leading to the counterpart of Theorem 5.2. Essentially, if $\varphi \geqslant 0$ and $\delta > 0$ is small, problem $(\mathrm{BVP})_\delta$ has as many positive solutions as the non-degenerate critical points of K with $a>0$. This is almost the same result for this problem contained in [37]. However, Theorem 5.4 represents a slight improvement because it permits to handle dimension $N=3$ and it provides an existence result (in any dimension) corresponding to the strict relative maxima of K.

With the aid of Theorem 5.4, we can provide an example where some highly oscillating boundary data produce a large number of solutions:

An example. Let $\Omega = B_1(0)$ be the unit open ball, n any positive integer. Let $y_j \in \partial B_1$, $j=1,\dots,n$ and $t>1$. We want to show that

$$-\Delta u = u^{\frac{N+2}{N-2}} \quad \text{in } B_1,$$

$$u = \delta \sum_{j=1}^n \frac{1}{|y - ty_j|^{N-2}} \quad \text{on } \partial B_1$$

has at least n positive solutions if $t < t_{\rho,n} := 1 + \frac{\rho^2}{4n^{\frac{2}{N-2}}}$, $\rho \leqslant \min_{i\neq j} \frac{|y_i - y_j|}{2}$ and δ smaller than some δ_t.

Denoted $a^t(y) := \sum_{j=1}^n \frac{1}{|y-ty_j|^{N-2}}$ and $K^t(y) := \frac{a^t(y)^2}{H(y)}$, it is enough to check, to apply Theorem 5.4, that

$$m_t := \max\{K^t(y) : y \in B_1(0),\ |y-y_j| \geqslant \rho\ \forall j\}$$
$$< \max\{K^t(y) : y \in B_1(0),\ |y-y_i| \leqslant \rho\} \quad \forall i = 1,\dots,n \text{ provided } t < t_{\rho,n}.$$

Appendix A

Here, we recall several kinds of estimates for

$$U_{\varepsilon,y}(x) = c_N \frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x-y|^2)^{\frac{N-2}{2}}}, \quad c_N = [N(N-2)]^{\frac{N-2}{4}}, \ \varepsilon > 0, \ y \in \mathbf{R}^N.$$

Also, $\int_{\mathbf{R}^N} |\nabla U_{\varepsilon,y}|^2 = \int_{\mathbf{R}^N} U_{\varepsilon,y}^{p+1} = S^{\frac{N}{2}}$ and

$$\frac{\partial U_{\varepsilon,y}}{\partial x_i}(x) = -c_N(N-2)\varepsilon^{\frac{N-2}{2}} \frac{x_i - y_i}{(\varepsilon^2 + |x-y|^2)^{\frac{N}{2}}}, \quad \left|\frac{\partial U_{\varepsilon,y}}{\partial x_i}(x)\right| \leqslant \frac{N-2}{2\varepsilon} U_{\varepsilon,y}(x), \tag{A.1}$$

$$\frac{\partial U_{\varepsilon,y}}{\partial \varepsilon}(x) = -c_N \frac{N-2}{2} \varepsilon^{\frac{N-4}{2}} \frac{\varepsilon^2 - |x-y|^2}{(\varepsilon^2 + |x-y|^2)^{\frac{N}{2}}}, \quad \left|\frac{\partial U_{\varepsilon,y}}{\partial \varepsilon}(x)\right| \leqslant \frac{N-2}{2\varepsilon} U_{\varepsilon,y}(x). \tag{A.2}$$

Direct computations give the following estimates.

Lemma A.1.

$$\int_\Omega U_{\varepsilon,y}^q = \begin{cases} O(\varepsilon^{N-\frac{N-2}{2}q}) & \text{if } q > \frac{N}{N-2}, \\ O\left(\varepsilon^{\frac{N}{2}} \log \frac{diam\, \Omega}{\varepsilon}\right) & \text{if } q = \frac{N}{N-2}, \\ O(\varepsilon^{\frac{N-2}{2}q} (diam\, \Omega)^{N-(N-2)q}) & \text{if } q < \frac{N}{N-2}, \end{cases}$$

$$\int_{B_r(y)^c} |x-y|^s U_{\varepsilon,y}^q = O\left(\frac{\varepsilon^{\frac{N-2}{2}q}}{r^{(N-2)q-N-s}}\right) \quad \text{if } q > \frac{N+s}{N-2},$$

where $r > 0$.

Now to get estimates for $PU_{\varepsilon,y}$ (recall that $\Delta PU_{\varepsilon,y} = \Delta U_{\varepsilon,y}$, $PU_{\varepsilon,y} \equiv 0$ on $\partial\Omega$), let us introduce

$$\psi_{\varepsilon,y} := U_{\varepsilon,y} - PU_{\varepsilon,y}, \quad f_{\varepsilon,y} := \psi_{\varepsilon,y} - c_N H(y,\cdot)\varepsilon^{\frac{N-2}{2}},$$

where $H(y,x)$ denotes the regular part of the Green's function, i.e., $\forall y \in \Omega$, $\Delta_x H(y,x) = 0$ in Ω and $H(y,x)|_{x\in\partial\Omega} = |x-y|^{-(N-2)}$. For any given $y \in \Omega$ we will denote $d := dist(y, \partial\Omega)$ and $H(y) = H(y,y)$. By the maximum principle:

$$0 \leqslant \psi_{\varepsilon,y} \leqslant U_{\varepsilon,y}, \quad ||\psi_{\varepsilon,y}||_\infty \leqslant \max_{x\in\partial\Omega} U_{\varepsilon,y}(x) \leqslant c_N \frac{\varepsilon^{\frac{N-2}{2}}}{d^{N-2}}.$$

In particular, $0 \leqslant U^p_{\varepsilon,y} - PU^p_{\varepsilon,y} \leqslant c_p \frac{\varepsilon^{\frac{N-2}{2}}}{d^{N-2}} U^{p-1}_{\varepsilon,y}$. We also have $f_{\varepsilon,y} = O(\frac{\varepsilon^{\frac{N+2}{2}}}{d^N})$ because $f_{\varepsilon,y}$ is harmonic in Ω with boundary data

$$f_{\varepsilon,y}(x) = c_N \varepsilon^{\frac{N-2}{2}} \left[\frac{1}{(\varepsilon^2 + |x-y|^2)^{\frac{N-2}{2}}} - \frac{1}{|x-y|^{N-2}} \right] = O\left(\frac{\varepsilon^{\frac{N+2}{2}}}{d^N} \right).$$

Similarly, one gets estimates for the derivatives of $\psi_{\varepsilon,y}$ and $f_{\varepsilon,y}$. Summarizing (see also [35] for more details)

Lemma A.2. *Given* $\varepsilon > 0$, $\psi_{\varepsilon,y}, f_{\varepsilon,y}$, d *as above*, *then*

$$\psi_{\varepsilon,y} = O\left(\frac{\varepsilon^{\frac{N-2}{2}}}{d^{N-2}} \right) \frac{\partial \psi_{\varepsilon,y}}{\partial y_i} = O\left(\frac{\varepsilon^{\frac{N-2}{2}}}{d^{N-1}} \right) \frac{\partial \psi_{\varepsilon,y}}{\partial \varepsilon} = O\left(\frac{\varepsilon^{\frac{N-4}{2}}}{d^{N-2}} \right), \tag{A.3}$$

$$\frac{\partial^2 \psi_{\varepsilon,y}}{\partial y_i \partial y_j} = O\left(\frac{\varepsilon^{\frac{N-2}{2}}}{d^N} \right) \frac{\partial^2 \psi_{\varepsilon,y}}{\partial y_i \partial \varepsilon} = O\left(\frac{\varepsilon^{\frac{N-4}{2}}}{d^{N-1}} \right) \frac{\partial^2 \psi_{\varepsilon,y}}{\partial \varepsilon^2} = O\left(\frac{\varepsilon^{\frac{N-6}{2}}}{d^{N-2}} \right), \tag{A.4}$$

$$f_{\varepsilon,y} = O\left(\frac{\varepsilon^{\frac{N+2}{2}}}{d^N} \right) \frac{\partial f_{\varepsilon,y}}{\partial y_i} = O\left(\frac{\varepsilon^{\frac{N+2}{2}}}{d^{N+1}} \right) \frac{\partial f_{\varepsilon,y}}{\partial \varepsilon} = O\left(\frac{\varepsilon^{\frac{N}{2}}}{d^N} \right). \tag{A.5}$$

We are now interested in some estimate for the L^{p+1}-norm of $\psi_{\varepsilon,y}$. Let us define

$$\tilde{\psi}_{\varepsilon,y}(x) := \begin{cases} \psi_{\varepsilon,y}(x) & \text{if } x \in \Omega, \\ U_{\varepsilon,y}(x) & \text{if } x \in \mathbf{R}^N \backslash \Omega. \end{cases}$$

We have that $\tilde{\psi}_{\varepsilon,y} \in D^{1,2}(\mathbf{R}^N)$, $D^{1,2}(\mathbf{R}^N)$ being the completion of $C_0^\infty(\mathbf{R}^N)$ with respect to the L^2-norm of the gradient, and, by Sobolev inequality,

$$\left(\int_{\mathbf{R}^N} \tilde{\psi}^{p+1}_{\varepsilon,y} \right)^{\frac{2}{p+1}} \leqslant \frac{1}{S} \int_{\mathbf{R}^N} |\nabla \tilde{\psi}_{\varepsilon,y}|^2,$$

where S is the Sobolev constant. For the r.h.s. we can obtain

$$\begin{aligned} \int_{\mathbf{R}^N} |\nabla \tilde{\psi}_{\varepsilon,y}|^2 &= \int_{\mathbf{R}^N} |\nabla U_{\varepsilon,y}|^2 - \int_\Omega |\nabla PU_{\varepsilon,y}|^2 \\ &= S^{\frac{N}{2}} - \int_\Omega U^{p+1}_{\varepsilon,y} + \int_\Omega U^p_{\varepsilon,y} \psi_{\varepsilon,y} = O\left(\left(\frac{\varepsilon}{d} \right)^{N-2} \right), \end{aligned}$$

because $\int_\Omega \nabla U_{\varepsilon,y}\nabla PU_{\varepsilon,y} = \int_\Omega |\nabla PU_{\varepsilon,y}|^2$. Hence,

$$\int_{\mathbf{R}^N} \tilde{\psi}_{\varepsilon,y}^{p+1} = \int_\Omega \psi_{\varepsilon,y}^{p+1} + O\left(\left(\frac{\varepsilon}{d}\right)^N\right) = O\left(\left(\frac{\varepsilon}{d}\right)^N\right)$$

which proves

Lemma A.3.

$$|\psi_{\varepsilon,y}|_{L^{p+1}(\Omega)} = O\left(\left(\frac{\varepsilon}{d}\right)^{\frac{N-2}{2}}\right). \tag{A.6}$$

Now, using estimates on $\psi_{\varepsilon,y}$ and its derivatives, we can get for the first and second derivatives of $PU_{\varepsilon,y}$:

Lemma A.4. *Let $\gamma > 0$. Then, for all $i \neq j$, we have*

$$\left\|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right\|^2 = \frac{c_1}{\varepsilon^2} + O(\varepsilon^{N-3}), \quad \left\|\frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon}\right\|^2 = \frac{c_2}{\varepsilon^2} + O(\varepsilon^{N-4}),$$

$$\left\langle \frac{\partial PU_{\varepsilon,y}}{\partial y_i}, \frac{\partial PU_{\varepsilon,y}}{\partial y_j}\right\rangle = O(\varepsilon^{N-3}), \quad \left\langle \frac{\partial PU_{\varepsilon,y}}{\partial y_i}, \frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon}\right\rangle = O(\varepsilon^{N-3}),$$

$$\left\|\frac{\partial^2 PU_{\varepsilon,y}}{\partial y_i \partial y_j}\right\| = O\left(\frac{1}{\varepsilon^2}\right), \quad \left\|\frac{\partial^2 PU_{\varepsilon,y}}{\partial y_i \partial \varepsilon}\right\| = O\left(\frac{1}{\varepsilon^2}\right), \quad \left\|\frac{\partial^2 PU_{\varepsilon,y}}{\partial \varepsilon^2}\right\| = O\left(\frac{1}{\varepsilon^2}\right)$$

uniformly for $y \in \Omega$ with $d(y, \partial\Omega) > \gamma$.

Proof. For the norm and scalar product of first derivatives, by Lemma A.1, Lemma A.2 and $\frac{\partial U_{\varepsilon,y}}{\partial \tau_i} = O\left(\frac{U_{\varepsilon,y}}{\varepsilon}\right)$, we get, for $i \neq j$,

$$\left\|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right\|^2 = p \int_{\mathbf{R}^N} U_{\varepsilon,y}^{p-1} \left(\frac{\partial U_{\varepsilon,y}}{\partial y_i}\right)^2 + O(\varepsilon^{N-3}) = \frac{c_1}{\varepsilon^2} + O(\varepsilon^{N-3}),$$

$$\left\|\frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon}\right\|^2 = p \int_{\mathbf{R}^N} U_{\varepsilon,y}^{p-1} \left(\frac{\partial U_{\varepsilon,y}}{\partial \varepsilon}\right)^2 + O(\varepsilon^{N-4}) = \frac{c_2}{\varepsilon^2} + O(\varepsilon^{N-4}),$$

$$\left\langle \frac{\partial PU_{\varepsilon,y}}{\partial y_i}, \frac{\partial PU_{\varepsilon,y}}{\partial y_j}\right\rangle = O\left(\int_{\Omega \setminus B_\gamma(y)} U_{\varepsilon,y}^{p-1} \left|\frac{\partial U_{\varepsilon,y}}{\partial y_i}\right| \left|\frac{\partial U_{\varepsilon,y}}{\partial y_j}\right| + \varepsilon^{N-3}\right) = O(\varepsilon^{N-3}),$$

$$\left\langle \frac{\partial PU_{\varepsilon,y}}{\partial y_i}, \frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon}\right\rangle = O\left(\int_{\Omega \setminus B_\gamma(y)} U_{\varepsilon,y}^{p-1} \left|\frac{\partial U_{\varepsilon,y}}{\partial y_i}\right| \left|\frac{\partial U_{\varepsilon,y}}{\partial \varepsilon}\right| + \varepsilon^{N-3}\right) = O(\varepsilon^{N-3}).$$

For the second derivatives, by Lemma A.2, we get for the first relation

$$\int_\Omega \left|\nabla \frac{\partial^2 PU_{\varepsilon,y}}{\partial y_i \partial y_j}\right|^2 = O\left(\int_\Omega \left[U_{\varepsilon,y}^{p-2} \left|\frac{\partial U_{\varepsilon,y}}{\partial y_i}\right| \left|\frac{\partial U_{\varepsilon,y}}{\partial y_j}\right| + U_{\varepsilon,y}^{p-1} \left|\frac{\partial^2 U_{\varepsilon,y}}{\partial y_i \partial y_j}\right|\right] \left|\frac{\partial^2 PU_{\varepsilon,y}}{\partial y_i \partial y_j}\right|\right)$$
$$= O\left(\frac{1}{\varepsilon^4} + \frac{1}{\varepsilon^2}\varepsilon^{\frac{N-2}{2}} \int_\Omega U_{\varepsilon,y}^p\right) = O\left(\frac{1}{\varepsilon^4}\right),$$

because $\frac{\partial^2 U_{\varepsilon,y}}{\partial y_i \partial y_j} = O\left(\frac{U_{\varepsilon,y}}{\varepsilon^2}\right)$. We proceed in an analogous way for the remaining relations.

Now, we carry out a more subtle analysis with the aid of the expansion of $\psi_{\varepsilon,y}$ in term of the regular part of Green's function.

Lemma A.5. *Let* $D = c_N^{\frac{2N}{N-2}} \int_{\mathbf{R}^N} \frac{dx}{(1+|x|^2)^{\frac{N+2}{2}}}$ *and* $\gamma > 0$. *Then*

$$||PU_{\varepsilon,y}||^2 = \int_\Omega |\nabla PU_{\varepsilon,y}|^2 = S^{\frac{N}{2}} - DH(y)\varepsilon^{N-2} + O(\varepsilon^{N-1}),$$
$$\int_\Omega PU_{\varepsilon,y}^{p+1} = S^{\frac{N}{2}} - (p+1)DH(y)\varepsilon^{N-2} + O(\varepsilon^{N-1}),$$
$$\left\langle PU_{\varepsilon,y}, \frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right\rangle = -D\frac{\partial H}{\partial y_i}(y,y)\varepsilon^{N-2} + O(\varepsilon^{N-1}),$$
$$\left\langle PU_{\varepsilon,y}, \frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon}\right\rangle = -\frac{N-2}{2}DH(y)\varepsilon^{N-3} + O(\varepsilon^{N-2}),$$
$$\int_\Omega PU_{\varepsilon,y}^p \frac{\partial PU_{\varepsilon,y}}{\partial y_i} = -2D\frac{\partial H}{\partial y_i}(y,y)\varepsilon^{N-2} + O\left(\varepsilon^{N-1}\log\frac{1}{\varepsilon}\right),$$
$$\int_\Omega PU_{\varepsilon,y}^p \frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon} = -(N-2)DH(y)\varepsilon^{N-3} + O(\varepsilon^{N-2}),$$

uniformly for $y \in \Omega$ *with* $d(y, \partial\Omega) > \gamma$.

Proof. Let us recall that $\int_{\mathbf{R}^N} |\nabla U_{\varepsilon,y}|^2 = \int_{\mathbf{R}^N} U_{\varepsilon,y}^{p+1} = S^{\frac{N}{2}}$. Now, for the first relation, by Lemma A.1, Lemma A.2 and using Taylor expansion for $H(y,x)$, we get

$$\int_\Omega |\nabla PU_{\varepsilon,y}|^2 = \int_\Omega U_{\varepsilon,y}^{p+1} - \int_\Omega U_{\varepsilon,y}^p \psi_{\varepsilon,y}$$
$$= \int_{\mathbf{R}^N} U_{\varepsilon,y}^{p+1} - c_N \varepsilon^{\frac{N-2}{2}} \int_\Omega U_{\varepsilon,y}^p [H(y) + O(|x-y|)] + O(\varepsilon^N)$$
$$= S^{\frac{N}{2}} - DH(y)\varepsilon^{N-2} + O(\varepsilon^{N-1}),$$

because

$$\int_\Omega U_{\varepsilon,y}^p |x-y| = O(\varepsilon^{\frac{N}{2}}).$$

Similarly, for the second one we have

$$\begin{aligned}\int_\Omega PU_{\varepsilon,y}^{p+1} &= \int_\Omega U_{\varepsilon,y}^{p+1} - (p+1) \int_\Omega U_{\varepsilon,y}^p \psi_{\varepsilon,y} + O(\varepsilon^{N-1}) \\ &= S^{\frac{N}{2}} - (p+1) DH(y)\varepsilon^{N-2} + O(\varepsilon^{N-1}).\end{aligned}$$

Next, by Lemma A.1, Lemma A.2 and Taylor expansion for $\frac{\partial H}{\partial y_i}(y,x)$:

$$\begin{aligned}\left\langle PU_{\varepsilon,y}, \frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right\rangle &= -c_N \varepsilon^{\frac{N-2}{2}} \int_\Omega U_{\varepsilon,y}^p \left[\frac{\partial H}{\partial y_i}(y,y) + O(|x-y|)\right] + O(\varepsilon^{N-1}) \\ &= -D \frac{\partial H}{\partial y_i}(y,y)\varepsilon^{N-2} + O(\varepsilon^{N-1}),\end{aligned}$$

because

$$\frac{1}{p+1}\int_{\mathbf{R}^N} U_{\varepsilon,y}^{p+1} = cost. \Rightarrow \int_\Omega U_{\varepsilon,y}^p \frac{\partial U_{\varepsilon,y}}{\partial y_i} = -\int_{\mathbf{R}^N\setminus\Omega} U_{\varepsilon,y}^p \frac{\partial U_{\varepsilon,y}}{\partial y_i} = O(\varepsilon^{N-1}).$$

Similarly,

$$\begin{aligned}\left\langle PU_{\varepsilon,y}, \frac{\partial PU_{\varepsilon,y}}{\partial \varepsilon}\right\rangle &= -\frac{N-2}{2} c_N \varepsilon^{\frac{N-4}{2}} \int_\Omega U_{\varepsilon,y}^p [H(y) + O(|x-y|)] + O(\varepsilon^{N-1}) \\ &= -\frac{N-2}{2} DH(y)\varepsilon^{N-3} + O(\varepsilon^{N-2}),\end{aligned}$$

because, as above,

$$\int_\Omega U_{\varepsilon,y}^p \frac{\partial U_{\varepsilon,y}}{\partial \varepsilon} = -\int_{\mathbf{R}^N\setminus\Omega} U_{\varepsilon,y}^p \frac{\partial U_{\varepsilon,y}}{\partial \varepsilon} = O(\varepsilon^{N-1}).$$

For the last but one relation, we get

$$\begin{aligned}\int_\Omega PU_{\varepsilon,y}^p \frac{\partial PU_{\varepsilon,y}}{\partial y_i} &= \int_\Omega U_{\varepsilon,y}^p \frac{\partial PU_{\varepsilon,y}}{\partial y_i} - p \int_\Omega U_{\varepsilon,y}^{p-1} \frac{\partial PU_{\varepsilon,y}}{\partial y_i}\psi_{\varepsilon,y} \\ &\quad + O\left(\int_\Omega U_{\varepsilon,y}^{p-2}\left|\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\right|\psi_{\varepsilon,y}^2\right).\end{aligned}$$

Now, oddness implies $\int_{B_\gamma(y)} U_{\varepsilon,y}^{p-1} \frac{\partial U_{\varepsilon,y}}{\partial y_i} = 0$, and hence, using Lemmas A.1 and A.2,

$$\begin{aligned} p \int_\Omega & U_{\varepsilon,y}^{p-1} \frac{\partial P U_{\varepsilon,y}}{\partial y_i} \psi_{\varepsilon,y} \\ &= p \int_{B_\gamma(y)} U_{\varepsilon,y}^{p-1} \frac{\partial U_{\varepsilon,y}}{\partial y_i} \psi_{\varepsilon,y} + O(\varepsilon^{N-1}) \\ &= p c_N \varepsilon^{\frac{N-2}{2}} \int_{B_\gamma(y)} U_{\varepsilon,y}^{p-1} \frac{\partial U_{\varepsilon,y}}{\partial y_i} \left[\sum_j \frac{\partial H}{\partial y_j}(y,y)(x_j - y_j) + O(|x-y|^2) \right] + O(\varepsilon^{N-1}) \\ &= D \frac{\partial H}{\partial y_i}(y,y) \varepsilon^{N-2} + O\left(\varepsilon^{N-1} \log \frac{1}{\varepsilon} \right), \end{aligned}$$

because

$$\begin{aligned} p c_N \int_{\mathbf{R}^N} U_{\varepsilon,y}^{p-1} \frac{\partial U_{\varepsilon,y}}{\partial y_i} (x_j - y_j) &= - c_N \int_{\mathbf{R}^N} \frac{\partial}{\partial x_i} (U_{\varepsilon,y}^p)(x_j - y_j) \\ &= c_N \int_{\mathbf{R}^N} U_{\varepsilon,y}^p \delta_{ij} = D \varepsilon^{\frac{N-2}{2}} \delta_{ij}. \end{aligned}$$

For the remainder term, by Lemmas A.1 and A.2, we get

$$\begin{aligned} \int_\Omega U_{\varepsilon,y}^{p-2} \left| \frac{\partial P U_{\varepsilon,y}}{\partial y_i} \right| \psi_{\varepsilon,y}^2 &= O\left(\varepsilon^{N-2} \int_\Omega U_{\varepsilon,y}^{p-2} \left(\left| \frac{\partial U_{\varepsilon,y}}{\partial y_i} \right| + \varepsilon^{\frac{N-2}{2}} \right) \right) \\ &= O\left(\varepsilon^{2N-5} \int_0^{\frac{diam\,\Omega}{\varepsilon}} \frac{\rho^N}{(1+\rho^2)^3} + \varepsilon^N \log \frac{1}{\varepsilon} \right) = O\left(\varepsilon^N \log \frac{1}{\varepsilon} \right). \end{aligned}$$

Thus, from the third relation of this Lemma A.5, we obtain the requested expansion. Finally, we have

$$\int_\Omega P U_{\varepsilon,y}^p \frac{\partial P U_{\varepsilon,y}}{\partial \varepsilon} = \int_\Omega U_{\varepsilon,y}^p \frac{\partial P U_{\varepsilon,y}}{\partial \varepsilon} - p \int_{B_\gamma(y)} U_{\varepsilon,y}^{p-1} \frac{\partial U_{\varepsilon,y}}{\partial \varepsilon} \psi_{\varepsilon,y} + O\left(\varepsilon^{N-1} \log \frac{1}{\varepsilon} \right),$$

because, as above,

$$\begin{aligned} \int_\Omega U_{\varepsilon,y}^{p-2} \left| \frac{\partial P U_{\varepsilon,y}}{\partial \varepsilon} \right| \psi_{\varepsilon,y}^2 &= O\left(\varepsilon^{N-2} \int_\Omega U_{\varepsilon,y}^{p-2} \left(\left| \frac{\partial U_{\varepsilon,y}}{\partial \varepsilon} \right| + \varepsilon^{\frac{N-4}{2}} \right) \right) \\ &= O\left(\varepsilon^{2N-5} \int_0^{\frac{diam\,\Omega}{\varepsilon}} \frac{\rho^{N-1}}{(1+\rho^2)^2} + \varepsilon^{N-1} \log \frac{1}{\varepsilon} \right) = O\left(\varepsilon^{N-1} \log \frac{1}{\varepsilon} \right). \end{aligned}$$

Once again, we need to estimate the different terms.

$$p\int_{B_\gamma(y)} U_{\varepsilon,y}^{p-1}\frac{\partial U_{\varepsilon,y}}{\partial\varepsilon}\psi_{\varepsilon,y} = pc_N\varepsilon^{\frac{N-2}{2}}\int_{B_\gamma(y)} U_{\varepsilon,y}^{p-1}\frac{\partial U_{\varepsilon,y}}{\partial\varepsilon}[H(y)+O(|x-y|)]+O(\varepsilon^{N-1})$$
$$=\frac{N-2}{2}DH(y)\varepsilon^{N-3}+O(\varepsilon^{N-2}).$$

Finally, from the fourth relation in this Lemma A.5, we obtain

$$\int_\Omega PU_{\varepsilon,y}^p\frac{\partial PU_{\varepsilon,y}}{\partial\varepsilon} = -(N-2)DH(y)\varepsilon^{N-3}+O(\varepsilon^{N-2}). \qquad \square$$

We conclude this appendix by showing that all the manifolds Z considered in the paper are "non-degenerate almost critical manifold" for the functional $E(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\frac{1}{p+1}\int_\Omega|u|^{p+1}, u\in H_0^1(\Omega)$.

Lemma A.6. *Let $d:=dist(y,\partial\Omega)$, $diam\,\Omega\leqslant R$. Then*

$$\text{(i)}\ \exists\alpha_N:||\nabla E(\alpha PU_{\varepsilon,y})||\leqslant\begin{cases}\alpha_N\left(\frac{\varepsilon^{\frac{N+2}{2}}}{d^{N-2}}R^{\frac{N-6}{2}}+|1-\alpha|\right) & \text{if } N>6,\\ \alpha_6\left(\frac{\varepsilon^4}{d^4}(\log\frac{R}{\varepsilon})^{\frac{2}{3}}+|1-\alpha|\right) & \text{if } N=6,\\ \alpha_N\left((\frac{\varepsilon}{d})^{N-2}+|1-\alpha|\right) & \text{if } 3\leqslant N<6,\end{cases}$$

for α bounded. Furthermore, $\exists 0<\varepsilon_0<1$, $c>0$:

(ii) $||\pi_z^\perp E''(z)w||\geqslant c||w||$, $z=\alpha PU_{\varepsilon,y}$, *for any $w\in T_1:=\{w\in H_0^1(\Omega):\langle w,PU_{\varepsilon,y}\rangle=\langle w,\frac{\partial PU_{\varepsilon,y}}{\partial\varepsilon}\rangle=\langle w,\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\rangle=0\ \forall i=1,\dots,N\}$ and for $0<\varepsilon<\varepsilon_0 d$, $1-\varepsilon_0<\alpha<1+\varepsilon_0$;*

(iii) $||\pi_z^\perp E''(z)w||\geqslant c||w||$, $z=PU_{\varepsilon,y}$, *for any $w\in T_2:=\{w\in H_0^1(\Omega):\langle w,\frac{\partial PU_{\varepsilon,y}}{\partial\varepsilon}\rangle=\langle w,\frac{\partial PU_{\varepsilon,y}}{\partial y_i}\rangle=0\ \forall i=1,\dots,N\}$ and for $0<\varepsilon<\varepsilon_0 d$.*

Proof. (i) Since $\int_\Omega\nabla PU_{\varepsilon,y}\nabla\varphi=\int_\Omega U_{\varepsilon,y}^p\varphi\ \forall\varphi\in H_0^1(\Omega)$ and $\int_\Omega U_{\varepsilon,y}^{p+1}=S^{\frac{N}{2}}$, we have

$$|\langle\nabla E(\alpha PU_{\varepsilon,y}),\varphi\rangle|=\left|\alpha\int_\Omega\nabla PU_{\varepsilon,y}\nabla\varphi-\alpha^p\int_\Omega PU_{\varepsilon,y}^p\varphi\right|$$
$$\leqslant\alpha S^{-\frac{1}{2}}||\varphi||\left(\int_\Omega(U_{\varepsilon,y}^p-PU_{\varepsilon,y}^p)^{\frac{p+1}{p}}\right)^{\frac{p}{p+1}}+|\alpha-\alpha^p|S^{\frac{N+2}{4}}||\varphi||$$
$$\leqslant p\alpha S^{-\frac{1}{2}}||\varphi||\,||\psi_{\varepsilon,y}||_\infty\left(\int_\Omega U_{\varepsilon,y}^{\frac{(p-1)(p+1)}{p}}\right)^{\frac{p}{p+1}}+|\alpha-\alpha^p|S^{\frac{N+2}{4}}||\varphi||.$$

By Lemma A.1 and (A.3), estimate (i) follows.

It is well known that (see Appendix D in [35])

$$\int_\Omega |\nabla w|^2 - p\int_\Omega U_{\varepsilon,y}^{p-1}w^2 \geqslant \frac{4}{N+4}\int_\Omega |\nabla w|^2 \tag{A.7}$$

for any $w \in T_1$. Hence, we get

$$\begin{aligned}||\pi_{T_1}E''(\alpha PU_{\varepsilon,y})w|| &\geqslant \frac{1}{||w||}\langle E''(\alpha PU_{\varepsilon,y})w, w\rangle \\ &= \frac{1}{||w||}\left[\int_\Omega |\nabla w|^2 - p\alpha^{p-1}\int_\Omega PU_{\varepsilon,y}^{p-1}w^2\right] \geqslant \frac{2}{N+4}||w||\end{aligned}$$

for any $w \in T_1$ and for $0<\varepsilon<\varepsilon_0 d$, $1-\varepsilon_0<\alpha<1+\varepsilon_0$. Hence (ii) holds.

We can write any $w \in T_2$ in the form $w = \lambda\pi_{T_2}PU_{\varepsilon,y} + v$, $v \in T_1$, $\lambda = \frac{\langle w, PU_{\varepsilon,y}\rangle}{\langle PU_{\varepsilon,y}, \pi_{T_2}PU_{\varepsilon,y}\rangle}$. Since $\pi_{T_2}PU_{\varepsilon,y} = PU_{\varepsilon,y} + o(1)$ as $\frac{\varepsilon}{d} \to 0$ in view of Lemma A.4, setting $w_1 = -\lambda\pi_{T_2}PU_{\varepsilon,y} + v$, we can get

$$\begin{aligned}\int_\Omega \nabla w\nabla w_1 - p\int_\Omega PU_{\varepsilon,y}^{p-1}ww_1 &= \lambda^2\left[p\int_\Omega PU_{\varepsilon,y}^{p+1} - \int_\Omega |\nabla PU_{\varepsilon,y}|^2\right] \\ &\quad + \int_\Omega |\nabla v|^2 - p\int_\Omega PU_{\varepsilon,y}^{p-1}v^2 + o(||w||^2) \\ &\geqslant (p-1)S^{\frac{N}{2}}\lambda^2 + \frac{4}{N+4}\int_\Omega |\nabla v|^2 + o(||w||^2) \\ &\geqslant c||w||||w_1||\end{aligned}$$

for $\frac{\varepsilon}{d}$ small, c a positive constant. Finally, we can conclude that

$$||\pi_{T_2}E''(PU_{\varepsilon,y})w|| \geqslant \frac{1}{||w_1||}\left[\int_\Omega \nabla w\nabla w_1 - p\int_\Omega PU_{\varepsilon,y}^{p-1}ww_1\right] \geqslant c||w||$$

for $\frac{\varepsilon}{d}$ small, $w \in T_2$, and then (iii). □

Appendix B

In this appendix, we give the proofs of all facts needed in the expansion of Pohozaev identities.

Proposition 2.1 gives a decomposition of u_δ in the form $u_\delta = \alpha_\delta PU_{\varepsilon_\delta,y_\delta} + w_\delta$, $w_\delta \in T_{\alpha_\delta PU_{\varepsilon_\delta,y_\delta}}$, $w_\delta \to 0$ as $\delta \to 0$ (from now on, we will omit for simplicity the dependence on δ), but it does not give any information about the rate of convergence of w. However, assuming $w \to 0$, $\alpha \to 1$ and using the equation for w, we can gain something more:

Lemma B.1. *Let* $\hat{q}=\min\{\frac{N}{2},N-2\}$. *Then*

$$||w|| = O\left(\left(\frac{\varepsilon}{d}\right)^{\hat{q}} + \delta\varepsilon\right). \tag{B.1}$$

Proof. In fact, the function w solves

$$\begin{aligned} -\Delta w &= [(\alpha PU_{\varepsilon,y}+w)^p - \alpha U_{\varepsilon,y}^p] + \delta a(x)(\alpha PU_{\varepsilon,y}+w)^p \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{B.2}$$

Using

$$(a+b)^p - a^p = O(a^{p-1}|b| + |b|^p),$$

$$(a+b)^p - a^p - pa^{p-1}b = O(|b|^p + a^{p-2}|b|^2(\text{if } p>2))$$

for $a\geqslant 0$, $a+b\geqslant 0$, we can get, by multiplying (B.2) for w and integrating,

$$\begin{aligned} \int_\Omega |\nabla w|^2 &= (\alpha^p-\alpha)\int_\Omega U_{\varepsilon,y}^p w + p\alpha^{p-1}\int_\Omega U_{\varepsilon,y}^{p-1}w^2 \\ &\quad + \delta\alpha^p a(y)\int_\Omega U_{\varepsilon,y}^p w + O\bigg(\int_\Omega U_{\varepsilon,y}^{p-1}|w|\psi_{\varepsilon,y} + |w|(|w|^p + |\psi_{\varepsilon,y}|^p) \\ &\quad + \int_\Omega U_{\varepsilon,y}^{p-2}|w|(w^2+\psi_{\varepsilon,y}^2)(\text{if } p>2) \\ &\quad + \delta\int_\Omega |x-y|U_{\varepsilon,y}^p|w| + \delta\int_\Omega U_{\varepsilon,y}^{p-1}w^2\bigg). \end{aligned}$$

By Lemma A.1 and (A.3), (A.6), for the term $\int_\Omega U_{\varepsilon,y}^{p-1}|w|\psi_{\varepsilon,y}$ we can get

$$\begin{aligned} \int_\Omega U_{\varepsilon,y}^{p-1}|w|\psi_{\varepsilon,y} &= \int_{B_d(y)} U_{\varepsilon,y}^{p-1}|w|\psi_{\varepsilon,y} + \int_{\Omega\setminus B_d(y)} U_{\varepsilon,y}^{p-1}|w|\psi_{\varepsilon,y} \\ &= O\left(\frac{\varepsilon^{\frac{N-2}{2}}}{d^{N-2}}\left(\int_\Omega U_{\varepsilon,y}^{\frac{(p-1)(p+1)}{p}}\right)^{\frac{p}{p+1}} + \left(\frac{\varepsilon}{d}\right)^{\frac{N-2}{2}}\left(\int_{\Omega\setminus B_d(y)} U_{\varepsilon,y}^{p+1}\right)^{\frac{p-1}{p+1}}\right)||w|| \\ &= O\left(\left(\frac{\varepsilon}{d}\right)^{\hat{q}}||w||\right). \end{aligned}$$

Hence, from $\int_\Omega U_{\varepsilon,y}^p w = \int_\Omega \nabla PU_{\varepsilon,y}\nabla w = 0$, $\alpha\to 1$ and (A.6) we derive

$$(1+o(1))\int_\Omega |\nabla w|^2 - p\int_\Omega U_{\varepsilon,y}^{p-1}w^2 = O\left(\left(\frac{\varepsilon}{d}\right)^{\hat{q}} + \delta\varepsilon\right)||w||.$$

In view of (A.7) we get the estimate

$$||w|| = O\left(\left(\frac{\varepsilon}{d}\right)^{\hat{q}} + \delta\varepsilon\right). \qquad \square$$

Now we give crucial estimates for expanding the Pohozaev identities for u_δ.

Lemma B.2. *Let $n(x)$ be the unit outer normal to $\partial\Omega$ in x, D as in Section 2. Then*

$$\int_{\partial\Omega}\left(\frac{\partial PU_{\varepsilon,y}}{\partial n}\right)^2 \langle x-y, n(x)\rangle = (N-2)\varepsilon^{N-2}H(y)D + O\left(\left(\frac{\varepsilon}{d}\right)^{N-1}\right), \tag{B.3}$$

$$\int_{\partial\Omega}\left(\frac{\partial PU_{\varepsilon,y}}{\partial n}\right)^2 n_j(x) = 2\varepsilon^{N-2}D\partial_j H(y) + O\left(\frac{\varepsilon^{N-1}}{d^N}\right), \quad j=1,\dots,N. \tag{B.4}$$

Proof. Multiplying

$$-\Delta PU_{\varepsilon,y} = U^p_{\varepsilon,y} \quad \text{in } \Omega,$$
$$PU_{\varepsilon,y} = 0 \quad \text{on } \partial\Omega$$

for $\langle x-y, \nabla PU_{\varepsilon,y}\rangle$ and $\partial_{x_j}PU_{\varepsilon,y}$, we can get by some integration by parts

$$\begin{aligned}
&\frac{N-2}{2}\int_\Omega U^p_{\varepsilon,y}PU_{\varepsilon,y} + \frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial PU_{\varepsilon,y}}{\partial n}\right)^2\langle x-y, n(x)\rangle \\
&= \int_\Omega \Delta PU_{\varepsilon,y}\langle x-y, \nabla PU_{\varepsilon,y}\rangle \\
&= -\int_\Omega U^p_{\varepsilon,y}\langle x-y, \nabla PU_{\varepsilon,y}\rangle \\
&= \frac{N-2}{2}\int_\Omega U^p_{\varepsilon,y}PU_{\varepsilon,y} - p\varepsilon\int_\Omega U^{p-1}_{\varepsilon,y}PU_{\varepsilon,y}\partial_\varepsilon U_{\varepsilon,y},
\end{aligned}$$

because $\langle x-y, \nabla_y U_{\varepsilon,y}\rangle = \frac{N-2}{2}U_{\varepsilon,y} + \varepsilon\partial_\varepsilon U_{\varepsilon,y}$, and

$$\begin{aligned}
-\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial PU_{\varepsilon,y}}{\partial n}\right)^2 n_j(x) &= \int_\Omega -\Delta PU_{\varepsilon,y}\partial_{x_j}PU_{\varepsilon,y} \\
&= \int_\Omega U^p_{\varepsilon,y}\partial_{x_j}PU_{\varepsilon,y} = p\int_\Omega U^{p-1}_{\varepsilon,y}PU_{\varepsilon,y}\partial_{y_j}U_{\varepsilon,y},
\end{aligned}$$

respectively. So, by the first equality we get

$$\begin{aligned}\int_{\partial\Omega}\left(\frac{\partial PU_{\varepsilon,y}}{\partial n}\right)^2\langle x-y,n(x)\rangle &= -2p\varepsilon\int_\Omega U_{\varepsilon,y}^{p-1}PU_{\varepsilon,y}\partial_\varepsilon U_{\varepsilon,y}\\ &=2\varepsilon^{\frac{N}{2}}c_NH(y)\left[\partial_\varepsilon\left(\int_{\mathbf{R}^N}U_{\varepsilon,y}^p\right)+\frac{1}{\varepsilon}\int_{\mathbf{R}^N\setminus\Omega}U_{\varepsilon,y}^p\right]\\ &\quad+O\left(\int_\Omega\left(|f_{\varepsilon,y}|+\varepsilon^{\frac{N-2}{2}}\frac{|x-y|}{d^{N-1}}\right)U_{\varepsilon,y}^p\right)\\ &\quad+O\left(\int_{\mathbf{R}^N\setminus\Omega}U_{\varepsilon,y}^{p+1}\right)\\ &=(N-2)\varepsilon^{N-2}H(y)D+O\left(\left(\frac{\varepsilon}{d}\right)^{N-1}\right),\end{aligned}$$

where we have used Lemma A.1 and the estimates $H(y)+d|\nabla H(y)|=O(\frac{1}{d^{N-2}})$, $\partial_\varepsilon U_{\varepsilon,y}=O\left(\frac{U_{\varepsilon,y}}{\varepsilon}\right)$ and (A.5). Hence (B.3) holds. Finally, by the second equality we derive

$$\begin{aligned}\int_{\partial\Omega}\left(\frac{\partial PU_{\varepsilon,y}}{\partial n}\right)^2 n_j(x) &= -2p\int_\Omega U_{\varepsilon,y}^{p-1}PU_{\varepsilon,y}\partial_{y_j}U_{\varepsilon,y}\\ &=2p\varepsilon^{\frac{N-2}{2}}c_N\int_\Omega\Bigg[H(y)+\langle\nabla H(y,y),x-y\rangle\\ &\quad+O\left(\frac{|x-y|^2}{d^N}\right)\Bigg]U_{\varepsilon,y}^{p-1}\partial_{y_j}U_{\varepsilon,y}+\frac{1}{\varepsilon}O\left(\int_\Omega|f_{\varepsilon,y}|U_{\varepsilon,y}^p+\int_{\mathbf{R}^N\setminus\Omega}U_{\varepsilon,y}^{p+1}\right)\\ &=2\frac{N+2}{N}\varepsilon^{N-2}c_N^{p+1}\partial_jH(y,y)\int_{\mathbf{R}^N}\frac{|x|^2}{(1+|x|^2)^{\frac{N+4}{2}}}dx+O\left(\frac{\varepsilon^{N-1}}{d^N}\right),\end{aligned}$$

where we have used Lemma A.1 and the estimates $H(y)+d|\nabla H(y)|+d^2|D_{ij}H(y)|=O(\frac{1}{d^{N-2}})$, $\partial_{y_j}U_{\varepsilon,y}=O(\frac{U_{\varepsilon,y}}{\varepsilon})$, (A.5) and

$$\int_\Omega|x-y|^2U_{\varepsilon,y}^{p-1}|\partial_{y_j}U_{\varepsilon,y}|=O\left(\varepsilon^{\frac{N}{2}}\right).$$

Hence (B.4) follows because, by an integration by parts,

$$\int_{\mathbf{R}^N}\frac{|x|^2}{(1+|x|^2)^{\frac{N+4}{2}}}dx=\frac{N}{N+2}\int_{\mathbf{R}^N}\frac{dx}{(1+|x|^2)^{\frac{N+2}{2}}}.\qquad\square$$

Lemma B.3. *There holds*

$$\frac{N-2}{N}\delta\int_\Omega \langle x-y,\nabla a(x)\rangle(\alpha PU_{\varepsilon,y}+w)^{p+1}$$

$$=\frac{1}{N}\alpha^{p+1}S^{\frac{N}{2}}\delta\varepsilon^2\Delta a(y)+O\left(\delta\left(\frac{\varepsilon}{d}\right)^N+\delta\varepsilon\left(\frac{\varepsilon}{d}\right)^{\frac{N-2}{2}}+\delta\varepsilon^3\ln\frac{1}{\varepsilon}+\delta^2\varepsilon^2\right), \quad \text{(B.5)}$$

$$\frac{N-2}{N}\delta\int_\Omega \partial_j a(x)(\alpha PU_{\varepsilon,y}+w)^{p+1}$$

$$=\frac{N-2}{N}\alpha^{p+1}S^{\frac{N}{2}}\delta\partial_j a(y)+O\left(\delta\left(\frac{\varepsilon}{d}\right)^{N-2}+\delta\left(\frac{\varepsilon}{d}\right)^{\frac{N}{2}}+\delta\varepsilon^2\right), \quad j=1,\dots,N. \quad \text{(B.6)}$$

Proof. Using

$$(a+b)^{p+1}-a^{p+1}=O(a^p|b|+|b|^{p+1}),$$

$$(a+b)^{p+1}-a^{p+1}-pa^pb=O(a^{p-1}b^2+|b|^{p+1}),$$

for $a\geqslant 0$, $a+b\geqslant 0$, we can get by Lemma A.1

$$\int_\Omega \langle x-y,\nabla a(x)\rangle(\alpha PU_{\varepsilon,y}+w)^{p+1}$$

$$=\alpha^{p+1}\int_\Omega \langle x-y,\nabla a(x)\rangle U_{\varepsilon,y}^{p+1}$$

$$+O\left(\int_\Omega |x-y|U_{\varepsilon,y}^p(|w|+\psi_{\varepsilon,y})+\|w\|^{p+1}+\int_\Omega \psi_{\varepsilon,y}^{p+1}\right)$$

$$=\frac{1}{N}\alpha^{p+1}c_N^{p+1}\varepsilon^2\Delta a(y)\int_{\mathbf{R}^N}\frac{|x|^2}{(1+|x|^2)^N}\,dx$$

$$+O\left(\left(\frac{\varepsilon}{d}\right)^N+\varepsilon^3\ln\frac{1}{\varepsilon}+\varepsilon\|w\|+\varepsilon|\psi_{\varepsilon,y}|_{L^{p+1}(\Omega)}+\|w\|^{p+1}+|\psi_{\varepsilon,y}|_{L^{p+1}(\Omega)}^{p+1}\right)$$

because

$$\langle x-y,\nabla a(x)\rangle=\langle x-y,\nabla a(y)\rangle+\langle D^2a(y)(x-y),x-y\rangle+O(|x-y|^3),$$

and

$$\int_\Omega \partial_j a(x)(\alpha PU_{\varepsilon,y}+w)^{p+1}$$
$$=\alpha^{p+1}\int_\Omega [\partial_j a(y)+\langle \nabla\partial_j a(y), x-y\rangle + O(|x-y|^2)]\,U_{\varepsilon,y}^{p+1}$$
$$+p\alpha^p\int_\Omega [\partial_j a(y)+O(|x-y|)]\,U_{\varepsilon,y}^p(w-\alpha\psi_{\varepsilon,y}) + O\left(||w||^2+\left(\int_\Omega \psi_{\varepsilon,y}^{p+1}\right)^{\frac{2}{p+1}}\right)$$
$$=\alpha^{p+1}\partial_j a(y) S^{\frac{N}{2}} + O\left(\left(\frac{\varepsilon}{d}\right)^N+\varepsilon^2+\varepsilon||w||+|\psi_{\varepsilon,y}|_\infty\int_\Omega U_{\varepsilon,y}^p + ||w||^2 + |\psi_{\varepsilon,y}|^2_{L^{p+1}(\Omega)}\right),$$

because $\int_\Omega U_{\varepsilon,y}^p\, w = 0$.

Using now (A.3), (A.6) and (B.1) in the above expansions, we conclude the proof. □

Let us remark that, by an integration by parts,

$$\int_{\mathbf{R}^N}\frac{|x|^2}{(1+|x|^2)^N}\,dx = \frac{N}{2(N-1)}\int_{\mathbf{R}^N}\frac{dx}{(1+|x|^2)^{N-1}}$$
$$=\frac{N}{2(N-2)}\left[\int_{\mathbf{R}^N}\frac{1}{(1+|x|^2)^N}\,dx+\int_{\mathbf{R}^N}\frac{|x|^2}{(1+|x|^2)^N}\,dx\right].$$

Let us introduce a smooth cut-off function ξ on $\mathbf{R}^N$ such that $0\leqslant\xi\leqslant 1$, $\xi=0$ on $B_{\frac{1}{2}}(0)$ and $\xi=1$ on $B_1(0)^c$. Set $\eta(x) := \xi(\frac{x-y}{d})$.

For $\gamma\in\{0,1\}$, we consider the function $z(x) := \eta(x)|x-y|^\gamma w(x)$ which solves

$$-\Delta z = g(x) \quad \text{in } \Omega,$$
$$z=0 \quad \text{on } \partial\Omega, \tag{B.7}$$

with

$$g(x) := -\eta(x)|x-y|^\gamma \Delta w(x) - \Delta\eta(x)|x-y|^\gamma w(x) - \gamma(N+\gamma-2)|x$$
$$-\ y|^{\gamma-2}\eta(x)w(x) - 2\gamma\langle\nabla\eta(x), x-y\rangle |x-y|^{\gamma-2}w(x)$$
$$-\ 2|x-y|^\gamma\langle\nabla\eta(x),\nabla w(x)\rangle - 2\gamma\eta(x)|x-y|^{\gamma-2}\langle\nabla w(x), x-y\rangle.$$

Similarly, we define $v(x) := \eta(x)|x-y|^\gamma PU_{\varepsilon,y}(x)$ which solves

$$-\Delta v = h(x) \quad \text{in } \Omega,$$
$$v=0 \quad \text{on } \partial\Omega, \tag{B.8}$$

with

$$
\begin{aligned}
h(x) &:= \eta(x)|x-y|^{\gamma} U_{\varepsilon,y}^{p} - \Delta\eta(x)|x-y|^{\gamma} PU_{\varepsilon,y}(x) \\
&- \gamma(N+\gamma-2)|x-y|^{\gamma-2}\eta(x)PU_{\varepsilon,y}(x) \\
&- 2\gamma\langle\nabla\eta(x), x-y\rangle|x-y|^{\gamma-2}PU_{\varepsilon,y}(x) \\
&- 2|x-y|^{\gamma}\langle\nabla\eta(x), \nabla PU_{\varepsilon,y}(x)\rangle \\
&- 2\gamma\eta(x)|x-y|^{\gamma-2}\langle\nabla PU_{\varepsilon,y}(x), x-y\rangle.
\end{aligned}
$$

By elliptic regularity theory and the theory of traces, we have the inequalities

$$
\begin{aligned}
\left| |x-y|^{\gamma} \frac{\partial w}{\partial n} \right|^2_{L^2(\partial\Omega)} &= \left| \frac{\partial}{\partial n}(\eta|x-y|^{\gamma} w) \right|^2_{L^2(\partial\Omega)} \leqslant C|g|^2_{L^q(\Omega)} \\
\left| |x-y|^{\gamma} \frac{\partial PU_{\varepsilon,y}}{\partial n} \right|^2_{L^2(\partial\Omega)} &= \left| \frac{\partial}{\partial n}(\eta|x-y|^{\gamma} PU_{\varepsilon,y}) \right|^2_{L^2(\partial\Omega)} \leqslant C|h|^2_{L^q(\Omega)}
\end{aligned}
$$

for some constant $C>0$ and $q := \frac{2N}{N+1}$.

Remark B.4. With the function z, we are cutting $|x-y|^{\gamma}w$ to be zero in a small neighbourhood $B_{\frac{d}{2}}(y)$ of the concentration point y. In this way, we will expect that the estimate for $||x-y|^{\gamma}\frac{\partial w}{\partial n}|_{L^2(\partial\Omega)}$ becomes sharper. This idea is already present in [35] where an estimate for $|\frac{\partial w}{\partial n}|_{L^2(\partial\Omega)}$ is obtained: it corresponds to the choice $\gamma = 0$ but this estimate is not enough for our purposes.

Multiplying $\eta(x)w$ also for $|x-y|$, we can expect to gain in the estimate some power of d as a multiplying factor. It is just what happens and it will be crucial in the proof of Theorem 2.2. We apply the same method also to obtain some estimate for $||x-y|^{\gamma}\frac{\partial PU_{\varepsilon,y}}{\partial n}|_{L^2(\partial\Omega)}$.

We are now in position to prove

Lemma B.5. *There holds*

$$
\left| \frac{\partial w}{\partial n} \right|^2_{L^2(\partial\Omega)} = o\left(\frac{\varepsilon^{N-2}}{d^{N-1}} + \delta\frac{\varepsilon^2}{d} \right), \tag{B.9}
$$

$$
\left| |x-y| \frac{\partial w}{\partial n} \right|^2_{L^2(\partial\Omega)} = o\left(\left(\frac{\varepsilon}{d}\right)^{N-2} + \delta\varepsilon^2 \right). \tag{B.10}
$$

Proof. It is enough to estimate each term of $g(x)$ in L^q-norm, $q = \frac{2N}{N+1}$. Taking into account that $|\Delta w| = O(U^p_{\varepsilon,y} + |w|^p)$, it is easy to see that

$$\left(\int_\Omega (\eta |x-y|^\gamma U^p_{\varepsilon,y})^q \right)^{\frac{2}{q}} = O\left(\varepsilon^{2\gamma-1} \left(\int_{\frac{d}{2\varepsilon}}^{+\infty} \frac{r^{q\gamma+N-1}}{(1+r^2)^{\frac{(N+2)q}{2}}} dr \right)^{\frac{N+1}{N}} \right) = O\left(\varepsilon^{2\gamma-1} \left(\frac{\varepsilon}{d} \right)^{N+3-2\gamma} \right),$$

$$\left(\int_\Omega (|\Delta\eta||x-y|^\gamma |w|)^q \right)^{\frac{2}{q}} = O\left(d^{2\gamma-4} |B_d(y)|^{\frac{3}{N}} \left(\int_\Omega |w|^{p+1} \right)^{\frac{2}{p+1}} \right) = O(d^{2\gamma-1}||w||^2),$$

$$|\gamma| \left(\int_\Omega (\eta|x-y|^{\gamma-2}|w|)^q \right)^{\frac{2}{q}} = |\gamma| O\left(\left(\int_\Omega |x-y|^{\frac{2N(\gamma-2)}{3}} \right)^{\frac{3}{N}} \left(\int_\Omega |w|^{p+1} \right)^{\frac{2}{p+1}} \right) = O(||w||^2),$$

$$\left(\int_\Omega (|\nabla\eta||x-y|^{\gamma-1}|w|)^q \right)^{\frac{2}{q}} = O\left(d^{2\gamma-4} |B_d(y)|^{\frac{3}{N}} \left(\int_\Omega |w|^{p+1} \right)^{\frac{2}{p+1}} \right) = O(d^{2\gamma-1}||w||^2),$$

$$\left(\int_\Omega (|\nabla\eta||x-y|^\gamma |\nabla w|)^q \right)^{\frac{2}{q}} = O(d^{2\gamma-2}|B_d(y)|^{\frac{1}{N}}||w||^2) = O(d^{2\gamma-1}||w||^2),$$

$$|\gamma| \left(\int_\Omega (\eta|x-y|^{\gamma-1}|\nabla w|)^q \right)^{\frac{2}{q}} = |\gamma| O\left(\left(\int_\Omega |x-y|^{2N(\gamma-1)} \right)^{\frac{1}{N}} ||w||^2 \right) = O(||w||^2).$$

It remains to estimate $(\int_\Omega (\eta|x-y|^\gamma |w|^p)^q)^{\frac{2}{q}}$, the most difficult because $pq > p+1$. We multiply $-\Delta w$ for $\eta^{\frac{2(N-2)}{N+1}}|x-y|^{\frac{2\gamma(N-2)}{N+1}}|w|^{\frac{2}{N+1}}w$ and, integrating by parts, with some manipulations, we can get

$$\begin{aligned} &\int_\Omega -\Delta w \eta^{\frac{2(N-2)}{N+1}} |x-y|^{\frac{2\gamma(N-2)}{N+1}} |w|^{\frac{2}{N+1}} w \\ &\quad = \frac{(N+1)(N+3)}{(N+2)^2} \int_\Omega |\nabla(\eta^{\frac{N-2}{N+1}}|x-y|^{\frac{\gamma(N-2)}{N+1}}|w|^{\frac{N+2}{N+1}})|^2 \\ &\qquad + O\bigg(\int_\Omega |\nabla w|\,|\nabla \eta|\,|x-y|^{\frac{2\gamma(N-2)}{N+1}} |w|^{\frac{N+3}{N+1}} + |\gamma| \int_\Omega |\nabla w|\,|x-y|^{\frac{2\gamma(N-2)}{N+1}-1} |w|^{\frac{N+3}{N+1}} \\ &\qquad + \int_\Omega |\nabla\eta|^2 |x-y|^{\frac{2\gamma(N-2)}{N+1}} |w|^{\frac{2(N+2)}{N+1}} + |\gamma| \int_\Omega |x-y|^{\frac{2\gamma(N-2)}{N+1}-2} |w|^{\frac{2(N+2)}{N+1}} \bigg). \end{aligned}$$

Since

$$\int_\Omega |\nabla w|\,|\nabla \eta|\,|x-y|^{\frac{2\gamma(N-2)}{N+1}}|w|^{\frac{N+3}{N+1}} = O\left(d^{\frac{(2\gamma-1)(N-2)}{N+1}}||w||^{\frac{2(N+2)}{N+1}}\right),$$
$$|\gamma|\int_\Omega |\nabla w||x-y|^{\frac{2\gamma(N-2)}{N+1}-1}|w|^{\frac{N+3}{N+1}} = |\gamma| O\left(||w||^{\frac{2(N+2)}{N+1}}\right),$$
$$\int_\Omega |\nabla \eta|^2|x-y|^{\frac{2\gamma(N-2)}{N+1}}|w|^{\frac{2(N+2)}{N+1}} = O\left(d^{\frac{(2\gamma-1)(N-2)}{N+1}}||w||^{\frac{2(N+2)}{N+1}}\right),$$
$$|\gamma|\int_\Omega |x-y|^{\frac{2\gamma(N-2)}{N+1}-2}|w|^{\frac{2(N+2)}{N+1}} = |\gamma| O\left(||w||^{\frac{2(N+2)}{N+1}}\right),$$

and using $|\Delta w| = O(U^p_{\varepsilon,y} + |w|^p)$, by the Sobolev inequality we get

$$\begin{aligned}
&S\left(\int_\Omega (\eta|x-y|^\gamma|w|^p)^q\right)^{\frac{2}{p+1}} + O\left(d^{\frac{(2\gamma-1)(N-2)}{N+1}}||w||^{\frac{2(N+2)}{N+1}}\right) + |\gamma| O\left(||w||^{\frac{2(N+2)}{N+1}}\right)\\
&\quad\leqslant \int_\Omega |\Delta w|\eta^{\frac{2(N-2)}{N+1}}|x-y|^{\frac{2\gamma(N-2)}{N+1}}|w|^{\frac{N+3}{N+1}}\\
&\quad= O\left(\int_\Omega (|w|^{p-1})\left(\eta^{\frac{2(N-2)}{N+1}}|x-y|^{\frac{2\gamma(N-2)}{N+1}}|w|^{\frac{2(N+2)}{N+1}}\right) + \int_\Omega U^p_{\varepsilon,y}\eta^{\frac{2(N-2)}{N+1}}|x-y|^{\frac{2\gamma(N-2)}{N+1}}|w|^{\frac{N+3}{N+1}}\right)\\
&\quad= O\left(||w||^{p-1}\left(\int_\Omega (\eta|x-y|^\gamma|w|^p)^q\right)^{\frac{2}{p+1}}\right)\\
&\qquad+ O\left(\varepsilon^{\frac{(2\gamma-1)(N-2)}{N+1}}||w||^{\frac{N+3}{N+1}}\left(\int_{\frac{d}{2\varepsilon}}^{+\infty}\frac{r^{4\gamma\frac{N(N-2)}{N^2+N+6}+N-1}}{(1+r^2)^{\frac{N(N+1)(N+2)}{N^2+N+6}}}\,dr\right)^{\frac{N^2+N+6}{2N(N+1)}}\right)\\
&\quad= o\left(\left(\int_\Omega (\eta|x-y|^\gamma|w|^p)^q\right)^{\frac{2}{p+1}}\right) + O\left(||w||^{\frac{N+3}{N+1}}\varepsilon^{\frac{(2\gamma-1)(N-2)}{N+1}}\left(\frac{\varepsilon}{d}\right)^{\frac{N^2+5N-2}{2(N+1)}-2\gamma\frac{N-2}{N+1}}\right).
\end{aligned}$$

It follows that

$$\left(\int_\Omega (\eta|x-y|^\gamma|w|^p)^q\right)^{\frac{2}{q}} = O\left(d^{2\gamma-1}||w||^{\frac{2(N+2)}{N-2}} + |\gamma|||w||^{\frac{2(N+2)}{N-2}} + \varepsilon^{2\gamma-1}\left(\frac{\varepsilon}{d}\right)^{\frac{N^2+5N-2}{2(N-2)}-2\gamma}||w||^{\frac{N+3}{N-2}}\right).$$

Resuming all this estimates, we get that for $\gamma = 0$

$$\left|\frac{\partial w}{\partial n}\right|^2_{L^2(\partial\Omega)} = O\left(\frac{\varepsilon^{N+2}}{d^{N+3}} + \frac{||w||^2}{d} + \frac{||w||^{\frac{N+3}{N-2}}}{\varepsilon}\left(\frac{\varepsilon}{d}\right)^{\frac{N^2+5N-2}{2(N-2)}}\right)$$

and for $\gamma = 1$

$$\left| |x-y| \frac{\partial w}{\partial n} \right|^2_{L^2(\partial\Omega)} = O\left(\varepsilon \left(\frac{\varepsilon}{d}\right)^{N+1} + ||w||^2 + ||w||^{\frac{N+3}{N-2}} \varepsilon \left(\frac{\varepsilon}{d}\right)^{\frac{N^2+N+6}{2(N-2)}} \right).$$

Inserting (B.1), using $\frac{N^2+5N-2}{2(N-2)} = \frac{N^2+N+6}{2(N-2)} + 2 \geqslant \frac{N}{2}$ and

$$\frac{(\delta\varepsilon)^{\frac{N+3}{N-2}}}{\varepsilon} \left(\frac{\varepsilon}{d}\right)^{\frac{N^2+5N-2}{2(N-2)}} = O\left(\delta \left(\frac{\varepsilon}{d}\right)^{\frac{N}{2}} \right) = O\left(\delta^{\frac{3}{4}} \frac{\varepsilon}{d^{\frac{1}{2}}} \delta^{\frac{1}{4}} \frac{\varepsilon^{\frac{N-2}{2}}}{d^{\frac{N-1}{2}}} \right) = O\left(\delta^{\frac{3}{2}} \frac{\varepsilon^2}{d} + d^{\frac{1}{2}} \frac{\varepsilon^{N-2}}{d^{N-1}} \right),$$

we can obtain the required estimates. $\square$

Similarly, we can proceed to prove

Lemma B.6. *There holds*

$$\left| \frac{\partial PU_{\varepsilon,y}}{\partial n} \right|^2_{L^2(\partial\Omega)} = O\left(\frac{\varepsilon^{N-2}}{d^{N-1}} \right), \tag{B.11}$$

$$\left| |x-y| \frac{\partial PU_{\varepsilon,y}}{\partial n} \right|^2_{L^2(\partial\Omega)} = O\left(\left(\frac{\varepsilon}{d}\right)^{N-2} \right). \tag{B.12}$$

Proof. We need to estimate h in L^q-norm, $q = \frac{2N}{N+1}$. By Lemma A.1, we have that

$$\left(\int_\Omega (\eta |x-y|^\gamma U^p_{\varepsilon,y})^q \right)^{\frac{2}{q}} = O\left(\varepsilon^{2\gamma-1} \left(\int_{\frac{d}{2\varepsilon}}^{+\infty} \frac{r^{\frac{2N}{N+1}\gamma+N-1}}{(1+r^2)^{\frac{N(N+2)}{N+1}}} dr \right)^{\frac{N+1}{N}} \right)$$
$$= O\left(\varepsilon^{2\gamma-1} \left(\frac{\varepsilon}{d}\right)^{N+3-2\gamma} \right),$$

$$\left(\int_\Omega (|\Delta\eta| |x-y|^\gamma PU_{\varepsilon,y})^q \right)^{\frac{2}{q}} = O\left(d^{2\gamma-1} \left(\int_{\Omega \setminus B_{\frac{d}{2}}(y)} U^{p+1}_{\varepsilon,y} \right)^{\frac{2}{p+1}} \right) = O\left(d^{2\gamma-1} \left(\frac{\varepsilon}{d}\right)^{N-2} \right),$$

$$|\gamma| \left(\int_\Omega (|x-y|^{\gamma-2} \eta PU_{\varepsilon,y})^q \right)^{\frac{2}{q}} = |\gamma| O\left(\left(\int_{\Omega \setminus B_{\frac{d}{2}}(y)} U^{p+1}_{\varepsilon,y} \right)^{\frac{2}{p+1}} \right) = |\gamma| O\left(\left(\frac{\varepsilon}{d}\right)^{N-2} \right),$$

$$|\gamma|\left(\int_\Omega (|\nabla\eta|\,|x-y|^{\gamma-1}PU_{\varepsilon,y})^q\right)^{\frac{2}{q}} = |\gamma| O\left(d^{2\gamma-1}\left(\int_{\Omega\setminus B_{\frac{d}{2}}(y)} U_{\varepsilon,y}^{p+1}\right)^{\frac{2}{p+1}}\right)$$
$$= |\gamma| O\left(d^{2\gamma-1}\left(\frac{\varepsilon}{d}\right)^{N-2}\right),$$

$$\left(\int_\Omega (|x-y|^\gamma|\nabla\eta(x)||\nabla PU_{\varepsilon,y}|)^q\right)^{\frac{2}{q}} = O\left(d^{2\gamma-1}\int_{\Omega\setminus B_{\frac{d}{2}}(y)} |\nabla PU_{\varepsilon,y}|^2\right)$$
$$= O\left(d^{2\gamma-1}\left(\frac{\varepsilon}{d}\right)^{N-2}\right),$$

$$|\gamma|\left(\int_\Omega (\eta|x-y|^{\gamma-1}|\nabla PU_{\varepsilon,y}|)^q\right)^{\frac{2}{q}} = |\gamma| O\left(\int_{\Omega\setminus B_{\frac{d}{2}}(y)} |\nabla PU_{\varepsilon,y}|^2\right) = |\gamma| O\left(\left(\tfrac{\varepsilon}{d}\right)^{N-2}\right),$$

where we have used

$$\int_{\Omega\setminus B_{\frac{d}{2}}(y)} |\nabla PU_{\varepsilon,y}|^2 \leqslant 2\left(\int_{\Omega\setminus B_{\frac{d}{2}}(y)} |\nabla U_{\varepsilon,y}|^2 + \int_{\Omega\setminus B_{\frac{d}{2}}(y)} |\nabla\psi_{\varepsilon,y}|^2\right) = O\left(\left(\frac{\varepsilon}{d}\right)^{N-2}\right),$$

in view of

$$\int_\Omega |\nabla\psi_{\varepsilon,y}|^2 = \int_\Omega |\nabla U_{\varepsilon,y}|^2 - \int_\Omega U_{\varepsilon,y}^p PU_{\varepsilon,y} = O\left(\left(\frac{\varepsilon}{d}\right)^{N-2}\right).$$

It follows that

$$\left|\frac{\partial PU_{\varepsilon,y}}{\partial n}\right|^2_{L^2(\partial\Omega)} = O\left(\frac{\varepsilon^{N-2}}{d^{N-1}}\right), \quad \left||x-y|\frac{\partial PU_{\varepsilon,y}}{\partial n}\right|^2_{L^2(\partial\Omega)} = O\left(\left(\frac{\varepsilon}{d}\right)^{N-2}\right).$$

By Lemmas (B.5) and (B.6), it can be easily deduced that

Lemma B.7. *There holds*

$$\int_{\partial\Omega}\left|\frac{\partial PU_{\varepsilon,y}}{\partial n}\right|\left|\frac{\partial w}{\partial n}\right| = o\left(\frac{\varepsilon^{N-2}}{d^{N-1}} + \delta\frac{\varepsilon^2}{d}\right), \tag{B.13}$$

$$\int_{\partial\Omega} |x-y|^2 \left|\frac{\partial PU_{\varepsilon,y}}{\partial n}\right| \left|\frac{\partial w}{\partial n}\right| = o\left(\left(\frac{\varepsilon}{d}\right)^{N-2} + \delta\varepsilon^2\right). \tag{B.14}$$

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