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The quasi-linear Brezis-Nirenberg problem in low dimensions ☆

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We discuss existence results for a quasi-linear elliptic equation of critical Sobolev growth [3,14] in the low-dimensional case, where the problem has a global character which is encoded in sign properties of the "regular" part for the corresponding Green's function as in [9,11].

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$. Given $1 < p < N$ and $\lambda < \lambda_1$, let us discuss existence issues for the quasilinear problem

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$$\begin{cases} -\Delta_p u = \lambda u^{p-1} + u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p(\cdot) = \operatorname{div}\big(|\nabla(\cdot)|^{p-2}\nabla(\cdot)\big)$ is the p-Laplace operator, $p^* = \frac{Np}{N-p}$ is the so-called critical Sobolev exponent and λ_1 is the first eigenvalue of $-\Delta_p$ given by

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}.$$

Since $W_0^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ is a continuous but non-compact embedding, standard variational methods fail to provide solutions of (1.1) by minimization of the Rayleigh quotient

$$Q_\lambda(u) = \frac{\int_\Omega |\nabla u|^p - \lambda \int_\Omega |u|^p}{(\int_\Omega |u|^{p^*})^{\frac{p}{p^*}}}, \quad u \in W_0^{1,p}(\Omega) \setminus \{0\}.$$

Setting

$$S_\lambda = \inf \left\{ Q_\lambda(u)\colon u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\},$$

it is known that S_0 coincides with the best Sobolev constant for the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$ and then is never attained since independent of Ω. Moreover, by a Pohozaev identity $(1.1)_{\lambda=0}$ is not solvable on star-shaped domains, see [3,14]. The presence of the perturbation term λu^{p-1} in (1.1) can possibly restore compactness and produce minimizers for Q_λ, as shown for all $\lambda > 0$ first by Brezis and Nirenberg [3] in the semi-linear case when $N \geq 4$ and then by Guedda and Veron [14] when $N \geq p^2$.

Let us discuss now the low-dimensional case $p < N < p^2$. In the semi-linear situation $p = 2$ it corresponds to $N = 3$ and displays the following special features: according to [3], problem (1.1) is solvable on a ball precisely for $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$ and then, for the minimization problem on a general domain Ω, there holds

$$\lambda_* = \inf\{\lambda \in (0,\lambda_1)\colon S_\lambda < S_0\} \geq \frac{1}{4}\lambda_1(B) = \frac{\pi^2}{4}\Big(\frac{3|\Omega|}{4\pi}\Big)^{-\frac{2}{3}}$$

through a re-arrangement argument, where B is the ball having the same measure of Ω. In particular, for $\lambda \leq \frac{\lambda_1}{4}$ a general non-existence result on B follows from an integral identity of Pohozaev type, obtained by testing the equation against $\psi(|x|)u'$ for a suitable smooth function ψ with $\psi(0) = 0$. An integration by parts for the term

$$\int_0^1 r^{N-1}|u'|^{p-2}u'u\Big[\frac{p-1}{p}\psi'' - \frac{N-1}{p}\frac{\psi'}{r} + \frac{N-1}{p}\frac{\psi}{r^2}\Big]$$

is required to eliminate the dependence on the derivatives of u, which is possible in general just for $p = 2$. The property $\lambda^* > 0$ then requires a different proof for $p \neq 2$.

Since S_λ decreases in a continuous way from S_0 to 0 as λ ranges in $[0, \lambda_1)$, notice that $S_\lambda = S_0$ for $\lambda \in [0, \lambda_*]$, $S_\lambda < S_0$ for $\lambda \in (\lambda_*, \lambda_1)$ and S_λ is not attained for $\lambda \in [0, \lambda_*)$. A natural question concerns the case $\lambda = \lambda_*$ and the following general answer

$$S_{\lambda_*} \text{ is not achieved} \tag{1.2}$$

has been given by Druet [9], with an elegant proof which unfortunately seems not to work for $p \neq 2$. A complete characterization for the critical parameter λ_* then follows through a blow-up approach crucially based on (1.2).

We use here some of the results in [1] - precisely reported in Section 2 for reader's convenience - as a crucial ingredient to treat the quasilinear Brezis-Nirenberg problem (1.1) in the low-dimensional case $p < N < p^2$. Given $x_0 \in \Omega$ and $\lambda < \lambda_1$, introduce the Green function $G_\lambda(\cdot, x_0)$ as a positive solution to

$$\begin{cases} -\Delta_p G - \lambda G^{p-1} = \delta_{x_0} & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Since uniqueness of $G_\lambda(\cdot, x_0)$ is just known for $p \geq 2$, hereafter we will just consider the case $p \geq 2$. If ω_N denotes the measure of the unit ball in $\mathbb{R}^N$, recall that the fundamental solution

$$\Gamma(x, x_0) = C_0 |x - x_0|^{-\frac{N-p}{p-1}}, \quad C_0 = \frac{p-1}{N-p} (N\omega_N)^{-\frac{1}{p-1}}, \tag{1.4}$$

solves $-\Delta_p \Gamma = \delta_{x_0}$ in $\mathbb{R}^N$. The function

$$H_\lambda(x, x_0) = G_\lambda(x, x_0) - \Gamma(x, x_0) \tag{1.5}$$

is usually referred to as the "regular" part of $G_\lambda(\cdot, x_0)$ but is just expected to be less singular than $\Gamma(x, x_0)$ at x_0.

The complete characterization in [9] for λ_* (see also [11] for an alternative proof) still holds in the quasi-linear case, as stated by the following main result.

Theorem 1.1. *Let $2 \leq p < N < 2p$ and $0 < \lambda < \lambda_1$. The implications (i) ⇒ (ii) ⇒ (iii) do hold, where*

(i) *there exists $x_0 \in \Omega$ such that $H_\lambda(x_0, x_0) > 0$*
(ii) $S_\lambda < S_0$
(iii) *S_λ is attained.*

Moreover, the implication (iii) ⇒ (i) does hold under the assumption (1.2) and in particular $\lambda_* > 0$.

Some comments are in order. Assumption $N < 2p$ is crucial here to guarantee that $H_\lambda(\cdot, x_0)$ is Hölder continuous at x_0, see [1]. When $2p \leq N < p^2$ we conjecture $H_\lambda(x, x_0)$ to be mildly but still singular at x_0, with a behavior like $\frac{m_\lambda(x_0)}{|x-x_0|^\alpha}$ for an appropriate $0 < \alpha < \frac{N-p}{p-1}$, and $m_\lambda(x_0)$ to play the same role as $H_\lambda(x_0, x_0)$ in Theorem 1.1. The quantity $m_\lambda(x_0)$ is usually referred to as the mass associated to $G_\lambda(\cdot, x_0)$ and appears in several contexts, see for example [12,13,18–20]. Notice that in the semilinear case $p = 2$ the range $2p \leq N < p^2$ is empty and such a situation doesn't show up in [9].

The implication $(iii) \Rightarrow (i)$ follows by a blow-up argument once (1.2) is assumed. To this aim, we first extend the pointwise blow-up theory in [10] to the quasi-linear context, a fundamental tool in the description of blow-up phenomena whose relevance goes beyond Theorem 1.1 and which completely settles some previous partial results [2,7,8] in this direction. Once sharp pointwise blow-up estimates are established, a major difficulty appears in the classical use of Pohozaev identities: written on small balls around the blow-up point as the radius tends to zero, they rule both the blow-up speed and the blow-up location since boundary terms in such identities can be controlled thanks to the property $\nabla H_\lambda(\cdot, x_0) \in L^\infty(\Omega)$. Clearly valid in the semi-linear situation, such gradient L^∞-bound is completely missing in the quasi-linear context but surprisingly the correct answer can still be found by a different approach, based on a suitable approximation scheme for $G_\lambda(\cdot, x_0)$. At the same time, we provide a different proof of some facts in [9] in order to avoid some rough arguments concerning the limiting problems on halfspaces, when dealing with boundary blow-up.

Under the assumption (1.2), in the proof of Theorem 1.1 we will show that $H_{\lambda_*}(x_0, x_0) = 0$ for some $x_0 \in \Omega$, a stronger property than the validity of the implication $(iii) \Rightarrow (i)$ since $H_\lambda(x, x)$ is strictly increasing in λ for all $x \in \Omega$. Since S_0 is not attained, notice that (1.2) always holds if $\lambda_* = 0$ and then $\lambda_* > 0$ follows by the property $H_0(x_0, x_0) < 0$ for all $x_0 \in \Omega$. Moreover, since

$$\sup_{x\in\Omega} H_{\lambda_*}(x, x) = \max_{x\in\Omega} H_{\lambda_*}(x, x) = 0, \tag{1.6}$$

by monotonicity of H_λ in λ and under the assumption (1.2) the critical parameter λ_* is the first unique value of $\lambda > 0$ attaining (1.6) and can be re-written as

$$\lambda_* = \sup\{\lambda \in (0, \lambda_1)\colon H_\lambda(x, x) < 0 \text{ for all } x \in \Omega\}.$$

In Section 2 we recall some facts from [1] that will be used throughout the paper and prove some useful convergence properties. The implication $(i) \Rightarrow (ii)$ is established in Section 3 by the expansion of $Q_\lambda(PU_{\epsilon,x_0})$ along the “bubble” PU_{ϵ,x_0} concentrating at x_0 as $\epsilon \to 0$ and integral identities of Pohozaev type for $G_\lambda(\cdot, x_0)$, crucial for a fine asymptotic analysis, are also derived. Section 4 is devoted to develop the blow-up argument along with sharp pointwise estimates to establish the final part in Theorem 1.1.

2. Some preliminary facts

For reader's convenience, let us collect here some of the results in [1]. To give the statement of Theorem 1.1 a full meaning, we need a general theory for problem (1.3), as stated in the following result.

Theorem 2.1. *[1] Let $1 < p \leq N$ and $\lambda < \lambda_1$. Assume $p \geq 2$ and $N < 2p$ if $\lambda \neq 0$. Then problem (1.3) has a positive solution $G_\lambda(\cdot, x_0)$ so that $H_\lambda(x, x_0)$ in (1.5) satisfies*

$$\nabla H_\lambda(\cdot, x_0) \in L^{\bar{q}}(\Omega), \quad \bar{q} = \frac{N(p-1)}{N-1}, \tag{2.1}$$

which is unique when either $\lambda = 0$ or $\lambda \neq 0$ and (2.1) holds. Moreover

- *given $M > 0$, $q_0 > \frac{N}{p}$ and $p_0 \geq 1$ there exists $C > 0$ so that*

$$\|H + c\|_{\infty, B_r(x_0)} \leq C(r^{-\frac{N}{p_0}} \|H + c\|_{p_0, B_{2r}(x_0)} + r^{\frac{pq_0 - N}{q_0(p-1)}} \|f\|_{q_0, B_{2r}(x_0)}^{\frac{1}{p-1}}) \tag{2.2}$$

  for all $\epsilon, r, c \in \mathbb{R}$, $f \in L^{q_0}(\Omega)$ and solution $G = \Gamma + H$, with $H \in L^\infty(\Omega)$ and $\nabla H \in L^{\bar{q}}(\Omega)$, to

$$-\Delta_p G + \Delta_p \Gamma = f \qquad \text{in } \Omega \setminus \{x_0\} \tag{2.3}$$

  so that $\epsilon^{p-1} \leq r \leq \frac{1}{4} dist(x_0, \partial\Omega)$, $\frac{|x-x_0|^{\frac{1}{p-1}}}{M(\epsilon^p + |x-x_0|^{\frac{p}{p-1}})^{\frac{N}{p}}} \leq |\nabla \Gamma| \leq M|\nabla \Gamma|(x, x_0)$, $|c| + \|H\|_\infty + \|f\|_{q_0}^{\frac{1}{p-1}} \leq M$, where $\Gamma(\cdot, x_0)$ is given by (1.4);
- *$\lambda G_\lambda^{p-1} \in L^{q_0}(\Omega)$ for $q_0 > \frac{N}{p}$ and $H_\lambda(\cdot, x_0)$ is a continuous function in $\overline{\Omega}$ satisfying*

$$|H_\lambda(x, x_0) - H_\lambda(x_0, x_0)| \leq C|x - x_0|^\alpha \quad \forall\, x \in \Omega \tag{2.4}$$

  for some $C > 0$, $\alpha \in (0, 1)$ with $H_\lambda(x_0, x_0)$ strictly increasing in λ.

Notice that the first part in Theorem 2.1 has been established in [15]. Let us stress that the condition $f \in L^{q_0}(\Omega)$ for some $q_0 > \frac{N}{p}$, which is valid for $f = \lambda G_\lambda^{p-1}$ when $N < 2p$ if $\lambda \neq 0$, is a natural condition on the R.H.S. of the difference equation (2.3) to prove L^∞-bounds on H as it arises for instance in the Moser iterative argument adopted in [22]. In this respect, observe that also in the semilinear case $H_\lambda(\cdot, x_0)$ is no longer regular at x_0 when $4 = 2p \leq N$.

The following a-priori estimates are the basis of Theorem 2.1 and will be crucially used here to establish some accurate pointwise blow-up estimates.

Proposition 2.2. *[1] Let $2 \leq p \leq N$. Assume that $a_n \in L^\infty(\Omega)$, $f_n \in L^1(\Omega)$ and $g_n, \hat{g}_n$ satisfy*

$$g_n, \hat{g}_n \in L^\infty(\Omega) \cap W^{1,p}(\Omega) \text{ } p\text{-harmonic in } \Omega, \; g_n, \hat{g}_n \text{ non-constant unless } 0$$

and

$$\lim_{n \to +\infty} \|a_n - a\|_\infty = 0 \text{ with } \sup_\Omega a < \lambda_1, \quad \sup_{n \in \mathbb{N}} [\|f_n\|_1 + \|g_n\|_\infty + \|\hat{g}_n\|_\infty] < +\infty.$$

If $u_n \in W^{1,p}_{g_n}(\Omega)$ *solves* $-\Delta_p u_n - a_n |u_n|^{p-2} u_n = f_n$ *in* Ω, *then* $\sup_{n \in \mathbb{N}} \|u_n\|_{p-1} < +\infty$ *and, if* $g_n = g$, *the sequence* u_n *is pre-compact in* $W^{1,q}(\Omega)$ *for all* $1 \le q < \bar{q}$. *Moreover, if* $N < 2p$, $a_n = \lambda_n \in \mathbb{R}$ *and* $\hat{u}_n \in W^{1,p}_{\hat{g}_n}(\Omega)$ *solves* $-\Delta_p \hat{u}_n = f_n$ *in* Ω, *then* $\sup_{n \in \mathbb{N}} \|u_n - \hat{u}_n\|_\infty < \infty$.

We will also make use of the following general form of comparison principle.

Proposition 2.3. *[1] Let* $2 \le p \le N$ *and* $a, f_1, f_2 \in L^\infty(\Omega)$. *Let* $u_i \in C^1(\bar{\Omega})$, $i = 1, 2$, *be solutions to*

$$-\Delta_p u_i - a u_i^{p-1} = f_i \quad \text{in } \Omega$$

so that

$$u_i > 0 \text{ in } \Omega, \quad \frac{u_1}{u_2} \le C \text{ near } \partial\Omega$$

for some $C > 0$. *If* $f_1 \le f_2$ *with* $f_2 \ge 0$ *in* Ω *and* $u_1 \le u_2$ *on* $\partial\Omega$, *then* $u_1 \le u_2$ *in* Ω.

Let us introduce now a special approximation scheme for $G_\lambda(\cdot, x_0)$, which is particularly suited for the problem we are interested in. Given $C_1 = N^{\frac{N-p}{p^2}} \left(\frac{N-p}{p-1}\right)^{\frac{(p-1)(N-p)}{p^2}}$, the so-called standard bubbles

$$U_{\epsilon,x_0}(x) = C_1 \left(\frac{\epsilon}{\epsilon^p + |x - x_0|^{\frac{p}{p-1}}} \right)^{\frac{N-p}{p}} \qquad \epsilon > 0, \; x_0 \in \mathbb{R}^N, \tag{2.5}$$

are the extremals of the Sobolev inequality

$$S_0 \left(\int_{\mathbb{R}^N} |u|^{p^*} \right)^{\frac{p}{p^*}} \le \int_{\mathbb{R}^N} |\nabla u|^p, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N),$$

and the unique entire solutions in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ of

$$-\Delta_p U = U^{p^*-1} \quad \text{in } \mathbb{R}^N, \tag{2.6}$$

see [5,21,25]. For $\lambda < \lambda_1$ consider its projection PU_{ϵ,x_0} in Ω, as the solution of

$$\begin{cases} -\Delta_p PU_{\epsilon,x_0} = \lambda PU_{\epsilon,x_0}^{p-1} + U_{\epsilon,x_0}^{p^*-1} & \text{in } \Omega \\ PU_{\epsilon,x_0} > 0 & \text{in } \Omega \\ PU_{\epsilon,x_0} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Letting $G_{\epsilon,x_0} = \frac{C_0}{C_1}\epsilon^{-\frac{N-p}{p}} PU_{\epsilon,x_0}$ with C_0 given by (1.4), decompose it as $G_{\epsilon,x_0} = \Gamma_{\epsilon,x_0} + H_{\epsilon,x_0}$, where

$$\Gamma_{\epsilon,x_0} = \frac{C_0}{C_1}\epsilon^{-\frac{N-p}{p}} U_{\epsilon,x_0} = \frac{C_0}{(\epsilon^p + |x-x_0|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \to \Gamma(x,x_0) \tag{2.8}$$

in $C^1_{\text{loc}}(\bar\Omega \setminus \{x_0\})$ as $\epsilon \to 0$. Since

$$f_{\epsilon,x_0} := -\Delta_p \Gamma_{\epsilon,x_0} = \Big(\frac{C_0}{C_1}\epsilon^{-\frac{N-p}{p}}\Big)^{p-1} U_{\epsilon,x_0}^{p^*-1} = \frac{C_0^{p-1} C_1^{\frac{p^2}{N-p}} \epsilon^p}{(\epsilon^p + |x-x_0|^{\frac{p}{p-1}})^{N-\frac{N-p}{p}}} \to 0 \tag{2.9}$$

in $C_{\text{loc}}(\bar\Omega \setminus \{x_0\})$ and

$$\int_\Omega f_{\epsilon,x_0} = -\int_{\partial\Omega} |\nabla \Gamma_{\epsilon,x_0}|^{p-2}\partial_\nu \Gamma_{\epsilon,x_0} \to -\int_{\partial\Omega} |\nabla\Gamma|^{p-2}(x,x_0)\partial_\nu\Gamma(x,x_0)d\sigma(x) = 1$$

as $\epsilon \to 0$ in view of (2.6) and (2.8), notice that $f_{\epsilon,x_0} \rightharpoonup \delta_{x_0}$ weakly in the sense of measures in Ω as $\epsilon \to 0$ and G_{ϵ,x_0} solves

$$\begin{cases} -\Delta_p G_{\epsilon,x_0} = \lambda G_{\epsilon,x_0}^{p-1} + f_{\epsilon,x_0} & \text{in } \Omega \\ G_{\epsilon,x_0} > 0 & \text{in } \Omega \\ G_{\epsilon,x_0} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Thanks to Theorem 2.1 and Proposition 2.2 we can now establish the following convergence result.

Proposition 2.4. *Let $2 \le p \le N$ and assume $N < 2p$ if $\lambda \neq 0$. Then there holds*

$$H_{\epsilon,x_0} \to H_\lambda(\cdot,x_0) \quad \text{in } C(\bar\Omega) \tag{2.11}$$

as $\epsilon \to 0$.

Proof. By Proposition 2.2 we can find a subsequence $\epsilon_n \to 0$ so that $G_{\epsilon_n,x_0} \to G$ in $W_0^{1,q}(\Omega)$ as $n \to +\infty$ for all $1 \le q < \bar q$, where $G = \Gamma(x,x_0) + H$ is a solution of (1.3) for some H in view of (2.8) and (2.10). In particular, if $\lambda \neq 0$ by the Sobolev embedding theorem there holds

$$G_{\epsilon_n,x_0} \to G \quad \text{in } L^p(\Omega) \text{ as } n \to +\infty \tag{2.12}$$

thanks to $\bar{q}^* > p$ in view of $N < 2p \le p^2$. Moreover, let us rewrite (2.10) in the equivalent form:

$$\begin{cases} -\Delta_p(\Gamma_{\epsilon,x_0} + H_{\epsilon,x_0}) + \Delta_p \Gamma_{\epsilon,x_0} = \lambda G_{\epsilon,x_0}^{p-1} & \text{in } \Omega \\ H_{\epsilon,x_0} = -\Gamma_{\epsilon,x_0} & \text{on } \partial\Omega. \end{cases} \tag{2.13}$$

Let us denote the solution of $(2.10)_{\lambda=0}$ by G^0_{ϵ,x_0} and set $H^0_{\epsilon,x_0} = G^0_{\epsilon,x_0} - \Gamma_{\epsilon,x_0}$. By the uniqueness part in Theorem 2.1 with $\lambda = 0$ we have that

$$G^0_{\epsilon,x_0} \to G_0(\cdot, x_0) \text{ in } W^{1,q}_0(\Omega)$$

as $\epsilon \to 0$, for all $1 \le q < \bar{q}$. Moreover, since $|H^0_{\epsilon,x_0}| \le M$ on $\partial\Omega$, by integrating (2.13) against $(H^0_{\epsilon,x_0} \mp M)_\pm$ we deduce that

$$|H^0_{\epsilon,x_0}| \le M \quad \text{in } \Omega \tag{2.14}$$

in an uniform way and then G^0_{ϵ,x_0} is locally uniformly bounded in $\bar{\Omega} \setminus \{x_0\}$. By elliptic estimates [6,16,22,23] and $(2.10)_{\lambda=0}$ we deduce that

$$G^0_{\epsilon,x_0} \text{ uniformly bounded in } C^{1,\alpha}_{\text{loc}}(\bar{\Omega} \setminus \{x_0\}) \tag{2.15}$$

for some $\alpha \in (0,1)$. Integrating $(2.13)_{\lambda=0}$ against $\eta^p H^0_{\epsilon,x_0}$, $0 \le \eta \in C^\infty_0(\Omega)$, we get that

$$\int_\Omega \eta^p |\nabla H^0_{\epsilon,x_0}|^p \le p \int_\Omega \eta^{p-1} |\nabla \eta| (|\nabla \Gamma_{\epsilon,x_0}|^{p-2} + |\nabla H^0_{\epsilon,x_0}|^{p-2}) |H^0_{\epsilon,x_0}| |\nabla H^0_{\epsilon,x_0}|$$

and then (2.14) and Young's inequality imply that

$$\nabla H^0_{\epsilon,x_0} \text{ uniformly bounded in } L^p(\Omega) \tag{2.16}$$

in view of (2.15).

Let us consider now the case $\lambda \neq 0$. Since

$$-\Delta_p(\Gamma_{\epsilon,x_0} + H_{\epsilon,x_0}) + \Delta_p(\Gamma_{\epsilon,x_0} + H^0_{\epsilon,x_0}) = \lambda G^{p-1}_{\epsilon,x_0} \quad \text{in } \Omega$$

with $H_{\epsilon,x_0} - H^0_{\epsilon,x_0} = 0$ on $\partial\Omega$, an integration against $H_{\epsilon,x_0} - H^0_{\epsilon,x_0}$ gives that

$$\int_\Omega |\nabla \left(H_{\epsilon,x_0} - H^0_{\epsilon,x_0} \right)|^p \le |\lambda| \int_\Omega G^{p-1}_{\epsilon,x_0} |H_{\epsilon,x_0} - H^0_{\epsilon,x_0}| \le |\lambda| \|G_{\epsilon,x_0}\|_p^{p-1} \|H_{\epsilon,x_0} - H^0_{\epsilon,x_0}\|_p$$

thanks to the Hölder's inequality and the coercivity properties of the p-Laplace operator, and then

$$\nabla \left(H^0_{\epsilon_n, x_0} - H^0_{\epsilon_n, x_0} \right) \text{ uniformly bounded in } L^p(\Omega) \tag{2.17}$$

in view of (2.12) and Poincaré inequality. A combination of (2.16) and (2.17) lead to a uniform L^p-bound on $\nabla H^0_{\epsilon_n, x_0}$, showing by Fatou's lemma that $\nabla H \in L^p(\Omega)$. By Theorem 2.1 we have that $G = G_\lambda(\cdot, x_0)$ and then

$$G_{\epsilon, x_0} \to G_\lambda(\cdot, x_0) \text{ in } W^{1,q}_0(\Omega) \tag{2.18}$$

as $\epsilon \to 0$, for all $1 \le q < \bar{q}$.

To extend (2.14) to the case $\lambda \neq 0$, observe that (2.10) and $-\Delta_p \Gamma_{\epsilon, x_0} = f_{\epsilon, x_0}$ in Ω imply $\|H_{\epsilon, x_0}\|_\infty \le C$ for all $\epsilon > 0$ thanks to Proposition 2.2 in view of $N < 2p$ when $\lambda \neq 0$. Since $f = \lambda G^{p-1}_{\epsilon, x_0}$ is uniformly bounded in $L^{q_0}(\Omega)$ for some $q_0 > \frac{N}{p}$ in view of $\frac{\bar{q}^*}{p-1} > \frac{N}{p}$ when $N < 2p$ and

$$|\nabla \Gamma_{\epsilon, x_0}| = \frac{C_0(N-p)}{p-1} \frac{|x - x_0|^{\frac{1}{p-1}}}{(\epsilon^p + |x - x_0|^{\frac{p}{p-1}})^{\frac{N}{p}}} \le M |\nabla \Gamma|(x, x_0),$$

we can apply (2.2) in Theorem 2.1 to H_{ϵ, x_0} as a solution to (2.13) by getting

$$|H_{\epsilon, x_0}(x) - H_\lambda(x_0, x_0)| \le C \left(r^{-\frac{N}{p-1}} \|H_{\epsilon, x_0} - H_\lambda(x_0, x_0)\|_{p-1, B_{2r}(x_0)} + r^{\frac{pq_0 - N}{q_0(p-1)}} \right) \tag{2.19}$$

for all $x \in B_r(x_0)$ and $\epsilon^{p-1} \le r \le \frac{1}{4}\text{dist}(x_0, \partial\Omega)$.

By contradiction assume that (2.11) does not hold. Then there exist sequences $\epsilon_n \to 0$ and $x_n \in \Omega$ so that $|H_{\epsilon_n, x_0}(x_n) - H_\lambda(x_n, x_0)| \ge 2\delta > 0$. Since by elliptic estimates [6,16,22,23] there holds

$$G_{\epsilon, x_0} \to G_\lambda(\cdot, x_0) \text{ in } C^1_{\text{loc}}(\bar{\Omega} \setminus \{x_0\}) \tag{2.20}$$

as $\epsilon \to 0$ in view of (2.10) and (2.18), we have that $\bar{x} = x_0$ and then

$$|H_{\epsilon_n, x_0}(x_n) - H_\lambda(x_0, x_0)| \ge \delta \tag{2.21}$$

thanks to $H_\lambda(\cdot, x_0) \in C(\bar{\Omega})$. Since by the Sobolev embedding theorem $H_{\epsilon, x_0} \to H_\lambda(\cdot, x_0)$ in $L^{p-1}(\Omega)$ as $\epsilon \to 0$ in view of (2.18) and $\bar{q}^* > p - 1$, we can insert (2.21) into (2.19) and get as $n \to +\infty$

$$\delta \le C \left(r^{-\frac{N}{p-1}} \|H_\lambda(\cdot, x_0) - H_\lambda(x_0, x_0)\|_{p-1, B_{2r}(x_0)} + r^{\frac{pq_0 - N}{q_0(p-1)}} \right) \tag{2.22}$$

for all $0 < r \le \frac{1}{4}\text{dist}(x_0, \partial\Omega)$. Since

$$r^{-\frac{N}{p-1}} \|H_\lambda(\cdot, x_0) - H_\lambda(x_0, x_0)\|_{p-1, B_{2r}(x_0)} \le C r^\alpha \to 0$$

as $r \to 0$ thanks to (2.4), estimate (2.22) leads to a contradiction and the proof is complete. □

As a by-product we have the following useful result.

Corollary 2.5. *Let $2 \le p \le N$ and assume $N < 2p$ if $\lambda \neq 0$. Then the expansion*

$$PU_{\epsilon,x_0} = U_{\epsilon,x_0} + \frac{C_1}{C_0}\epsilon^{\frac{N-p}{p}} H_\lambda(\cdot, x_0) + o\big(\epsilon^{\frac{N-p}{p}}\big) \tag{2.23}$$

does hold uniformly in Ω as $\epsilon \to 0$.

3. Energy expansions and Pohozaev identities

We are concerned with the discussion of implication $(i) \Rightarrow (ii)$ in Theorem 1.1, whereas the proof of $(ii) \Rightarrow (iii)$ in Theorem 1.1 is rather classical and can be found in [14].

Let $0 < \lambda < \lambda_1$ and $x_0 \in \Omega$ so that $H_\lambda(x_0, x_0) > 0$. In order to show $S_\lambda < S_0$ let us expand $Q_\lambda(PU_{\epsilon,x_0})$ for $\epsilon > 0$ small. Since PU_{ϵ,x_0} solves (2.7), we have that

$$\begin{aligned}\int_\Omega |\nabla PU_{\epsilon,x_0}|^p - \lambda \int_\Omega (PU_{\epsilon,x_0})^p &= \int_\Omega U_{\epsilon,x_0}^{p^*-1} PU_{\epsilon,x_0} = \int_\Omega U_{\epsilon,x_0}^{p^*} \\ &+ \frac{C_1}{C_0}\epsilon^{\frac{N-p}{p}} \int_\Omega U_{\epsilon,x_0}^{p^*-1}\left[H_\lambda(x, x_0) + o(1)\right]\end{aligned} \tag{3.1}$$

as $\epsilon \to 0$ in view of (2.23). Given $\Omega_\epsilon = \frac{\Omega - x_0}{\epsilon^{p-1}}$ observe that

$$\int_\Omega U_{\epsilon,x_0}^{p^*} = \int_{\Omega_\epsilon} U_1^{p^*} = \int_{\mathbb{R}^N} U_1^{p^*} + O(\epsilon^N) \tag{3.2}$$

and

$$\begin{aligned}\int_\Omega U_{\epsilon,x_0}^{p^*-1}[H_\lambda(x, x_0) + o(1)] &= \int_\Omega U_{\epsilon,x_0}^{p^*-1}[H_\lambda(x_0, x_0) + O(|x - x_0|^\alpha) + o(1)] \\ &= \epsilon^{\frac{(N-p)(p-1)}{p}} \int_{\Omega_\epsilon} U_1^{p^*-1}[H_\lambda(x_0, x_0) + O(\epsilon^{\alpha(p-1)}|y|^\alpha) + o(1)] \\ &= \epsilon^{\frac{(N-p)(p-1)}{p}} H_\lambda(x_0, x_0) \int_{\mathbb{R}^N} U_1^{p^*-1} + o(\epsilon^{\frac{(N-p)(p-1)}{p}})\end{aligned} \tag{3.3}$$

in view of (2.4) and $\int_{\mathbb{R}^n} U_1^{p^*-1}|y|^\alpha < +\infty$. Inserting (3.2)-(3.3) into (3.1) we deduce

$$\int_\Omega |\nabla PU_{\epsilon,x_0}|^p - \lambda \int_\Omega (PU_{\epsilon,x_0})^p = \int_{\mathbb{R}^N} U_1^{p^*} + \epsilon^{N-p}\frac{C_1}{C_0} H_\lambda(x_0, x_0) \int_{\mathbb{R}^N} U_1^{p^*-1} + o(\epsilon^{N-p}). \tag{3.4}$$

By the Taylor expansion

$$(PU_{\epsilon,x_0})^{p^*} = U_{\epsilon,x_0}^{p^*} + \epsilon^{\frac{N-p}{p}} \frac{C_1}{C_0} p^* U_{\epsilon,x_0}^{p^*-1}[H_\lambda(x,x_0) + o(1)] + O(\epsilon^{2\frac{N-p}{p}} U_{\epsilon,x_0}^{p^*-2} + \epsilon^N)$$

in view of (2.23) and $\|H_\lambda(\cdot,x_0)\|_\infty < +\infty$, we obtain

$$\int_\Omega (PU_{\epsilon,x_0})^{p^*} = \int_{\mathbb{R}^N} U_1^{p^*} + \epsilon^{N-p}\frac{C_1}{C_0} p^* H_\lambda(x_0,x_0) \int_{\mathbb{R}^N} U_1^{p^*-1} + o(\epsilon^{N-p}) \tag{3.5}$$

thanks to (3.2)-(3.3) and

$$\int_\Omega U_{\epsilon,x_0}^{p^*-2} = \epsilon^{2\frac{(N-p)(p-1)}{p}} \int_{\Omega_\epsilon} U_1^{p^*-2} = O(\epsilon^{2\frac{(N-p)(p-1)}{p}})$$

for $N < 2p$. Expansions (3.4)-(3.5) now yield

$$Q_\lambda(PU_{\epsilon,x_0}) = S_0 - (p-1) S_0^{\frac{p-N}{p}} (\int_{\mathbb{R}^N} U_1^{p^*-1}) \frac{C_1}{C_0} \epsilon^{N-p} H_\lambda(x_0,x_0) + o(\epsilon^{N-p})$$

in view of (2.6) and

$$S_0 = \frac{\int_{\mathbb{R}^N} |\nabla U_1|^p}{(\int_{\mathbb{R}^N} U_1^{p^*})^{\frac{p}{p^*}}} = (\int_{\mathbb{R}^N} U_1^{p^*})^{\frac{p}{N}}.$$

Then, for $\epsilon > 0$ small we obtain that $S_\lambda < S_0$ thanks to $H_\lambda(x_0,x_0) > 0$.

As already discussed in the Introduction, a fundamental tool is represented by the Pohozaev identity. Derived [4] for autonomous PDE's involving the p-Laplace operator, it extends to the non-autonomous case and writes, in the situation of our interest, as follows: if $u \in C^{1,\alpha}(\bar D)$ solves $-\Delta_p u = \lambda u^{p-1} + c u^{p^*-1} + f$ in D for $f \in C^1(\bar D)$ and $c \in \{0,1\}$, given $x_0 \in \mathbb{R}^N$ there holds

$$\begin{aligned} &\int_D [NH - f\langle x - x_0, \nabla u\rangle - \frac{N-p}{p}|\nabla u|^p] \\ &= \int_{\partial D} \langle x - x_0, -\frac{|\nabla u|^p}{p}\nu + |\nabla u|^{p-2}\partial_\nu u \nabla u + H\nu\rangle \end{aligned} \tag{3.6}$$

with $H(u) = \frac{\lambda}{p}u^p + \frac{c}{p^*}u^{p^*}$ and

$$\int_D |\nabla u|^p = \int_D [\lambda u^p + c u^{p^*} + fu] + \int_{\partial D} u|\nabla u|^{p-2}\partial_\nu u. \tag{3.7}$$

An integral identity of Pohozaev type for $G_\lambda(\cdot,x_0)$ like (3.8) below is of fundamental importance since $H_\lambda(x_0,x_0)$ appears as a sort of residue. In the semi-linear case such identity (3.8) holds in the limit of (3.6)-(3.7) on $B_\delta(x_0)$ as $\delta \to 0$ thanks to $\nabla H_\lambda(\cdot, x_0) \in L^\infty(\Omega)$, a property far from being obvious in the quasi-linear context where just integral bounds on $\nabla H_\lambda(\cdot, x_0)$ like (2.1) are available. Instead, we can use the special approximating sequence G_{ϵ,x_0} to derive the following result.

Proposition 3.1. *Let $2 \le p < N$ and assume $N < 2p$ if $\lambda \neq 0$. Given $x_0 \in \Omega$, $0 < \delta <$ dist $(x_0, \partial\Omega)$ and $\lambda < \lambda_1$, there holds*

$$\begin{aligned} &C_0 H_\lambda(x_0, x_0) \\ &\quad = \lambda \int\limits_{B_\delta(x_0)} G_\lambda^p(x, x_0)dx + \int\limits_{\partial B_\delta(x_0)} \Big(\frac{\delta}{p} |\nabla G_\lambda(x, x_0)|^p - \delta |\nabla G_\lambda(x, x_0)|^{p-2} (\partial_\nu G_\lambda(x, x_0))^2 \\ &\qquad - \frac{\lambda\delta}{p} G_\lambda^p(x, x_0) - \frac{N-p}{p} G_\lambda(x, x_0) |\nabla G_\lambda(x, x_0)|^{p-2} \partial_\nu G_\lambda(x, x_0) \Big) d\sigma(x) \end{aligned} \tag{3.8}$$

for some $C_0 > 0$.

Proof. Since by elliptic regularity theory [6,16,22,23] $G_{\epsilon,x_0} \in C^{1,\alpha}(\bar\Omega)$ for some $\alpha \in (0,1)$ in view of (2.10), we can apply the Pohozaev identity (3.6) to G_{ϵ,x_0} with $c = 0$ and $f = f_{\epsilon,x_0}$ on $D = B_\delta(x_0) \subset \Omega$ to get

$$\begin{aligned} &\int\limits_{\partial B_\delta(x_0)} \Big(-\frac{\delta}{p} |\nabla G_{\epsilon,x_0}|^p + \delta |\nabla G_{\epsilon,x_0}|^{p-2} (\partial_\nu G_{\epsilon,x_0})^2 + \frac{\lambda\delta}{p} G_{\epsilon,x_0}^p \\ &\qquad + \frac{N-p}{p} G_{\epsilon,x_0} |\nabla G_{\epsilon,x_0}|^{p-2} \partial_\nu G_{\epsilon,x_0} \Big) \\ &= \int\limits_{B_\delta(x_0)} \Big(\lambda G_{\epsilon,x_0}^p - \frac{N-p}{p} f_{\epsilon,x_0} G_{\epsilon,x_0} - f_{\epsilon,x_0} \langle x - x_0, \nabla G_{\epsilon,x_0} \rangle \Big) \end{aligned} \tag{3.9}$$

in view of (3.7). The approximating sequence G_{ϵ,x_0} has the key property that $\nabla G_{\epsilon,x_0}$ and f_{ϵ,x_0} are at main order multiples of $\nabla U_{\epsilon,x_0}$ and $U_{\epsilon,x_0}^{p^*-1}$, respectively, in such a way that $f_{\epsilon,x_0} \nabla G_{\epsilon,x_0}$ allows for a further integration by parts of the R.H.S. in (3.9). The function H_{ϵ,x_0} appears in the remaining lower-order terms and explains why in the limit $\epsilon \to 0$ an additional term containing $H_\lambda(x_0, x_0)$ will appear in (3.8). The identity

$$\begin{aligned} &\int\limits_{B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-2} G_{\epsilon,x_0} \langle x - x_0, \nabla U_{\epsilon,x_0} \rangle \\ &= \int\limits_{B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-1} \langle x - x_0, \nabla G_{\epsilon,x_0} - \nabla H_{\epsilon,x_0} + H_{\epsilon,x_0} \frac{\nabla U_{\epsilon,x_0}}{U_{\epsilon,x_0}} \rangle \\ &= \int\limits_{B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-1} \langle x - x_0, \nabla G_{\epsilon,x_0} + p^* H_{\epsilon,x_0} \frac{\nabla U_{\epsilon,x_0}}{U_{\epsilon,x_0}} \rangle \end{aligned}$$

$$-\delta \int_{\partial B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-1} H_{\epsilon,x_0} + N \int_{B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-1} H_{\epsilon,x_0}$$

does hold thanks to $G_{\epsilon,x_0} = \Gamma_{\epsilon,x_0} + H_{\epsilon,x_0}$ and $\Gamma_{\epsilon,x_0} \nabla U_{\epsilon,x_0} = U_{\epsilon,x_0}(\nabla G_{\epsilon,x_0} - \nabla H_{\epsilon,x_0})$, which inserted into

$$\begin{aligned}
&\int_{B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-1} \langle x - x_0, \nabla G_{\epsilon,x_0} \rangle \\
&= \delta \int_{\partial B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-1} G_{\epsilon,x_0} - (p^*-1) \int_{B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-2} G_{\epsilon,x_0} \langle x - x_0, \nabla U_{\epsilon,x_0} \rangle \\
&\quad - N \int_{B_\delta(x_0)} U_{\epsilon,x_0}^{p^*-1} G_{\epsilon,x_0}
\end{aligned}$$

leads to

$$\begin{aligned}
\int_{B_\delta(x_0)} f_{\epsilon,x_0} \langle x - x_0, \nabla G_{\epsilon,x_0} \rangle = -(p^*-1) \int_{B_\delta(x_0)} f_{\epsilon,x_0} H_{\epsilon,x_0} \Big[\langle x - x_0, \frac{\nabla U_{\epsilon,x_0}}{U_{\epsilon,x_0}} \rangle + \frac{N-p}{p} \Big] \\
- \frac{N-p}{p} \int_{B_\delta(x_0)} f_{\epsilon,x_0} G_{\epsilon,x_0} + o_\epsilon(1) \qquad (3.10)
\end{aligned}$$

as $\epsilon \to 0$ in view of (2.8)-(2.9) and (2.11). Since there holds

$$\begin{aligned}
&\frac{p(p-1)}{N-p} C_0^{1-p} C_1^{-\frac{p^2}{N-p}} \int_{B_\delta(x_0)} f_{\epsilon,x_0} H_{\epsilon,x_0} \Big[\langle x - x_0, \frac{\nabla U_{\epsilon,x_0}}{U_{\epsilon,x_0}} \rangle + \frac{N-p}{p} \Big] \\
&= \epsilon^p \int_{B_\delta(x_0)} H_{\epsilon,x_0} \frac{(p-1)\epsilon^p - |x-x_0|^{\frac{p}{p-1}}}{(\epsilon^p + |x-x_0|^{\frac{p}{p-1}})^{N+2-\frac{N}{p}}} \\
&= \int_{B_{\frac{\delta}{\epsilon^{p-1}}}(0)} H_{\epsilon,x_0}(\epsilon^{p-1} y + x_0) \frac{(p-1) - |y|^{\frac{p}{p-1}}}{(1+|y|^{\frac{p}{p-1}})^{N+2-\frac{N}{p}}} \\
&\to \int_{\mathbb{R}^N} \frac{(p-1) - |y|^{\frac{p}{p-1}}}{(1+|y|^{\frac{p}{p-1}})^{N+2-\frac{N}{p}}} H_\lambda(x_0, x_0)
\end{aligned}$$

as $\epsilon \to 0$ in view of (2.4), (2.11) and the Lebesgue convergence Theorem, we can insert (3.10) into (3.9) and as $\epsilon \to 0$ get the validity of

$$C_0 H_\lambda(x_0, x_0) = \int_{B_\delta(x_0)} \lambda G_\lambda(x, x_0)^p dx$$

$$+\int_{\partial B_\delta(x_0)} \Big(\frac{\delta}{p}|\nabla G_\lambda(x,x_0)|^p - \delta |\nabla G_\lambda(x,x_0)|^{p-2}(\partial_\nu G_\lambda(x,x_0))^2$$
$$-\frac{\lambda\delta}{p}G_\lambda^p(x,x_0) - \frac{N-p}{p}G_\lambda(x,x_0)|\nabla G_\lambda(x,x_0)|^{p-2}\partial_\nu G_\lambda(x,x_0)\Big)\, d\sigma(x)$$

in view of (2.20) and $\lim_{\epsilon\to 0} G_{\epsilon,x_0} = G_\lambda(\cdot,x_0)$ in $L^p(\Omega)$ if $\lambda \neq 0$, as it follows by (2.18) and $\bar q^* > p$ thanks to $N < 2p \le p^2$, where

$$C_0 = (p^*-1)\frac{N-p}{p(p-1)} C_0^{p-1} C_1^{\frac{p^2}{N-p}} \int_{\mathbb{R}^N} \frac{|y|^{\frac{p}{p-1}} - (p-1)}{(1+|y|^{\frac{p}{p-1}})^{N+2-\frac{N}{p}}}.$$

Concerning the sign of the constant C_0, observe that

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|y|^{\frac{p}{p-1}}}{(1+|y|^{\frac{p}{p-1}})^{N+2-\frac{N}{p}}} &= -\frac{p-1}{pN+p-N}\int_{\mathbb{R}^N} \langle y, \nabla (1+|y|^{\frac{p}{p-1}})^{\frac{N}{p}-N-1}\rangle \\ &= \frac{N(p-1)}{pN+p-N}\int_{\mathbb{R}^N} (1+|y|^{\frac{p}{p-1}})^{\frac{N}{p}-N-1}\end{aligned}$$

and then

$$\int_{\mathbb{R}^N} \frac{|y|^{\frac{p}{p-1}}}{(1+|y|^{\frac{p}{p-1}})^{N+2-\frac{N}{p}}} = \frac{N(p-1)}{p}\int_{\mathbb{R}^N} \frac{1}{(1+|y|^{\frac{p}{p-1}})^{N+2-\frac{N}{p}}},$$

which implies $C_0 > 0$ in view of

$$\int_{\mathbb{R}^N} \frac{|y|^{\frac{p}{p-1}} - (p-1)}{(1+|y|^{\frac{p}{p-1}})^{N+2-\frac{N}{p}}} = \frac{(N-p)(p-1)}{p}\int_{\mathbb{R}^N} (1+|y|^{\frac{p}{p-1}})^{\frac{N}{p}-N-2} > 0.$$

The proof of (3.8) is complete. □

4. The blow-up approach

Following [9] let us introduce the following blow-up procedure. Letting $\lambda_n = \lambda_* + \frac{1}{n}$, we have that $S_{\lambda_n} < S_0 = S_{\lambda_*}$ and then S_{λ_n} is achieved by a nonnegative $u_n \in W_0^{1,p}(\Omega)$ which, up to a normalization, satisfies

$$-\Delta_p u_n = \lambda_n u_n^{p-1} + u_n^{p^*-1} \text{ in } \Omega, \qquad \int_\Omega u_n^{p^*} = S_{\lambda_n}^{\frac{N}{p}}. \tag{4.1}$$

Since $\lambda_* < \lambda_1$, by (4.1) the sequence u_n is uniformly bounded in $W_0^{1,p}(\Omega)$ and then, up to a subsequence, $u_n \rightharpoonup u_0 \ge 0$ in $W_0^{1,p}(\Omega)$ and a.e. in Ω as $n \to +\infty$. Since

$$Q_{\lambda_n}(u) = Q_{\lambda_*}(u) - \frac{1}{n}\frac{\|u_n\|_p^p}{\|u_n\|_{p^*}^p} \geq S_0 - \frac{C}{n}$$

for some $C > 0$ thanks to the Hölder's inequality, we deduce that

$$\lim_{n\to+\infty} S_{\lambda_n} = S_0. \tag{4.2}$$

By letting $n \to +\infty$ in (4.1) we deduce that $u_0 \in W_0^{1,p}(\Omega)$ solves

$$-\Delta_p u_0 = \lambda_* u_0^{p-1} + u_0^{p^*-1} \text{ in } \Omega, \quad \int_\Omega u_0^{p^*} \leq S_0^{\frac{N}{p}},$$

thanks to $u_n \to u_0$ a.e. in Ω as $n \to +\infty$ and the Fatou convergence Theorem, and then

$$S_0 \leq Q_{\lambda_*}(u_0) = (\int_\Omega u_0^{p^*})^{\frac{p}{N}} \leq S_0$$

if $u_0 \neq 0$. Since $S_{\lambda_*} = S_0$ would be achieved by u_0 if $u_0 \neq 0$, assumption (1.2) is crucial to guarantee $u_0 = 0$ and then

$$u_n \rightharpoonup 0 \text{ in } W_0^{1,p}(\Omega), \quad u_n \to 0 \text{ in } L^q(\Omega) \text{ for } 1 \leq q < p^* \text{ and a.e. in } \Omega \tag{4.3}$$

in view of the Sobolev embedding Theorem. Since by elliptic regularity theory [6,16,22,23] and the strong maximum principle [24] $0 < u_n \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$, we can start a blow-up approach to describe the behavior of u_n since $\|u_n\|_\infty \to +\infty$ as $n \to +\infty$, as it follows by (4.3) and $\int_\Omega u_n^{p^*} = S_{\lambda_n}^{\frac{N}{p}} \to S_0^{\frac{N}{p}}$ as $n \to +\infty$.

Letting $x_n \in \Omega$ so that $u_n(x_n) = \max_\Omega u_n$, define the blow-up speed as $\mu_n = [u_n(x_n)]^{-\frac{p}{N-p}} \to 0$ as $n \to +\infty$ and the blow-up profile

$$U_n(y) = \mu_n^{\frac{N-p}{p}} u_n(\mu_n y + x_n), \quad y \in \Omega_n = \frac{\Omega - x_n}{\mu_n}, \tag{4.4}$$

which satisfies

$$-\Delta_p U_n = \lambda_n \mu_n^p U_n^{p-1} + U_n^{p^*-1} \text{ in } \Omega_n, \quad U_n = 0 \text{ on } \partial\Omega_n \tag{4.5}$$

with $0 < U_n \leq U_n(0) = 1$ in Ω_n and

$$\sup_{n\in\mathbb{N}} \left[\int_{\Omega_n} |\nabla U_n|^p + \int_{\Omega_n} U_n^{p^*}\right] < +\infty.$$

Since U_n is uniformly bounded in $C^{1,\alpha}(A \cap \Omega_n)$ for all $A \subset\subset \mathbb{R}^N$ by elliptic estimates [6,16,22,23], we get that, up to a subsequence, $U_n \to U$ in $C^1_{\text{loc}}(\bar{\Omega}_\infty)$, where Ω_∞ is an half-space with $\text{dist}(0, \partial\Omega_\infty) = L \in (0, \infty]$ in view of $1 = U_n(0) - U_n(y) \leq C|y|$ for $y \in B_2(0) \cap \partial\Omega_n$ and $U \in D^{1,p}(\Omega_\infty)$ solves

$$-\Delta_p U = U^{p^*-1} \text{ in } \Omega_\infty, \quad U = 0 \text{ on } \partial\Omega_\infty, \quad 0 < U \leq U(0) = 1 \text{ in } \Omega_\infty.$$

Since $L < +\infty$ would provide $U \in D_0^{1,p}(\Omega_\infty)$, by [17] one would get $U = 0$, in contradiction with $U(0) = 1$. Since

$$\lim_{n\to+\infty} \frac{\text{dist}(x_n, \partial\Omega)}{\mu_n} = \lim_{n\to+\infty} \text{dist}(0, \partial\Omega_n) = +\infty, \tag{4.6}$$

by [5,21,25] we have that U coincides with $U_\infty = (1 + \Lambda|y|^{\frac{p}{p-1}})^{-\frac{N-p}{p}}$, $\Lambda = C_1^{-\frac{p^2}{(N-p)(p-1)}}$ (by (2.5) with $x_0 = 0$ and $\epsilon = C_1^{\frac{p}{(N-p)(p-1)}}$ to have $U_\infty(0) = 1$). Since

$$U_n(y) = \mu_n^{\frac{N-p}{p}} u_n(\mu_n y + x_n) \to (1 + \Lambda|y|^{\frac{p}{p-1}})^{-\frac{N-p}{p}} \text{ uniformly in } B_R(0) \tag{4.7}$$

as $n \to +\infty$ for all $R > 0$, in particular there holds

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \int_{B_{R\mu_n}(x_n)} u_n^{p^*} = \int_{\mathbb{R}^N} U_\infty^{p^*} = S_0^{\frac{N}{p}}. \tag{4.8}$$

Contained in (4.1)-(4.2), the energy information $\lim_{n\to+\infty} \int_\Omega u_n^{p^*} = S_0^{\frac{N}{p}}$ combines with (4.8) to give

$$\lim_{R\to+\infty} \lim_{n\to+\infty} \int_{\Omega\setminus B_{R\mu_n}(x_n)} u_n^{p^*} = 0, \tag{4.9}$$

a property which will simplify the blow-up description of u_n. Up to a subsequence, let us assume $x_n \to x_0 \in \bar{\Omega}$ as $n \to +\infty$.

The proof of the implication $(iii) \Rightarrow (i)$ in Theorem 1.1 proceeds through the 5 steps that will be developed below. The main technical point is to establish a comparison between u_n and the bubble

$$U_n(x) = \frac{\mu_n^{\frac{N-p}{p(p-1)}}}{(\mu_n^{\frac{p}{p-1}} + \Lambda|x - x_n|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}$$

in the form $u_n \leq CU_n$ in Ω, no matter x_n tends to $\partial\Omega$ or not. Thanks to such a fundamental estimate, we will first apply some Pohozaev identity in the whole Ω_n to exclude the boundary blow-up $d_n = \text{dist}\,(x_n, \partial\Omega) \to 0$ as $n \to +\infty$. In the interior case, still by a Pohozaev identity on $B_\delta(x_n)$ as $n \to +\infty$ and $\delta \to 0$, we will obtain an information on the limiting blow-up point $x_0 = \lim_{n\to+\infty} x_n \in \Omega$ in the form $H_{\lambda_*}(x_0, x_0) = 0$ and then the property $H_\lambda(x_0, x_0) > H_{\lambda_*}(x_0, x_0) = 0$ for $\lambda > \lambda_*$ will follow by the monotonicity of $H_\lambda(x_0, x_0)$.

Step 1. There holds $u_n \to 0$ in $C_{\rm loc}(\bar\Omega \setminus \{x_0\})$ as $n \to +\infty$, where $x_0 = \lim_{n\to+\infty} x_n \in \bar\Omega$.

First observe that

$$u_n \to 0 \text{ in } L^{p^*}_{\rm loc}(\bar\Omega \setminus \{x_0\}) \tag{4.10}$$

as $n \to +\infty$ in view of (4.9) and we are then concerned with establishing the uniform convergence by a Moser iterative argument. Given a compact set $K \subset \bar\Omega \setminus \{x_0\}$, consider $\eta \in C_0^\infty(\mathbb{R}^N \setminus \{x_0\})$ be a cut-off function with $0 \le \eta \le 1$ and $\eta = 1$ in K. Since $u_n = 0$ on $\partial\Omega$, use $\eta^p u_n^\beta$, $\beta \ge 1$, as a test function in (4.1) to get

$$\frac{\beta p^p}{(\beta-1+p)^p}\int_\Omega \eta^p |\nabla w_n|^p \le \frac{p^p}{(\beta-1+p)^{p-1}}\int_\Omega \eta^{p-1}|\nabla\eta| w_n |\nabla w_n|^{p-1} + \int_\Omega \lambda_n \eta^p w_n^p + \int_\Omega \eta^p u_n^{p^*-p} w_n^p$$

in terms of $w_n = u_n^{\frac{\beta-1+p}{p}}$ and then by the Young inequality

$$\int_\Omega \eta^p |\nabla w_n|^p \le C\beta^p \left(\int_\Omega |\nabla \eta|^p w_n^p + \int_\Omega \eta^p w_n^p + \int_\Omega \eta^p u_n^{p^*-p} w_n^p \right) \tag{4.11}$$

for some $C > 0$. Since by the Hölder inequality

$$\int_\Omega \eta^p u_n^{p^*-p} w_n^p \le C\Big(\int_{\Omega\cap supp\ \eta} u_n^{p^*}\Big)^{\frac{p}{N}} \|\eta w_n\|_{p^*}^p = o(\|\eta w_n\|_{p^*}^p)$$

as $n \to +\infty$ in view of (4.10) and $\Omega \cap supp\ \eta \subset\subset \bar\Omega \setminus \{x_0\}$, by (4.11) and the Sobolev embedding Theorem we deduce that

$$\|\eta w_n\|_{p^*}^p \le C\|w_n\|_p^p = C\int_\Omega u_n^{\beta-1+p} \to 0$$

for all $1 \le \beta < p^* - p + 1$ in view of (4.3) and then $u_n \to 0$ in $L^q(K)$ for all $1 \le q < \frac{Np^*}{N-p}$ as $n \to +\infty$. We have then established that

$$u_n \to 0 \text{ in } L^q_{\rm loc}(\bar\Omega \setminus \{x_0\}) \tag{4.12}$$

as $n \to +\infty$ for all $1 \le q < \frac{Np^*}{N-p}$. Since $\frac{N}{p}(p^* - p) = p^* < \frac{Np^*}{N-p}$, observe that (4.12) now provides that the R.H.S. in the equation (4.1) can be written as $(\lambda_n + u_n^{p^*-p})u_n^{p-1}$ with a bound on the coefficient $\lambda_n + u_n^{p^*-p}$ in $L^{q_0}_{\text{loc}}(\bar\Omega \setminus \{x_0\})$ for some $q_0 > \frac{N}{p}$. Given compact sets $K \subset \tilde K \subset \bar\Omega \setminus \{x_0\}$ with dist $(K, \partial \tilde K) > 0$, by [22] we have the estimate $\|u_n\|_{\infty,K} \le C\|u_n\|_{p,\tilde K}$ and then $u_n \to 0$ in $C(K)$ as $n \to +\infty$ in view of (4.12) and $p < \frac{Np^*}{N-p}$. The convergence $u_n \to 0$ in $C_{\text{loc}}(\bar\Omega \setminus \{x_0\})$ has been then established as $n \to +\infty$.

Step 2. The following pointwise estimates

$$\lim_{n\to+\infty} \max_{\Omega} |x - x_n|^{\frac{N-p}{p}} u_n < \infty, \quad \lim_{R\to+\infty} \lim_{n\to+\infty} \max_{\Omega\setminus B_{R\mu_n}(x_n)} |x - x_n|^{\frac{N-p}{p}} u_n = 0 \quad (4.13)$$

do hold.

By contradiction and up to a subsequence, assume the existence of $y_n \in \Omega$ such that either

$$|x_n - y_n|^{\frac{N-p}{p}} u_n(y_n) = \max_{\Omega} |x - x_n|^{\frac{N-p}{p}} u_n \to +\infty \quad (4.14)$$

as $n \to +\infty$ or

$$\max_{\Omega} |x - x_n|^{\frac{N-p}{p}} u_n \le C_0, \quad |x_n - y_n|^{\frac{N-p}{p}} u_n(y_n) = \max_{\Omega\setminus B_{R_n\mu_n}(x_n)} |x - x_n|^{\frac{N-p}{p}} u_n \ge \delta > 0 \quad (4.15)$$

for some $R_n \to +\infty$ as $n \to +\infty$. Setting $\nu_n = [u_n(y_n)]^{-\frac{p}{N-p}}$, there hold $\frac{|x_n-y_n|}{\nu_n} \to +\infty$ in case (4.14), $\frac{|x_n-y_n|}{\nu_n} \in [\delta^{\frac{p}{N-p}}, C_0^{\frac{p}{N-p}}]$ in case (4.15) and $\nu_n \to 0$ as $n \to +\infty$, since $x_n - y_n \to 0$ as $n \to +\infty$ when (4.15) holds thanks to Step 1. Up to a further subsequence, let us assume that $\frac{x_n-y_n}{\nu_n} \to p$ as $n \to +\infty$, where $p = +\infty$ in case (4.14) and $p \in \mathbb{R}^N \setminus \{0\}$ in case (4.15). Since $(\frac{|x_n-y_n|}{\mu_n})^{\frac{N-p}{p}} \ge (\frac{|x_n-y_n|}{\mu_n})^{\frac{N-p}{p}} U_n(\frac{y_n-x_n}{\mu_n}) = |x_n - y_n|^{\frac{N-p}{p}} u_n(y_n)$ in view of (4.5), where U_n is given by (4.4), then $\frac{|x_n-y_n|}{\mu_n} \to +\infty$ as $n \to +\infty$ also in case (4.14). Setting $V_n(y) = \nu_n^{\frac{N-p}{p}} u_n(\nu_n y + y_n)$ for $y \in \tilde\Omega_n = \frac{\Omega-y_n}{\nu_n}$, then $V_n(0) = 1$ and in $\tilde\Omega_n$ there hold:

$$\begin{aligned} V_n(y) &\le \nu_n^{\frac{N-p}{p}} |\nu_n y + y_n - x_n|^{-\frac{N-p}{p}} |x_n - y_n|^{\frac{N-p}{p}} u_n(y_n) = (\frac{|x_n - y_n|}{|\nu_n y + y_n - x_n|})^{\frac{N-p}{p}} \\ &\le 2^{\frac{N-p}{p}} \end{aligned} \quad (4.16)$$

for $|y| \le \frac{1}{2}\frac{|x_n-y_n|}{\nu_n}$ in case (4.14) and

$$|y - \frac{x_n - y_n}{\nu_n}|^{\frac{N-p}{p}} V_n(y) = |\nu_n y + y_n - x_n|^{\frac{N-p}{p}} u_n(\nu_n y + y_n) \le C_0 \quad (4.17)$$

in case (4.15). Since

$$-\Delta_p V_n = \lambda_n \nu_n^p V_n^{p-1} + V_n^{p^*-1} \text{ in } \tilde{\Omega}_n, \quad V_n = 0 \text{ on } \partial\tilde{\Omega}_n,$$

by (4.16)-(4.17) and standard elliptic estimates [6,16,22,23] we get that V_n is uniformly bounded in $C^{1,\alpha}(A \cap \tilde{\Omega}_n)$ for all $A \subset\subset \mathbb{R}^N \setminus \{p\}$. Up to a subsequence, we have that $V_n \to V$ in $C^1_{\text{loc}}(\bar{\Omega}_\infty \setminus \{p\})$, where Ω_∞ is an half-space with $\text{dist}(0, \partial\Omega_\infty) = L$. Since $p \neq 0$, there hold $B_{\frac{|p|}{2}}(0) \subset\subset \mathbb{R}^N \setminus \{p\}$ and $1 = V_n(0) - V_n(y) \leq C|y|$ for $y \in B_{\frac{|p|}{2}}(0) \cap \partial\tilde{\Omega}_n$, leading to $L \in (0, \infty]$. Since $V \geq 0$ solves $-\Delta_p V = V^{p^*-1}$ in Ω_∞, by the strong maximum principle [24] we deduce that $V > 0$ in Ω_∞ in view of $V(0) = 1$ thanks to $0 \in \Omega_\infty$. Setting $M = \min\{L, |p|\}$, by $\frac{|x_n - y_n|}{\mu_n} \to +\infty$ as $n \to +\infty$ we have that $B_{\frac{M}{2}\nu_n}(y_n) \subset \Omega \setminus B_{R\mu_n}(x_n)$ for all $R > 0$ provided n is sufficiently large (depending on R) and then

$$\int_{\Omega \setminus B_{R\mu_n}(x_n)} u_n^{p^*} \geq \int_{B_{\frac{M}{2}\nu_n}(y_n)} u_n^{p^*} = \int_{B_{\frac{M}{2}}(0)} V_n^{p^*} \to \int_{B_{\frac{M}{2}}(0)} V^{p^*} > 0,$$

in contradiction with (4.9). The proof of (4.13) is complete.

Step 3. There exists $C > 0$ so that

$$u_n \leq \frac{C \mu_n^{\frac{N-p}{p(p-1)}}}{(\mu_n^{\frac{p}{p-1}} + \Lambda |x - x_n|^{\frac{p}{p-1}})^{\frac{N-p}{p}}} \quad \text{in } \Omega \tag{4.18}$$

does hold for all $n \in \mathbb{N}$.

Since (4.18) does already hold in $B_{R\mu_n}(x_n)$ for all $R > 0$ thanks to (4.7), notice that (4.18) is equivalent to establish the estimate

$$u_n \leq \frac{C \mu_n^{\frac{N-p}{p(p-1)}}}{|x - x_n|^{\frac{N-p}{p-1}}} \quad \text{in } \Omega \setminus B_{R\mu_n}(x_n) \tag{4.19}$$

for some $C, R > 0$ and all $n \in N$. Let us first prove the following weaker form of (4.19): given $0 < \eta < \frac{N-p}{p(p-1)}$ there exist $C, R > 0$ so that

$$u_n \leq \frac{C \mu_n^{\frac{N-p}{p(p-1)} - \eta}}{|x - x_n|^{\frac{N-p}{p-1} - \eta}} \quad \text{in } \Omega \setminus B_{R\mu_n}(x_n) \tag{4.20}$$

does hold for all $n \in N$. Since $|x - x_n|^{\eta - \frac{N-p}{p-1}}$ satisfies

$$-\Delta_p |x - x_n|^{\eta - \frac{N-p}{p-1}} = \eta(p-1)(\frac{N-p}{p-1} - \eta)^{p-1} |x - x_n|^{\eta(p-1) - N},$$

we have that $\Phi_n = C \frac{\mu_n^{\frac{N-p}{p(p-1)} - \eta} + M_n}{|x - x_n|^{\frac{N-p}{p-1} - \eta}}$, where $\rho, C > 0$ and $M_n = \sup\limits_{\Omega \cap \partial B_\rho(x_0)} u_n$, satisfies

$$
\begin{aligned}
&-\Delta_p \Phi_n - (\lambda_n + \frac{\delta}{|x-x_n|^p})\Phi_n^{p-1} \\
&= \left[\eta(p-1)(\frac{N-p}{p-1}-\eta)^{p-1} - (\lambda_n|x-x_n|^p+\delta)\right]\frac{\Phi_n^{p-1}}{|x-x_n|^p} \\
&\geq 0 \quad \text{in } \Omega\cap B_\rho(x_0)\setminus\{x_n\}
\end{aligned}
$$

provided ρ and δ are sufficiently small (depending on η). Taking $R>0$ large so that $u_n^{p^*-p} \leq \frac{\delta}{|x-x_n|^p}$ in $\Omega\setminus B_{R\mu_n}(x_n)$ for all n large thanks to (4.13), we have that

$$
-\Delta_p u_n - (\lambda_n + \frac{\delta}{|x-x_n|^p})u_n^{p-1} = (u_n^{p^*-p} - \frac{\delta}{|x-x_n|^p})u_n^{p-1} \leq 0 \quad \text{in } \Omega\setminus B_{R\mu_n}(x_n).
$$

By (4.7) on $\partial B_{R\mu_n}(x_n)$ it is easily seen that $u_n \leq \Phi_n$ on the boundary of $\Omega\cap B_\rho(x_0)\setminus B_{R\mu_n}(x_n)$ for some $C>0$, and then by Proposition 2.3 one deduces the validity of

$$
u_n \leq C\frac{\mu_n^{\frac{N-p}{p(p-1)}-\eta}+M_n}{|x-x_n|^{\frac{N-p}{p-1}-\eta}} \tag{4.21}
$$

in $\Omega\cap B_\rho(x_0)\setminus B_{R\mu_n}(x_n)$. Setting $A=\Omega\setminus B_\rho(x_0)$, observe that the function $v_n=\frac{u_n}{M_n}$ satisfies

$$
-\Delta_p v_n - \lambda_n v_n^{p-1} = f_n \text{ in } \Omega, \quad v_n=0 \text{ on } \partial\Omega, \quad \sup_{\Omega\cap\partial B_\rho(x_0)} v_n = 1, \tag{4.22}
$$

where $f_n = \frac{u_n^{p^*-1}}{M_n^{p-1}} = u_n^{\frac{p^2}{N-p}} v_n^{p-1}$. Letting g_n be the p-harmonic function in A so that $g_n=v_n$ on ∂A, observe that $\|g_n\|_\infty=1$ in view of $0\leq v_n\leq 1$ on ∂A. Since by Step 1 there holds

$$
a_n = \lambda_n + u_n^{\frac{p^2}{N-p}} \to \lambda_* \quad \text{in } L^\infty(A)
$$

as $n\to+\infty$ with $\lambda_*<\lambda_1(\Omega)<\lambda_1(A)$, by Proposition 2.2 we deduce that $\sup_{n\in\mathbb{N}}\|v_n\|_{p-1,A}<+\infty$ and then $\sup_{n\in\mathbb{N}}\|f_n\|_{1,A}<+\infty$ in view of Step 1. Letting w_n the solution of

$$
-\Delta_p w_n = f_n \text{ in } A, \quad w_n = 0 \text{ on } \partial A,
$$

by Proposition 2.2 we also deduce that $\sup_{n\in\mathbb{N}}\|v_n-w_n\|_{\infty,A}<+\infty$ thanks to $N<2p$. Since by the Sobolev embedding Theorem $\sup_{n\in\mathbb{N}}\|w_n\|_{q,A}<+\infty$ for all $1\leq q<\bar{q}^*$ in view of Proposition 2.2 and $\sup_{n\in\mathbb{N}}\|f_n\|_{1,A}<+\infty$, similar estimates hold for v_n and then $\sup_{n\in\mathbb{N}}\|f_n\|_{q_0,A}<+\infty$ for some $q_0>\frac{N}{p}$ in view of $N<2p$. By elliptic estimates [22] we get that $\sup_{n\in\mathbb{N}}\|w_n\|_{\infty,A}<+\infty$ and in turn $\sup_{n\in\mathbb{N}}\|v_n\|_{\infty,A}<+\infty$, or equivalently

$$\sup_{\Omega\setminus B_\rho(x_0)} u_n \le C \sup_{\Omega\cap\partial B_\rho(x_0)} u_n \tag{4.23}$$

for some $C > 0$. Thanks to (4.23) one can extend the validity of (4.21) from $\Omega\cap B_\rho(x_0)\setminus B_{R\mu_n}(x_n)$ to $\Omega\setminus B_{R\mu_n}(x_n)$. In order to establish (4.20), we claim that M_n in (4.21) satisfies

$$M_n = o(\mu_n^{\frac{N-p}{p(p-1)}-\eta}) \tag{4.24}$$

for all $0 < \eta < \frac{N-p}{p(p-1)}$.

Indeed, by contradiction assume that there exist $0 < \bar\eta < \frac{N-p}{p(p-1)}$ and a subsequence so that

$$\mu_n^{\frac{N-p}{p(p-1)}-\bar\eta} \le CM_n \tag{4.25}$$

for some $C > 0$. Since $v_n = O(|x-x_n|^{-\frac{N-p}{p-1}+\bar\eta})$ uniformly in $\Omega\setminus B_{R\mu_n}(x_n)$ in view of (4.21) and (4.25), we have that v_n and then $f_n = u_n^{\frac{p^2}{N-p}} v_n^{p-1}$ are uniformly bounded in $C_{\rm loc}(\bar\Omega\setminus\{x_0\})$ and by elliptic estimates [6,16,22,23] $v_n \to v$ in $C^1_{\rm loc}(\bar\Omega\setminus\{x_0\})$ as $n\to+\infty$, up to a further subsequence, where $v\neq 0$ in view of $\sup\limits_{\Omega\cap\partial B_\rho(x_0)} v = \lim\limits_{n\to+\infty}\sup\limits_{\Omega\cap\partial B_\rho(x_0)} v_n = 1$. Moreover, notice that $\lim\limits_{n\to+\infty}\|f_n\|_1 = 0$ would imply $v_n\to v$ in $W_0^{1,q}(\Omega)$ for all $1\le q<\bar q$ and in $L^s(\Omega)$ for all $1\le s<\bar q^*$ as $n\to+\infty$ in view of Proposition 2.2, where v is a solution of

$$-\Delta_p v - \lambda_* v^{p-1} = 0 \quad \text{in } \Omega. \tag{4.26}$$

Letting

$$T_l(s) = \begin{cases} |s| & \text{if } |s|\le l \\ \pm l & \text{if } \pm s > l \end{cases}$$

and using $T_l(v_n)\in W_0^{1,p}(\Omega)$ as a test function in (4.22), one would get

$$\int\limits_{\{|v_n|\le l\}} |\nabla v_n|^p \le \lambda_n \int\limits_\Omega v_n^p + l\|f_n\|_1 \to \lambda_* \int\limits_\Omega v^p$$

as $n\to+\infty$ in view of $\bar q^* > p$ and then deduce

$$\int\limits_\Omega |\nabla v|^p \le \lambda_* \int\limits_\Omega v^p < +\infty$$

as $l\to+\infty$. Since $v\in W_0^{1,p}(\Omega)$ solves (4.26) with $\lambda_* < \lambda_1$, one would have $v=0$, in contradiction with $\sup\limits_{\Omega\cap\partial B_\rho(x_0)} v = 1$. Once

$$\liminf_{n\to+\infty}\|f_n\|_1>0 \tag{4.27}$$

has been established, by (4.4), (4.7) and (4.21) observe that

$$\int_{B_{R\mu_n}(x_n)} f_n=\int_{B_{R\mu_n}(x_n)}\frac{u_n^{p^*-1}}{M_n^{p-1}}=\frac{\mu_n^{\frac{N-p}{p}}}{M_n^{p-1}}\int_{B_R(0)}U_n^{p^*-1}=O\left(\frac{\mu_n^{\frac{N-p}{p}}}{M_n^{p-1}}\right) \tag{4.28}$$

and

$$\int_{\Omega\setminus B_{R\mu_n}(x_n)} f_n=L_n^{\frac{p^2}{N-p}}(\frac{L_n}{M_n})^{p-1}O\left(\mu_n^{(p^*-1)\eta-\frac{p}{p-1}}\log\frac{1}{\mu_n}+1\right) \tag{4.29}$$

where $L_n=\mu_n^{\frac{N-p}{p(p-1)}-\eta}+M_n$. Setting $\eta_0=\frac{p}{(p-1)(p^*-1)}$, then (4.24) necessarily holds for $\eta\in(\eta_0,\frac{N-p}{p(p-1)})$ since otherwise $L_n=O(M_n)$ and (4.28)-(4.29) would provide $\lim\limits_{n\to+\infty}\|f_n\|=0$ along a subsequence thanks to $\lim\limits_{n\to+\infty}M_n=0$, in contradiction with (4.27). Notice that (4.24) holds for $\eta=\eta_0$ too, since otherwise the conclusion $\lim\limits_{n\to+\infty}\|f_n\|=0$ would follow as above thanks to $L_n=O(\mu_n^{\frac{N-p}{p(p-1)}-\eta})$ for $\eta\in(\eta_0,\frac{N-p}{p(p-1)})$. Setting $\eta_k=(\frac{p^2}{Np-N+p})^k\eta_0$, arguing as above (4.24) can be established for $\eta\in[\eta_{k+1},\eta_k)$, $k\geq 0$, by using the validity of (4.24) for $\eta\in[\eta_k,\frac{N-p}{p-1})$ in view of the relation

$$(p^*-1)\eta_{k+1}-\frac{p}{p-1}+\frac{p^2}{N-p}\left[\frac{N-p}{p(p-1)}-\eta_k\right]=0.$$

Since $\frac{p^2}{Np-N+p}<1$ for $p<N$, we have that $\eta_k\to 0$ as $k\to+\infty$ and then (4.24) is proved for all $0<\eta<\frac{N-p}{p(p-1)}$, in contradiction with (4.25). Therefore, we have established (4.24) and the validity of (4.20) follows.

In order to establish (4.19), let us repeat the previous argument for $v_n=\mu_n^{-\frac{N-p}{p(p-1)}}u_n$, where v_n solves

$$-\Delta_p v_n-\lambda_n v_n^{p-1}=f_n \text{ in } \Omega,\quad v_n=0 \text{ on } \partial\Omega, \tag{4.30}$$

with $f_n=\mu_n^{-\frac{N-p}{p}}u_n^{p^*-1}$. Notice that f_n satisfies

$$f_n\leq\frac{C_0\mu_n^{\frac{p}{p-1}-(p^*-1)\eta}}{|x-x_n|^{N+\frac{p}{p-1}-(p^*-1)\eta}}\quad \text{in } \Omega\setminus B_{R\mu_n}(x_n) \tag{4.31}$$

for some $C_0>0$ in view of (4.20) and then, by arguing as in (4.28),

$$\int_\Omega f_n=O(1)+O\Big(\int_{\Omega\setminus B_{R\mu_n}(x_n)}\frac{\mu_n^{\frac{p}{p-1}-(p^*-1)\eta}}{|x-x_n|^{N+\frac{p}{p-1}-(p^*-1)\eta}}\Big)=O(1) \tag{4.32}$$

for $0<\eta<\frac{p}{(p-1)(p^*-1)}=\frac{p(N-p)}{(p-1)(Np-N+p)}$. Letting h_n be the solution of

$$-\Delta_p h_n = f_n \text{ in } \Omega, \quad h_n = 0 \text{ on } \partial\Omega,$$

by (4.32) and Proposition 2.2 we deduce that $\sup\limits_{n\in\mathbb{N}} \|v_n - h_n\|_\infty < +\infty$ thanks to $N<2p$, or equivalently

$$\|u_n - \mu_n^{\frac{N-p}{p(p-1)}} h_n\|_\infty = O(\mu_n^{\frac{N-p}{p(p-1)}}). \tag{4.33}$$

For $\alpha > N$ the radial function

$$W(y) = (\alpha - N)^{-\frac{1}{p-1}} \int_{|y|}^{\infty} \frac{(t^{\alpha-N}-1)^{\frac{1}{p-1}}}{t^{\frac{\alpha-1}{p-1}}} dt$$

is a positive and strictly decreasing solution of $-\Delta_p W = |y|^{-\alpha}$ in $\mathbb{R}^N \setminus B_1(0)$ so that

$$\lim_{|y|\to\infty} |y|^{\frac{N-p}{p-1}} W(y) = \frac{p-1}{N-p} (\alpha - N)^{-\frac{1}{p-1}} > 0. \tag{4.34}$$

Taking $0<\eta<\frac{p}{(p-1)(p^*-1)}$ to ensure $\alpha := N + \frac{p}{p-1} - (p^*-1)\eta > N$, then $w_n(x) = \mu_n^{-\frac{N-p}{p-1}} W(\frac{x-x_n}{\mu_n})$ satisfies

$$-\Delta_p w_n = \frac{\mu_n^{\frac{p}{p-1}-(p^*-1)\eta}}{|x-x_n|^{N+\frac{p}{p-1}-(p^*-1)\eta}} \quad \text{in } \mathbb{R}^N \setminus B_1(x_n).$$

Since

$$h_n(x) = \mu_n^{-\frac{N-p}{p(p-1)}} u_n(x) + O(1) = \mu_n^{-\frac{N-p}{p-1}} U_n(\frac{x-x_n}{\mu_n}) + O(1) \le C_1 w_n(x)$$

for some $C_1>0$ and for all $x \in \partial B_{R\mu_n}(x_n)$ in view of (4.7), (4.33) and $W(R)>0$, we have that $\Phi_n = C w_n$ satisfies

$$-\Delta_p \Phi_n \ge f_n \text{ in } \Omega \setminus B_{R\mu_n}(x_n), \quad \Phi_n \ge h_n \text{ on } \partial\Omega \cup \partial B_{R\mu_n}(x_n)$$

for $C = C_0^{\frac{1}{p-1}} + C_1$ thanks to (4.31), and then by weak comparison principle we deduce that

$$h_n \le \Phi_n \le \frac{C}{|x-x_n|^{\frac{N-p}{p-1}}} \quad \text{in } \Omega \setminus B_{R\mu_n}(x_n) \tag{4.35}$$

for some $C>0$ in view of (4.34). Inserting (4.35) into (4.33) we finally deduce the validity of (4.19)

Step 4. There holds $x_0 \notin \partial\Omega$.

Assume by contradiction $x_0 \in \partial\Omega$ and set $\hat{x} = x_0 - \nu(x_0)$. Let us apply the Pohozaev identity (3.6) to u_n with $c = 1$, $f = 0$ and $x_0 = \hat{x}$ on $D = \Omega$, together with (3.7), to get

$$\int_{\partial\Omega} |\nabla u_n|^p \langle x - \hat{x}, \nu\rangle = \frac{p}{p-1}\lambda_n \int_{\Omega} u_n^p \tag{4.36}$$

in view of $u_n = 0$ and $\nabla u_n = (\partial_\nu u_n)\nu$ on $\partial\Omega$. Since $v_n = \mu_n^{-\frac{N-p}{p(p-1)}} u_n$ solves (4.30) and v_n, f_n are uniformly bounded in $C_{\text{loc}}(\bar{\Omega} \setminus \{x_0\})$ in view of (4.18) and (4.31) with $\eta = 0$, by elliptic estimates [6,16,22,23] we deduce that v_n is uniformly bounded in $C^1_{\text{loc}}(\bar{\Omega} \setminus \{x_0\})$. Fixing $\rho > 0$ small so that $\langle x - \hat{x}, \nu(x)\rangle \geq \frac{1}{2}$ for all $x \in \partial\Omega \cap B_\rho(x_0)$, by (4.18), (4.36) and the C^1-bound on v_n we have that

$$\int_{\partial\Omega\cap B_\rho(x_0)} |\nabla u_n|^p = O(\lambda_n \int_{\Omega} u_n^p + \int_{\partial\Omega\setminus B_\rho(x_0)} |\nabla u_n|^p) = O(\mu_n^{\frac{N-p}{p-1}}) \tag{4.37}$$

since $\frac{p(N-p)}{p-1} < N$ thanks $N < 2p \leq p^2$. Setting $d_n = \text{dist}(x_n, \partial\Omega)$ and $W_n(y) = d_n^{\frac{N-p}{p}} u_n(d_n y + x_n)$ for $y \in \Omega_n = \frac{\Omega - x_n}{d_n}$, we have that $d_n \to 0$ and $\Omega_n \to \Omega_\infty$ as $n \to +\infty$ where Ω_∞ is an halfspace containing 0 with $\text{dist}(0, \partial\Omega_\infty) = 1$. Setting $\delta_n = \frac{\mu_n}{d_n} \to 0$ as $n \to +\infty$ in view of (4.6), the function $G_n = \delta_n^{-\frac{N-p}{p(p-1)}} W_n = \mu_n^{-\frac{N-p}{p(p-1)}} d_n^{\frac{N-p}{p-1}} u_n(d_n y + x_n) \geq 0$ solves

$$-\Delta_p G_n - \lambda_n d_n^p G_n^{p-1} = \tilde{f}_n \text{ in } \Omega_n, \quad G_n = 0 \text{ on } \partial\Omega_n, \tag{4.38}$$

with $\tilde{f}_n = \mu_n^{-\frac{N-p}{p}} d_n^N u_n^{p^*-1}(d_n y + x_n) = d_n^N f_n(d_n y + x_n)$ so that

$$\tilde{f}_n \leq \frac{C\delta_n^{\frac{p}{p-1}}}{|y|^{N+\frac{p}{p-1}}}, \; G_n \leq \frac{C}{|y|^{\frac{N-p}{p-1}}} \quad \text{in } \Omega_n \tag{4.39}$$

in view of (4.18) and (4.31) with $\eta = 0$. By (4.39) and elliptic estimates [6,16,22,23] we deduce that $G_n \to G$ in $C^1_{\text{loc}}(\bar{\Omega}_\infty \setminus \{0\})$ as $n \to +\infty$, where $G \geq 0$ does solve

$$-\Delta_p G = \left(\int_{\mathbb{R}^N} U^{p^*-1}\right) \delta_0 \text{ in } \Omega_\infty, \quad G = 0 \text{ on } \partial\Omega_\infty,$$

in view of (4.38) and

$$\lim_{n\to+\infty} \int_{B_\epsilon(0)} \tilde{f}_n = \lim_{n\to+\infty} \int_{B_{\epsilon\frac{d_n}{\mu_n}}(0)} U_n^{p^*-1} = \int_{\mathbb{R}^N} U^{p^*-1} \tag{4.40}$$

for all $\epsilon > 0$ in view of (4.6)-(4.7) and (4.18). By the strong maximum principle [24] we then have that $G > 0$ in Ω_∞ and $\partial_\nu G < 0$ on $\partial\Omega_\infty$. On the other hand, for any $R > 0$ there holds

$$\int_{\partial\Omega_n \cap B_R(0)} |\nabla G_n|^p = \mu_n^{-\frac{N-p}{p-1}} d_n^{\frac{N-1}{p-1}} \int_{\partial\Omega \cap B_{Rd_n}(x_n)} |\nabla u_n|^p = O(d_n^{\frac{N-1}{p-1}})$$

in view of (4.37) and then as $n \to +\infty$

$$\int_{\partial\Omega_\infty \cap B_R(0)} |\nabla G|^p = 0.$$

We end up with the contradictory conclusion $\nabla G = 0$ on $\partial\Omega_\infty$, and then $x_0 \notin \partial\Omega$.

Step 5. There holds $H_{\lambda_*}(x_0, x_0) = 0$.

Let us apply the Pohozaev identity (3.6) to u_n with $c = 1$ and $f = 0$ on $D = B_\delta(x_0) \subset \Omega$ and (3.7) to get

$$\lambda_n \int_{B_\delta(x_0)} u_n^p + \int_{\partial B_\delta(x_0)} \Big(\frac{\delta}{p} |\nabla u_n|^p - \delta |\nabla u_n|^{p-2} (\partial_\nu u_n)^2 - \frac{\lambda_n \delta}{p} u_n^p - \frac{N-p}{p} u_n |\nabla u_n|^{p-2} \partial_\nu u_n \Big) - \frac{N-p}{Np} \delta \int_{\partial B_\delta(x_0)} u_n^{p^*} = 0. \quad (4.41)$$

As in the previous Step, up to a subsequence, there holds $G_n = \mu_n^{-\frac{N-p}{p(p-1)}} u_n \to G$ in $C^1_{\mathrm{loc}}(\bar\Omega \setminus \{x_0\})$ as $n \to +\infty$, where $G \geq 0$ satisfies

$$-\Delta_p G - \lambda_* G^{p-1} = \Big(\int_{\mathbb{R}^N} U^{p^*-1} \Big) \delta_{x_0} \text{ in } \Omega, \quad G = 0 \text{ on } \partial\Omega,$$

as it follows by (4.38) and (4.40) with $d_n = 1$. Arguing as in Proposition 2.4, we can prove that $H = G - \Gamma$ satisfies (2.1) and by Theorem 2.1 it follows that $G = \big(\int_{\mathbb{R}^N} U^{p^*-1}\big)^{\frac{1}{p-1}} G_{\lambda_*}(\cdot, x_0)$. Since $\mu_n^{-\frac{N-p}{p(p-1)}} u_n \to \big(\int_{\mathbb{R}^N} U^{p^*-1}\big)^{\frac{1}{p-1}} G_{\lambda_*}(\cdot, x_0)$ in $C^1_{\mathrm{loc}}(\bar\Omega \setminus \{x_0\})$ as $n \to +\infty$, by letting $n \to +\infty$ in (4.41) we finally get

$$\lambda_* \int_{B_\delta(x_0)} G^p_{\lambda_*}(x, x_0) dx + \int_{\partial B_\delta(x_0)} \Big(\frac{\delta}{p} |\nabla G_{\lambda_*}(x, x_0)|^p - \delta |\nabla G_{\lambda_*}(x, x_0)|^{p-2} (\partial_\nu G_{\lambda_*}(x, x_0))^2 - \frac{\lambda_* \delta}{p} G^p_{\lambda_*}(x, x_0) - \frac{N-p}{p} G_{\lambda_*}(x, x_0) |\nabla G_{\lambda_*}(x, x_0)|^{p-2} \partial_\nu G_{\lambda_*}(x, x_0) \Big) d\sigma(x) = 0$$

and then $H_{\lambda_*}(x_0, x_0) = 0$ by (3.8).

Data availability

No data was used for the research described in the article.

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