ON A QUASILINEAR MEAN FIELD EQUATION WITH AN EXPONENTIAL NONLINEARITY

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Abstract. The mean field equation involving the $N-$Laplace operator and an exponential nonlinearity is considered in dimension $N \geq 2$ on bounded domains with homogeneous Dirichlet boundary condition. By a detailed asymptotic analysis we derive a quantization property in the non-compact case, yielding to the compactness of the solutions set in the so-called non-resonant regime. In such a regime, an existence result is then provided by a variational approach.

1. Introduction

We are concerned with the following quasilinear mean field equation

$$\begin{cases} -\Delta_N u = \lambda \frac{Ve^u}{\int_\Omega Ve^u dx} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$, where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ denotes the $N-$Laplace operator, V is a smooth nonnegative function and $\lambda \in \mathbb{R}$. In the sequel, (1.1) will be referred to as the N-mean field equation.

In terms of λ or $\rho = \frac{\lambda}{\int Ve^u}$, the planar case $N = 2$ on Euclidean domains or on closed Riemannian surfaces has strongly attracted the mathematical interest, as it arises in conformal geometry [18, 19, 44], in statistical mechanics [16, 17, 20, 46], in the study of turbulent Euler flows [29, 64] and in connection with self-dual condensates for some Chern-Simons-Higgs model [25, 28, 32, 37, 51, 52, 58].

For $N = 2$ Brézis and Merle [15] initiated the study of the asymptotic behavior for solutions of (1.1) by providing a concentration-compactness result in Ω without requiring any boundary condition. A quantization property for concentration masses has been later given in [48], and a very refined asymptotic description has been achieved in [23, 47]. A first natural question concerns the validity of a similar asymptotic behavior in the quasilinear case $N > 2$, where the nonlinearity of the differential operator creates an additional difficulty. The only available result is a concentration-compactness result [2, 61], which provides a too weak compactness property towards existence issues for (1.1). Since a complete classification for the limiting problem

$$\begin{cases} -\Delta_N U = e^U \text{ in } \mathbb{R}^N \\ \int_{\mathbb{R}^N} e^U < \infty \end{cases} \tag{1.2}$$

is not available for $N > 2$ (except for extremals of the corresponding Moser-Trudinger's inequality [43, 50]) as opposite to the case $N = 2$ [21], the starting point of Li-Shafrir's analysis [48] fails and a general quantization property is completely missing. Under a "mild" control on the boundary values of u, Y.Y.Li and independently Wolanski have proposed for $N = 2$ an alternative approach based on Pohozaev identities, successfully applied also in other contexts [6, 7, 66]. The typical assumption on V is the following:

$$\frac{1}{C_0} \leq V(x) \leq C_0 \text{ and } |\nabla V(x)| \leq C_0 \qquad \forall\, x \in \Omega \tag{1.3}$$

for some $C_0 > 0$.

Pushing the analysis of [2, 61] up to the boundary and making use of the above approach, our first main result is the following:

Theorem 1.1. *Let $u_k \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, be a sequence of weak solutions to*

$$-\Delta_N u_k = V_k e^{u_k} \qquad \text{in } \Omega, \tag{1.4}$$

where V_k satisfies (1.3) for all $k \in \mathbb{N}$. Assume that

$$\sup_{k\in\mathbb{N}} \int_\Omega e^{u_k} < +\infty \tag{1.5}$$

and

$$osc_{\partial\Omega} u_k = \sup_{\partial\Omega} u_k - \inf_{\partial\Omega} u_k \leq M$$

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for some $M \in \mathbb{R}$. *Then, up to a subsequence,* u_k *verifies one of the following alternatives: either*
(i) u_k *is uniformly bounded in* $L^\infty_{loc}(\Omega)$
or
(ii) $u_k \to -\infty$ *as* $k \to +\infty$ *uniformly in* $L^\infty_{loc}(\Omega)$
or
(iii) there exists a finite, non-empty set $S = \{p_1, ..., p_m\} \subset \Omega$ *such that* $u_k \to -\infty$ *uniformly in* $L^\infty_{loc}(\Omega \setminus S)$ *and*

$$V_k e^{u_k} \rightharpoonup c_N \sum_{i=1}^m \delta_{p_i} \tag{1.6}$$

weakly in the sense of measures in Ω *as* $k \to +\infty$, *where* $c_N = N\left(\frac{N^2}{N-1}\right)^{N-1}\omega_N$ *with* $\omega_N = |B_1(0)|$. *In addition, if* $osc_{\partial\Omega} u_k = 0$ *for all* k, *alternatives (i)-(iii) do hold in* $\overline{\Omega}$, *with* $S \subset \Omega$ *in case (iii).*

Without an uniform control on the oscillation of u_k on $\partial\Omega$, in general the concentration mass α_i in (1.6) at each p_i, $i = 1, \dots, m$, just satisfies $\alpha_i \geq N^N \omega_N$, see [2, 61] for details. Moreover, the assumption $\text{osc}_{\partial\Omega} u_k = 0$ is used here to rule out boundary blow-up. For strictly convex domains, one could simply use the moving-plane method to exclude maximum points of u_k near $\partial\Omega$ as in [61]. For $N = 2$ this extra assumption can be removed by using the Kelvin transform to take care of non-convex domains, see [54, 60]. Although $N-$harmonic functions in $\mathbb{R}^N$ are invariant under Kelvin transform, such a property does not carry over to (1.4) due to the nonlinearity of $-\Delta_N$. To overcome such a difficulty, we still make use of the Pohozaev identity near boundary points, to exclude the boundary blow-up as in [56, 62].

Problem (1.2) has a $(N+1)-$dimensional family of explicit solutions $U_{\epsilon,p}(x) = U(\frac{x-p}{\epsilon}) - N \log \epsilon$, $\epsilon > 0$ and $p \in \mathbb{R}^N$, where

$$U(x) = \log \frac{F_N}{(1 + |x|^{\frac{N}{N-1}})^N}, \quad x \in \mathbb{R}^N, \tag{1.7}$$

with $F_N = N\left(\frac{N^2}{N-1}\right)^{N-1}$. As $\epsilon \to 0^+$, a description of the blow-up behavior at p is well illustrated by $U_{\epsilon,p}$. Since

$$\int_{\mathbb{R}^N} e^{U_{\epsilon,p}} = c_N,$$

in analogy with Li-Shafrir's result it is expected that the concentration mass α_i in (1.6) at each p_i, $i = 1, ..., m$, should be an integer multiple of c_N. The additional assumption $\sup_k \text{osc}_{\partial\Omega} u_k < +\infty$ allows us to prove that all the blow-up points p_i, $i = 1, \dots, m$, are "simple" in the sense $\alpha_i = c_N$.

Concerning the N-mean field equation (1.1), as a simple consequence of Theorem 1.1 we deduce the following crucial compactness property:

Corollary 1.2. *Let* $\Lambda \subset [0, +\infty) \setminus c_N\mathbb{N}$ *be a compact set. Then, there exists a constant* $C > 0$ *such that* $\|u\|_\infty \leq C$ *does hold for all* $\lambda \in \Lambda$, *all weak solution* $u \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, *of* (1.1) *and all* V *satisfying* (1.3).

In the sequel, we will refer to the case $\lambda \neq c_N\mathbb{N}$ as the *non-resonant regime*. Existence issues can be attacked by variational methods: solutions of (1.1) can be found as critical points of

$$J_\lambda(u) = \frac{1}{N} \int_\Omega |\nabla u|^N - \lambda \log\left(\int_\Omega V e^u\right), \ u \in W_0^{1,N}(\Omega). \tag{1.8}$$

The Moser-Trudinger inequality [57] guarantees that the functional J_λ is well-defined and C^1-Fréchet differentiable on $W_0^{1,N}(\Omega)$ for any $\lambda \in \mathbb{R}$. Moreover, if $\lambda < c_N$ the functional J_λ is coercive and then attains the global minimum. For $\lambda = c_N$ J_λ still has a lower bound but is not coercive anymore: in general, in the resonant regime $\lambda \in c_N\mathbb{N}$ existence issues are very delicate. When $\lambda > c_N$ the functional J_λ is unbounded both from below and from above, and critical points have to be found among saddle points. Moreover, the *Palais-Smale condition* for J_λ is not globally available, see [53], but holds only for bounded sequences in $W_0^{1,N}(\Omega)$.

The second main result is the following:

Theorem 1.3. *Assume that the space of formal barycenters* $\mathfrak{B}_m(\overline{\Omega})$ *of* $\overline{\Omega}$ *with order* $m \geq 1$ *is non contractible. Then equation* (1.1) *has a solution in* $C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, *for all* $\lambda \in \big(c_N m, c_N(m+1)\big)$.

For mean-field equations, such a variational approach has been introduced in [33] and fully exploited later by Djadli and Malchiodi [35] in their study of constant Q-curvature metrics on four manifolds. It has revelead to be very powerful in many contexts, see for example [1, 8, 34, 55] and refences therein. Alternative approaches are available: the computation of the corresponding Leray-Schauder degree [23, 24], based on a very refined asymptotic analysis of blow-up solutions; perturbative constructions of Lyapunov-Schimdt in the almost resonant regime [5, 24, 28, 29, 30, 37, 38, 52]. For our problem a refined asymptotic analysis for blow-up solutions is still missing, and perturbation arguments are very difficult due to the nonlinearity of Δ_N. A variational approach is the only reasonable way to attack existence issues, and in this way the analytic problem is reduced to a topological one concerning the non-contractibility of a model space, the so-called *space of formal barycenters*, characterizing the very low sublevels of J_λ. We refer to Section 3 for a definition of $\mathfrak{B}_m(\overline{\Omega})$. To have non-contractibility of $\mathfrak{B}_m(\overline{\Omega})$ for domains Ω homotopically equivalent to a finite simplicial complex, a sufficient condition is the non-triviality of the $\mathbb{Z}$-homology, see [41]. Let us emphasize that the variational approach produces solutions a.e. $\lambda \in \big(c_N m, c_N(m+1)\big)$, $m \geq 1$, and Corollary 1.2 is crucial to get the validity of Theorem 1.3 for all λ in such a range.

The paper is organized as follows. In Section 2 we show how to push the concentration-compactness analysis [2, 61] up to the boundary, by discussing boundary blow-up and mass quantization. Section 3 is devoted to Theorem 1.3 and some comments concerning $\mathfrak{B}_m(\overline{\Omega})$. In the appendix, we collect some basic results that will be used frequently throughout the paper.

2. Concentration-Compactness analysis

Even though representation formulas are not available for Δ_N, the Brézis-Merle's inequality [15] can be extended to $N > 2$ by different means:

Lemma 2.1. [2, 61] *Let $u \in C^{1,\alpha}(\overline{\Omega})$ be a weak solution of*

$$-\Delta_N u = f \quad \text{in } \Omega$$

with $f \in L^1(\Omega)$. Let φ be a N-harmonic function in Ω with $\varphi = u$ on $\partial\Omega$. Then, for every $\alpha \in (0, \alpha_N)$ there exists a constant $C = C(\alpha, |\Omega|)$ such that

$$\int_\Omega \exp\left[\frac{\alpha|u(x)-\varphi(x)|}{\|f\|_{L^1}^{\frac{1}{N-1}}}\right] \leq C, \tag{2.1}$$

where $\alpha_N = (N^N d_N \omega_N)^{\frac{1}{N-1}}$ and

$$d_N = \inf_{X \neq Y \in \mathbb{R}^N} \frac{< |X|^{N-2}X - |Y|^{N-2}Y, X-Y >}{|X-Y|^N} > 0.$$

In addition, if $u = 0$ on $\partial\Omega$ inequality (2.1) does hold with $\alpha_N = (N^N \omega_N)^{\frac{1}{N-1}}$.

Under some smallness uniform condition on the nonlinear term, a-priori estimates hold true as follows:

Lemma 2.2. *Let $u_k \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, be a sequence of weak solutions to (1.4), where V_k satisfies (1.3) for all $k \in \mathbb{N}$. Assume that*

$$\sup_k \int_{\Omega \cap B_{4R}} V_k e^{u_k} < N^N d_N \omega_N \tag{2.2}$$

does hold for some $R > 0$, and u_k satisfies $u_k = c_k$ on $\partial\Omega \cap \overline{B_{4R}}$, $u_k \geq c_k$ in $\Omega \cap B_{4R}$ for $c_k \in \mathbb{R}$ if $\partial\Omega \cap \overline{B_{4R}} \neq \emptyset$. Then

$$\sup_k \|u_k^+\|_{L^\infty(\Omega \cap B_R)} < +\infty. \tag{2.3}$$

Proof. Let φ_k be the $N-$harmonic function in $\Omega \cap B_{4R}$ so that $\varphi_k = u_k$ on $\partial(\Omega \cap B_{4R})$. Choosing

$$\alpha \in \left((\sup_k \int_{\Omega \cap B_{4R}} V_k e^{u_k})^{\frac{1}{N-1}}, \alpha_N \right)$$

in view of (2.2), by Lemma 2.1 we get that $e^{|u_k - \varphi_k|}$ is uniformly bounded in $L^q(\Omega \cap B_{4R})$, for some $q > 1$. Since $V_k \geq 0$, by the weak comparison principle we get that $c_k \leq \varphi_k \leq u_k$ in $\Omega \cap B_{4R}$. Since $\varphi_k = c_k$ on $\partial\Omega \cap \overline{B_{4R}}$ and

$$\sup_k \|\varphi_k^+\|_{L^N(\Omega \cap B_{4R})} \leq \sup_k \|u_k^+\|_{L^N(\Omega \cap B_{4R})} < +\infty \tag{2.4}$$

in view of (1.3) and (2.2), by Theorem A.1 we get that $\varphi_k \leq C_0$ in $\Omega \cap B_{2R}$ uniformly in k, for some C_0. Since $e^{u_k} \leq e^{C_0} e^{|u_k - \varphi_k|}$, we get that e^{u_k} is uniformly bounded in $L^q(\Omega \cap B_{2R})$. Since $q > 1$, by Theorem A.1 we deduce the validity of (2.3) in view of (2.4). □

We can now prove our first main result:

Proof (of Theorem 1.1).
First of all, by (1.3) for V_k and (1.5) we deduce that $V_k e^{u_k}$ is uniformly bounded in $L^1(\Omega)$. Up to a subsequence, by the Prokhorov Theorem we can assume that $V_k e^{u_k} \rightharpoonup \mu \in \mathcal{M}^+(\overline{\Omega})$ as $k \to +\infty$ in the sense of measures in $\overline{\Omega}$, i.e.

$$\int_\Omega V_k e^{u_k} \varphi \to \int_\Omega \varphi d\mu \text{ as } k \to +\infty \qquad \forall\, \varphi \in C(\overline{\Omega}).$$

A point $p \in \overline{\Omega}$ is said a *regular point* for μ if $\mu(\{p\}) < N^N \omega_N$, and let us denote the set of non-regular points as:

$$\Sigma = \{p \in \overline{\Omega} : \ \mu(\{p\}) \geq N^N \omega_N\}.$$

Since μ is a bounded measure, it follows that Σ is a finite set. We complete the argument through the following five steps.

Step 1 Letting

$$S = \big\{p \in \overline{\Omega} : \ \limsup_{k \to +\infty} \sup_{\Omega \cap B_R(p)} u_k = +\infty \ \forall R > 0\big\},$$

there holds $S \cap \Omega = \Sigma \cap \Omega$ ($S = \Sigma$ if $osc_{\partial\Omega} u_k = 0$ for all k).

Letting $p_0 \in S$, assume that $p_0 \in \Omega$ or $u_k = c_k$ on $\partial\Omega$ for some $c_k \in \mathbb{R}$. In the latter case, notice that $u_k \geq c_k$ in Ω in view of the weak comparison principle. Setting

$$\Sigma' = \left\{ p \in \overline{\Omega} : \ \mu(\{p\}) \geq N^N d_N \omega_N \right\},$$

by Lemma 2.2 we know that $p_0 \in \Sigma'$. Indeed, if $p_0 \notin \Sigma'$, then (2.2) would hold for some $R > 0$ small, and then by Lemma 2.2 it would follow that u_k is uniformly bounded from above in $\Omega \cap B_R(p_0)$, contradicting $p_0 \in S$. To show that $p_0 \in \Sigma$, the key point is to recover a good control of u_k on $\partial\big(\Omega \cap B_R(p_0)\big)$, for some $R > 0$, in order to drop d_N. Assume that $p_0 \notin \Sigma$, in such a way that

$$\sup_k \int_{\Omega \cap B_{2R}(p_0)} V_k e^{u_k} < N^N \omega_N \tag{2.5}$$

for some $R > 0$ small. Since Σ' is a finite set, up to take R smaller, let us assume that $\partial\big(\Omega \cap B_{2R}(p_0)\big) \cap \Sigma' \subset \{p_0\}$, and then by compactness we have that

$$u_k \leq M \quad \text{in } \partial\big(\Omega \cap B_{2R}(p_0)\big) \setminus \ B_R(p_0) \tag{2.6}$$

in view of $S \cap \Omega \subset \Sigma' \cap \Omega$ and $S \subset \Sigma'$ if $osc_{\partial\Omega} u_k = 0$ for all k. If $p_0 \in \Omega$, we can also assume that $\overline{B_{2R}(p_0)} \subset \Omega$. If $p_0 \in \partial\Omega$, $u_k = c_k$ on $\partial\Omega$ yields to $c_k \leq M$ in view of (2.6). In both cases, we have shown that (2.6) does hold in the stronger way:

$$u_k \leq M \quad \text{in } \partial\big(\Omega \cap B_{2R}(p_0)\big). \tag{2.7}$$

Letting $w_k \in W_0^{1,N}(\Omega \cap B_{2R}(p_0))$ be the weak solution of

$$\begin{cases} -\Delta_N w_k = V_k e^{u_k} & \text{in } \Omega \cap B_{2R}(p_0) \\ w_k = 0 & \text{on } \partial\big(\Omega \cap B_{2R}(p_0)\big), \end{cases}$$

by (2.7) and the weak comparison principle we get that

$$u_k \leq w_k + M \quad \text{in } \Omega \cap B_{2R}(p_0).$$

Applying Lemma 2.1 to w_k in view of (2.5), it follows that

$$\int_{\Omega \cap B_{2R}(p_0)} e^{q u_k} \leq e^{qM} \int_{\Omega \cap B_{2R}(p_0)} e^{q w_k} \leq C$$

for all k, for some $q > 1$ and $C > 0$. In particular, u_k^+ is uniformly bounded in $L^N\big(\Omega \cap B_{2R}(p_0)\big)$ and $V_k e^{u_k}$ is uniformly bounded in $L^q\big(\Omega \cap B_{2R}(p_0)\big)$. By Theorem A.1 it follows that u_k is uniformly bounded from above in $\Omega \cap B_R(p_0)$, in contradiction with $p_0 \notin S$. So, we have shown that $p_0 \in \Sigma$, which yields to $S \cap \Omega \subset \Sigma \cap \Omega$ and $S \subset \Sigma$ if $osc_{\partial\Omega} u_k = 0$ for all k.
Conversely, let $p_0 \in \Sigma$. If $p_0 \notin S$, one could find $R_0 > 0$ so that $u_k \leq M$ in $\Omega \cap B_{R_0}(p_0)$, for some $M \in \mathbb{R}$, yielding to

$$\int_{\Omega \cap B_R(p_0)} V_k e^{u_k} \leq C_0 e^M R^N, \ R \leq R_0,$$

in view of (1.3). In particular, $\mu(\{p_0\}) = 0$, contradicting $p_0 \in \Sigma$. Hence $\Sigma \subset S$, and the proof of Step 1 is complete.

Step 2 $S \cap \Omega = \emptyset$ ($S = \emptyset$) implies the validity of alternative (i) or (ii) in Ω (in $\overline{\Omega}$ if $osc_{\partial\Omega} u_k = 0$ for all k).

Since u_k is uniformly bounded from above in $L^\infty_{loc}(\Omega)$, then either u_k is uniformly bounded in $L^\infty_{loc}(\Omega)$ or there exists, up to a subsequence, a compact set $K \subset \Omega$ so that $\min_K u_k \to -\infty$ as $k \to +\infty$. The set $\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \delta\}$ is a compact connected set so that $K \subset \Omega_\delta$, for $\delta > 0$ small. Since $u_k \leq M$ in Ω for some $M > 0$, the function $s_k = M - u_k$ is a nonnegative weak solution of $-\Delta_N s_k = -V_k e^{u_k}$ in Ω. By the Harnack inequality in Theorem A.2, we have that

$$\max_{\Omega_\delta} s_k \leq C\big(\min_{\Omega_\delta} s_k + 1\big)$$

in view of

$$\|V_k e^{u_k}\|_{L^\infty(\Omega)} \leq C_0 e^M.$$

In terms of u_k, it reads as

$$\max_{\Omega_\delta} u_k \leq M\big(1 - \frac{1}{C}\big) + 1 + \frac{1}{C} \min_K u_k \to -\infty$$

as $k \to +\infty$ for all $\delta > 0$ small, yielding to the validity of alternative (ii) in Ω. Assume in addition that $u_k = c_k$ on $\partial\Omega$ for some $c_k \in \mathbb{R}$. Notice that $c_k \leq u_k \leq M$ in Ω for all k. If alternative (i) does not hold in $\overline{\Omega}$, up to a subsequence, we get that $c_k \to -\infty$. Since $V_k e^{u_k}$ is uniformly bounded in Ω, we apply Corollary A.3 to $s_k = u_k - c_k$, a nonnegative solution of $-\Delta_N s_k = V_k e^{u_k}$ with $s_k = 0$ on $\partial\Omega$, to get $s_k \leq M'$ in Ω for some $M' \in \mathbb{R}$. Hence, $u_k \leq M' + c_k \to -\infty$ in Ω as $k \to +\infty$, yielding to the validity of alternative (ii) in $\overline{\Omega}$. The proof of Step 2 is complete.

Step 3 $S \cap \Omega \neq \emptyset$ implies the validity of alternative (iii) in Ω (in $\overline{\Omega}$ if $osc_{\partial\Omega} u_k = 0$ for all k) with (1.6) replaced by the property:

$$V_k e^{u_k} \rightharpoonup \sum_{i=1}^{m} \alpha_i \delta_{p_i} \tag{2.8}$$

weakly in the sense of measures in Ω (in $\overline{\Omega}$) as $k \to +\infty$, with $\alpha_i \geq N^N \omega_N$ and $S \cap \Omega = \{p_1, \dots, p_m\}$ ($S = \{p_1, \dots, p_m\}$).

Let us first consider the case that u_k is uniformly bounded in $L^\infty_{loc}(\Omega \setminus S)$. Fix $p_0 \in S$ and $R > 0$ small so that $\overline{B_R(p_0)} \cap S = \{p_0\}$. Arguing as in (2.6)-(2.7), we have that $u_k \geq m$ on $\partial(\Omega \cap B_R(p_0))$ for some $m \in \mathbb{R}$. Since u_k is uniformly bounded in $L^\infty_{loc}(\Omega \setminus S)$, by Theorem A.4 it follows that u_k is uniformly bounded in $C^{1,\alpha}_{loc}(\overline{\Omega \cap B_R(p_0)} \setminus \{p_0\})$, for some $\alpha \in (0,1)$, and, up to a subsequence and a diagonal process, we can assume that $u_k \to u$ in $C^1_{loc}(\overline{\Omega \cap B_R(p_0)} \setminus \{p_0\})$ as $k \to +\infty$. By (1.3) on each V_k, we can also assume that $V_k \to V$ uniformly in Ω as $k \to +\infty$. Hence, there holds

$$V_k e^{u_k} \rightharpoonup \mu = V e^u \, dx + \alpha_0 \delta_{p_0} \tag{2.9}$$

weakly in the sense of measures in $\overline{\Omega \cap B_R(p_0)}$ as $k \to +\infty$, where $\alpha_0 \geq N^N \omega_N$. Since

$$\lim_{k \to +\infty} \int_{\Omega \cap B_R(p_0)} V_k e^{u_k} = \int_{\Omega \cap B_R(p_0)} V e^u + \alpha_0 > \alpha_0$$

in view of (2.9), for k large we can find a unique $0 < r_k < R$ so that

$$\int_{\Omega \cap B_{r_k}(p_0)} V_k e^{u_k} = \alpha_0. \tag{2.10}$$

Notice that $r_k \to 0$ as $k \to +\infty$. Indeed, if $r_k \geq \delta > 0$ were true along a subsequence, one would reach the contradiction

$$\alpha_0 \geq \int_{\Omega \cap B_\delta(p_0)} V_k e^{u_k} \to \int_{\Omega \cap B_\delta(p_0)} V e^u + \alpha_0 > \alpha_0$$

as $k \to +\infty$ in view of (2.9)-(2.10). Denoting by χ_A the characteristic function of a set A, we have the following crucial property:

$$\chi_{B_{r_k}(p_0)} V_k e^{u_k} \rightharpoonup \alpha_0 \delta_{p_0}$$

weakly in the sense of measures in $\overline{\Omega \cap B_R(p_0)}$ as $k \to +\infty$, as it easily follows by (2.10) and $\lim\limits_{k \to +\infty} r_k = 0$.

We can now specialize the argument to deal with the case $p_0 \in S \cap \Omega$. Assume that R is small so that $\overline{B_R(p_0)} \subset \Omega$. Letting $w_k \in W^{1,N}_0(B_R(p_0))$ be the weak solution of

$$\begin{cases} -\Delta_N w_k = \chi_{B_{r_k}(p_0)} V_k e^{u_k} & \text{in } B_R(p_0) \\ w_k = 0 & \text{on } \partial B_R(p_0), \end{cases}$$

by the weak comparison principle there holds $0 \leq w_k \leq u_k - m$ in $B_R(p_0)$ in view of $0 \leq \chi_{B_{r_k}(p_0)} V_k e^{u_k} \leq V_k e^{u_k}$. Arguing as before, up to a subsequence, by Theorem A.4 we can assume that $w_k \to w$ in $C^1_{loc}(\overline{B_R(p_0)} \setminus \{p_0\})$ as $k \to +\infty$, where $w \geq 0$ is a $N-$harmonic and continous function in $B_R(p_0) \setminus \{p_0\}$ which solves

$$-\Delta_N w = \alpha_0 \delta_{p_0} \quad \text{in } B_R(p_0)$$

in a distributional sense. By Theorem A.5 we deduce that

$$w \geq (N\omega_N)^{-\frac{1}{N-1}} \alpha_0^{\frac{1}{N-1}} \log \frac{1}{|x - p_0|} + C \geq N \log \frac{1}{|x - p_0|} + C \qquad \text{in } B_r(p_0) \tag{2.11}$$

in view of $\alpha_0 \geq N^N \omega_N$, for some $C \in \mathbb{R}$ and $0 < r \leq \min\{1, R\}$. Since

$$\int_{B_R(p_0)} e^{w_k} \leq e^{-m} \sup_k \int_\Omega e^{u_k} < +\infty$$

in view of (1.5), as $k \to +\infty$ we get that $\int_{B_R(p_0)} e^w < +\infty$, in contradiction with (2.11):

$$\int_{B_R(p_0)} e^w \geq e^C \int_{B_r(p_0)} \frac{1}{|x - p_0|^N} = +\infty.$$

Since u_k is uniformly bounded from above and not from below in $L^\infty_{loc}(\Omega \setminus S)$, there exists, up to a subsequence, a compact set $K \subset \Omega \setminus S$ so that $\min_K u_k \to -\infty$ as $k \to +\infty$. Arguing as in Step 2 by simply replacing $\mathrm{dist}(\cdot, \partial\Omega)$ with $\mathrm{dist}(\cdot, \partial\Omega \cap S)$, we can show that $u_k \to -\infty$ in $L^\infty_{loc}(\Omega \setminus S)$ as $k \to +\infty$, and (2.8) does hold in Ω with $\{p_1, \dots, p_m\} = S \cap \Omega$. If in addition $u_k = c_k$ on $\partial\Omega$ for some $c_k \in \mathbb{R}$, we can argue as in the end of Step 2 (by using Theorem A.2 instead of Corollary A.3) to get that $u_k \to -\infty$ in $L^\infty_{loc}(\overline{\Omega} \setminus S)$ as $k \to +\infty$, yielding to the validity of (2.8) in $\overline{\Omega}$ with $\{p_1, \dots, p_m\} = S$. The proof of Step 3 is complete.

To proceed further we make use of Pohozaev identities. Let us emphasize that $u_k \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, and the classical Pohozaev identities usually require more regularity. In [27] a self-contained proof is provided in the quasilinear case, which reads in our case as:

Lemma 2.3. *Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain, f be a locally Lipschitz continuous function and $0 \leq V \in C^1(\overline{\Omega})$. Then, there holds*

$$\int_\Omega \big[N\ V + \langle x-y, \nabla V\rangle\big] F(u) = \int_{\partial\Omega} V\ F(u)\langle x-y,\nu\rangle + |\nabla u|^{N-2}\langle x-y,\nabla u\rangle \partial_\nu u - \frac{|\nabla u|^N}{N}\langle x-y,\nu\rangle$$

for all weak solution $u \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, of $-\Delta_N u = Vf(u)$ in Ω and all $y \in \mathbb{R}^N$, where $F(t) = \int_{-\infty}^t f(s)ds$ and ν is the unit outward normal vector at $\partial\Omega$.

Thanks to Lemma 2.3, in the next two Steps we can now describe the interior blow-up phenomenon and exclude the occurence of boundary blow-up:

Step 4 If $osc_{\partial\Omega} u_k \leq M$ for some $M \in \mathbb{R}$, then $\alpha_i = c_N$ for all $p_i \in S \cap \Omega$.

Since $0 \leq u_k - \inf_{\partial\Omega} u_k \leq M$ on $\partial\Omega$, we have that $s_k = u_k - \inf_{\partial\Omega} u_k \geq 0$ satisfies

$$\begin{cases} -\Delta_N s_k = W_k e^{s_k} & \text{in } \Omega \\ 0 \leq s_k \leq M & \text{on } \partial\Omega, \end{cases}$$

where $W_k = V_k e^{\inf_{\partial\Omega} u_k}$. Letting now φ_k be the $N-$harmonic function in Ω with $\varphi_k = s_k$ on $\partial\Omega$, by the weak comparison principle we have that $0 \leq \varphi_k \leq M$ in Ω. Since $\sup_k \int_\Omega W_k e^{s_k} < +\infty$ and $e^{\gamma s} \geq \delta s^N$ for all $s \geq 0$, for some $\delta > 0$, by Lemma 2.1 we deduce that $s_k - \varphi_k$ and then s_k are uniformly bounded in $L^N(\Omega)$. Since $W_k e^{s_k} = V_k e^{u_k}$ is uniformly bounded in $L^\infty_{loc}(\overline{\Omega} \setminus S)$, by Theorem A.4 it follows as in Step 3 that, up to a subsequence, $s_k \to s$ in $C^1_{loc}(\Omega \setminus S)$. Fix $p_0 \in S \cap \Omega$ and take $R_0 > 0$ small so that $B = B_{R_0}(p_0) \subset\subset \Omega$ and $\overline{B} \cap S = \{p_0\}$. The limiting function $s \geq 0$ is a N-harmonic and continuous function in $B \setminus \{p_0\}$ which solves

$$-\Delta_N s = \alpha_0 \delta_{p_0} \quad \text{in } B,$$

where $\alpha_0 \geq N^N \omega_N$. By Theorem A.5 we have that $s = \alpha_0^{\frac{1}{N-1}} \Gamma(|x-p_0|) + H$, where $H \in L^\infty_{loc}(B)$ does satisfy

$$\lim_{x \to p_0} |x-p_0||\nabla H(x)| = 0. \tag{2.12}$$

Applying the Pohozaev identity to s_k on $B_R(p_0)$, $0 < R \leq R_0$, with $y = p_0$, we get that

$$\int_{B_R(p_0)} \big[NW_k + \langle x-p_0, \nabla W_k\rangle\big] e^{s_k} = R \int_{\partial B_R(p_0)} \left[W_k e^{s_k} + |\nabla s_k|^{N-2}(\partial_\nu s_k)^2 - \frac{|\nabla s_k|^N}{N}\right].$$

Since $S \cap \Omega \neq \emptyset$ and $V_k e^{u_k} = W_k e^{s_k}$, by Step 3 we get that $\int_{\partial B_R(p_0)} W_k e^{s_k} \to 0$ and

$$\int_{B_R(p_0)} \big[NW_k + \langle x-p_0, \nabla W_k\rangle\big] e^{s_k} = N \int_{B_R(p_0)} V_k e^{u_k} + O\left(\int_{B_R(p_0)} |x-p_0| V_k e^{u_k}\right) \to N\alpha_0$$

as $k \to +\infty$. Letting $k \to \infty$ we get that

$$\begin{aligned} N\alpha_0 &= R\int_{\partial B_R(p_0)} |\nabla H - (\frac{\alpha_0}{N\omega_N})^{\frac{1}{N-1}} \frac{x-p_0}{|x-p_0|^2}|^{N-2} [\partial_\nu H - (\frac{\alpha_0}{N\omega_N})^{\frac{1}{N-1}} \frac{1}{|x-p_0|}]^2 \\ &\quad - \frac{R}{N} \int_{\partial B_R(p_0)} |\nabla H - (\frac{\alpha_0}{N\omega_N})^{\frac{1}{N-1}} \frac{x-p_0}{|x-p_0|^2}|^N \\ &= R\frac{N-1}{N} \int_{\partial B_R(p_0)} \left[(\frac{\alpha_0}{N\omega_N})^{\frac{2}{N-1}} \frac{1}{|x-p_0|^2} + O(\frac{1}{|x-p_0|}|\nabla H| + |\nabla H|^2)\right]^{\frac{N}{2}} \\ &= R\frac{N-1}{N} \int_{\partial B_R(p_0)} (\frac{\alpha_0}{N\omega_N})^{\frac{N}{N-1}} \frac{1}{|x-p_0|^N} \left[1 + O(|x-p_0||\nabla H| + |x-p_0|^2|\nabla H|^2)\right] \end{aligned}$$

in view of $s_k \to s = \alpha_0^{\frac{1}{N-1}} \Gamma(|x-p_0|) + H$ in $C^1_{loc}(\overline{B} \setminus \{p_0\})$ as $k \to +\infty$. Letting $R \to 0$ we get that

$$N\alpha_0 = \frac{N-1}{N} (\frac{\alpha_0}{N\omega_N})^{\frac{N}{N-1}} N\omega_N,$$

in view of (2.12). Therefore, there holds

$$\alpha_0 = N(\frac{N^2}{N-1})^{N-1} \omega_N = c_N$$

for all $p_0 \in S \cap \Omega$, and the proof of Step 4 is complete.

Step 5 If $osc_{\partial\Omega} u_k = 0$ for all k, then $S \subset \Omega$.

Assume now that $u_k = c_k$ on $\partial\Omega$. Since by the weak comparison principle $c_k \leq u_k$ in Ω for all k, the function $s_k = u_k - c_k$ is a nonnegative weak solution of

$$\begin{cases} -\Delta_N s_k = W_k e^{s_k} & \text{in } \Omega \\ s_k = 0 & \text{on } \partial\Omega, \end{cases}$$

where $W_k = V_k e^{c_k}$. Since $W_k e^{s_k} = V_k e^{u_k}$ is uniformly bounded in $L^1(\Omega)$, by Lemma 2.1 we have that s_k is uniformly bounded in $L^N(\Omega)$. Since $W_k e^{s_k} = V_k e^{u_k}$ is uniformly bounded in $L^\infty_{loc}(\overline{\Omega} \setminus S)$, arguing as in Step 3, by Theorem A.4 it follows that s_k is uniformly bounded in $C^{1,\alpha}_{loc}(\overline{\Omega} \setminus S)$, $\alpha \in (0,1)$, and, up to a subsequence, $s_k \to s$ in $C^1_{loc}(\overline{\Omega} \setminus S)$. We claim that $s \in C^1(\overline{\Omega})$.

If $c_k \to -\infty$, we have that $s \in C^1_{loc}(\overline{\Omega} \setminus S)$ is a nonnegative N-harmonic function in $\Omega \setminus S$ with $s = 0$ on $\partial\Omega \setminus S$. By Theorem A.2 we deduce that $s = 0$ in Ω, and then $s \in C^1(\overline{\Omega})$. Up to a subsequence, we can now assume that $c_k \to c \in \mathbb{R}$ as $k \to +\infty$ and $S = \{p_1, \dots, p_m\} \subset \partial\Omega$ in view of Step 3. By [12, 13] $s \in W^{1,q}_0(\Omega)$ for all $q < N$ and is a distributional solution of

$$\begin{cases} -\Delta_N s = We^s & \text{in } \Omega \\ s = 0 & \text{on } \partial\Omega \end{cases} \tag{2.13}$$

(referred to as SOLA, Solution Obtained as Limit of Approximations), where $W = Ve^c$ and $We^s \in L^1(\Omega)$. By considering different L^1-approximations or even L^1-weak approximations of $We^s \in L^1(\Omega)$ one always get the same limiting SOLA [26], which is then unique in the sense explained right now. Unfortunately, the sequence $W_k e^{s_k}$ does not converge L^1-weak to We^s as $k \to +\infty$ since it keeps track that some mass is concentrating near the boundary points $p_1, \dots, p_m$. Given $p = p_i \in S$ and $\alpha = \alpha_i$, arguing as in (2.10) we can find a radius $r_k \to 0$ as $k \to +\infty$ so that

$$\int_{\Omega \cap B_{r_k}(p)} W_k e^{s_k} = \alpha. \tag{2.14}$$

Let $w_k \in W^{1,N}_0(\Omega \cap B_R(p))$ be the weak solution of

$$\begin{cases} -\Delta_N w_k = \chi_{\Omega \cap B_{r_k}(p)} W_k e^{s_k} & \text{in } \Omega \cap B_R(p) \\ w_k = 0 & \text{on } \partial(\Omega \cap B_R(p)), \end{cases}$$

where $R < \frac{1}{2}\text{dist } (p, S \setminus \{p\})$. Arguing as in Step 3, up to a subsequence, we have that $w_k \to w$ in $C^1_{loc}(\overline{\Omega \cap B_R(p)} \setminus \{p\})$ as $k \to +\infty$, where $w \geq 0$ is $N-$harmonic and continous in $\overline{\Omega \cap B_R(p)} \setminus \{p\}$. If $w > 0$ in $\Omega \cap B_R(p)$, by [11, 14] we have that

$$\lim_{r \to 0^+} rw(\sigma r + p) = -\langle \sigma, \nu(p) \rangle \tag{2.15}$$

uniformly for σ with $\langle \sigma, \nu(p) \rangle \leq -\delta < 0$. Thanks to (2.15), as in Step 3 we still end up with the contradiction $\int_{\Omega \cap B_R(p)} e^w = +\infty$. Therefore, by the strong maximum principle we necessarily have that $w = 0$ in $\Omega \cap B_R(p)$. Since w_k is the part of s_k which carries the information on the concentration phenomenon at p and tends to disappear as $k \to +\infty$, we can expect that s_k in the limit does not develop any singularities. We aim to show that $e^s \in L^q(\Omega \cap B_R(p))$ for all $q \geq 1$, by mimicking some arguments in [2]. Letting φ_k be the $N-$harmonic extension in $\Omega \cap B_R(p)$ of $s_k \mid_{\partial(\Omega \cap B_R(p))}$, for $M, a > 0$ we have that

$$\begin{aligned} &\int_{\Omega \cap B_R(p)} \langle |\nabla s_k|^{N-2} \nabla s_k - |\nabla w_k|^{N-2} \nabla w_k - |\nabla \varphi_k|^{N-2} \nabla \varphi_k, \nabla [T_{M+a}(s_k - w_k - \varphi_k) - T_M(s_k - w_k - \varphi_k)] \rangle \\ &= \int_{\Omega \cap B_R(p)} (1 - \chi_{\Omega \cap B_{r_k}(p)}) W_k e^{s_k} [T_{M+a}(s_k - w_k - \varphi_k) - T_M(s_k - w_k - \varphi_k)] \\ &\leq a \int_{\{|s_k - w_k - \varphi_k| > M\}} (1 - \chi_{\Omega \cap B_{r_k}(p)}) W_k e^{s_k}, \end{aligned} \tag{2.16}$$

where the truncature operator T_M, $M > 0$, is defined as

$$T_M(u) = \begin{cases} -M & \text{if } u < -M \\ u & \text{if } |u| \leq M \\ M & \text{if } u > M. \end{cases}$$

The crucial property we will take advantage of is the following:

$$\sup_k \int_{\{|s_k - w_k - \varphi_k| > M\}} (1 - \chi_{\Omega \cap B_{r_k}(p)}) W_k e^{s_k} \to 0 \qquad \text{as } M \to +\infty. \tag{2.17}$$

Indeed, by [49] notice that, up to a subsequence, we can assume that $\varphi_k \to \varphi$ in $C^1(\overline{\Omega \cap B_R(p)})$ as $k \to +\infty$, where φ is the $N-$harmonic function in $\Omega \cap B_R(p)$ with $\varphi = s$ on $\partial(\Omega \cap B_R(p))$. Since $s_k - w_k - \varphi_k \to s - \varphi$ uniformly in $\Omega \cap (B_R(p) \setminus B_r(p))$ as $k \to +\infty$ for any given $r \in (0, R)$, we can find $M_r > 0$ large so that

$$\cup_k \{|s_k - w_k - \varphi_k| > M\} \subset \Omega \cap B_r(p) \qquad \forall\, M \geq M_r,$$

and then

$$\sup_k \int_{\{|s_k - w_k - \varphi_k| > M\}} (1 - \chi_{\Omega \cap B_{r_k}(p)}) W_k e^{s_k} \leq \sup_k \int_{\Omega \cap B_r(p)} (1 - \chi_{\Omega \cap B_{r_k}(p)}) W_k e^{s_k}$$

for all $M \geq M_r$. Since by (2.9) and (2.14)

$$\int_{\Omega \cap B_r(p)} (1 - \chi_{\Omega \cap B_{r_k}(p)}) W_k e^{s_k} \to \int_{\Omega \cap B_r(p)} We^s$$

as $k \to +\infty$ and $We^s \in L^1(\Omega)$, for all $\epsilon > 0$ we can find $r_\epsilon > 0$ small so that

$$\sup_k \int_{\Omega \cap B_{r_\epsilon}(p)} (1 - \chi_{\Omega \cap B_{r_k}(p)}) W_k e^{s_k} \le \epsilon,$$

yielding to the validity of (2.17). Inserting (2.17) into (2.16) we get that, for all $\epsilon > 0$, there exists M_ϵ so that

$$\int_{\{M < |s_k - w_k - \varphi_k| \le M + a\}} \langle |\nabla s_k|^{N-2} \nabla s_k - |\nabla w_k|^{N-2} \nabla w_k - |\nabla \varphi_k|^{N-2} \nabla \varphi_k, \nabla (s_k - w_k - \varphi_k) \rangle \le a\epsilon \tag{2.18}$$

for all $M \ge M_\epsilon$ and $a > 0$. Recall that $w_k \to 0$, $s_k \to s$ in $C^1_{loc}(\overline{\Omega \cap B_R(p)} \setminus \{p\})$ and in $W^{1,q}(\Omega \cap B_R(p))$ for all $q < N$ as $k \to +\infty$ in view of [12, 13]. Since

$$\langle |\nabla s_k|^{N-2} \nabla s_k - |\nabla w_k|^{N-2} \nabla w_k, \nabla (s_k - w_k) \rangle \ge 0$$

and $\nabla \varphi_k$ behaves well, we can let $k \to +\infty$ in (2.18) and by the Fatou Lemma get

$$\frac{d_N}{a} \int_{\{M < |s - \varphi| \le M + a\}} |\nabla (s - \varphi)|^N \le \frac{1}{a} \int_{\{M < |s - \varphi| \le M + a\}} \langle |\nabla s|^{N-2} \nabla s - |\nabla \varphi|^{N-2} \nabla \varphi, \nabla (s - \varphi) \rangle \le \epsilon \tag{2.19}$$

for some $d_N > 0$ and all $M \ge M_\epsilon$. Introducing $H_{M,a}(s) = \frac{T_{M+a}(s-\varphi) - T_M(s-\varphi)}{a}$ and the distribution $\Phi_{s-\varphi}(M) = |\{x \in \Omega \cap B_R(p) : |s - \varphi|(x) > M\}$ of $|s - \varphi|$, we have that

$$\begin{aligned} \Phi_{s-\varphi}(M+a)^{\frac{N-1}{N}} &\le \left(\int_{\Omega \cap B_R(p)} |H_{M,a}(s)|^{\frac{N}{N-1}} \right)^{\frac{N-1}{N}} \le (N^N \omega_N)^{-\frac{1}{N}} \int_{\Omega \cap B_R(p)} |\nabla H_{M,a}(s)| \\ &\le (N^N \omega_N)^{-\frac{1}{N}} \frac{1}{a} \int_{\{M < |s - \varphi| \le M + a\}} |\nabla (s - \varphi)| \end{aligned}$$

in view of the Sobolev embedding $W^{1,1}_0(\Omega \cap B_R(p)) \hookrightarrow L^{\frac{N}{N-1}}(\Omega \cap B_R(p))$ with sharp constant $(N^N \omega_N)^{-\frac{1}{N}}$, see [39]. By the Hölder inequality and (2.19) we then deduce that

$$\Phi_{s-\varphi}(M+a) \le (\frac{N^N d_N \omega_N}{\epsilon})^{-\frac{1}{N-1}} \frac{\Phi_{s-\varphi}(M) - \Phi_{s-\varphi}(M+a)}{a}$$

for all $M \ge M_\epsilon$. By letting $a \to 0^+$ it follows that

$$\Phi_{s-\varphi}(M) \le -(\frac{N^N d_N \omega_N}{\epsilon})^{-\frac{1}{N-1}} \Phi'_{s-\varphi}(M)$$

for a.e. $M \ge M_\epsilon$, and by integration in (M_ϵ, M)

$$\Phi_{s-\varphi}(M) \le |\Omega \cap B_R(p)| \exp \left[-(\frac{N^N d_N \omega_N}{\epsilon})^{\frac{1}{N-1}} M \right]$$

for all $M \ge M_\epsilon$, in view of $\Phi_{s-\varphi}(M_\epsilon) \le |\Omega \cap B_R(p)|$. Given $q \ge 1$ we can argue as follows:

$$\begin{aligned} &\int_{\Omega \cap B_R(p)} e^{q|s-\varphi|} - |\Omega \cap B_R(p)| = q \int_{\Omega \cap B_R(p)} dx \int_0^{|s(x) - \varphi(x)|} e^{qM} dM = q \int_0^\infty e^{qM} \Phi_{s-\varphi}(M) dM \\ &\le |\Omega \cap B_R(p)| \left[e^{qM_\epsilon} + q \int_{M_\epsilon}^\infty \exp \left((q - (\frac{N^N d_N \omega_N}{\epsilon})^{\frac{1}{N-1}}) M \right) \right] dM < +\infty \end{aligned}$$

by taking ϵ sufficiently small. Since $\varphi \in C^1(\overline{\Omega \cap B_R(p)})$, we get that e^s is a L^q-function near any $p \in S$, and then $e^s \in L^q(\Omega)$ for all $q \ge 1$. By the uniqueness result in [36] and by Theorems A.1, A.4 we get that $s \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$.

Remark 2.4. *The proof of $s \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, might be carried over in a shorter way. Indeed, the function $We^s \in L^1(\Omega)$ can be approximated either in a strong L^1-sense or in a weak measure-sense. In the former case, the limiting function z is an entropy solution of*

$$\begin{cases} -\Delta_N z = We^s & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

while in the latter we end up with s by choosing $W_k e^{s_k}$ as the approximation in measure-sense. As consequence of the impressive uniqueness result in [36]*, $s = z$ and then s is a entropy solution of* (2.13) *(see* [2, 10] *for the definition of entropy solution). Lemma 2.1 is proved in* [2] *for entropy solutions, and has been used there, among other things, to show that a entropy solution s of* (2.13) *is necessarily in $C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. We have preferred a longer proof to give a self-contained argument which does not require to introduce special notions of distributional solutions (like SOLA, entropy and renormalized solutions, just to quote some of them).*

Fix any $p_0 \in \partial\Omega$ and take $R_0 > 0$ small so that $\overline{B_{R_0}(p_0)} \cap S = \{p_0\}$. Setting $y_k = p_0 + \rho_{k,R}\nu(p_0)$ with $0 < R \le R_0$ and

$$\rho_{k,R} = \frac{\int_{\partial\Omega\cap B_R(p_0)} \langle x - p_0, \nu\rangle |\nabla u_k|^N}{\int_{\partial\Omega\cap B_R(p_0)} \langle \nu(p_0), \nu\rangle |\nabla u_k|^N},$$

we have that

$$\int_{\partial\Omega\cap B_R(p_0)} \langle x - y_k, \nu\rangle |\nabla u_k|^N = 0. \tag{2.20}$$

Up to take R_0 smaller, we can assume that $|\rho_{k,R}| \le 2R$. Applying Lemma 2.3 to s_k on $\Omega \cap B_R(p_0)$ with $y = y_k$, we obtain that

$$\begin{aligned}\int_{\Omega\cap B_R(p_0)} [NW_k + \langle x - y_k, \nabla W_k\rangle] e^{s_k} &= \int_{\partial(\Omega\cap B_R(p_0))} W_k e^{s_k} \langle x - y_k, \nu\rangle \\ &+ \int_{\partial(\Omega\cap B_R(p_0))} \Big[|\nabla s_k|^{N-2} \langle x - y_k, \nabla s_k\rangle \partial_\nu s_k - \frac{|\nabla s_k|^N}{N} \langle x - y_k, \nu\rangle \Big].\end{aligned} \tag{2.21}$$

We would like to let $k \to +\infty$, but $\partial(\Omega \cap B_R(p_0))$ contains the portion $\partial\Omega \cap B_R(p_0)$ where the convergence $s_k \to s$ might fail. The clever choice of $\rho_{k,R}$, as illustrated by (2.20), leads to

$$\int_{\partial\Omega\cap B_R(p_0)} \Big[|\nabla s_k|^{N-2} \langle x - y_k, \nabla s_k\rangle \partial_\nu s_k - \frac{|\nabla s_k|^N}{N} \langle x - y_k, \nu\rangle \Big] = (1 - \frac{1}{N}) \int_{\partial\Omega\cap B_R(p_0)} |\nabla u_k|^N \langle x - y_k, \nu\rangle = 0$$

in view of $\nabla s_k = \nabla u_k$ and $\nabla s_k = -|\nabla s_k|\nu$ on $\partial\Omega$ by means of $s_k = 0$ on $\partial\Omega$. Hence, (2.21) reduces to

$$\begin{aligned}N \int_{\Omega\cap B_R(p_0)} V_k e^{u_k} &= -\int_{\Omega\cap B_R(p_0)} \langle x - y_k, \frac{\nabla V_k}{V_k}\rangle V_k e^{u_k} + \int_{\partial(\Omega\cap B_R(p_0))} V_k e^{u_k} \langle x - y_k, \nu\rangle \\ &+ \int_{\Omega\cap\partial B_R(p_0)} \Big[|\nabla s_k|^{N-2} \langle x - y_k, \nabla s_k\rangle \partial_\nu s_k - \frac{|\nabla s_k|^N}{N} \langle x - y_k, \nu\rangle \Big].\end{aligned} \tag{2.22}$$

Since $|x - y_k| \le 3R$ and $|\frac{\nabla V_k}{V_k}| \le C_0^2$ in $\Omega \cap B_R(p_0)$ in view of (1.3), by letting $k \to +\infty$ in (2.22) we get that

$$N\mu(\Omega \cap B_R(p_0)) \le 3RC_0^2 \mu(\Omega \cap B_R(p_0)) + 3C_0 Re^M |\partial(\Omega \cap B_R(p_0))| + 3R(1 + \frac{1}{N}) \int_{\Omega\cap\partial B_R(p_0)} |\nabla s|^N$$

in view of $s_k \to s$ in $C^1_{loc}(\overline{\Omega} \setminus S)$. Since $s \in C^1(\overline{\Omega})$, by letting $R \to 0$ we deduce that $\mu(\{p_0\}) = 0$, and then $p_0 \notin \Sigma = S$. Since this is true for all $p_0 \in \partial\Omega$, we have shown that $S \subset \Omega$, and the proof of Step 5 is complete.

The combination of the previous 5 Steps provides us with a complete proof of Theorem 1.1.

□

Once Theorem 1.1 has been established, we can derive the following:
Proof (of Corollary 1.2).
By contradiction, assume the existence of sequences $\lambda_k \in \Lambda$, V_k satisfying (1.3) and $u_k \in C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, weak solutions to (1.1) so that $\|u_k\|_\infty \to +\infty$ as $k \to +\infty$. First of all, we can assume $\lambda_k > 0$ (otherwise $u_k = 0$) and

$$\max_{\Omega} V_k e^{u_k - \alpha_k} \to +\infty \tag{2.23}$$

as $k \to +\infty$ in view of Corollary A.3, where $\alpha_k = \log(\frac{\int_\Omega V_k e^{u_k}}{\lambda_k})$. The function $\hat{u}_k = u_k - \alpha_k$ solves

$$\begin{cases} -\Delta_N \hat{u}_k = V_k e^{\hat{u}_k} & \text{in } \Omega, \\ \hat{u}_k = -\alpha_k & \text{on } \partial\Omega. \end{cases}$$

Since $\lambda_k \in \Lambda$ and Λ is a compact set, we have that $\sup_k \int_\Omega V_k e^{\hat{u}_k} = \sup_k \lambda_k < +\infty$, and then $\sup_k \int_\Omega e^{\hat{u}_k} < +\infty$ in view of (1.3). Since $\operatorname{osc}_{\partial\Omega}(\hat{u}_k) = 0$, we can apply Theorem 1.1 to $\hat{u}_k$. Since $\max_\Omega \hat{u}_k \to +\infty$ as $k \to +\infty$ in view of (1.3) and (2.23), alternative (iii) in Theorem 1.1 occurs for $\hat{u}_k$. By (1.6) we get that

$$\lambda_k = \int_\Omega V_k e^{\hat{u}_k} \to c_N m$$

as $k \to +\infty$, for some $m \in \mathbb{N}$. Hence, $c_N m \in \Lambda$, in contradiction with $\Lambda \subset [0, +\infty) \setminus c_N\mathbb{N}$.

□

3. A general existence result

The Moser-Trudinger inequality [57] states that, for some $C_\Omega > 0$, there holds

$$\int_\Omega \exp(\alpha |u|^{\frac{N}{N-1}}) dx \le C_\Omega \tag{3.1}$$

for all $u \in W_0^{1,N}(\Omega)$ with $\|u\|_{W_0^{1,N}(\Omega)} \le 1$ and all $\alpha \le \alpha_N = (N^N \omega_N)^{\frac{1}{N-1}}$, whereas (3.1) is false when $\alpha > \alpha_N$. A simple consequence of (3.1), always referred to as the Moser-Trudinger inequality, is the following:

$$\log\left(\int_\Omega e^u dx\right) \le \frac{1}{N c_N} \|u\|^N_{W_0^{1,N}(\Omega)} + \log C_\Omega \tag{3.2}$$

for all $u \in W_0^{1,N}(\Omega)$, where c_N is defined in Theorem 1.1. Indeed, (3.2) follows by (3.1) by noticing

$$u \le [(\frac{N\alpha_N}{N-1})^{-\frac{N-1}{N}} \|u\|_{W_0^{1,N}(\Omega)}] \times [(\frac{N\alpha_N}{N-1})^{\frac{N-1}{N}} \frac{|u|}{\|u\|_{W_0^{1,N}(\Omega)}}] \le \frac{1}{Nc_N} \|u\|^N_{W_0^{1,N}(\Omega)} + \alpha_N |\frac{u}{\|u\|_{W_0^{1,N}(\Omega)}}|^{\frac{N}{N-1}}$$

in view of the Young's inequality. By (3.2) it follows that:

$$J_\lambda(u) \ge \frac{1}{N}(1 - \frac{\lambda}{c_N}) \|u\|^N_{W_0^{1,N}(\Omega)} - \lambda \log(C_0 C_\Omega)$$

for all $u \in W_0^{1,N}(\Omega)$ in view of (1.3), where J_λ is given in (1.8). Hence, J_λ is bounded from below for $\lambda \le c_N$ and coercive for $\lambda < c_N$. Since the map $u \in W_0^{1,N}(\Omega) \to V e^u \in L^1(\Omega)$ is compact in view of (3.2) and the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, for $\lambda < c_N$ we have that J_λ attains the global minimum in $W_0^{1,N}(\Omega)$, and then (1.1) is solvable. In Theorem 1.3 we just consider the difficult case $\lambda > c_N$. Notice that a solution $u \in W_0^{1,N}(\Omega)$ of (1.1) belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, in view of (3.2) and Theorems A.1, A.4.

The constant $\frac{1}{Nc_N}$ in (3.2) is optimal as it follows by evaluating the inequality along

$$U(\frac{x-p}{\epsilon}) - \frac{N^2}{N-1} \log \epsilon, \quad p \in \Omega,$$

as $\epsilon \to 0$, up to make a cut-off away from p so to have a function in $W_0^{1,N}(\Omega)$. The function U is given in (1.7) and, as already mentioned in the Introduction, satisfies

$$\int_{\mathbb{R}^N} e^U = c_N.$$

Indeed, the equation $-\Delta_N U = e^U$ does hold pointwise in $\mathbb{R}^N \setminus \{0\}$, and then can be integrated in $B_R(0) \setminus B_\epsilon(0)$, $0 < \epsilon < R$, to get

$$\int_{B_R(0)\setminus B_\epsilon(0)} e^U = -\int_{\partial B_R(0)} |\nabla U|^{N-2} \langle \nabla U, \nu \rangle + \int_{\partial B_\epsilon(0)} |\nabla U|^{N-2} \langle \nabla U, \nu \rangle,$$

where $\nu(x) = \frac{x}{|x|}$. Letting $\epsilon \to 0$ and $R \to +\infty$, we get that

$$\int_{\mathbb{R}^N} e^U = N(\frac{N^2}{N-1})^{N-1} \omega_N = c_N$$

in view of

$$\nabla U = -\frac{N^2}{N-1} \frac{|x|^{\frac{N}{N-1}-2} x}{1 + |x|^{\frac{N}{N-1}}}.$$

Since $\frac{1}{Nc_N}$ in (3.2) is optimal, the functional J_λ is unbounded from below for $\lambda > c_N$, and our goal is to develop a global variational strategy to find a critical point of saddle type. The classical Morse theory states that a sublevel is a deformation retract of an higher sublevel unless there are critical points in between, and the crucial assumption on the functional is the validity of the so-called Palais-Smale condition. Unfortunately, in our context such assumption fails since J_λ admits unbounded Palais-Smale sequences for $\lambda \ge c_N$, see [40, 53]. This technical difficulty can be overcome by using a method introduced by Struwe that exploits the monotonicity of the functional $\frac{J_\lambda}{\lambda}$ in λ. An alternative approach has been found in [53], which provides a deformation between two sublevels unless J_{λ_k} has critical points in the energy strip for some sequence $\lambda_k \to \lambda$. Thanks to the compactness result in Corollary 1.2 and the a-priori estimates in Theorem A.4, we have at hands the following crucial tool:

Lemma 3.1. *Let $\lambda \in (c_N, +\infty) \setminus c_N\mathbb{N}$. If J_λ has no critical levels u with $a \le J_\lambda(u) \le b$, then J_λ^a is a deformation retract of J_λ^b, where*

$$J_\lambda^t = \{u \in W_0^{1,N}(\Omega) : J_\lambda(u) \le t\}.$$

To attack existence issues for (1.1) when $\lambda \in (c_N, +\infty) \setminus c_N\mathbb{N}$, it is enough to find any two sublevels J_λ^a and J_λ^b which are not homotopically equivalent.

Hereafter, the parameter λ is fixed in $(c_N, +\infty) \setminus c_N\mathbb{N}$. By Corollary 1.2 and Theorem A.4 we have that J_λ does not have critical points with large energy. Exactly as in [55], Lemma 3.1 can be used to construct a deformation retract of $W_0^{1,N}(\Omega)$ onto very high sublevels of J_λ. More precisely, we have the following

Lemma 3.2. *There exists $L>0$ large so that J_λ^L is a deformation retract of $W_0^{1,N}(\Omega)$. In particular, J_λ^L is contractible.*

For the sake of completeness, we give some details of the proof.

Proof. Take $L \in \mathbb{N}$ large so that J_λ has no critical points u with $J_\lambda(u) \geq L$. By Lemma 3.1 J_λ^n is a deformation retract of J_λ^{n+1} for all $n \geq L$, and η_n will denote the corresponding retraction map. Given $u \in W_0^{1,N}(\Omega)$ with $J_\lambda(u) > L$, by setting recursively

$$\begin{cases} \eta^{1,n}(s,u) = \eta_n(s,u) \\ \eta^{2,n}(s,u) = \eta_{n-1}(s-1,\eta_n(1,u)) \\ \vdots \\ \eta^{k+1,n} = \eta_{n-k}(s-k,\eta^{(k)}(k,u)), \end{cases}$$

for $s \geq 0$ we consider the following map

$$\hat{\eta}(s,u) = \begin{cases} \eta^{k+1,n}(s,u) & \text{if } n < J_\lambda(u) \leq n+1 \text{ for } n \geq L, s \in [k,k+1] \\ u & \text{if } J_\lambda(u) \leq L. \end{cases}$$

Next, define s_u as the first $s>0$ such that $J_\lambda(\hat{\eta}(s,u)) = L$ if $J_\lambda(u) > L$ and as 0 if $J_\lambda(u) \leq L$. The map $\eta(t,u) = \hat{\eta}(ts_u,u) : [0,1] \times W_0^{1,N}(\Omega) \to W_0^{1,N}(\Omega)$ satisfies $\eta(1,u) \in J_\lambda^L$ for $u \in W_0^{1,N}(\Omega)$ and $\eta(t,u) = u$ for $(t,u) \in [0,1] \times J_\lambda^L$. Since s_u depends continuously in u, the map η is continuous in both variables, providing us with the required deformation retract. □

Thanks to Lemmas 3.1 and 3.2, we are led to study the topology of sublevels for J_λ with very low energy. The real core of such a global variational approach is an improved form [22] of the Moser-Trudinger inequality for functions $u \in W_0^{1,N}(\Omega)$ with a measure $\frac{Ve^u}{\int_\Omega Ve^u}$ concentrated on several subomains in Ω. As a consequence, when $\lambda \in (c_N m, c_N(m+1))$ and $J_\lambda(u)$ is very negative, the measure $\frac{Ve^u}{\int_\Omega Ve^u}$ can be concentrated near at most m points of $\overline{\Omega}$, and can be naturally associated to an element $\sigma \in \mathcal{B}_m(\overline{\Omega})$, where

$$\mathfrak{B}_m(\overline{\Omega}) := \{\sum_{i=1}^m t_i \delta_{p_i} : \ t_i \geq 0, \ \sum_{i=1}^m t_i = 1, \ p_i \in \overline{\Omega}\}$$

has been first introduced by Bahri and Coron in [3, 4] and is known in literature as the *space of formal barycenters* of $\overline{\Omega}$ with order m. The topological structure of J_λ^{-L}, $L>0$ large, is completely characterized in terms of $\mathcal{B}_m(\overline{\Omega})$. The non-contractibility of $\mathcal{B}_m(\overline{\Omega})$ let us see a change in topology between J_λ^L and J_λ^{-L} for $L>0$ large, and by Lemma 3.1 we obtain the existence result claimed in Theorem 1.3. Notice that our approach is simpler than the one in [33, 34, 35] (see also [9]), by using [53] instead of the Struwe's monotonicity trick to bypass the general failure of PS-condition for J_λ.

As already explained, the key point is the following improvement of the Moser-Trudinger inequality:

Lemma 3.3. *Let Ω_i, $i = 1,\dots,l+1$, be subsets of $\overline{\Omega}$ so that dist$(\Omega_i,\Omega_j) \geq \delta_0 > 0$, for $i \neq j$, and $\gamma_0 \in (0,\frac{1}{l+1})$. Then, for any $\epsilon > 0$ there exists a constant $C = C(\epsilon,\delta_0,\gamma_0)$ such that there holds*

$$\log(\int_\Omega Ve^u dx) \leq \frac{1}{Nc_N(l+1-\epsilon)} \|u\|^N_{W_0^{1,N}(\Omega)} + C$$

for all $u \in W_0^{1,N}(\Omega)$ with

$$\frac{\int_{\Omega_i} Ve^u}{\int_\Omega Ve^u} \geq \gamma_0 \quad i = 1,\dots,l+1. \tag{3.3}$$

Proof. Let $g_1,\dots,g_{l+1}$ be cut-off functions so that $0 \leq g_i \leq 1$, $g_i = 1$ in Ω_i, $g_i = 0$ in $\{\text{dist}(x,\Omega_i) \geq \frac{\delta_0}{4}\}$ and $\|g_i\|_{C^2(\overline{\Omega})} \leq C_{\delta_0}$. Since g_i, $i = 1,\dots,l$, have disjoint supports, for all $u \in W_0^{1,N}(\Omega)$ there exists $i = 1,\dots,l+1$ such that

$$\int_\Omega (g_i|\nabla u|)^N \leq \frac{1}{l+1} \int_{\cup_{i=1}^{l+1} \text{supp} g_i} |\nabla u|^N \leq \frac{1}{l+1} \|u\|^N_{W_0^{1,N}(\Omega)}. \tag{3.4}$$

Since by the Young's inequality

$$\begin{aligned} |\nabla(g_i u)|^N &\leq (g_i|\nabla u| + |\nabla g_i||u|)^N \leq (g_i|\nabla u|)^N + C_1\left[(g_i|\nabla u|)^{N-1}|\nabla g_i||u| + (|\nabla g_i||u|)^N\right] \\ &\leq [1 + \frac{\epsilon}{(l+1)(3l+3-\epsilon)}](g_i|\nabla u|)^N + C_2(|\nabla g_i||u|)^N \end{aligned}$$

for all $\epsilon > 0$ and some $C_1 > 0$, $C_2 = C_2(\epsilon) > 0$, we have that

$$\|g_i u\|^N_{W_0^{1,N}(\Omega)} \leq \int_\Omega (g_i|\nabla u|)^N + \frac{\epsilon}{(l+1)(3l+3-\epsilon)} \|u\|_{W_0^{1,N}(\Omega)} + Nc_N C_3 \|u\|^N_{L^N(\Omega)},$$

where $C_3 = \frac{C_2 C_{\delta_0}^N}{N c_N}$. Since $g_i u \in W_0^{1,N}(\Omega)$, by (3.2) and (3.4) it follows that

$$\int_\Omega e^{g_i u} \leq C_\Omega \exp\left(\frac{3}{N c_N (3l+3-\epsilon)} \|u\|^N_{W_0^{1,N}(\Omega)} + C_3 \|u\|^N_{L^N(\Omega)}\right) \tag{3.5}$$

does hold for all $u \in W_0^{1,N}(\Omega)$ and some $i = 1, \ldots, l+1$.

Let $\eta \in (0, |\Omega|)$ be given. Since $\{|u| \geq 0\} = \Omega$ and $\lim\limits_{a \to +\infty} |\{|u| \geq a\}| = 0$, the set

$$A_\eta = \{a \geq 0 : \ |\{|u| \geq a\}| \geq \eta\}$$

is non-empty and bounded from above. Letting $a_\eta = \sup A_\eta$, we have that $a_\eta \geq 0$ is a finite number so that

$$|\{|u| \geq a_\eta\}| \geq \eta, \qquad |\{|u| \geq a\}| < \eta \quad \forall\, a > a_\eta \tag{3.6}$$

in view of the left-continuity of the map $a \to \ |\{|u| \geq a\}|$. Given $\eta > 0$ and $u \in W_0^{1,N}(\Omega)$ satisfying (3.3), we can fix $a = a_\eta$ and $i = 1, \ldots, l+1$ so that (3.5) applies to $(|u| - 2a)_+$ yielding to

$$\int_\Omega V e^u \leq \frac{1}{\gamma_0} \int_{\Omega_i} V e^{|u|} \leq \frac{C_0 e^{2a}}{\gamma_0} \int_\Omega e^{g_i (|u|-2a)_+} \leq \frac{C_0 C_\Omega}{\gamma_0} \exp\left(\frac{3}{N c_N (3l+3-\epsilon)} \|u\|^N_{W_0^{1,N}(\Omega)} + 2a + C_3 \|(|u|-2a)_+\|^N_{L^N(\Omega)}\right)$$

in view of (1.3). By the Poincaré and Young inequalities and the first property in (3.6) it follows that

$$2a \leq \frac{2}{\eta} \int_{\{|u| \geq a\}} |u| \leq \frac{C_5}{\eta} \|u\|_{W_0^{1,N}(\Omega)} \leq \frac{3\epsilon}{N c_N (3l+3-\epsilon)(3l+3-2\epsilon)} \|u\|^N_{W_0^{1,N}(\Omega)} + C_6$$

for some $C_5 > 0$ and $C_6 = C_6(\epsilon, \eta) > 0$, and there holds

$$\|(|u|-2a)_+\|^N_{L^N(\Omega)} \leq \eta^{\frac{1}{2}} \|(|u|-2a)_+\|^N_{L^{2N}(\Omega)} \leq C_4 \eta^{\frac{1}{2}} \|u\|^N_{W_0^{1,N}(\Omega)}$$

for some $C_4 > 0$ in view of the Hölder and Sobolev inequalities and the second property in (3.6). Choosing η small as

$$\eta = \left(\frac{\epsilon}{C_3 C_4 N c_N (3l+3-2\epsilon)(l+1-\epsilon)}\right)^2,$$

we finally get that

$$\int_\Omega V e^u \leq \frac{C_0 C_\Omega}{\gamma_0} \exp\left(\frac{1}{N c_N (l+1-\epsilon)} \|u\|^N_{W_0^{1,N}(\Omega)} + C\right)$$

for some $C = C(\epsilon, \delta_0, \gamma_0)$. □

A criterium for the occurrence of (3.3) is the following:

Lemma 3.4. *Let $l \in \mathbb{N}$ and $0 < \epsilon, r < 1$. There exist $\bar\epsilon > 0$ and $\bar r > 0$ such that, for every $0 \leq f \in L^1(\Omega)$ with*

$$\|f\|_{L^1(\Omega)} = 1\ , \quad \int_{\Omega \cap \bigcup_{i=1}^l B_r(p_i)} f < 1 - \epsilon \qquad \forall\, p_1, \ldots, p_l \in \overline{\Omega}, \tag{3.7}$$

there exist $l+1$ points $\bar p_1, \ldots, \bar p_{l+1} \in \overline{\Omega}$ so that

$$\int_{\Omega \cap B_{\bar r}(\bar p_i)} f \geq \bar\epsilon\ , \qquad B_{2\bar r}(\bar p_i) \cap B_{2\bar r}(\bar p_j) = \emptyset \quad \forall\, i \neq j.$$

Proof. By contradiction, for all $\bar\epsilon, \bar r > 0$ we can find $0 \leq f \in L^1(\Omega)$ satisfying (3.7) such that, for every $(l+1)$-tuple of points $p_1, ..., p_{l+1} \in \overline{\Omega}$ the statement

$$\int_{\Omega \cap B_{\bar r}(p_i)} f \geq \bar\epsilon\ , \qquad B_{2\bar r}(p_i) \cap B_{2\bar r}(p_j) = \emptyset \quad \forall\, i \neq j \tag{3.8}$$

is false. Setting $\bar r = \frac{r}{8}$, by compactness we can find h points $x_i \in \overline{\Omega}$, $i = 1, \ldots, h$, such that $\overline{\Omega} \subset \bigcup_{i=1}^h B_{\bar r}(x_i)$. Setting $\bar\epsilon = \frac{\epsilon}{2h}$, there exists $i = 1, ..., h$ such that $\int_{\Omega \cap B_{\bar r}(x_i)} f \geq \bar\epsilon$. Let $\{\tilde x_1, ..., \tilde x_j\} \subseteq \{x_1, ..., x_h\}$ be the maximal set with respect to the property $\int_{\Omega \cap B_{\bar r}(\tilde x_i)} f \geq \bar\epsilon$. Set $j_1 = 1$ and let X_1 denote the set

$$X_1 = \Omega \cap \bigcup_{i \in \Lambda_1} B_{\bar r}(\tilde x_i) \subseteq \Omega \cap B_{6\bar r}(\tilde x_{j_1}), \quad \Lambda_1 = \{i = 1, ..., j : \ B_{2\bar r}(\tilde x_i) \cap B_{2\bar r}(\tilde x_{j_1}) \neq \emptyset\}.$$

If non empty, choose $j_2 \in \{1, ..., j\} \setminus \Lambda_1$, i.e. $B_{2\bar r}(\tilde x_{j_2}) \cap B_{2\bar r}(\tilde x_{j_1}) = \emptyset$. Let X_2 denote the set

$$X_2 = \Omega \cap \bigcup_{i \in \Lambda_2} B_{\bar r}(\tilde x_i) \subseteq \Omega \cap B_{6\bar r}(\tilde x_{j_2}), \quad \Lambda_2 = \{i = 1, ..., j : \ B_{2\bar r}(\tilde x_i) \cap B_{2\bar r}(\tilde x_{j_2}) \neq \emptyset\}.$$

Iterating this process, if non empty, at the $l-$th step we choose $j_l \in \{1, ..., j\} \setminus \bigcup_{j=1}^{l-1} \Lambda_j$, i.e. $B_{2\bar r}(\tilde x_{j_l}) \cap B_{2\bar r}(\tilde x_{j_i}) = \emptyset$ for all $i = 1, \ldots, l-1$, and we define

$$X_l = \Omega \cap \bigcup_{i \in \Lambda_l} B_{\bar r}(\tilde x_i) \subseteq \Omega \cap B_{6\bar r}(\tilde x_{j_l}), \quad \Lambda_l = \{i = 1, ..., j : \ B_{2\bar r}(\tilde x_i) \cap B_{2\bar r}(\tilde x_{j_l}) \neq \emptyset\}.$$

By (3.8) the process has to stop at the $s-$th step with $s \le l$. By the definition of $\bar r$ we obtain

$$\Omega \cap \bigcup_{i=1}^{j} B_{\bar r}(\tilde x_i) \subset \bigcup_{i=1}^{s} X_i \subset \Omega \cap \bigcup_{i=1}^{s} B_{6\bar r}(\tilde x_{j_i}) \subset \Omega \cap \bigcup_{i=1}^{s} B_r(\tilde x_{j_i})$$

in view of $\{1,...,j\} = \bigcup_{i=1}^{s} \Lambda_i$. Therefore, we have that

$$\int_{\Omega \setminus \bigcup_{i=1}^{s} B_r(\tilde x_{j_i})} f \le \int_{\Omega \setminus \bigcup_{i=1}^{j} B_{\bar r}(\tilde x_i)} f = \int_{(\Omega \cap \bigcup_{i=1}^{h} B_{\bar r}(x_i)) \setminus (\bigcup_{i=1}^{j} B_{\bar r}(\tilde x_i))} f < (h-j)\bar\epsilon < \frac{\epsilon}{2}$$

in view of the definition of $\tilde x_1, \dots, \tilde x_j$. Define p_i as $\tilde x_{j_i}$ for $i = 1, \dots, s$ and as $\tilde x_{j_s}$ for $i = s+1, \dots, l$. Since $\int_{\Omega \setminus \bigcup_{i=1}^{l} B_r(p_i)} f < \frac{\epsilon}{2}$, we deduce that

$$\int_{\Omega \cap \bigcup_{i=1}^{l} B_r(p_i)} f = \int_\Omega f - \int_{\Omega \setminus \bigcup_{i=1}^{l} B_r(p_i)} f > 1 - \frac{\epsilon}{2} > 1 - \epsilon,$$

contradicting the second property in (3.7). The proof is complete. □

As a consequence, we get that

Lemma 3.5. *Let $\lambda \in \big(c_N m, c_N(m+1)\big)$, $m \in \mathbb{N}$. For any $0 < \epsilon, r < 1$ there exists a large $L = L(\epsilon, r) > 0$ such that, for every $u \in W_0^{1,N}(\Omega)$ with $J_\lambda(u) \le -L$, we can find m points $p_{i,u} \in \overline{\Omega}$, $i = 1, \dots, m$, satisfying*

$$\int_{\Omega \setminus \cup_{i=1}^{m} B_r(p_{i,u})} V e^u \le \epsilon \int_\Omega V e^u.$$

Proof. By contradiction there exist ϵ, $r \in (0,1)$ and functions $u_k \in W_0^{1,N}(\Omega)$ so that $J_\lambda(u_k) \to -\infty$ as $k \to +\infty$ and

$$\int_{\Omega \setminus \cup_{i=1}^{m} B_r(p_i)} V e^{\hat u_k} > \epsilon \tag{3.9}$$

for all $p_1, ..., p_m \in \overline{\Omega}$, where $\hat u_k = u_k - \log \int_\Omega V e^{u_k}$. Since

$$\int_{\Omega \setminus \cup_{i=1}^{m} B_r(p_i)} V e^{\hat u_k} = \int_\Omega V e^{\hat u_k} - \int_{\Omega \cap \cup_{i=1}^{m} B_r(p_i)} V e^{\hat u_k} = 1 - \int_{\Omega \cap \cup_{i=1}^{m} B_r(p_i)} V e^{\hat u_k},$$

by (3.9) we get that

$$\int_{\Omega \cap \cup_{i=1}^{m} B_r(p_i)} V e^{\hat u_k} < 1 - \epsilon$$

for all m-tuple $p_1, \dots, p_m \in \overline{\Omega}$. Applying Lemma 3.4 with $l = m$ and $f = V e^{\hat u_k}$, we find $\bar\epsilon, \bar r > 0$ and $\bar p_1, \dots, \bar p_{m+1} \in \overline{\Omega}$ so that

$$\int_{\Omega \cap B_{\bar r}(\bar p_i)} V e^{u_k} \ge \bar\epsilon \int_\Omega V e^{u_k}, \qquad B_{2\bar r}(\bar p_i) \cap B_{2\bar r}(\bar p_j) = \emptyset \quad \forall\, i \ne j.$$

Applying Lemma 3.3 with $\Omega_i = \Omega \cap B_{\bar r}(\bar p_i)$ for $i = 1, \dots, m+1$, $\delta_0 = 2\bar r$ and $\gamma_0 = \bar\epsilon$, it now follows that

$$\log\left(\int_\Omega V e^{u_k}\right) \le \frac{1}{N c_N (m+1-\eta)} \|u\|^N_{W_0^{1,N}(\Omega)} + C$$

for all $\eta > 0$, for some $C = C(\eta, \delta_0, \gamma_0, a, b)$. Since $\lambda < c_N(m+1)$, we get that

$$J_\lambda(u_k) = \frac{1}{N} \|u_k\|^N_{W_0^{1,N}(\Omega)} - \lambda \log\left(\int_\Omega V e^{u_k} dx\right) \ge \frac{1}{N}\left(1 - \frac{\lambda}{c_N(m+1-\eta)}\right) \|u_k\|^N_{W_0^{1,N}(\Omega)} - C\lambda \ge -C\lambda$$

for $\eta > 0$ small, in contradiction with $J_\lambda(u_k) \to -\infty$ as $k \to +\infty$. □

The set $\mathcal{M}(\overline{\Omega})$ of all Radon measures on $\overline{\Omega}$ is a metric space with the Kantorovich-Rubinstein distance, which is induced by the norm

$$\|\mu\|_* = \sup_{\|\phi\|_{Lip(\overline{\Omega})} \le 1} \int_\Omega \phi d\mu, \qquad \mu \in \mathcal{M}(\overline{\Omega}).$$

Lemma 3.5 can be re-phrased as

Lemma 3.6. *Let $\lambda \in \big(c_N m, c_N(m+1)\big)$, $m \in \mathbb{N}$. For any $\epsilon > 0$ small there exists a large $L = L(\varepsilon) > 0$ such that, for every $u \in W_0^{1,N}(\Omega)$ with $J_\lambda(u) \le -L$, we have*

$$dist\left(\frac{V e^u}{\int_\Omega V e^u}, \mathfrak{B}_m(\overline{\Omega})\right) \le \epsilon. \tag{3.10}$$

Proof. Given $\epsilon \in (0,2)$ and $r = \frac{\epsilon}{4}$, let $L = L(\frac{\epsilon}{4}, r) > 0$ be as given in Lemma 3.5. For all $u \in W_0^{1,N}(\Omega)$ with $J_\lambda(u) \le -L$, let us denote for simplicity as $p_1, \dots, p_m \in \overline{\Omega}$ the corresponding points $p_{1,u}, \dots, p_{n,u}$ such that

$$\int_{\Omega \setminus \bigcup_{i=1}^m B_r(p_i)} Ve^u \le \frac{\epsilon}{4} \int_\Omega Ve^u. \tag{3.11}$$

Define $\sigma \in \mathfrak{B}_m(\overline{\Omega})$ as

$$\sigma = \sum_{i=1}^m t_i \delta_{p_i}, \qquad t_i = \frac{\int_{A_{r,i}} Ve^u}{\int_{\Omega \cap \bigcup_{i=1}^m B_r(p_i)} Ve^u},$$

where $A_{r,i} = (\Omega \cap B_r(p_i)) \setminus \bigcup_{j=1}^{i-1} B_r(p_j)$. Since $A_{r,i}$, $i = 1, \dots, m$, are disjoint sets with $\bigcup_{i=1}^m A_{r,i} = \Omega \cap \bigcup_{i=1}^m B_r(p_i)$, we have that $\sum_{i=1}^m t_i = 1$ and

$$\begin{aligned}
\left| \int_\Omega \phi \left[Ve^u dx - (\int_\Omega Ve^u) d\sigma \right] \right| &\le \left| \int_{\Omega \setminus \bigcup_{i=1}^m B_r(p_i)} Ve^u \phi \right| + \left| \int_{\Omega \cap \bigcup_{i=1}^m B_r(p_i)} Ve^u \phi - (\int_\Omega Ve^u) \sum_{i=1}^m t_i \phi(p_i) \right| \\
&\le \frac{\epsilon}{4} \int_\Omega Ve^u + \sum_{i=1}^m \left| \int_{A_{r,i}} Ve^u \phi - (\int_\Omega Ve^u) t_i \phi(p_i) \right| \\
&\le \frac{\epsilon}{4} \int_\Omega Ve^u + \sum_{i=1}^m \int_{A_{r,i}} Ve^u |\phi - \phi(p_i)| + \left| \frac{\int_\Omega Ve^u}{\int_{\Omega \cap \bigcup_{i=1}^m B_r(p_i)} Ve^u} - 1 \right| \sum_{i=1}^m \int_{A_{r,i}} Ve^u \\
&\le \left(\frac{\epsilon}{4} + r + \frac{\epsilon}{4 - \epsilon} \right) \int_\Omega Ve^u
\end{aligned}$$

in view of (3.11), $\|\phi\|_{Lip(\overline{\Omega})} \le 1$ and

$$\left| \frac{\int_\Omega Ve^u}{\int_{\Omega \cap \bigcup_{i=1}^m B_r(p_i)} Ve^u} - 1 \right| \le \frac{\epsilon}{4 - \epsilon}.$$

Since there holds

$$\left| \int_\Omega \phi \left[\frac{Ve^u dx}{\int_\Omega Ve^u} - d\sigma \right] \right| \le \epsilon$$

for all $\phi \in Lip(\overline{\Omega})$ with $\|\phi\|_{Lip(\overline{\Omega})} \le 1$, we have that

$$\| \frac{Ve^u}{\int_\Omega Ve^u} - \sigma \|_* \le \epsilon$$

for some $\sigma \in \mathfrak{B}_m(\overline{\Omega})$, and then

$$\text{dist}\left(\frac{Ve^u}{\int_\Omega Ve^u}, \mathfrak{B}_m(\overline{\Omega}) \right) \le \epsilon.$$

The proof is complete. □

When (3.10) does hold, one would like to project $\frac{Ve^u}{\int_\Omega Ve^u}$ onto $\mathfrak{B}_m(\overline{\Omega})$. To avoid boundary points (which cause troubles in the construction of the map Φ below) we replace $\overline{\Omega}$ by its retract of deformation $K = \{x \in \Omega : \text{ dist}(x, \partial\Omega) \ge \delta\}$, $\delta > 0$ small. Since $\mathfrak{B}_m(K)$ is a retract of deformation of $\mathfrak{B}_m(\overline{\Omega})$, by [8] there exists a projection map

$$\Pi_m : \{\sigma \in \mathcal{M}(\overline{\Omega}) : \ dist(\sigma, \mathfrak{B}_m(\overline{\Omega})) < \epsilon_0\} \ \to \ \mathfrak{B}_m(K), \quad \epsilon_0 > 0 \text{ small},$$

which is continuous with respect to the Kantorovich-Rubinstein distance. Thanks to Π_m and Lemma 3.6, for $\epsilon \le \epsilon_0$ there exist $L = L(\epsilon) > 0$ large and a continuous map

$$\begin{array}{rccc} \Psi : & J_\lambda^{-L} & \to & \mathfrak{B}_m(K) \\ & u & \to & \Pi_m(\frac{Ve^u}{\int_\Omega Ve^u}). \end{array}$$

The key point now is to construct a continuous map $\Phi : \mathfrak{B}_m(K) \to J_\lambda^{-L}$ so that $\Psi \circ \Phi$ is homotopically equivalent to $\text{Id}_{\mathfrak{B}_m(K)}$. When $\mathfrak{B}_m(\overline{\Omega})$ is non contractible, the same is true for $\mathfrak{B}_m(K)$ and then for J_λ^{-L} for $L > 0$ large. Theorem 1.3 then follows by Lemmas 3.1 and 3.2.

The construction of Φ relies on an appropriate choice of a one-parameter family of functions $\varphi_{\epsilon,\sigma}$, $\sigma \in \mathfrak{B}_m(K)$, modeled on the standard bubbles $U_{\epsilon,p}$, see (1.7). Letting $\chi \in C_0^\infty(\Omega)$ be so that $\chi = 1$ in $\Omega_{\frac{\delta}{2}} = \{x \in \Omega : \text{ dist}(x, \partial\Omega) > \frac{\delta}{2}\}$, we define

$$\varphi_{\epsilon,\sigma}(x) = \chi(x) \log \sum_{i=1}^m t_i \left(\frac{F_N}{(\epsilon^{\frac{N}{N-1}} + |x - p_i|^{\frac{N}{N-1}})^N V(p_i)} \right),$$

where $\sigma = \sum_{i=1}^{m} t_i \delta_{p_i} \in \mathfrak{B}_m(K)$ and $\epsilon > 0$. Since $\varphi_{\epsilon,\sigma} \in W_0^{1,N}(\Omega)$, the map Φ can be constructed as Φ_{ϵ_0}, $\epsilon_0 > 0$ small, where

$$\begin{array}{rccc} \Phi_\epsilon : & \mathfrak{B}_m(K) & \to & J_\lambda^{-L} \\ & \sigma & \to & \varphi_{\epsilon,\sigma}. \end{array}$$

To map $\mathfrak{B}_m(K)$ into the very low sublevel J_λ^{-L}, the difficult point is to produce uniform estimates in σ as $\epsilon \to 0$. We have

Lemma 3.7. *There hold*

(1) *there exist $C_0 > 0$ and $\epsilon_0 > 0$ so that*

$$\left\| \frac{Ve^{\varphi_{\epsilon,\sigma}}}{\int_\Omega Ve^{\varphi_{\epsilon,\sigma}}} - \sigma \right\|_* \le C_0 \epsilon$$

for all $0 < \epsilon \le \epsilon_0$ and $\sigma \in \mathfrak{B}_m(K)$;

(2) *$J_\lambda(\varphi_{\epsilon,\sigma}) \to -\infty$ as $\epsilon \to 0$ uniformly in $\sigma \in \mathfrak{B}_m(K)$.*

Proof. Recall that

$$U_{\epsilon,p}(x) = \log \left(\frac{F_N \epsilon^{\frac{N}{N-1}}}{(\epsilon^{\frac{N}{N-1}} + |x-p|^{\frac{N}{N-1}})^N} \right).$$

Fix $\phi \in Lip(\overline{\Omega})$ with $\|\phi\|_{Lip(\overline{\Omega})} \le 1$. Since $\varphi_{\epsilon,\sigma}$ is bounded from above in $\Omega \setminus \Omega_{\frac{\delta}{2}}$ uniformly in σ, we have that

$$\begin{aligned} \int_\Omega Ve^{\varphi_{\epsilon,\sigma}} \phi &= \epsilon^{-\frac{N}{N-1}} \sum_{i=1}^m \int_{\Omega_{\frac{\delta}{2}}} \frac{t_i V\phi}{V(p_i)} e^{U_{\epsilon,p_i}} + O(1) = \epsilon^{-\frac{N}{N-1}} \sum_{i=1}^m \int_{B_{\frac{\delta}{2}}(p_i)} \frac{t_i V\phi}{V(p_i)} e^{U_{\epsilon,p_i}} + O(1) \\ &= \epsilon^{-\frac{N}{N-1}} \left(c_N \int_\Omega \phi d\sigma + O(\epsilon) \right) \end{aligned} \tag{3.12}$$

as $\epsilon \to 0$ uniformly in ϕ and σ. We have used that

$$\int_{B_{\frac{\delta}{2}}(p_i)} \frac{V\phi}{V(p_i)} e^{U_{\epsilon,p_i}} = \int_{B_{\frac{\delta}{2\epsilon}}(0)} (\phi(p_i) + O(\epsilon|y|)) e^U = c_N \phi(p_i) + O(\epsilon)$$

does hold as $\epsilon \to 0$, uniformly in ϕ and σ, in view of (1.3). Therefore, there holds

$$\left| \int_\Omega \phi \left(\frac{Ve^{\varphi_{\epsilon,\sigma}}}{\int_\Omega Ve^{\varphi_{\epsilon,\sigma}}} dx - d\sigma \right) \right| \le C_0 \epsilon$$

for all $\phi \in Lip(\overline{\Omega})$ with $\|\phi\|_{Lip(\overline{\Omega})} \le 1$, and then

$$\| \frac{Ve^{\varphi_{\epsilon,\sigma}}}{\int_\Omega Ve^{\varphi_{\epsilon,\sigma}}} - \sigma \|_* \le C_0 \epsilon$$

for all $\sigma \in \mathfrak{B}_m(K)$. Part (1) is proved.

For part (2), it is enough to show that

$$\log \int_\Omega Ve^{\varphi_{\epsilon,\sigma}} = \frac{N}{N-1} \log \frac{1}{\epsilon} + O(1) \tag{3.13}$$

$$\frac{1}{N} \int_\Omega |\nabla \varphi_{\epsilon,\sigma}|^N \le \frac{N}{N-1} c_N m \log \frac{1}{\epsilon} + O(1) \tag{3.14}$$

as $\epsilon \to 0$ uniformly in $\sigma \in \mathfrak{B}_m(K)$, in view of $\lambda > mc_N$. Estimate (3.13) follows by (3.12) with $\phi = 1$. As far as (3.14) is concerned, let us set $\varphi_{\epsilon,\sigma} = \chi \tilde{\varphi}_{\epsilon,\sigma}$. All the estimates below are uniform in σ. Since

$$\nabla \tilde{\varphi}_{\epsilon,\sigma} = -\frac{N^2}{N-1} \frac{\sum_{i=1}^m t_i V(p_i)^{-1} (\epsilon^{\frac{N}{N-1}} + |x-p_i|^{\frac{N}{N-1}})^{-(N+1)} |x-p_i|^{\frac{N}{N-1}-2} (x-p_i)}{\sum_{i=1}^m t_i V(p_i)^{-1} (\epsilon^{\frac{N}{N-1}} + |x-p_i|^{\frac{N}{N-1}})^{-N}},$$

we have that $\|\tilde{\varphi}_{\epsilon,\sigma}\|_{C^1(\Omega \setminus \Omega_{\frac{\delta}{2}})} = O(1)$ and then

$$|\nabla \varphi_{\epsilon,\sigma}| = O(1)$$

in $\Omega \setminus \Omega_{\frac{\delta}{2}}$. Therefore we can write that

$$\frac{1}{N} \int_\Omega |\nabla \varphi_{\epsilon,\sigma}|^N = \frac{1}{N} \int_{\Omega_{\frac{\delta}{2}}} |\nabla \tilde{\varphi}_{\epsilon,\sigma}|^N + O(1). \tag{3.15}$$

We estimate $|\nabla\tilde{\varphi}_{\epsilon,\sigma}|$ in two different ways:
(i) $|\nabla\tilde{\varphi}_{\epsilon,\sigma}|(x) \leq \frac{N^2}{N-1}\frac{1}{d(x)}$, where $d(x)=\min\{|x-p_i|:, i=1,...,m\}$;
(ii) $|\nabla\tilde{\varphi}_{\epsilon,\sigma}| \leq \frac{N^2}{N-1}C_0\epsilon^{-1}$ in view of

$$\frac{\epsilon|x-p_i|^{\frac{N}{N-1}-1}}{\epsilon^{\frac{N}{N-1}}+|x-p_i|^{\frac{N}{N-1}}} \leq C_0$$

by the Young's inequality. By estimate (ii) we have that

$$\int_{\Omega_{\frac{\delta}{2}}}|\nabla\tilde{\varphi}_{\epsilon,\sigma}|^N = \int_{\Omega_{\frac{\delta}{2}}\setminus\bigcup_{j=1}^m B_\epsilon(p_j)}|\nabla\tilde{\varphi}_{\epsilon,\sigma}|^N + O(1) \leq \sum_{j=1}^m\int_{A_j\setminus B_\epsilon(p_j)}|\nabla\tilde{\varphi}_{\epsilon,\sigma}|^N + O(1) \tag{3.16}$$

in view of $\Omega_{\frac{\delta}{2}}\setminus\bigcup_{j=1}^m B_\epsilon(p_j)\subset\bigcup_{j=1}^m\Big(A_j\setminus B_\epsilon(p_j)\Big)$, where $A_j=\{x\in\Omega_{\frac{\delta}{2}}:\ |x-p_j|=d(x)\}$. Since by estimate (i) we have that

$$\int_{A_j\setminus B_\epsilon(p_j)}|\nabla\tilde{\varphi}_{\epsilon,\sigma}|^N \leq (\frac{N^2}{N-1})^N\int_{A_j\setminus B_\epsilon(p_j)}\frac{1}{|x-p_j|^N} \leq (\frac{N^2}{N-1})^N\int_{B_R(0)\setminus B_\epsilon(0)}\frac{1}{|x|^N}+O(1) = \frac{N^2}{N-1}c_N\log\frac{1}{\epsilon}+O(1)$$

in terms of $R=\operatorname{diam}\Omega$, by (3.15)-(3.16) we deduce the validity of (3.14). The proof is complete. □

In order to prove that $\Psi\circ\Phi$ is homotopically equivalent to $\mathrm{Id}_{\mathfrak{B}_m(K)}$, we construct an explicit homotopy H as follows

$$H:(0,1]\longrightarrow C\big((\mathfrak{B}_m(K),\|\cdot\|_*);(\mathfrak{B}_m(K),\|\cdot\|_*)\big),\ t\mapsto H(t)=\Psi\circ\Phi_{t\varepsilon_0}.$$

The map H is continuous in $(0,1]$ with respect to the norm $\|\cdot\|_{\infty,\mathfrak{B}_m(K)}$. In order to conclude, we need to prove that there holds

$$\lim_{t\to0}\|H(t)-\mathrm{Id}_{\mathfrak{B}_m(K)}\|_{\infty,\mathfrak{B}_m(K)} = \lim_{\epsilon\to0}\sup_{\sigma\in\mathfrak{B}_m(K)}\|\Psi\circ\Phi_\epsilon(\sigma)-\sigma\|_*=0,$$

where $\epsilon=t\epsilon_0$. Since $\Pi_m(\sigma)=\sigma$ and $\mathfrak{B}_m(K)$ is a compact set in $\big(\mathcal{M}(\overline{\Omega}),\|\cdot\|_*\big)$, by the continuity of Π_m in $\|\cdot\|_*$ and Lemma 3.7-(1) we deduce that

$$\|\Psi\circ\Phi_\epsilon(\sigma)-\sigma\|_* = \|\Pi_m\left(\frac{Ve^{\varphi_{\epsilon,\sigma}}}{\int_\Omega Ve^{\varphi_{\epsilon,\sigma}}}\right)-\Pi_m(\sigma)\|_*\to0$$

as $\epsilon\to0$, uniformly in $\sigma\in\mathfrak{B}_m(K)$. Finally, we extend $H(t)$ at $t=0$ in a continuous way by setting $H(0)=id_{\mathfrak{B}_m(K)}$.

Let us now discuss the main assumption in Theorem 1.3. In [1] it is claimed that $\mathfrak{B}_m(\Omega)$ is non contractible for all $m\geq1$ if Ω is non contractible too, as it arises for closed manifolds [35]. However, by the techniques in [42] it is shown in [41] that $\mathfrak{B}_m(X)$ is contractible for all $m\geq1$, for a non contractible topological and acyclic (i.e. with trivial $\mathbb{Z}-$homology) space X. A concrete example is represented by the punctured Poincaré sphere, and it is enough to take a tubular neighborhood Ω of it to find a counterexample to the claim in [1]. A sufficient condition for the main assumption in Theorem 1.3 is the following:

Theorem 3.8. [41] *Assume that X is homotopically equivalent to a finite simplicial complex. Then $\mathfrak{B}_m(X)$ is non contractible for all $m\geq2$ if and only if X is not acyclic (i.e. with non trivial $\mathbb{Z}$-homology).*

Appendix

Let us collect here some useful regularity estimates which have been frequently used throughout the paper. Concerning $L^\infty-$estimates, the general interior estimates in [63] are used here to derive also boundary estimates for solutions $u\in W_c^{1,N}(\Omega)=\{u\in W^{1,N}(\Omega):\ u\mid_{\partial\Omega}=c\}$, $c\in\mathbb{R}$, through the *Schwarz reflection principle*.

Given $x_0\in\partial\Omega$, we can find a smooth diffeomorphism ψ from a small ball $B\subset\mathbb{R}^N$, $0\in B$, into a neighborhood V of x_0 in $\mathbb{R}^N$ so that $\psi(B\cap\{y_N=0\})=V\cap\partial\Omega$ and $\psi(B^+)=V\cap\Omega$, where $B^+=B\cap\{y_N>0\}$. Letting $u_0\in W_c^{1,N}(\Omega)$ be a critical point of

$$\frac{1}{p}\int_\Omega|\nabla u|^N-\int_\Omega fu,\quad u\in W_c^{1,N}(\Omega),$$

then $v_0=u_0\circ\psi$ is a critical point of

$$I(v)=\int_{B^+}\left[\frac{1}{N}|A(y)\nabla v|^N-fv\right]|\det\nabla\psi|,\quad v\in\mathcal{V},$$

in view of $|\nabla u|^N\circ\psi=|A\nabla v|^N$ in B^+ for $v=u\circ\psi$, where $A(y)=(D\psi^{-1})^t(\psi(y))$ is an invertible $N\times N$ matrix for all $y\in B^+$ and

$$\mathcal{V}=\{v\in W^{1,N}(B^+):v=c\text{ on }y_N=0\text{ and }v=u_0\circ\psi\text{ on }\partial B\cap\{y_N>0\}\}.$$

In the sequel, $g_\sharp$ and $g^\sharp$ denote the odd and even extension in B of a function g defined on B^+, respectively. Decomposing the matrix A as

$$A=\left(\begin{array}{c|c}A' & a_1\\\hline a_2 & a_{NN}\end{array}\right)$$

with $a_1, a_2 : B^+ \to \mathbb{R}^{N-1}$, for $y \in B$ let us introduce

$$A^\sharp = \left(\begin{array}{c|c} (A')^\sharp & (a_1)_\sharp \\ \hline (a_2)_\sharp & (a_{NN})^\sharp \end{array} \right).$$

The odd reflection $(v_0 - c)_\sharp + c \in W^{1,N}(B)$ is a weak solution in B of

$$-\text{div } \mathcal{A}(y, \nabla v) = (f|\det \nabla\psi|)_\sharp,$$

where $\mathcal{A} : (y,p) \in B \times \mathbb{R}^N \to |\det \nabla\psi|^\sharp |A^\sharp(y)p|^{N-2}[(A^\sharp)^t A^\sharp](y)p \in \mathbb{R}^N$. In view of the invertibility of $A(y)$ for all $y \in B^+$, the map $\mathcal{A}$ satisfies

$$|\mathcal{A}(y,p)| \le a|p|^{N-1}, \quad \langle p, \mathcal{A}(y,p)\rangle \ge a^{-1}|p|^N \tag{A.1}$$

for all $y \in B$ and $p \in \mathbb{R}^N$, for some $a > 0$. Since $2c - u \le u$ when $u \ge c$, thanks to (A.1) we can now apply the general local interior estimates of J. Serrin in [63] to get:

Theorem A.1. *Let $u \in W^{1,N}_{loc}(\Omega)$ be a weak solution of*

$$-\Delta_N u = f \quad \text{in } \Omega. \tag{A.2}$$

Assume that $f \in L^{\frac{N}{N-\epsilon}}(\Omega \cap B_{2R})$, $0 < \epsilon \le 1$, and $u \in W^{1,N}(\Omega \cap B_{2R})$ satisfies $u = c$ on $\partial\Omega \cap \overline{B_{2R}}$, $u \ge c$ in $\Omega \cap B_{2R}$ for some $c \in \mathbb{R}$ if $\partial\Omega \cap \overline{B_{2R}} \neq \emptyset$. Then, the following estimates do hold:

$$\|u^+\|_{L^\infty(\Omega\cap B_R)} \le C(\|u^+\|_{L^N(\Omega\cap B_{2R})} + 1)$$
$$\|u\|_{L^\infty(\Omega\cap B_R)} \le C(\|u\|_{L^N(\Omega\cap B_{2R})} + 1) \quad (\text{if } c = 0)$$

for some $C = C\left(N, a, \epsilon, R, \|f\|_{L^{\frac{N}{N-\epsilon}}(\Omega\cap B_{2R})}\right)$.

Since the Harnack inequality in [63] is very general, it can be applied in particular when $\mathcal{A}$ satisfies (A.1), by allowing us to treat also boundary points through the *Schwarz reflection principle*. The following statement is borrowed from [59]:

Theorem A.2. *Let $u \in W^{1,N}_{loc}(\Omega)$ be a nonnegative weak solution of (A.2), where $f \in L^{\frac{N}{N-\epsilon}}(\Omega)$, $0 < \epsilon \le 1$. Let $\Omega' \subset \Omega$ be a sub-domain of Ω. Assume that $u \in W^{1,N}(\Omega \cap \Omega')$ satisfies $u = 0$ on $\partial\Omega \cap \overline{\Omega'}$. Then, there exists $C = C(N, \epsilon, \Omega')$ so that*

$$\sup_{\Omega'} u \le C\left(\inf_{\Omega'} u + \|f\|^{\frac{1}{N-1}}_{L^{\frac{N}{N-\epsilon}}(\Omega)}\right).$$

By choosing $\Omega' = \Omega$ we deduce that

Corollary A.3. *Let $u \in W^{1,N}_0(\Omega)$ be a weak solution of $-\Delta_N u = f$ in Ω, where $f \in L^{\frac{N}{N-\epsilon}}(\Omega)$, $0 < \epsilon \le 1$. Then, there exists a constant $C = C(N, \epsilon, \Omega)$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C\|f\|^{\frac{1}{N-1}}_{L^{\frac{N}{N-\epsilon}}(\Omega)}.$$

Thanks to Theorem A.1, by the estimates in [31, 49, 65] we now have that

Theorem A.4. *Let $u \in W^{1,N}_{loc}(\Omega)$ be a weak solution of (A.2). Assume that $f \in L^\infty(\Omega\cap B_{2R})$, and $u \in W^{1,N}(\Omega\cap B_{2R})$ satisfies $u = 0$ on $\partial\Omega \cap B_{2R}$. Then, there holds $\|u\|_{C^{1,\alpha}(\Omega\cap B_R)} \le C = C = C(N, a, R, \|f\|_{\infty,\Omega\cap B_{2R}}, \|u\|_{L^N(\Omega\cap B_{2R})})$, for some $\alpha \in (0,1)$.*

We will now consider (A.2) with a Dirac measure δ_{p_0} as R.H.S. In our situation, the fundamental solution Γ takes the form

$$\Gamma(|x|) = (N\omega_N)^{-\frac{1}{N-1}} \log \frac{1}{|x|}.$$

In a very general framework, Serrin has described in [63] the behavior of solutions near a singularity. In particular, every N-harmonic and continuous function u in $\Omega \setminus \{0\}$, which is bounded from below in Ω, has either a removable singularity at 0 or there holds

$$\frac{1}{C}\Gamma \le u \le C\Gamma \tag{A.3}$$

in a neighborhood of 0, for some $C \ge 1$. For the $p-$Laplace operator Kichenassamy and Veron [45] have later improved (A.3) by expressing u in terms of Γ. A combination of [45, 63] leads in our situation to:

Theorem A.5. *Let u be a N-harmonic continuous function in $\Omega - \{0\}$, which is bounded from below in Ω. Then there exists $\gamma \in \mathbb{R}$ such that*

$$u - \gamma\Gamma \in L^\infty_{loc}(\Omega)$$

and u is a distributional solution in Ω of

$$-\Delta_N u = \gamma|\gamma|^{N-2}\delta_0$$

with $|\nabla u|^{N-1} \in L^1_{loc}(\Omega)$. *Moreover, for* $\gamma \neq 0$ *there holds*

$$\lim_{x \to 0} |x|^{|\alpha|} D^{|\alpha|}(u - \gamma \Gamma)(x) = 0$$

for all multi-indices $\alpha = (\alpha_1, ..., \alpha_N)$ *with length* $|\alpha| = \alpha_1 + ... + \alpha_N \geq 1$.

References

[1] L. Abatangelo, A. Portaluri A., *Morse theory for a fourth order elliptic equation with exponential nonlinearity.* NoDEA Nonlinear Differential Equations Appl. **18** (2011), no. 1, 27–43.

[2] J.A. Aguilar Crespo, I. Peral Alonso, *Blow-up behavior for solutions of* $-\Delta_N u = V(x)e^u$ *in bounded domains in* $\mathbb{R}^N$. Nonlinear Anal. **29** (1997), no. 4, 365–384.

[3] A. Bahri, J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent.* C. R. Acad. Sci. Paris Sér. I Math. **301** (1985), no. 7, 345–348.

[4] A. Bahri, J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain.* Comm. Pure Appl. Math. **41** (1988), no. 3, 253–294.

[5] S. Baraket, F. Pacard, *Construction of singular limits for a semilinear elliptic equation in dimension* 2. Calc. Var. Partial Differential Equations **6** (1998), no. 1, 1–38.

[6] D. Bartolucci, E. Montefusco, *Blow-up analysis, existence and qualitative properties of solutions for the two-dimensional Emden-Fowler equation with singular potential.* Math. Methods Appl. Sci. **30** (2007), no. 18, 2309–2327.

[7] D. Bartolucci, G. Tarantello, *Liouville type equations with singular data and their applications to periodic multivortices for the Electroweak Theory.* Comm. Math. Phys. **229** (2002), no. 1, 3–47.

[8] L. Battaglia, A. Jevnikar, A. Malchiodi, D. Ruiz, *A general existence result for the Toda system on compact surfaces*, arxiv: 1306.5404v2.

[9] M. Ben Ayed, M. Ould Ahmedou, *Existence and multiplicity results for a fourth order mean field equation.* J. Funct. Anal. **258** (2010), no. 9, 3165–3194.

[10] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vázquez, *An* L^1*-theory of existence and uniqueness of solutions of nonlinear elliptic equations.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **22** (1995), no. 2, 241–273.

[11] M.F. Bidaut-Véron, R. Borghol, L. Véron, *Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations.* Calc. Var. Partial Differential Equations **27** (2006), no. 2, 159–177.

[12] L. Boccardo, T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data.* J. Funct. Anal. **87** (1989), no. 1, 149–169.

[13] L. Boccardo, T. Gallouët, *Nonlinear elliptic equations with right hand side measures.* Comm. Partial Differential Equations **17** (1992), no. 3-4, 641–655.

[14] R. Borghol, L. Véron, *Boundary singularities of* $N-$*harmonic functions.* Comm. Partial Differential Equations **32** (2007), no. 6, 1001–1015.

[15] H. Brézis, F. Merle, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions.* Comm. Partial Differential Equations **16** (1991), no. 8-9, 1223–1253.

[16] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics descriptions.* Comm. Math. Phys. **143** (1992), no. 3, 501–525.

[17] E. Caglioti, P.L. Lions, C. Marchioro, M. Pulvirenti, *A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics descriptions. Part II.* Comm. Math. Phys. **74** (1995), no. 2, 229–260.

[18] S.-Y. Chang, M.J. Gursky, P.C. Yang, *The scalar curvature equation on* $2-$ *and* $3-$*spheres.* Calc. Var. Partial Differential Equations **1** (1993), no. 2, 205–229.

[19] S.-Y. Chang, P.C. Yang, *Prescribing Gaussian curvature on* $\mathbb{S}^2$. Acta Math. **159** (1987), no. 3-4, 215–259.

[20] S. Chanillo, M. Kiessling, *Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry.* Comm. Math. Phys. **160** (1994), no. 2, 217-238.

[21] W. Chen, C. Li, *Classification of solutions of some nonlinear elliptic equations.* Duke Math. J. **63** (1991), no. 3, 615–623.

[22] W. Chen, C. Li, *Prescribing Gaussian curvatures on surfaces with conical singularities.* J. Geom. Anal. **1** (1991), no. 4, 359–372.

[23] C.C. Chen, C.S. Lin, *Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces.* Comm. Pure Appl. Math. **55** (2002), no. 6, 728–771.

[24] C.C. Chen, C.S. Lin, *Topological degree for a mean field equation on Riemann surfaces.* Comm. Pure Appl. Math **56** (2003), no. 12, 1667–1727.

[25] C.C. Chen, C.S. Lin, G. Wang, *Concentration phenomena of two-vortex solutions in a Chern-Simons model.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), 367–397.

[26] A. Dall'Aglio, *Approximated solutions of equations with* L^1 *data. Application to the* $H-$*convergence of quasi-linear parabolic equations.* Ann. Mat. Pura Appl. (4) **170** (1996), 207–240.

[27] L. Damascelli, A. Farina, B. Sciunzi, E. Valdinoci, *Liouville results for m-Laplace equations of Lame-Emden-Fowler type.* Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 4, 1099–1119.

[28] M. del Pino, P. Esposito, P. Figueroa, M. Musso, *Non-topological condensates for the self-dual Chern-Simons-Higgs model*, to appear on Comm. Pure Appl. Math.

[29] M. del Pino, P. Esposito, M. Musso, *Two-dimensional Euler flows with concentrated vorticities.* Trans. Amer. Math. Soc. **362** (2012), no. 12, 6381–6395.

[30] M. del Pino, M. Kowalczyk, M. Musso, *Singular limits in Liouville-type equations.* Calc. Var. Partial Differential Equations **24** (2005), no. 1, 47–81.

[31] E. Dibenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations.* Nonlinear Anal. **7** (1983), no. 8, 827–850.

[32] W. Ding, J. Jost, J. Li, G. Wang, *An analysis of the two-vortex case in the Chern-Simons-Higgs model.* Calc. Var. Partial Differential Equations **7** (1998), no. 1, 87-97.

[33] W. Ding, J. Jost, J. Li, G. Wang, *Existence results for mean field equations.* Ann. Inst. H. Poincaré Anal. Non Linéaire **16** (1999), no. 5, 653-666.

[34] Z. Djadli, *Existence result for the mean field problem on Riemann surfaces of all genus.* Commun. Contemp. Math. **10** (2008), no. 2, 205–220.

[35] Z. Djadli, A. Malchiodi, *Existence of conformal metrics with constant Q-curvature.* Ann. of Math. (2) **168** (2008), no. 3, 813–858.

[36] G. Dolzmann, N. Hungerbhler, S. Müller, *Uniqueness and maximal regularity for nonlinear elliptic systems of n-Laplace type with measure valued right hand side.* J. Reine Angew. Math. **520** (2000), 1–35.

[37] P. Esposito, P. Figueroa, *Singular mean field equations on compact Riemann surfaces*, to appear on Nonlinear Anal.

[38] P. Esposito, M. Grossi, A. Pistoia, *On the existence of blowing-up solutions for a mean field equation.* Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 2, 227–257.

[39] H. Federer, W.H. Fleming, *Normal and integral currents.* Ann. of Math. (2) **72** (1960), 458–520.

[40] J. Horák, M. Lucia, *A minmax theorem in the presence of unbounded Palais-Smale sequences.* Israel J. Math. **172** (2009), 125–143.

[41] S. Kallel, *On the contractibility of spaces of formal barycenters and symmetric smash products.* Preprint.

[42] S. Kallel, R. Karoui, *Symmetric joins and weighted barycenters.* Adv. Nonlinear Stud. **11** (2011), no. 1, 117–143.

[43] B. Kawohl, M. Lucia, *Best constants in some exponential Sobolev inequalities.* Indiana Univ. Math. J. **57** (2008), no. 4, 1907–1927.

[44] J.L. Kazdan, F.W. Warner, *Curvature functions for compact* $2-$*manifolds*. Ann. of Math. (2) **99** (1974), 14-47.
[45] S. Kichenassamy, L. Veron, *Singular solutions of the p-Laplace equation*. Math. Ann. **275** (1986), no. 4, 599–615.
[46] M.K.H. Kiessling, *Statistical mechanics of classical particles with logarithmic interactions*. Comm. Pure Appl. Math. **46** (1993), no. 1, 27–56.
[47] Y.Y. Li, *Harnack type inequality: the Method of Moving Planes*. Comm. Math. Phys. **200** (1999), no. 2, 421–444.
[48] Y.Y. Li, I. Shafrir, *Blow-up analysis for solutions of* $-\Delta u = Ve^u$ *in dimension two*. Indiana Univ. Math. J. **43** (1994), no. 4, 1255–1270.
[49] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*. Nonlinear Anal. **12** (1988), no. 11, 1203–1219.
[50] K.-C. Lin, *Extremal functions for Moser's inequality*. Trans. Amer. Math. Soc. **348** (1996), no. 7, 2663–2671.
[51] C.S. Lin, C.-L. Wang, *Elliptic functions, Green functions and the mean field equations on tori*. Ann. of Math. (2) **172** (2010), no. 2, 911–954.
[52] C.S. Lin, S. Yan, *Existence of Bubbling Solutions for Chern-Simons Model on a Torus*. Arch. Ration. Mech. Anal. **207** (2013), no. 2, 353–392.
[53] M. Lucia, *A deformation Lemma for a Moser-Trudinger type functional*. Nonlinear Anal. **63** (2005), no. 2, 282–299.
[54] L. Ma, J. Wei, *Convergence for a Liouville equation*. Comment. Math. Helv. **76** (2001), 506–514.
[55] A. Malchiodi, *Morse theory and a scalar field equation on compact surfaces*. Adv. Differential Equations **13** (2008), no. 11-12, 1109–1129.
[56] L. Martinazzi, M. Petrache, *Asymptotics and quantization for a mean-field equation of higher order*. Comm. Partial Differential Equations **35** (2010), no. 3, 443–464.
[57] J. Moser, *A sharp form of an inequality by N. Trudinger*. Indiana Univ. Math. J. **20** (1970/1971), 1077-1092.
[58] M. Nolasco, G. Tarantello, *Double vortex condensates in the Chern-Simons-Higgs theory*. Calc. Var. Partial Differential Equations **9** (1999), no. 1, 31–94.
[59] P. Pucci, J. Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and their Applications, 73. Birkhäuser, Basel, 2007.
[60] X. Ren, J. Wei, *On a two-dimensional elliptic problem with large exponent in nonlinearity*. Trans. Amer. Math. Soc. **343** (1994), 749–763.
[61] X. Ren, J. Wei, *Counting peaks of solutions to some quasilinear elliptic equations with large exponents*. J. Differential Equations **117** (1995), no. 1, 28–55.
[62] F. Robert, J. Wei, *Asymptotic behavior of a fourth order mean field equation with Dirichlet boundary condition*. Indiana Univ. Math. J. **57** (2008), no. 5, 2039–2060.
[63] J. Serrin, *Local behavior of solutions of quasilinear equations*. Acta Math. **111** (1964), 247–302.
[64] J.T. Stuart, *On finite amplitude oscillations in laminar mixing layers*. J. Fluid. Mech. **29** (1967), no. 3, 417–440.
[65] P. Tolksdorf, *Regularity for more general class of quasilinear elliptic equations*. J. Differential Equations **51** (1984), no. 1, 126–150.
[66] G. Wang, C. Xia, *Blow-up analysis of a Finsler-Liouville equation in two dimensions*. J. Differential Equations **252** (2012), no. 2, 1668–1700.

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