Uniqueness of solutions for an elliptic equation modeling MEMS

PIERPAOLO ESPOSITO * AND NASSIF GHOUSSOUB†

November 18, 2018

1 Introduction

We study the effect of the parameter λ, the dimension N, the profile f and the geometry of the domain $\Omega \subset \mathbb{R}^N$, on the question of uniqueness of the solutions to the following elliptic boundary value problem with a singular nonlinearity:

$$\begin{cases} -\Delta u = \frac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (S)_{\lambda,f}$$

This equation has been proposed as a model for a simple electrostatic Micro-Electromechanical System (MEMS) device consisting of a thin dielectric elastic membrane with boundary supported at 0 below a rigid ground plate located at height $z = 1$. See [10, 11]. A voltage – directly proportional to the parameter λ – is applied, and the membrane deflects towards the ground plate and a snap-through may occur when it exceeds a certain critical value λ^*, the pull-in voltage.

In [9] a fine ODE analysis of the radially symmetric case with a profile $f \equiv 1$ on a ball B, yields the following bifurcation diagram that describes the L^∞-norm of the solutions u – which in this case necessarily coincides with $u(0)$ – in terms of the corresponding voltage λ.

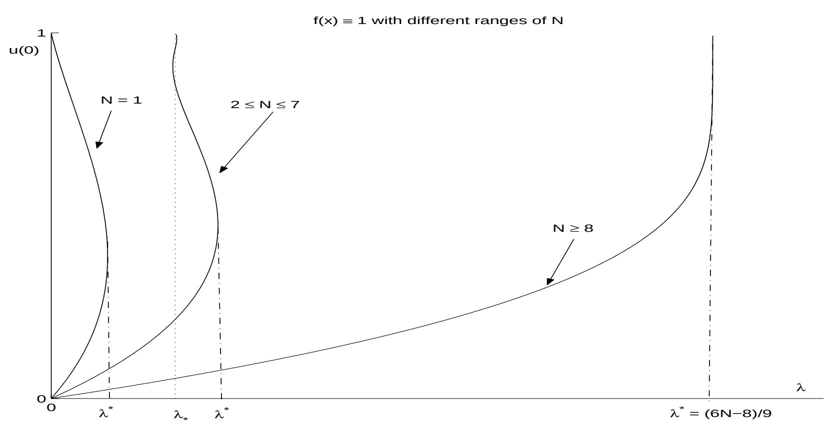

Figure 1: *Plots of $u(0)$ versus λ for profile $f(x) \equiv 1$ defined in the unit ball $B_1(0) \subset \mathbb{R}^N$ with different ranges of N. In the case $N \geq 8$, we have $\lambda^* = 2(3N-4)/9$.*

*Dipartimento di Matematica, Università degli Studi “Roma Tre", 00146 Rome, Italy. E-mail: esposito@mat.uniroma3.it. Research supported by M.U.R.S.T., project “Variational methods and nonlinear differential equations".

†Department of Mathematics, University of British Columbia, Vancouver, B.C. Canada V6T 1Z2. E-mail: nassif@math.ubc.ca. Research partially supported by the Natural Science and Engineering Research Council of Canada.

The question whether the diagram above describes realistically the set of all solutions in more general domains and for non-constant profiles, and whether rigorous mathematical proofs can be given for such a description, has been the subject of many recent investigations. See [3, 4, 5, 7, 8].

We summarize in the following two theorems some of the established results concerning the above diagram. First, for every solution u of $(S)_{\lambda,f}$, we consider the linearized operator

$$L_{u,\lambda} = -\Delta - \frac{2\lambda f}{(1-u)^3}$$

and its eigenvalues $\{\mu_{k,\lambda}(u); k = 1, 2, \dots\}$ (with the convention that eigenvalues are repeated according to their multiplicities). The Morse index $m(u, \lambda)$ of a solution u is the largest k for which $\mu_{k,\lambda}(u)$ is negative. A solution u of $(S)_{\lambda,f}$ is said to be *stable* (resp., *semi-stable*) if $\mu_{1,\lambda}(u) > 0$ (resp., $\mu_{1,\lambda}(u) \geq 0$).

A description of the first stable branch and of the higher unstable ones is given in the following.

Theorem A [3, 4, 5] *Suppose f is a smooth nonnegative function in Ω. Then, there exists a finite $\lambda^* > 0$ such that*

1. *If $0 \leq \lambda < \lambda^*$, there exists a (unique) minimal solution u_λ of $(S)_{\lambda,f}$ such that $\mu_{1,\lambda}(u_\lambda) > 0$. It is also unique in the class of all semi-stable solutions.*
2. *If $\lambda > \lambda^*$, there is no solution for $(S)_{\lambda,f}$.*
3. *If $1 \leq N \leq 7$, then $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a solution of $(S)_{\lambda^*,f}$ such that $\mu_{1,\lambda^*}(u^*) = 0$, and u^* – referred to as the extremal solution of problem $(S)_{\lambda,f}$ – is the unique solution.*
4. *If $1 \leq N \leq 7$, there exists λ_2^* with $0 < \lambda_2^* < \lambda^*$ such that for any $\lambda \in (\lambda_2^*, \lambda^*)$, problem $(S)_{\lambda,f}$ has a second solution U_λ with $\mu_{1,\lambda}(U_\lambda) < 0$ and $\mu_{2,\lambda}(U_\lambda) > 0$. Moreover, at $\lambda = \lambda_2^*$ there exists a second solution $U^* := \lim_{\lambda \downarrow \lambda_2^*} U_\lambda$ with*
$$\mu_{1,\lambda_2^*}(U^*) < 0 \quad \text{and} \quad \mu_{2,\lambda_2^*}(U^*) = 0.$$
5. *Given a more specific potential f in the form*
$$f(x) = \left(\prod_{i=1}^{k} |x - p_i|^{\alpha_i}\right) h(x)\,, \quad \inf_\Omega h > 0, \tag{1}$$
with points $p_i \in \Omega$, $\alpha_i \geq 0$, and given u_n a solution of $(S)_{\lambda_n,f}$, we have the equivalence
$$\|u_n\|_\infty \to 1 \quad \Longleftrightarrow \quad m(u_n, \lambda_n) \to +\infty$$
as $n \to +\infty$.

It was also shown in [4] that the profile f can dramatically change the bifurcation diagram, and totally alter the critical dimensions for compactness. Indeed, the following theorem summarizes the result related to the effect of power law profiles.

Theorem B [4] *Assume Ω is the unit ball B and f in the form*

$$f(x) = |x|^\alpha h(|x|)\,, \quad \inf_B h > 0.$$

Then we have

1. *If $N \geq 8$ and $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, the extremal solution u^* is again a classical solution of $(S)_{\lambda^*,f}$ such that $\mu_{1,\lambda^*}(u^*) = 0$.*
2. *If $N \geq 8$ and $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, the conclusion of Theorem A-(4) still holds true.*

3. *On the other hand, if either* $2 \le N \le 7$ *or* $N \ge 8$, $0 \le \alpha \le \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, *for* $f(x) = |x|^\alpha$ *necessarily we have that*
$$u^*(x) = 1 - |x|^{\frac{2+\alpha}{3}}, \qquad \lambda^* = \frac{(2+\alpha)(3N+\alpha-4)}{9}.$$

The bifurcation diagram suggests the following conjectures:

1. For $2 \le N \le 7$ there exists a curve $(\lambda(t), u(t))_{t\ge 0}$ in the solution set
$$\mathcal{V} = \Big\{(\lambda, u) \in (0, +\infty) \times C^1(\bar\Omega) : \ u \text{ is a solution of } (S)_{\lambda,f}\Big\}, \tag{2}$$
starting from $(0,0)$ at $t=0$ and going to "infinity": $\|u(t)\|_\infty \to 1$ as $t \to +\infty$, with infinitely many bifurcation or turning points in $\mathcal{V}$.

2. In dimension $N \ge 2$ and for any profile f, there exists a unique solution for small voltages λ.

3. For $2 \le N \le 7$ there exist exactly two solutions for λ in a small left neighborhhod of λ^*.

Conjectures 1 and 2 have been established for power law profiles in the radially symmetric case [7], and for the case where $f \equiv 1$, and Ω is a suitably symmetric domain in $\mathbb{R}^2$ [8]. Indeed, in these cases Guo and Wei first show that
$$\lambda_* = \inf\{\lambda > 0 : \ (S)_{\lambda,f} \text{ has a non-minimal solution}\} > 0,$$
and then apply the fine bifurcation theory developed by Buffoni, Dancer and Toland [1] to verify the validity of Conjecture 1 too. Property $\lambda_* > 0$ allows them to carry out some limiting argument and to prove that the Morse index of $u(t)$ blows up as $t \to +\infty$, which is crucial to show that infinitely many bifurcation or turning points occur along the curve. Thanks to Theorem A-(5), we shall be able in Section 2 to show the validity of Conjecture 1 in general domains Ω, by circumventing the need to prove that $\lambda_* > 0$. On the other hand, we shall prove in Section 3 that indeed $\lambda_* > 0$ for a large class of domains, and therefore we have uniqueness for small voltage. Our proofs simplify considerably those of Guo and Wei, and extend them to general star-shaped domains Ω and power law profiles $f(x) = |x|^\alpha$, $\alpha \ge 0$.

Conjecture 3 has been shown in [3] in the class of solutions u with $m(u,\lambda) \le k$, for every given $k \in \mathbb{N}$, and is still open in general.

2 A quenching branch of solutions

The first global result on the set of solutions in general domains was proved by the first author in [3]. By using a degree argument (repeated below), he showed the following result.

Theorem 2.1. *Assume* $2 \le N \le 7$ *and* f *be as in (1). There exist a sequence* $\{\lambda_n\}_{n\in\mathbb{N}}$ *and associated solution* u_n *of* $(S)_{\lambda_n,f}$ *so that*
$$m(u_n, \lambda_n) \to +\infty \qquad \text{as } n \to +\infty.$$

Let us introduce some notations according to Section 2.1 in [1]. Set
$$X = Y = \{u \in C^1(\bar\Omega) : \ u = 0 \text{ on } \partial\Omega\}, \quad U = (0, +\infty) \times \{u \in X : \|u\|_\infty < 1\},$$
and define the real analytic function $F : \mathbb{R} \times U \to Y$ as $F(\lambda, u) = u - \lambda K(u)$, where $K(u) = -\Delta^{-1}\big(f(x)(1-u)^{-2}\big)$ is a compact operator on every closed subset in $\{u \in X : \|u\|_\infty < 1\}$ and Δ^{-1} is the Laplacian resolvent with homogeneous Dirichlet boundary condition. The solution set $\mathcal{V}$ given in (2) rewrites as
$$\mathcal{V} = \{(\lambda, u) \in U : \ F(\lambda, u) = 0\},$$
and the projection of $\mathcal{V}$ onto X is defined as
$$\Pi_X \mathcal{V} = \{u \in X : \ \exists\, \lambda \text{ so that } (\lambda, u) \in \mathcal{V}\}.$$

Proof: In view of Theorem A-(5), we have the equivalence

$$\sup_{(\lambda,u)\in\mathcal{V}} \max_{\Omega} u = 1 \qquad \Longleftrightarrow \qquad \sup_{(\lambda,u)\in\mathcal{V}} m(u,\lambda) = +\infty.$$

Arguing by contradiction, we can assume that

$$\sup_{(\lambda,u)\in\mathcal{V}} \max_{\Omega} u \le 1-2\delta, \qquad \sup_{(\lambda,u)\in\mathcal{V}} m(u,\lambda) < +\infty \tag{3}$$

for some $\delta \in (0,\frac{1}{2})$. By Theorem 1.3 in [3] one can find $\lambda_1,\lambda_2 \in (0,\lambda^*)$, $\lambda_1<\lambda_2$, so that $(S)_{\lambda,f}$ possesses

- for λ_1, only the (non degenerate) minimal solution u_{λ_1} which satisfies $m(u_{\lambda_1},\lambda_1)=0$;
- for λ_2, only the two (non degenerate) solutions u_{λ_2}, U_{λ_2} satisfying $m(u_{\lambda_2},\lambda_2)=0$ and $m(U_{\lambda_2},\lambda_2)=1$, respectively.

Consider a δ-neighborhood $\mathcal{V}_\delta$ of $\Pi_X\mathcal{V}$:

$$\mathcal{V}_\delta := \{u\in X:\ \text{dist}_X(u,\Pi_X\mathcal{V})\le\delta\}.$$

Note that (3) gives that $\mathcal{V}$ is contained in a closed subset of $\{u\in X: \|u\|_\infty<1\}$:

$$\mathcal{V}_\delta \subset \{u\in X:\ \|u\|_\infty \le 1-\delta\}.$$

We can now define the Leray-Schauder degree d_λ of $F(\lambda,\cdot)$ on $\mathcal{V}_\delta$ with respect to zero, since by definition of $\Pi_X\mathcal{V}$ (the set of all solutions) $\partial\mathcal{V}_\delta$ does not contain any solution of $(S)_{\lambda,f}$ for any value of λ. Since d_λ is well defined for any $\lambda\in[0,\lambda^*]$, by homotopy $d_{\lambda_1}=d_{\lambda_2}$. To get a contradiction, let us now compute d_{λ_1} and d_{λ_2}. Since the only zero of $F(\lambda_1,\cdot)$ in $\mathcal{V}_\delta$ is u_{λ_1} with Morse index zero, we have $d_{\lambda_1}=1$. Since $F(\lambda_2,\cdot)$ has in $\mathcal{V}_\delta$ exactly two zeroes u_{λ_2} and U_{λ_2} with Morse index zero and one, respectively, we have $d_{\lambda_2}=1-1=0$. This contradicts $d_{\lambda_1}=d_{\lambda_2}$, and the proof is complete. ∎

We can now combine Theorem A-(5) with the fine bifurcation theory in [1] to establish a more precise multiplicity result. See also [2].
Observe that $\mathcal{A}_0 := \{(\lambda,u_\lambda):\ \lambda\in(0,\lambda^*)\}$ is a maximal arc-connected subset of

$$S := \{(\lambda,u)\in U:\ F(\lambda,u)=0 \text{ and } \partial_u F(\lambda,u): X\to Y \text{ is invertible}\}$$

with $\mathcal{A}_0\subset S$. Assume that the extremal solution u^* is a classical solution so to have $u^*\in(\bar{S}\cap U)\setminus S$. Assumption (C1) of Section 2.1 in [1] does hold in our case. As far as condition (C2):

$$\{(\lambda,u)\in U: F(\lambda,u)=0\} \text{ is open in } \{(\lambda,u)\in\mathbb{R}\times X: F(\lambda,u)=0\},$$

let us stress that it is a weaker statement than requiring U to be an open subset in $\mathbb{R}\times X$. In our case, the map $F(\lambda,u)$ is defined only in U (and not in the whole X), and then condition (C2) does not make sense. However, we can replace it with the new condition (C2):

$$U \text{ is an open set in } \mathbb{R}\times X,$$

which does hold in our context. Since (C2) is used only in Theorem 2.3-(iii) in [1] to show that S is open in $\bar{S}$, our new condtion (C2) does not cause any trouble in the arguments of [1].
Since $\partial_u F(\lambda,u)$ is a Fredholm operator of index 0, by a Lyapunov-Schmidt reduction we have that assumptions (C3)-(C5) do hold in our case (let us stress that these conditions are local and U is an open set in $\mathbb{R}\times X$).
Setting $\bar\lambda=0$ and defining the map $\nu: U\to[0,+\infty)$ as $\nu(\lambda,u)=\frac{1}{1-\|u\|_\infty}$, conditions (C6)-(C8) do hold in view of the property $\lambda\in[0,\lambda^*]$. Theorem 2.4 in [1] then applies and gives the following.

Theorem 2.2. *Assume u^* a classical solution of $(S)_{\lambda^*,f}$. Then there exists an analytic curve $(\hat\lambda(t),\hat u(t))_{t\ge0}$ in $\mathcal{V}$ starting from $(0,0)$ and so that $\|\hat u(t)\|_\infty\to1$ as $t\to+\infty$. Moreover, $\hat u(t)$ is a non-degenerate solution of $(S)_{\hat\lambda(t),f}$ except at isolated points.*

By the Implicit Function Theorem, the curve $(\hat{\lambda}(t), \hat{u}(t))$ can only have isolated intersections. If we now use the usual trick of finding a minimal continuum in $\{(\hat{\lambda}(t), \hat{u}(t)) : t \geq 0\}$ joining $(0,0)$ to "infinity", we obtain a continuous curve $(\lambda(t), u(t))$ in $\mathcal{V}$ with no self-intersections which is only piecewise analytic. Clearly, $\partial_u F(\lambda, u) : X \to Y$ is still invertible along the curve except at isolated points.

Let now $2 \leq N \leq 7$ and f be as in (1). By the equivalence in Theorem A-(5) we get that $m(\lambda(t), u(t)) \to +\infty$ as $t \to +\infty$, and then $\mu_{k,\lambda(t)}(u(t)) < 0$ for t large, for every $k \geq 1$. Since $\mu_{k,\lambda(0)}(u(0)) = \mu_{k,0}(0) > 0$ and $u(t)$ is a non-degenerate solution of $(S)_{\lambda(t),f}$ except at isolated points, we find $t_k > 0$ so that $\mu_{k,\lambda(t)}(u(t))$ changes from positive to negative sign across t_k. Since $\mu_{k+1,\lambda(t)}(u(t)) \geq \mu_{k,\lambda(t)}(u(t))$, we can choose t_k to be non-increasing in k and to have $t_k \to +\infty$ as $k \to +\infty$.
To study secondary bifurcations, we will use the gradient structure in the problem. Setting $(\lambda_k, u_k) := (\lambda(t_k), u(t_k))$, we have that $(\lambda_k, u_k) \notin S$. Choose $\delta > 0$ small so that $\|u_k\|_\infty < 1 - \delta$, and replace the nonlinearity $(1-u)^{-2}$ with a regularized one:

$$f_\delta(u) = \begin{cases} (1-u)^{-2} & \text{if } u \leq 1-\delta\,, \\ \delta^{-2} & \text{if } u \geq 1-\delta, \end{cases}$$

and the map $F(\lambda, u)$ with the corresponding one $F_\delta(\lambda, u)$. We replace X and Y with $H^2(\Omega) \cap H_0^1(\Omega)$ and $L^2(\Omega)$, respectively. The map $F_\delta(\lambda, u)$ can be considered as a map from $\mathbb{R} \times X \to Y$ with a gradient structure:

$$\partial_u \mathcal{J}_\delta(\lambda, u)[\varphi] = \langle F_\delta(\lambda, u), \varphi \rangle_{L^2(\Omega)}$$

for every $\lambda \in \mathbb{R}$ and $u, \varphi \in X$, where $\mathcal{J}_\delta : \mathbb{R} \times X \to \mathbb{R}$ is the functional given by

$$\mathcal{J}_\delta(\lambda, u) = \frac{1}{2} \int_\Omega |\nabla u|^2\, dx - \lambda \int_\Omega f(x) G_\delta(u)\, dx\,, \qquad G_\delta(u) = \int_0^u f_\delta(s) ds.$$

Assumptions (G1)-(G2) in Section 2.2 of [1] do hold. We have that $(\lambda(t), u(t)) \in S$ for t close to t_k and $m(\lambda(t), u(t))$ changes across t_k. If $\lambda(t)$ is injective, by Proposition 2.7 in [1] we have that $(\lambda(t_k), u(t_k))$ is a bifurcation point. Then we get the validity of Conjecture 1 as claimed below.

Theorem 2.3. *Assume $2 \leq N \leq 7$ and f be as in (1). Then there exists a continuous, piecewise analytic curve $(\lambda(t), u(t))_{t \geq 0}$ in $\mathcal{V}$, starting from $(0,0)$ and so that $\|\hat{u}(t)\|_\infty \to 1$ as $t \to +\infty$, which has either infinitely many turning points, i.e. points where $(\lambda(t), u(t))$ changes direction (the branch locally "bends back"), or infinitely many bifurcation points.*

Remark 2.1. In [7] the above analysis is performed in the radial setting to obtain a curve $(\lambda(t), u(t))_{t \geq 0}$, as given by Theorem 2.3, composed by radial solutions and so that $m_r(\lambda(t), u(t)) \to +\infty$ as $t \to +\infty$, $m_r(\lambda, u)$ being the radial Morse index of a solution (λ, u). In this way, it can be shown that bifurcation points can't occur and then $(\lambda(t), u(t))_{t \geq 0}$ exhibits infinitely many turning points. Moreover, they can also deal with the case where $N \geq 8$ and $\alpha > \alpha_N$.

3 Uniqueness of solutions for small voltage in star-shaped domains

We address the issue of uniqueness of solutions of the singular elliptic problem

$$\begin{cases} -\Delta u = \frac{\lambda |x|^\alpha}{(1-u)^2} & \text{in } \Omega \\ 0 < u < 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

for $\lambda > 0$ small, where $\alpha \geq 0$ and Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$. We shall make crucial use of the following extension of Pohozaev's identity due to Pucci and Serrin [12].

Proposition 3.1. *Let v be a solution of the boundary value problem*

$$\begin{cases} -\Delta v = f(x, v) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then for any $a \in \mathbb{R}$ and any $h \in C^2(\Omega;\mathbb{R}^N) \cap C^1(\bar{\Omega};\mathbb{R}^N)$, the following identity holds

$$\int_\Omega [\mathrm{div}(h)F(x,v) - avf(x,v) + \langle \nabla_x F(x,v), h\rangle]\ dx = \int_\Omega \left[(\frac{1}{2}\mathrm{div}(h) - a)|\nabla v|^2 - \langle Dh\nabla v, \nabla v\rangle\right]\ dx + \frac{1}{2}\int_{\partial\Omega} |\nabla v|^2 \langle h, \nu\rangle d\sigma, \tag{5}$$

where $F(x,s) = \int_0^s f(x,t)\,dt$.

An application of the method in [13] leads to the following result.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a star-shaped domain with respect to 0. If $N \geq 3$, then for λ small (4) has the unique solution u_λ.*

Proof: Since u_λ is the minimal solution of (4) for $\lambda \in (0,\lambda^*)$, setting $v = u - u_\lambda$ equation (4) rewrites equivalently as

$$\begin{cases} -\Delta v = \lambda |x|^\alpha g_\lambda(x,v) & \text{in } \Omega \\ 0 \leq v < 1 - u_\lambda & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where

$$g_\lambda(x,s) = \frac{1}{(1-u_\lambda(x)-s)^2} - \frac{1}{(1-u_\lambda(x))^2}\,. \tag{7}$$

It then suffices to prove that the solutions of (6) must be trivial for λ small enough. First compute $G_\lambda(x,s)$:

$$G_\lambda(x,s) = \int_0^s g_\lambda(x,t)\,dt = \frac{1}{1-u_\lambda(x)-s} - \frac{1}{1-u_\lambda(x)} - \frac{s}{(1-u_\lambda(x))^2}.$$

Since the validity of the relation

$$\nabla_x\Big(|x|^\alpha G_\lambda(x,s)\Big) = \alpha|x|^{\alpha-2}xG_\lambda(x,s) + |x|^\alpha \nabla_x G_\lambda(x,s),$$

for $h(x) = \frac{x}{N}$ and $f(x,v) = |x|^\alpha g_\lambda(x,v)$ we apply the Pohozaev identity (5) to a solution v of (6) to get

$$\begin{aligned} &\lambda\int_\Omega |x|^\alpha\big[(1+\frac{\alpha}{N})G_\lambda(x,v(x)) - av(x)g_\lambda(x,v(x)) + \langle\nabla_x G_\lambda(x,v(x)), \frac{x}{N}\rangle\big]\,dx \\ &= \int_\Omega \big[(\frac{1}{2}-a)|\nabla v|^2 - \langle D(\frac{x}{N})\nabla v, \nabla v\rangle\big]dx + \frac{1}{2N}\int_{\partial\Omega}|\nabla v|^2\langle x,\nu\rangle\,d\sigma \\ &\geq (\frac{1}{2} - a - \frac{1}{N})\int_\Omega |\nabla v|^2 dx. \end{aligned} \tag{8}$$

Since easy calculations show that

$$\frac{G_\lambda(x,s)}{g_\lambda(x,s)} = \frac{1-u_\lambda(x)-s-\frac{(1-u_\lambda(x)-s)^2(1-u_\lambda(x)+s)}{(1-u_\lambda(x))^2}}{1-\frac{(1-u_\lambda(x)-s)^2}{(1-u_\lambda(x))^2}}$$

and

$$\frac{\nabla_x G_\lambda(x,s)}{g_\lambda(x,s)} = \frac{1-\frac{(1-u_\lambda(x)-s)^2(1-u_\lambda(x)+2s)}{(1-u_\lambda(x))^3}}{1-\frac{(1-u_\lambda(x)-s)^2}{(1-u_\lambda(x))^2}}\nabla u_\lambda(x),$$

we obtain

$$\Big|\frac{G_\lambda(x,s)}{g_\lambda(x,s)}\Big| \leq C_0|1-u_\lambda(x)-s| \qquad \text{and} \qquad \Big|\frac{\nabla_x G_\lambda(x,s)}{g_\lambda(x,s)} - \nabla u_\lambda\Big| \leq C_0|1-u_\lambda(x)-s|^2|\nabla u_\lambda| \tag{9}$$

for some $C_0 > 0$, provided λ is away from λ^*. Since $u_\lambda \to 0$ in $C^1(\bar\Omega)$ as $\lambda \to 0^+$, for $a > 0$ from (9) we deduce that for any (x,s) satisfying $|1-u_\lambda(x)-s| \le \delta$

$$\begin{aligned}&(1+\frac{\alpha}{N})G_\lambda(x,s) - asg_\lambda(x,s) + \langle \nabla_x G_\lambda(x,s), \frac{x}{N}\rangle \\ &\le g_\lambda(x,s)\Big[C_0(1+\frac{\alpha}{N})\delta - a(1-u_\lambda(x)-\delta) + \langle \nabla u_\lambda, \frac{x}{N}\rangle + \frac{C_0}{N}\delta^2|\nabla u_\lambda||x|\Big] \le 0,\end{aligned} \tag{10}$$

provided δ and λ are sufficiently small (depending on a). Since $N \ge 3$, we can pick $0 < a < \frac{1}{2} - \frac{1}{N}$, and then by (8), (10) get that

$$\begin{aligned}&\lambda \int_{\{0\le v\le 1-u_\lambda-\delta\}} |x|^\alpha \big[(1+\frac{\alpha}{N})G_\lambda(x,v(x)) - av(x)g_\lambda(x,v(x)) + \langle \nabla_x G_\lambda(x,v(x)), \frac{x}{N}\rangle\big]dx \\ &\ge (\frac{1}{2} - a - \frac{1}{N})\int_\Omega |\nabla v|^2\,dx \ge C_s(\frac{1}{2}-a-\frac{1}{N})\int_\Omega v^2\,dx\end{aligned} \tag{11}$$

for δ and λ sufficiently small, where C_s is the best constant in the Sobolev embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$.

On the other hand, since $G_\lambda(x,s)$, $sg_\lambda(x,s)$ and $\nabla_x G_\lambda(x,s)$ are quadratic with respect to s as $s \to 0$ (uniformly in λ away from λ^*), there exists a constant $C_\delta > 0$ such that

$$(1+\frac{\alpha}{N})G_\lambda(x,v(x)) - avg_\lambda(x,v(x)) + \langle \nabla_x G_\lambda(x,v(x)), \frac{x}{N}\rangle \le C_\delta v^2(x) \tag{12}$$

for $x \in \{0 \le v \le 1-u_\lambda-\delta\}$, uniformly for λ away from λ^*. Combining (11) and (12) we get that

$$C_s(\frac{1}{2}-a-\frac{1}{N})\int_{\{0\le v\le 1-u_\lambda-\delta\}} v^2 dx \le \lambda C_\delta \int_{\{0\le v\le 1-u_\lambda-\delta\}} |x|^\alpha v^2 dx.$$

Therefore, for λ sufficiently small we conclude that $v \equiv 0$ in $\{0 \le v \le 1-u_\lambda-\delta\}$. This implies that $v \equiv 0$ in Ω for sufficiently small λ, and we are done. ■

We now refine the above argument so as to cover other situations. To this aim, we consider the – potentially empty – set

$$H(\Omega) = \Big\{h \in \mathcal{C}^1(\bar\Omega, \mathbb{R}^N) : \mathrm{div}(h) \equiv 1 \text{ and } \langle h, \nu\rangle \ge 0 \text{ on } \partial\Omega\Big\},$$

and the corresponding parameter

$$M(\Omega) := \inf\Big\{\sup_{x\in\Omega} \bar\mu(h,x) :\ h \in H(\Omega)\Big\},$$

where

$$\bar\mu(h,x) = \frac{1}{2}\sup_{|\xi|=1}\langle (Dh(x) + Dh(x)^T)\xi, \xi\rangle.$$

The following is an extension of Theorem 3.1.

Theorem 3.2. *Let Ω be a bounded domain in $\mathbb{R}^N$ such that $M(\Omega) < \frac{1}{2}$. Then, for λ small the minimal solution u_λ is the unique solution of problem (4), provided either $N \ge 3$ or $\alpha > 0$.*

Proof: As above, we shall prove that equation (6), with g_λ as in (7), has only trivial solutions for λ small. For a solution v of (6) the Pohozaev identity (5) with $h \in H(\Omega)$ yields

$$\begin{aligned}&\lambda \int_\Omega |x|^\alpha \big[G_\lambda(x,v(x))(1+\alpha\langle \frac{x}{|x|^2}, h\rangle) - av(x)g_\lambda(x,v(x)) + \langle \nabla_x G_\lambda(x,v(x)), h\rangle\big]\,dx \\ &= \int_\Omega \big[(\frac{1}{2}-a)|\nabla v|^2 - \frac{1}{2}\langle (Dh+Dh^T)\nabla v, \nabla v\rangle\big]dx + \frac{1}{2}\int_{\partial\Omega} |\nabla v|^2\langle h,\nu\rangle\,d\sigma \\ &\ge \int_\Omega (\frac{1}{2} - a - \bar\mu(h,x))|\nabla v|^2\,dx.\end{aligned} \tag{13}$$

Fix $0 < a < \frac{1}{2} - M(\Omega)$ and choose $h \in H(\Omega)$ such that

$$\frac{1}{2} - a - \sup_{x\in\Omega} \bar{\mu}(h, x) > 0.$$

It follows from (9) that for any (x, s) satisfying $|1 - u_\lambda(x) - s| \leq \delta|x|$ there holds

$$\begin{aligned} &G_\lambda(x,s)(1+\alpha\langle\frac{x}{|x|^2},h\rangle) - avg_\lambda(x,s) + \langle\nabla_x G_\lambda(x,s),h\rangle \\ &\leq g_\lambda(x,s)\big[C_0\delta|x| + \alpha C_0\delta|h| - a(1-u_\lambda-\delta|x|) + \langle\nabla u_\lambda,h\rangle + C_0\delta^2|x|^2|\nabla u_\lambda||h|\big] \leq 0 \end{aligned} \tag{14}$$

provided λ and δ are sufficiently small. It then follows from (13) and (14) that

$$\begin{aligned} &\lambda\int_{\{0\leq v\leq 1-u_\lambda-\delta|x|\}} |x|^\alpha \Big[G_\lambda(x,v(x))(1+\alpha\langle\frac{x}{|x|^2},h\rangle) - av(x)g_\lambda(x,v(x)) + \langle\nabla_x G_\lambda(x,v(x)),h\rangle\Big]\,dx \\ &\geq (\frac{1}{2} - a - \sup_{x\in\Omega}\bar{\mu}(h,x))\int_\Omega |\nabla v|^2\,dx. \end{aligned} \tag{15}$$

On the other hand, there exists a constant $C_\delta > 0$ such that

$$\begin{aligned} &G_\lambda(x,v(x))(1+\alpha\langle\frac{x}{|x|^2},h(x)\rangle) - av(x)g_\lambda(x,v(x)) + <\nabla_x G_\lambda(x,v(x)),h(x)> \\ &= \frac{v^2(x)}{(1-u_\lambda(x)-v(x))(1-u_\lambda(x))^2}(1+\alpha\langle\frac{x}{|x|^2},h(x)\rangle) + \frac{av^2(x)[v(x)-2+2u_\lambda(x))]}{(1-u_\lambda(x)-v(x))^2(1-u_\lambda(x))^2} \\ &+ \frac{v^2(x)(3-3u_\lambda(x)-2v(x))}{(1-u_\lambda(x)-v(x))^2(1-u_\lambda(x))^3} < \nabla u_\lambda(x),h(x) > \leq C_\delta\frac{v^2(x)}{|x|^2} \end{aligned}$$

for $x \in \{0 \leq v \leq 1 - u_\lambda - \delta|x|\}$, uniformly for λ away from λ^*.
If now $N \geq 3$, then Hardy's inequality combined with (15) implies

$$\frac{(N-2)^2}{4}(\frac{1}{2} - a - \sup_{x\in\Omega}\bar{\mu}(h,x))\int_{\{0\leq v\leq 1-u_\lambda-\delta|x|\}} \frac{v^2}{|x|^2}\,dx \leq \lambda C_\delta\int_{\{0\leq v\leq 1-u_\lambda-\delta|x|\}} \frac{v^2}{|x|^2}\,dx.$$

On the other hand, when $N = 2$ the space $H_0^1(\Omega)$ embeds continously into $L^p(\Omega)$ for every $p > 1$, and then, by Hölder inequality, for $\alpha > 0$ we get that

$$\int_\Omega \frac{v^2}{|x|^{2-\alpha}}\,dx \leq \left(\int_\Omega |x|^{-(2-\alpha)\frac{p}{p-2}}\,dx\right)^{\frac{p-2}{p}} \left(\int_\Omega |v|^p\,dx\right)^{\frac{2}{p}} \leq C_{N,\alpha}^{-1}\int_\Omega |\nabla v|^2\,dx$$

provided $(2-\alpha)\frac{p}{p-2} < 2$, which is true for p large depending on α (see [6] for some very general Hardy inequalities). It combines with (15) to yield

$$C_{N,\alpha}(\frac{1}{2} - a - \sup_{x\in\Omega}\bar{\mu}(h,x))\int_{\{0\leq v\leq 1-u_\lambda-\delta|x|\}} \frac{v^2}{|x|^{2-\alpha}}\,dx \leq \lambda C_\delta\int_{\{0\leq v\leq 1-u_\lambda-\delta|x|\}} \frac{v^2}{|x|^{2-\alpha}}\,dx\,.$$

In both cases, we can conclude that for λ sufficiently small $v \equiv 0$ for $x \in \{0 \leq v \leq 1 - u_\lambda - \delta|x|\}$, for some $\delta > 0$ small. Since we can assume δ and λ sufficiently small to have

$$1 - u_\lambda - \delta|x| \geq \frac{1}{2} \quad \text{in} \quad \big\{x \in \Omega :\ |x| \geq \frac{1}{2}\text{dist}(0,\partial\Omega)\big\},$$

we then have

$$v \equiv 0 \quad \text{in} \quad \big\{x \in \Omega :\ v(x) \leq \frac{1}{2}\big\} \cap \big\{x \in \Omega :\ |x| \geq \frac{1}{2}\text{dist}(0,\partial\Omega)\big\}.$$

Since $v = 0$ on $\partial\Omega$ and the domain $\{x \in \Omega :\ |x| \geq \frac{1}{2}\text{dist}(0,\partial\Omega)\}$ is connected, the continuity of v gives that

$$v \equiv 0 \quad \text{in} \quad \big\{x \in \Omega :\ |x| \geq \frac{1}{2}\text{dist}(0,\partial\Omega)\big\}.$$

Therefore, the maximum principle for elliptic equations implies $v \equiv 0$ in Ω, which completes the proof of Theorem 3.2. ■

Remark 3.1. In [13] examples of dumbell shaped domains $\Omega \subset \mathbb{R}^N$ which satisfy condition $M(\Omega) < \frac{1}{2}$ are given for $N \geq 3$. When $N \geq 4$, there even exist topologically nontrivial domains with this property. Let us stress that in both cases Ω is not starlike, which means that the assumption $M(\Omega) < \frac{1}{2}$ on a domain Ω is more general than being shar-shaped.

The remaining case $N = 2$ and $\alpha = 0$, is a bit more delicate. We have the following result.

Theorem 3.3. *If Ω is either a strictly convex or a symmetric domain in $\mathbb{R}^2$, then $(S)_{\lambda,1}$ has the unique solution u_λ for small λ.*

Proof: The crucial point here is the following inequality: for every solution v of (6) there holds

$$\int_{\partial\Omega} |\nabla v|^2 \, d\sigma \geq l(\partial\Omega)^{-1} \left(\int_\Omega |\Delta v| \, dx \right)^2 .$$

Indeed, we have that

$$\int_{\partial\Omega} |\nabla v|^2 \, d\sigma \geq l(\partial\Omega)^{-1} \left(\int_{\partial\Omega} |\nabla v| \, d\sigma \right)^2 = l(\partial\Omega)^{-1} \left(\int_{\partial\Omega} \partial_\nu v \, d\sigma \right)^2 = l(\partial\Omega)^{-1} \left(\int_\Omega |\Delta v| \, dx \right)^2 ,$$

where $l(\partial\Omega)$ is the length of $\partial\Omega$. Note that $-\Delta v = \lambda g_\lambda(x, v) \geq 0$ for every solution $u_\lambda + v$ of $(S)_{\lambda,1}$, in view of the minimality of u_λ.

By Lemma 4 in [13] for λ small there exists $x_\lambda \in \Omega$ so that

$$\langle \nabla u_\lambda(x), x - x_\lambda \rangle \leq 0 \qquad \forall \, x \in \Omega. \tag{16}$$

In particular, for λ small x_λ lies in a compact subset of Ω and, when Ω is symmetric, coincides exactly with the center of symmetries. In both situations, then we have that there exists $c_0 > 0$ so that

$$\langle x - x_\lambda, \nu(x) \rangle \geq c_0 \qquad \forall \, x \in \partial\Omega.$$

We use now the Pohozaev identity (5) with $a = 0$ and $h(x) = \frac{x - x_\lambda}{2}$. For every solution v of (6) it yields

$$\lambda \int_\Omega \left[G_\lambda(x, v(x)) + \langle \nabla_x G_\lambda(x, v(x)), \frac{x - x_\lambda}{2} \rangle \right] dx = \frac{1}{4} \int_{\partial\Omega} |\nabla v|^2 \langle x - x_\lambda, \nu \rangle \, d\sigma \geq \frac{c_0}{4} \left(\int_\Omega |\Delta v| \, dx \right)^2 . \tag{17}$$

Since

$$\nabla_x G_\lambda(x, s) = (1 - u_\lambda(x) - s)^{-2} \left[1 - \frac{(1 - u_\lambda(x) - s)^2 (1 - u_\lambda(x) + 2s)}{(1 - u_\lambda(x))^3} \right] \nabla u_\lambda(x),$$

by (16) we easily see that

$$\langle \nabla_x G_\lambda(x, s), x - x_\lambda \rangle \leq 0$$

for λ and δ small, provided (x, s) satisfies $|1 - u_\lambda(x) - s| \leq \delta$. Since $G_\lambda(x, s)$, $\nabla_x G_\lambda(x, s)$ are quadratic with respect to s as $s \to 0$ (uniformly in λ small), there exists a constant $C_\delta > 0$ such that

$$G_\lambda(x, v(x)) \leq C_\delta v^2(x) \, , \qquad \langle \nabla_x G_\lambda(x, v(x)), \frac{x - x_\lambda}{2} \rangle \leq C_\delta v^2(x)$$

for $x \in \{0 \leq v \leq 1 - u_\lambda - \delta\}$, uniformly for λ small.
Since on two-dimensional domains

$$\left(\int_\Omega |v|^p \, dx \right)^{\frac{1}{p}} \leq C_p \int_\Omega |\Delta v| \, dx$$

for every $p \geq 1$ and $v \in W^{2,1}(\Omega)$ so that $v = 0$ on $\partial\Omega$, we get that

$$\lambda \int_\Omega \langle \nabla_x G_\lambda(x, v(x)), \frac{x - x_\lambda}{2} \rangle \, dx \leq \lambda C_\delta \int_\Omega v^2 \, dx \leq \lambda C_\delta C_2^2 \left(\int_\Omega |\Delta v| \, dx \right)^2 . \tag{18}$$

As far as the term with $G_\lambda(x, v(x))$, fix $b \in (0,1)$ and split Ω as the disjoint union of $\Omega_1 = \{v \le b\}$ and $\Omega_2 = \{v > b\}$. On Ω_1 we have that

$$\lambda \int_{\Omega_1} G_\lambda(x, v(x))\, dx \le \lambda C_\delta \int_\Omega v^2\, dx \le \lambda C_\delta C_2^2 \left(\int_\Omega |\Delta v|\, dx \right)^2$$

provided λ and δ are small to satisfy $b \le 1 - u_\lambda - \delta$ in Ω_1.
Since for λ small

$$\frac{G_\lambda(x,s)^2}{g_\lambda(x,s)} \le C \quad \forall\, b \le s \le 1,$$

we have that

$$\begin{aligned} \lambda \int_{\Omega_2} G_\lambda(x, v(x))\, dx &\le \lambda D_1 \int_\Omega |v(x)|^{\frac{3}{2}} g_\lambda^{\frac{1}{2}}(x, v(x))\, dx \le \lambda D_2 \left(\int_\Omega |v|^3\, dx \right)^{\frac{1}{2}} \left(\int_\Omega g_\lambda(x, v(x))\, dx \right)^{\frac{1}{2}} \\ &\le \lambda^{\frac{1}{2}} D_3 \left(\int_\Omega |\Delta v|\, dx \right)^2 \end{aligned}$$

for some positive constants D_1, D_2 and D_3. So we get that

$$\lambda \int_\Omega G_\lambda(x, v(x))\, dx \le \left(\lambda C_\delta C_2^2 + \lambda^{\frac{1}{2}} D_3 \right) \left(\int_\Omega |\Delta v|\, dx \right)^2. \tag{19}$$

Inserting (18)-(19) into (17) finally we get that

$$\left(2\lambda C_\delta C_2^2 + \lambda^{\frac{1}{2}} D_3 - \frac{c_0}{4} \right) \left(\int_\Omega |\Delta v|\, dx \right)^2 \ge 0,$$

and then $v \equiv 0$ for λ small. ■

References

[1] B. Buffoni, E.N. Dancer and J.F. Toland, *The sub-harmonic bifurcation of Stokes waves*, Arch. Ration. Mech. Anal. **152** (2000), no. 3, 241–271.

[2] E.N. Dancer, *Infinitely many turning points for some supercritical problems*, Ann. Mat. Pura Appl. **178** (2000), no. 4, 225–233.

[3] P. Esposito, *Compactness of a nonlinear eigenvalue problem with a singular nonlinearity*, Commun. Contemp. Math. **10** (2008), no. 1, 17–45.

[4] P. Esposito, N. Ghoussoub and Y. Guo, *Compactness along the branch of semi-stable and unstable solutions for an elliptic problem with a singular nonlinearity*, Comm. Pure Appl. Math. **60** (2007), no. 12, 1731–1768.

[5] N. Ghoussoub and Y. Guo, *On the partial differential equations of electrostatic MEMS devices: stationary case*, SIAM J. Math. Anal. **38** (2006/2007), no. 5, 1423–1449.

[6] N. Ghoussoub and A. Moradifam, *On the best possible remaining term in the improved Hardy inequality*, Proc. Nat. Acad. Sci., vol. 105, no. 37 (2008) 13746-13751.

[7] Z. Guo and J. Wei, *Infinitely many turning points for an elliptic problem with a singular nonlinearity*, J. Lond. Math. Soc. (2), to appear.

[8] Z. Guo and J. Wei, *Asymptotic behavior of touch-down solutions and global bifurcations for an elliptic problem with a singular nonlinearity*, Commun. Pure Appl. Anal. **7** (2008), no. 4, 765–786.

[9] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1972/73), 241–269.

[10] J.A. Pelesko, *Mathematical modeling of electrostatic MEMS with tailored dielectric properties*, SIAM J. Appl. Math. **62** (2001/2002), no. 3, 888–908.

[11] J.A. Pelesko and D.H. Bernstein, *Modeling MEMS and NEMS*. Chapman & Hall/CRC, Boca Raton, FL, 2003.

[12] P. Pucci and J. Serrin, *A general variational identity,* Indiana Univ. Math. J. **35** (1986), no. 3, 681–703.

[13] R. Schaaf, *Uniqueness for semilinear elliptic problems: supercritical growth and domain geometry,* Adv. Differential Equations **5** (2000), no. 10-12, 1201–1220.