On some conjectures proposed by Haïm Brezis

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Abstract


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1. Introduction

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N, N \geq 3 \), and \( a(x) \) be a continuous function in \( \Omega \). We consider the problem

\[
(P) \quad \begin{cases}
-\Delta u + au = u^p & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( p = (N + 2)/(N - 2) \) is the critical Sobolev exponent. Solutions for Eq. (P) can be found by studying the minimization problem for the functional

\[
J_a(u) = \frac{\int_{\Omega} |\nabla u|^2 + \int_{\Omega} a(x)u^2}{(\int_{\Omega} |u|^{p+1})^{2/(p+1)}} \quad u \in H_0^1(\Omega) \setminus \{0\}.
\]

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In fact, by standard elliptic regularity theory, any minimum $u_a$ for $J_a$ provides a smooth solution for problem (P), up to rescaling the Lagrange multiplier and changing $u_a$ into $|u_a|$. Remarking that the solvability of (P) implies the coercivity of the operator $-\Delta + a$ on $H^1_0(\Omega)$, i.e. the first eigenvalue $\lambda_1$ is positive, we can assume from now on that $-\Delta + a$ is coercive on $H^1_0(\Omega)$.

Denoting $S_a$ the infimum of $J_a$, it is well known that $S_a \leq S$, $S = S_0$ being the Sobolev constant, and $S_a < S$ implies $S_a$ achieved (see for instance [2]).

For $N \geq 4$, in [2] it is also proved that $a(x)$ negative somewhere implies $S_a < S$ and, since $S$ is never attained, the properties

(i) $a(x)$ negative somewhere,
(ii) $S_a < S$,
(iii) $S_a$ is achieved

are equivalent. So the problem turns out to be completely characterized by the local nature of $a(x)$.

On the other hand, in dimension $N = 3$, the global nature of $a(x)$ becomes significative: in [2] it was discussed the particular case of the unit ball and $a(x) \equiv \text{const}$. Some notations are in order to state the general result: let us define $G_a(x, y)$, $x \in \Omega \setminus \{y\}$, the Green function in $y \in \Omega$ of $-\Delta + a$ in $\Omega$ with Dirichlet boundary condition, as the distributional solution for

\[
\begin{aligned}
-\Delta_x G_a(x, y) + a(x)G_a(x, y) &= \delta_y \quad \text{in } \Omega, \\
G_a(x, y) &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where $\delta_y$ is the Dirac measure in $y$, and let us set $H_a(x, y) = G_a(x, y) - 1/(\omega_2 |x - y|)$ the regular part of the Green function $G_a(x, y)$, where $H_a(x, y) \in C(\Omega \times \Omega)$ is a distributional solution for

\[
\begin{aligned}
-\Delta_x H_a(x, y) + a(x)G_a(x, y) &= 0 \quad \text{in } \Omega, \\
H_a(x, y) &= \frac{1}{\omega_2 |x - y|} \quad \text{on } \partial \Omega.
\end{aligned}
\]

For the general case, the following result was conjectured in [1] and proved in [4]:

**Theorem 1.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ and let $a(x) \in C(\bar{\Omega})$ be such that $-\Delta + a$ is coercive. The properties:

(i) $\exists y \in \Omega$ such that $H_a(y, y) > 0$,
(ii) $S_a < S$,
(iii) $S_a$ is achieved

are equivalent.

By test functions computations (see [13]), one gets (i) $\Rightarrow$ (ii). The fact that (ii) $\Rightarrow$ (iii) is well-known and is already present for instance in [2]. Brezis, in [1], proposed
the two following conjectures:

(a) the implication (iii) ⇒ (ii) does hold;
(b) the implication (ii) ⇒ (i) does hold.

Druet proved these two conjectures and thus Theorem 1.1 in [4]. The proof of the conjecture (a) makes use of the minimality condition $D^2 J_a(u_a) \geq 0$. Thanks to (a), the proof that (ii) ⇒ (i) is reduced in [4] to some asymptotic analysis in the following way: let $a \in C(\Omega)$ such that $S_a < S$. By continuity and monotonicity of $S_{a+\delta}$ with respect to $\delta > 0$, we get $S_{a+\delta} < S$ for $0 < \delta < \delta_0$ and $S_{a+\delta} = S$ for $\delta \geq \delta_0$, $\delta_0$ some positive real number. Since $S_{a+\delta} < S$ for $0 < \delta < \delta_0$, there exists $u_{a+\delta}$ a smooth positive function achieving $S_{a+\delta}$ for $0 < \delta < \delta_0$ and satisfying the renormalization $\int_{\Omega} u^6_{a+\delta} = (S_{a+\delta})^{3/2}$. By uniform coercivity of $-\Delta + (a + \delta)$ on $H^1_0(\Omega)$, it is easily seen that

$$\sup_{\delta \in (0, \delta_0)} \int_{\Omega} |\nabla u_{a+\delta}|^2 < +\infty$$

and then, up to a subsequence, we can assume $u_{a+\delta} \rightharpoonup u$ as $\delta \to \delta_0$ in $H^1_0(\Omega)$. Since, by (a), $S_{a+\delta_0}$ is not achieved, $u \equiv 0$ and then we have

$$u_{a+\delta} \to 0 \text{ as } \delta \to \delta_0 \text{ weakly but not strongly in } H^1_0(\Omega).$$

Hence $u_{a,\delta}$ must blow-up as $\delta \to \delta_0$. In view of the compactness of the embedding of $H^1_0(\Omega)$ into $L^2(\Omega)$, we have

$$\lim_{\delta \to \delta_0} \int_{\Omega} |\nabla u_{a,\delta}|^2 = \lim_{\delta \to \delta_0} \int_{\Omega} u^6_{a,\delta} = (S_{a+\delta_0})^{3/2} = S^{3/2}$$

and then we obtain $|\nabla u_{a,\delta}|^2 \to S^{3/2} \delta y_0$ as $\delta \to \delta_0$ weakly in the sense of measures for some $y_0 \in \tilde{\Omega}$ (see [14]). In [4], Druet carried over an asymptotic analysis based on pointwise estimates of $u_{a,\delta}$ as $\delta \to \delta_0$ and proved that $y_0 \in \Omega$ and $H_{a+\delta_0}(y_0, y_0) = 0$. This proves (b) thanks to the monotonicity $H_a(y, y) > H_{a+\delta_0}(y, y)$.

The aim of this paper is to give an alternative and more direct proof that $y_0 \in \Omega$ and $H_{a+\delta_0}(y_0, y_0) \geq 0$ by exploiting integral estimates and a careful expansion of $S_{a+\delta} = J_{a+\delta}(u_{a+\delta})$. Moreover, the same computations lead to the implication (i) ⇒ (ii): in particular we get $H_{a+\delta_0}(y_0, y_0) = 0$. Hence, assuming (ii) ⇔ (iii), we prove the equivalence (i) ⇔ (ii).

Related problems can be found in [3,5–12] concerning the Euclidean and Riemannian case.

2. Expansion of the energy functional

Replacing $a(x) + \delta_0$ with $a(x)$, we are led to study the asymptotic behaviour for $\{u_\delta\}$, where $u_\delta$ achieves $S_{a-\delta}$ and is a smooth positive solution of

$$\begin{cases}
-\Delta u_\delta + (a - \delta) u_\delta = u^5_\delta & \text{in } \Omega, \\
u_\delta = 0 & \text{on } \partial \Omega,
\end{cases}$$

such that $|\nabla u_\delta|^2 \to S^{3/2} \delta y_0$ as $\delta \to 0^+$ weakly in the sense of measures, $y_0 \in \tilde{\Omega}$. Moreover, we assume $-\Delta + a$ coercive on $H^1_0(\Omega)$. 

\[ \int_{\Omega} |\nabla u_{a+\delta}|^2 = \int_{\Omega} u^6_{a+\delta} = (S_{a+\delta_0})^{3/2} = S^{3/2} \]

and then we obtain $|\nabla u_{a+\delta}|^2 \to S^{3/2} \delta y_0$ as $\delta \to \delta_0$ weakly in the sense of measures for some $y_0 \in \tilde{\Omega}$ (see [14]). In [4], Druet carried over an asymptotic analysis based on pointwise estimates of $u_{a+\delta}$ as $\delta \to \delta_0$ and proved that $y_0 \in \Omega$ and $H_{a+\delta_0}(y_0, y_0) = 0$. This proves (b) thanks to the monotonicity $H_a(y, y) > H_{a+\delta_0}(y, y)$.

The aim of this paper is to give an alternative and more direct proof that $y_0 \in \Omega$ and $H_{a+\delta_0}(y_0, y_0) \geq 0$ by exploiting integral estimates and a careful expansion of $S_{a+\delta} = J_{a+\delta}(u_{a+\delta})$. Moreover, the same computations lead to the implication (i) ⇒ (ii): in particular we get $H_{a+\delta_0}(y_0, y_0) = 0$. Hence, assuming (ii) ⇔ (iii), we prove the equivalence (i) ⇔ (ii).

Related problems can be found in [3,5–12] concerning the Euclidean and Riemannian case.
Let us define for \((\varepsilon, y) \in (0, +\infty) \times \Omega\)
\[
U_{\varepsilon, y}(x) = \varepsilon^{-1/2} U \left( \frac{x - y}{\varepsilon} \right), \quad U(x) = \frac{3^{1/4}}{(1 + |x|^2)^{1/2}}
\]
and \(P : H^1(\Omega) \rightarrow H^1_0(\Omega)\) the orthogonal projection defined for \(\varphi \in H^1(\Omega)\) as
\[
\int_{\Omega} \nabla P \varphi \nabla \psi = \int_{\Omega} \nabla \varphi \nabla \psi \quad \forall \psi \in H^1_0(\Omega),
\]
where \(H^1_0(\Omega)\) is endowed with the inner product \(\langle u, v \rangle = \int_{\Omega} \nabla u \nabla v\) and the induced norm \(\|\cdot\|\).

Let us set \(\psi_{\varepsilon, y} = U_{\varepsilon, y} - PU_{\varepsilon, y}, f_{\varepsilon, y} = \psi_{\varepsilon, y} + 3^{1/4} \omega_2 \varepsilon^{1/2} H(\cdot, y)\) and \(H(x, y) := H_0(x, y)\), where \(H_0(x, y)\) is the regular part of the Green function of \(-\Delta\) in \(\Omega\) with Dirichlet boundary condition. We have the following properties (see the Appendix in [10]):
\[
0 \leq PU_{\varepsilon, y} \leq U_{\varepsilon, y}, \quad \|\psi_{\varepsilon, y}\|_\infty = O \left( \frac{\varepsilon^{1/2}}{d} \right), \quad \|\psi_{\varepsilon, y}\|_{L^\infty(\Omega)} = O \left( \left( \frac{\varepsilon}{d} \right)^{1/2} \right), \quad (3)
\]
\[
\|f_{\varepsilon, y}\|_\infty = O \left( \frac{\varepsilon^{5/2}}{d^3} \right), \quad \sup_{y \in \Omega} dH(y, y) < 0, \quad \sup_{x \in \Omega} \|\nabla H(x, y)\| = O \left( \frac{1}{d^2} \right), \quad (4)
\]
where \(d = \text{dist}(y, \partial \Omega)\).

We follow now [10]: for \(\delta\) small, we can decompose \(u_\delta\) in the form
\[
u_\delta = z_\delta PU_{\varepsilon_\delta, y_\delta} + w_\delta
\]
for \(z_\delta, \varepsilon_\delta > 0, y_\delta \in \Omega, w_\delta \in T_\delta\) such that
\[
z_\delta \to 1, \quad \varepsilon_\delta \to 0, \quad y_\delta \to y_0, \quad \frac{\varepsilon_\delta}{d_\delta} \to 0, \quad w_\delta \to 0 \quad \text{in } H^1_0(\Omega) \text{ as } \delta \to 0,
\]
where
\[
T_\delta = \text{Span} \left\{ PU_{\varepsilon_i, y_\delta}, \frac{\partial PU_{\varepsilon_i, y_\delta}}{\partial x}, \frac{\partial PU_{\varepsilon_i, y_\delta}}{\partial y} : i = 1, \ldots, 3 \right\}^\perp.
\]

From now on we will omit the dependence on \(\delta\). We need to estimate the rate of convergence of \(z\) and \(w\). First we prove

**Lemma 2.1.** There exists \(C > 0\) such that
\[
\int_{\Omega} |\nabla v|^2 + \int_{\Omega} a(x)v^2 - \frac{5}{2} \int_{\Omega} U_{\varepsilon, y}^4 v^2 \geq C \int_{\Omega} |\nabla v|^2, \quad \forall v \in T_\delta \quad (5)
\]
for \(\delta\) small.

**Proof.** The proof is based on a well-known inequality (see [10]):
\[
\int_{\Omega} |\nabla v|^2 - \frac{5}{7} \int_{\Omega} U_{\varepsilon, y}^4 v^2 \geq \frac{4}{7} \int_{\Omega} |\nabla v|^2, \quad \forall v \in T_\delta. \quad (6)
\]
We proceed in the following way. Setting
\[
C/\delta = \inf_{v \in T/\delta : |\nabla v|^2 = 1} \left\{ 1 + \int_{\Omega} a(x) v^2 - 5 \int_{\Omega} U^{4}_{e,y} v^2 \right\},
\]
we have that $C/\delta$ is attained if $C/\delta < 1$. In fact, let $C/\delta < 1$ and let $v_n$ be a minimizing sequence: up to a subsequence, we can assume that $v_n \rightharpoonup v/\delta$ as $n \to +\infty$ weakly in $H^1_\delta(\Omega)$ and $v_\delta \in T/\delta$, $\int_{\Omega} |\nabla v_\delta|^2 \leq 1$ and $1 + \int_{\Omega} a(x) v_\delta^2 - 5 \int_{\Omega} U^{4}_{e,y} v_\delta^2 = C/\delta$.

Since $C/\delta < 1$, we get $v_\delta \neq 0$ and the inequality
\[
(1 - C/\delta) \int_{\Omega} |\nabla v_\delta|^2 + \int_{\Omega} a(x) v_\delta^2 - 5 \int_{\Omega} U^{4}_{e,y} v_\delta^2 \leq (1 - C/\delta)
\]
holds. By the minimality of $C/\delta$, the previous inequality must be an equality and $\int_{\Omega} |\nabla v_\delta|^2 = 1$. Hence, $C/\delta$ is achieved by $v_\delta$.

Now we show that $\liminf_{\delta \to 0} C/\delta > 0$. Otherwise, we could find a subsequence of minimizers $v_\delta$ for $C/\delta$ such that $C/\delta \to L \leq 0$ and $v_\delta \rightharpoonup v$ as $\delta \to 0$, weakly in $H^1_\delta(\Omega)$. Hence, $v$ solves $-(1 - L) Dv + av = 0$ in $(H^1_\delta(\Omega))'$ and, by coercivity of $-D + a$, we get $v = 0$. In view of the compactness of the embedding of $H^1_\delta(\Omega)$ into $L^2(\Omega)$, by (6) we get
\[
C/\delta = \int_{\Omega} |\nabla v|^2 - 5 \int_{\Omega} U^{4}_{e,y} v^2 + o(\|v\|^2) \geq \frac{3}{7} \int_{\Omega} |\nabla v|^2 = \frac{3}{7}
\]
contradicting $L \leq 0$. Finally, we set $C = \frac{1}{2} \liminf_{\delta \to 0} C/\delta > 0$. □

From this Lemma, we derive now the exact behaviour of $w$:

**Lemma 2.2.** We have the estimate
\[
\|w\| = O \left( \frac{\epsilon}{\delta} + \epsilon^{1/2} \right)
\]
and there holds the formula
\[
\int_{\Omega} |\nabla w|^2 + \int_{\Omega} a(x) w^2 - 5 \int_{\Omega} U^{4}_{e,y} w^2 = - \int_{\Omega} a(x) PU_{e,y} w + o \left( \frac{\epsilon}{\delta} \right).
\]

**Proof.** The function $w$ satisfies the equation
\[
\begin{cases}
-\Delta w = (xPU_{e,y} + w)^3 - x PU_{e,y} - (a(x) - \delta)(x PU_{e,y} + w) & \text{in } \Omega \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(9)

By Sobolev inequality, multiplying (9) for $w$ and integrating by parts we get
\[
\int_{\Omega} |\nabla w|^2 + \int_{\Omega} a(x) w^2 - 5 \int_{\Omega} U^{4}_{e,y} w^2 = - \int_{\Omega} a(x) PU_{e,y} w
\]
\[
+ o(\|w\|^2) + O \left( \|w\| \|\psi_{e,y}\|_{L^2(\Omega)}^2 + \int_{\Omega} U^{4}_{e,y} \|\psi_{e,y}\| w \right)
\]
\[
(10)
\]
because $\int_{\Omega} U^{5}_{e,y} w = < PU_{e,y}, w > = 0$. \n
By (3) and Sobolev inequality we get
\[ \int_{\Omega} U_{e,y}^4 \psi_{e,y} |w| = O(\|w\|) \left[ \frac{\epsilon^{1/2}}{d} \left( \int_{B_\epsilon(y)} U_{e,y}^{24/5} \right)^{5/6} + \left( \frac{\epsilon}{d} \right)^{1/2} \left( \int_{\Omega \setminus B_\epsilon(y)} U_{e,y}^6 \right)^{2/3} \right] \]
\[ = O \left( \frac{\epsilon}{d} \|w\| \right) \]
and
\[ \int_{\Omega} PU_{e,y} |w| = O(\|w\|) \left( \int_{\Omega} U_{e,y}^{6/5} \right)^{5/6} = O(\epsilon^{1/2} \|w\|). \]

We insert these estimates in (10): using the coercivity (5), first we get (7) and in turn we obtain (8).

We are now in position to prove Theorem 1.1 expanding the energy functional.

**Proof of Theorem 1.1.** With the aid of (7) and (8), we expand now
\[ S_{a-\delta} = \int_{\Omega} \frac{\nabla u_\delta}{(\int_{\Omega} u_\delta^2)^{1/3}} \left[ \int_{\Omega} |\nabla PU_{e,y}|^2 + \int_{\Omega} a(x) PU_{e,y}^2 + 5 \int_{\Omega} U_{e,y}^4 w^2 + \int_{\Omega} a(x) PU_{e,y} w + o(\frac{\epsilon}{d}) \right] \]
\[ = \int_{\Omega} \frac{\nabla PU_{e,y} |w| \psi_{e,y}}{\|\psi_{e,y}\|_{L^1(\Omega)}} \left[ \int_{\Omega} U_{e,y}^6 - 6 \int_{\Omega} U_{e,y}^5 \psi_{e,y} + 15 \int_{\Omega} U_{e,y}^4 w^2 + o(\frac{\epsilon}{d}) \right]^{1/3} \]
because, as in Lemma 2.2,
\[ \int_{\Omega} U_{e,y}^4 |w| \psi_{e,y} = O \left( \frac{\epsilon}{d} \|w\| \right) = o \left( \frac{\epsilon}{d} \right) \]
and similarly
\[ \int_{\Omega} U_{e,y}^4 \psi_{e,y} = O \left( \frac{\epsilon}{d} \|\psi_{e,y}\|_{L^1(\Omega)} \right) = o \left( \frac{\epsilon}{d} \right). \]

Since \( \int_{\Omega} U_{e,y}^6 = S^{3/2} + o(\frac{\epsilon}{d}) \) and \( \int_{\Omega} |\nabla PU_{e,y}|^2 = S^{3/2} - \int_{\Omega} U_{e,y}^5 \psi_{e,y} + o(\frac{\epsilon}{d}) \), we obtain
\[ S_{a-\delta} = S + S^{-1/2} \left( \int_{\Omega} U_{e,y}^5 \psi_{e,y} + \int_{\Omega} a(x) PU_{e,y}^2 + \int_{\Omega} a(x) PU_{e,y} w \right) + o \left( \frac{\epsilon}{d} \right). \]

By (4) and a Taylor expansion for \( H(x, y) \) we get
\[ \psi_{e,y}(x) = -3^{1/4} \omega_2 \epsilon^{1/2} H(y, y) + O \left( \frac{\epsilon^{1/2} \sup_{x \in \Omega} |\nabla H(x, y)| + \epsilon^{5/2}}{d^3} \right) \]
\[ = -3^{1/4} \omega_2 \epsilon^{1/2} H(y, y) + O \left( \frac{\epsilon^{1/2} \sup_{x \in \Omega} |\nabla H(x, y)| + \epsilon^{5/2}}{d^2} \right) \]
and hence
\[ \int_{\Omega} U^5_{e,y} \psi_{e,y} = \int_{\Omega} U^5_{e,y} \left[ -3^{1/4} \omega_2 \varepsilon^{1/2} H(y, y) + O \left( \frac{\varepsilon^{1/2} |x - y|}{d^2} + \frac{\varepsilon^{5/2}}{d^3} \right) \right] \]
\[ = -3^{3/2} \omega_2 \varepsilon H(y, y) \int_{\mathbb{R}^3} \frac{dx}{(1 + |x|^2)^{5/2}} + o \left( \frac{\varepsilon}{d} \right). \]
If \( d \to 0 \), then
\[ S_{\varepsilon - \delta} = S - S^{-1/2} 3^{1/2} \omega_2^2 \varepsilon H(y, y) + o \left( \frac{\varepsilon}{d} \right) \]
because \( \int_{\mathbb{R}^3} \frac{dx}{(1 + |x|^2)^{5/2}} = \omega_2/3 \) and
\[ \int_{\Omega} a(x) PU_{e,y}^2 + \int_{\Omega} a(x) PU_{e,y}w = O(\varepsilon) = o \left( \frac{\varepsilon}{d} \right). \]
Since \( \sup_{y \in \Omega} dH(y, y) < 0 \) (see (4)), we conclude \( S_{\varepsilon - \delta} > S \). A contradiction.

So we can exclude the boundary concentration: \( y_0 \in \Omega \). The expansion for \( S_{\varepsilon - \delta} \) becomes
\[ S_{\varepsilon - \delta} = S + S^{-1/2} \varepsilon \left[ -3^{1/2} \omega_2^2 H(y_0, y_0) + \int_{\Omega} a(x) \left( \frac{PU_{e,y}}{\varepsilon^{1/2}} \right)^2 \right. \]
\[ + \left. \int_{\Omega} a(x) \left( \frac{PU_{e,y}}{\varepsilon^{1/2}} \right) \left( \frac{w}{\varepsilon^{1/2}} \right) \right] + o(\varepsilon). \]
By (4) we get
\[ \frac{PU_{e,y}}{\varepsilon^{1/2}} = \frac{3^{1/4}}{\varepsilon^{1/2} + |x - y|^2}^{1/2} + 3^{1/4} \omega_2 H(x, y) - \frac{f_{e,y}}{\varepsilon^{1/2}} \to 3^{1/4} \omega_2 G(x, y_0) \] as \( \delta \to 0 \)
in \( L^s(\Omega) \) for any \( s < \frac{3}{2} \) and uniformly in any compact set of \( \Omega \setminus \{0\} \), where \( G(x, y) := G_0(x, y) \). Moreover there holds
\[ \int_{\Omega} a(x) \frac{3^{1/2}}{\varepsilon^{1/2} + |x - y|^2} \to \int_{\Omega} a(x) \frac{3^{1/2}}{|x - y_0|^2} \] as \( \delta \to 0 \).
By (7), the functions \( \tilde{w} = w/\varepsilon^{1/2} \) are uniformly bounded in \( H_0^1(\Omega) \) and solve the equation
\[ \begin{cases} -\Delta \tilde{w} + \left( a(x) - \delta \right) \left( x \frac{PU_{e,y}}{\varepsilon^{1/2}} + \tilde{w} \right) = \varepsilon^2 \left( x \frac{PU_{e,y}}{\varepsilon^{1/2}} + \tilde{w} \right)^5 - \varepsilon^2 x \left( \frac{U_{e,y}}{\varepsilon^{1/2}} \right)^5 & \text{in } \Omega \\ \tilde{w} = 0 & \text{on } \partial \Omega. \end{cases} \]
Up to a subsequence, we can assume that \( \tilde{w} \to f \) as \( \delta \to 0 \) weakly in \( H_0^1(\Omega) \). Hence \( f \) satisfies for any \( \Phi \in C_0^\infty(\Omega \setminus \{y_0\}) \)
\[ \int_{\Omega} \nabla f \nabla \Phi + \int_{\Omega} a(x) f \Phi = -3^{1/4} \omega_2 \int_{\Omega} a(x) G(x, y_0) \Phi. \]
Since \( f \in H^1_0(\Omega) \), it can be easily seen that (11) holds for any \( \Phi \in H^1_0(\Omega) \) and hence \( f(x) = 3^{1/4} \omega_2 (H_0(x, y_0) - H(x, y_0)) \). In view of the compactness of the embedding of \( H^1_0(\Omega) \) into \( L^s(\Omega) \) for any \( 1 \leq s < 6 \), we get

\[
\int_{\Omega} a(x) P U_{x,y}^2 + \int_{\Omega} a(x) P U_{x,y} w = 3^{1/2} \omega_2^2 \int_{\Omega} a(x) G(x, y_0)^2 + 3^{1/4} \omega_2 \int_{\Omega} a(x) G(x, y_0) f + o(\varepsilon).
\]

Since \( f \in W^{2,s} \cap C(\tilde{\Omega}) \) and \( G(x, y_0) \in L^s \) for any \( 1 \leq s < 3 \), we get that \( G(x, y_0) \) is an admissible test function in (11) and then

\[
3^{1/4} \omega_2 \int_{\Omega} a(x) G(x, y_0)^2 + \int_{\Omega} a(x) G(x, y_0) f = \int_{\Omega} \Delta f G(x, y_0) = -f(y_0) = 3^{1/4} \omega_2 (H(y_0, y_0) - H_0(y_0, y_0)).
\]

Finally, we get

\[
S_{a-\delta} = S - 3^{1/2} \omega_2^2 S^{-1/2} \varepsilon H_0(y_0, y_0) + o(\varepsilon) < S
\]

and then \( H_0(y_0, y_0) \geq 0 \). Hence (ii) \( \Rightarrow \) (i).

To prove the converse, let us remark that the expansion for \( S_{a-\delta} = J_{a-\delta}(u_\delta), \ u_\delta = x_\delta P U_{x,y} + w_\delta \), is based on (7) and on

\[
\int_{\Omega} |\nabla w_\delta|^2 + \int_{\Omega} a(x) w_\delta^2 - \frac{5}{2} \int_{\Omega} W_{x,y}^4 w_\delta^2
\]

\[
= - 3^{1/4} \omega_2 \varepsilon_\delta \int_{\Omega} a(x) G(x, y_0) f + o(\varepsilon_\delta) \quad (8')
\]

and not on the equations satisfied by \( u_\delta \) and \( w_\delta \).

Let us assume \( H_0(y_0, y_0) > 0 \) and let us consider test functions in the form \( u_\varepsilon = P U_{x,y} + \varepsilon^{1/2} f, \ \varepsilon > 0 \). Since \( \int_{\Omega} U_{x,y}^4 f^2 = O(\|f\|_\infty^2 \int_\Omega U_{x,y}^4) \rightarrow 0 \) as \( \varepsilon \rightarrow 0 \), we get that (7) and (8') hold for \( w = \varepsilon^{1/2} f \) and then the expansion

\[
J_d(u_\varepsilon) = S - 3^{1/2} \omega_2^2 S^{-1/2} \varepsilon H_0(y_0, y_0) + o(\varepsilon)
\]

follows. Hence \( S_d < S \) and (i) \( \Rightarrow \) (ii) is proved. \( \square \)

References


