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On some conjectures proposed by Haïm Brezis☆

Pierpaolo Esposito

Dipartimento di Matematica, Università di Roma "Tor Vergata", Via della Ricerca Scientifica, Roma 00133, Italy

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Abstract

Druet (Ann. Inst. H. Poincaré Anal. Non Linèaire 19(2) (2002) 125) solved two conjectures proposed by Haïm Brezis (Comm. Pure Appl. Math. 39 (1986) 17) about "low"-dimension phenomena for some elliptic problem with critical Sobolev exponent. In Druet (Ann. Inst. H. Poincaré Anal. Non Linèaire 19(2) (2002) 125), the proof of one of the two conjectures is reduced to an asymptotic analysis which is carried over with very general techniques involving pointwise estimates. We propose here a different and simpler approach in the blow-up analysis based on integral estimates and on a careful expansion of the energy functional.

1. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, and $a(x)$ be a continuous function in $\bar{\Omega}$. We consider the problem

$$(\mathrm{P}) \quad \begin{cases} -\Delta u + au = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p = (N+2)/(N-2)$ is the critical Sobolev exponent. Solutions for Eq. (P) can be found by studying the minimization problem for the functional

$$J_a(u) = \frac{\int_\Omega |\nabla u|^2 + \int_\Omega a(x)u^2}{\left(\int_\Omega |u|^{p+1}\right)^{2/(p+1)}}, \quad u \in H_0^1(\Omega) \setminus \{0\}. \tag{1}$$

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E-mail address: pesposit@axp.mat.uniroma2.it (P. Esposito).

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In fact, by standard elliptic regularity theory, any minimum u_a for J_a provides a smooth solution for problem (P), up to rescaling the Lagrange multiplier and changing u_a into $|u_a|$. Remarking that the solvability of (P) implies the coercivity of the operator $-\Delta+a$ on $H_0^1(\Omega)$, i.e. the first eigenvalue λ_1 is positive, we can assume from now on that $-\Delta+a$ is coercive on $H_0^1(\Omega)$.

Denoting S_a the infimum of J_a, it is well known that $S_a \leqslant S$, $S=S_0$ being the Sobolev constant, and $S_a < S$ implies S_a achieved (see for instance [2]).

For $N \geqslant 4$, in [2] it is also proved that $a(x)$ negative somewhere implies $S_a < S$ and, since S is never attained, the properties

(i) $a(x)$ negative somewhere,
(ii) $S_a < S$,
(iii) S_a is achieved

are equivalent. So the problem turns out to be completely characterized by the local nature of $a(x)$.

On the other hand, in dimension $N=3$, the global nature of $a(x)$ becomes significative: in [2] it was discussed the particular case of the unit ball and $a(x)\equiv$ const. Some notations are in order to state the general result: let us define $G_a(x,y)$, $x\in\Omega\setminus\{y\}$, the Green function in $y\in\Omega$ of $-\Delta+a$ in Ω with Dirichlet boundary condition, as the distributional solution for

$$\begin{cases} -\Delta_x G_a(x,y)+a(x)G_a(x,y)=\delta_y & \text{in } \Omega,\\ G_a(x,y)=0 & \text{on } \partial\Omega, \end{cases}$$

where δ_y is the Dirac measure in y, and let us set $H_a(x,y)=G_a(x,y)-1/(\omega_2|x-y|)$ the regular part of the Green function $G_a(x,y)$, where $H_a(x,y)\in C(\Omega\times\Omega)$ is a distributional solution for

$$\begin{cases} -\Delta_x H_a(x,y)+a(x)G_a(x,y)=0 & \text{in } \Omega,\\ H_a(x,y)=-\frac{1}{\omega_2|x-y|} & \text{on } \partial\Omega. \end{cases}$$

For the general case, the following result was conjectured in [1] and proved in [4]:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^3$ and let $a(x)\in C(\bar\Omega)$ be such that $-\Delta+a$ is coercive. The properties*:

(i) $\exists y\in\Omega$ *such that* $H_a(y,y)>0$,
(ii) $S_a<S$,
(iii) S_a *is achieved*

are equivalent.

By test functions computations (see [13]), one gets (i) $\Rightarrow$ (ii). The fact that (ii) $\Rightarrow$ (iii) is well-known and is already present for instance in [2]. Brezis, in [1], proposed the two following conjectures:

(a) the implication (iii) $\Rightarrow$ (ii) does hold;
(b) the implication (ii) $\Rightarrow$ (i) does hold.

Druet proved these two conjectures and thus Theorem 1.1 in [4]. The proof of the conjecture (a) makes use of the minimality condition $D^2J_a(u_a) \geqslant 0$. Thanks to (a), the proof that (ii) $\Rightarrow$ (i) is reduced in [4] to some asymptotic analysis in the following way: let $a \in C(\bar{\Omega})$ such that $S_a < S$. By continuity and monotonicity of $S_{a+\delta}$ with respect to $\delta > 0$, we get $S_{a+\delta} < S$ for $0 < \delta < \delta_0$ and $S_{a+\delta} = S$ for $\delta \geqslant \delta_0$, δ_0 some positive real number. Since $S_{a+\delta} < S$ for $0 < \delta < \delta_0$, there exists $u_{a+\delta}$ a smooth positive function achieving $S_{a+\delta}$ for $0 < \delta < \delta_0$ and satisfying the renormalization $\int_\Omega u_{a+\delta}^6 = (S_{a+\delta})^{3/2}$. By uniform coercivity of $-\Delta + (a+\delta)$ on $H_0^1(\Omega)$, it is easily seen that $\sup_{\delta \in (0,\delta_0)} \int_\Omega |\nabla u_{a+\delta}|^2 < +\infty$ and then, up to a subsequence, we can assume $u_{a+\delta} \rightharpoonup u$ as $\delta \to \delta_0$ in $H_0^1(\Omega)$. Since, by (a), $S_{a+\delta_0}$ is not achieved, $u \equiv 0$ and then we have $u_{a+\delta} \rightharpoonup 0$ as $\delta \to \delta_0$ weakly but not strongly in $H_0^1(\Omega)$. Hence $u_{a+\delta}$ must blow-up as $\delta \to \delta_0$. In view of the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, we have

$$\lim_{\delta \to \delta_0} \int_\Omega |\nabla u_{a+\delta}|^2 = \lim_{\delta \to \delta_0} \int_\Omega u_{a+\delta}^6 = (S_{a+\delta_0})^{3/2} = S^{3/2}$$

and then we obtain $|\nabla u_{a+\delta}|^2 \rightharpoonup S^{3/2}\delta_{y_0}$ as $\delta \to \delta_0$ weakly in the sense of measures for some $y_0 \in \bar{\Omega}$ (see [14]). In [4], Druet carried over an asymptotic analysis based on pointwise estimates of $u_{a+\delta}$ as $\delta \to \delta_0$ and proved that $y_0 \in \Omega$ and $H_{a+\delta_0}(y_0, y_0) = 0$. This proves (b) thanks to the monotonicity $H_a(y,y) > H_{a+\delta_0}(y,y)$.

The aim of this paper is to give an alternative and more direct proof that $y_0 \in \Omega$ and $H_{a+\delta_0}(y_0,y_0) \geqslant 0$ by exploiting integral estimates and a careful expansion of $S_{a+\delta} = J_{a+\delta}(u_{a+\delta})$. Moreover, the same computations lead to the implication (i) $\Rightarrow$ (ii): in particular we get $H_{a+\delta_0}(y_0,y_0) = 0$. Hence, assuming (ii) $\Leftrightarrow$ (iii), we prove the equivalence (i) $\Leftrightarrow$ (ii).

Related problems can be found in [3,5–12] concerning the Euclidean and Riemannian case.

2. Expansion of the energy functional

Replacing $a(x) + \delta_0$ with $a(x)$, we are led to study the asymptotic behaviour for $\{u_\delta\}$, where u_δ achieves $S_{a-\delta}$ and is a smooth positive solution of

$$\begin{cases} -\Delta u_\delta + (a-\delta)u_\delta = u_\delta^5 & \text{in } \Omega, \\ u_\delta = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

such that $|\nabla u_\delta|^2 \rightharpoonup S^{3/2}\delta_{y_0}$ as $\delta \to 0^+$ weakly in the sense of measures, $y_0 \in \bar{\Omega}$. Moreover, we assume $-\Delta + a$ coercive on $H_0^1(\Omega)$.

Let us define for $(\varepsilon, y) \in (0, +\infty) \times \Omega$

$$U_{\varepsilon,y}(x) = \varepsilon^{-1/2} U\left(\frac{x-y}{\varepsilon}\right), \quad U(x) = \frac{3^{1/4}}{(1+|x|^2)^{1/2}}$$

and $P: H^1(\Omega) \to H_0^1(\Omega)$ the orthogonal projection defined for $\varphi \in H^1(\Omega)$ as

$$\int_\Omega \nabla P\varphi \nabla\psi = \int_\Omega \nabla\varphi\nabla\psi \quad \forall \psi \in H_0^1(\Omega),$$

where $H_0^1(\Omega)$ is endowed with the inner product $< u, v > = \int_\Omega \nabla u \nabla v$ and the induced norm $\|\cdot\|$.

Let us set $\psi_{\varepsilon,y} = U_{\varepsilon,y} - PU_{\varepsilon,y}$, $f_{\varepsilon,y} = \psi_{\varepsilon,y} + 3^{1/4}\omega_2\varepsilon^{1/2}H(\cdot, y)$ and $H(x,y) := H_0(x,y)$, where $H_0(x,y)$ is the regular part of the Green function of $-\Delta$ in Ω with Dirichlet boundary condition. We have the following properties (see the Appendix in [10]):

$$0 \leqslant PU_{\varepsilon,y} \leqslant U_{\varepsilon,y}, \quad \|\psi_{\varepsilon,y}\|_\infty = O\left(\frac{\varepsilon^{1/2}}{d}\right), \quad \|\psi_{\varepsilon,y}\|_{L^6(\Omega)} = O\left(\left(\frac{\varepsilon}{d}\right)^{1/2}\right), \tag{3}$$

$$\|f_{\varepsilon,y}\|_\infty = O\left(\frac{\varepsilon^{5/2}}{d^3}\right), \quad \sup_{y\in\Omega} \mathrm{d}H(y,y) < 0, \quad \sup_{x\in\Omega} |\nabla H(x,y)| = O\left(\frac{1}{d^2}\right), \tag{4}$$

where $d = \mathrm{dist}(y, \partial\Omega)$.

We follow now [10]: for δ small, we can decompose u_δ in the form

$$u_\delta = \alpha_\delta PU_{\varepsilon_\delta, y_\delta} + w_\delta$$

for $\alpha_\delta,\ \varepsilon_\delta > 0,\ y_\delta \in \Omega,\ w_\delta \in T_\delta$ such that

$$\alpha_\delta \to 1,\ \varepsilon_\delta \to 0,\ y_\delta \to y_0,\ \frac{\varepsilon_\delta}{d_\delta} \to 0,\ w_\delta \to 0 \quad \text{in } H_0^1(\Omega) \text{ as } \delta \to 0,$$

where

$$T_\delta = \mathrm{Span}\left\{PU_{\varepsilon_\delta,y_\delta}, \frac{\partial PU_{\varepsilon_\delta,y_\delta}}{\partial\varepsilon}, \frac{\partial PU_{\varepsilon_\delta,y_\delta}}{\partial y_i}:\ i = 1,\ldots,3\right\}^\perp.$$

From now on we will omit the dependence on δ. We need to estimate the rate of convergence of α and w. First we prove

Lemma 2.1. *There exists $C > 0$ such that*

$$\int_\Omega |\nabla v|^2 + \int_\Omega a(x)v^2 - 5\int_\Omega U_{\varepsilon,y}^4 v^2 \geqslant C\int_\Omega |\nabla v|^2, \quad \forall v \in T_\delta \tag{5}$$

for δ small.

Proof. The proof is based on a well-known inequality (see [10]):

$$\int_\Omega |\nabla v|^2 - 5\int_\Omega U_{\varepsilon,y}^4 v^2 \geqslant \frac{4}{7}\int_\Omega |\nabla v|^2, \quad \forall v \in T_\delta. \tag{6}$$

We proceed in the following way. Setting

$$C_\delta = \inf_{v \in T_\delta : \int_\Omega |\nabla v|^2 = 1} \left\{ 1 + \int_\Omega a(x) v^2 - 5 \int_\Omega U_{\varepsilon,y}^4 v^2 \right\},$$

we have that C_δ is attained if $C_\delta < 1$. In fact, let $C_\delta < 1$ and let v_n be a minimizing sequence: up to a subsequence, we can assume that $v_n \rightharpoonup v_\delta$ as $n \to +\infty$ weakly in $H_0^1(\Omega)$ and $v_\delta \in T_\delta$, $\int_\Omega |\nabla v_\delta|^2 \leqslant 1$ and $1 + \int_\Omega a(x) v_\delta^2 - 5 \int_\Omega U_{\varepsilon,y}^4 v_\delta^2 = C_\delta$.

Since $C_\delta < 1$, we get $v_\delta \neq 0$ and the inequality

$$(1 - C_\delta) \int_\Omega |\nabla v_\delta|^2 + \int_\Omega a(x) v_\delta^2 - 5 \int_\Omega U_{\varepsilon,y}^4 v_\delta^2 \leqslant (1 - C_\delta) + \int_\Omega a(x) v_\delta^2 - 5 \int_\Omega U_{\varepsilon,y}^4 v_\delta^2 = 0$$

holds. By the minimality of C_δ, the previous inequality must be an equality and $\int_\Omega |\nabla v_\delta|^2 = 1$. Hence, C_δ is achieved by v_δ.

Now we show that $\liminf_{\delta \to 0} C_\delta > 0$. Otherwise, we could find a subsequence of minimizers v_δ for C_δ such that $C_\delta \to L \leqslant 0$ and $v_\delta \rightharpoonup v$, as $\delta \to 0$, weakly in $H_0^1(\Omega)$. Hence, v solves $-(1 - L)\Delta v + av = 0$ in $(H_0^1(\Omega))'$ and, by coercivity of $-\Delta + a$, we get $v = 0$. In view of the compactness of the embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, by (6) we get

$$C_\delta = \int_\Omega |\nabla v_\delta|^2 - 5 \int_\Omega U_{\varepsilon,y}^4 v_\delta^2 + o(\|v_\delta\|^2) \geqslant \frac{3}{7} \int_\Omega |\nabla v_\delta|^2 = \frac{3}{7}$$

contradicting $L \leqslant 0$. Finally, we set $C = \frac{1}{2} \liminf_{\delta \to 0} C_\delta > 0$. □

From this Lemma, we derive now the exact behaviour of w:

Lemma 2.2. *We have the estimate*

$$\|w\| = O\left(\frac{\varepsilon}{d} + \varepsilon^{1/2}\right) \tag{7}$$

and there holds the formula

$$\int_\Omega |\nabla w|^2 + \int_\Omega a(x) w^2 - 5 \int_\Omega U_{\varepsilon,y}^4 w^2 = -\int_\Omega a(x) PU_{\varepsilon,y} w + o\left(\frac{\varepsilon}{d}\right). \tag{8}$$

Proof. The function w satisfies the equation

$$\begin{cases} -\Delta w = (\alpha PU_{\varepsilon,y} + w)^5 - \alpha U_{\varepsilon,y}^5 - (a(x) - \delta)(\alpha PU_{\varepsilon,y} + w) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

By Sobolev inequality, multiplying (9) for w and integrating by parts we get

$$\int_\Omega |\nabla w|^2 + \int_\Omega a(x) w^2 - 5 \int_\Omega U_{\varepsilon,y}^4 w^2 = -\alpha \int_\Omega (a(x) - \delta) PU_{\varepsilon,y} w + o(\|w\|^2) + O\left(\|w\| \|\psi_{\varepsilon,y}\|_{L^6(\Omega)}^2 + \int_\Omega U_{\varepsilon,y}^4 |\psi_{\varepsilon,y}| |w| \right) \tag{10}$$

because $\int_\Omega U_{\varepsilon,y}^5 w = < PU_{\varepsilon,y}, w > = 0$.

By (3) and Sobolev inequality we get

$$\int_\Omega U^4_{\varepsilon,y}\psi_{\varepsilon,y}|w| = O(\|w\|)\left[\frac{\varepsilon^{1/2}}{d}\left(\int_{B_d(y)} U^{24/5}_{\varepsilon,y}\right)^{5/6} + \left(\frac{\varepsilon}{d}\right)^{1/2}\left(\int_{\Omega\setminus B_d(y)} U^6_{\varepsilon,y}\right)^{2/3}\right]$$
$$= O\left(\frac{\varepsilon}{d}\|w\|\right)$$

and

$$\int_\Omega PU_{\varepsilon,y}|w| = O(\|w\|)\left(\int_\Omega U^{6/5}_{\varepsilon,y}\right)^{5/6} = O(\varepsilon^{1/2}\|w\|).$$

We insert these estimates in (10): using the coercivity (5), first we get (7) and in turn we obtain (8). □

We are now in position to prove Theorem 1.1 expanding the energy functional.

Proof of Theorem 1.1. With the aid of (7) and (8), we expand now

$$S_{a-\delta} = \frac{\int_\Omega |\nabla u_\delta|^2 + \int_\Omega (a(x)-\delta)u_\delta^2}{\left(\int_\Omega u_\delta^6\right)^{1/3}}$$
$$= \frac{\alpha^2\int_\Omega |\nabla PU_{\varepsilon,y}|^2 + \int_\Omega a(x)PU^2_{\varepsilon,y} + 5\int_\Omega U^4_{\varepsilon,y}w^2 + \int_\Omega a(x)PU_{\varepsilon,y}w + o(\frac{\varepsilon}{d})}{\left[\alpha^6\int_\Omega U^6_{\varepsilon,y} - 6\int_\Omega U^5_{\varepsilon,y}\psi_{\varepsilon,y} + 15\int_\Omega U^4_{\varepsilon,y}w^2 + o(\frac{\varepsilon}{d})\right]^{1/3}}$$

because, as in Lemma 2.2,

$$\int_\Omega U^4_{\varepsilon,y}|w|\psi_{\varepsilon,y} = O\left(\frac{\varepsilon}{d}\|w\|\right) = o\left(\frac{\varepsilon}{d}\right)$$

and similarly

$$\int_\Omega U^4_{\varepsilon,y}\psi^2_{\varepsilon,y} = O\left(\frac{\varepsilon}{d}\|\psi_{\varepsilon,y}\|_{L^6(\Omega)}\right) = o\left(\frac{\varepsilon}{d}\right).$$

Since $\int_\Omega U^6_{\varepsilon,y} = S^{3/2} + o(\frac{\varepsilon}{d})$ and $\int_\Omega |\nabla PU_{\varepsilon,y}|^2 = S^{3/2} - \int_\Omega U^5_{\varepsilon,y}\psi_{\varepsilon,y} + o(\frac{\varepsilon}{d})$, we obtain

$$S_{a-\delta} = S + S^{-1/2}\left(\int_\Omega U^5_{\varepsilon,y}\psi_{\varepsilon,y} + \int_\Omega a(x)PU^2_{\varepsilon,y} + \int_\Omega a(x)PU_{\varepsilon,y}w\right) + o\left(\frac{\varepsilon}{d}\right).$$

By (4) and a Taylor expansion for $H(x,y)$ we get

$$\psi_{\varepsilon,y}(x) = -3^{1/4}\omega_2\varepsilon^{1/2}H(y,y) + O\left(\varepsilon^{1/2}|x-y|\sup_{x\in\Omega}|\nabla H(x,y)| + \frac{\varepsilon^{5/2}}{d^3}\right)$$
$$= -3^{1/4}\omega_2\varepsilon^{1/2}H(y,y) + O\left(\frac{\varepsilon^{1/2}|x-y|}{d^2} + \frac{\varepsilon^{5/2}}{d^3}\right)$$

and hence

$$\int_\Omega U^5_{\varepsilon,y}\psi_{\varepsilon,y} = \int_\Omega U^5_{\varepsilon,y}\left[-3^{1/4}\omega_2\varepsilon^{1/2}H(y,y) + O\left(\frac{\varepsilon^{1/2}|x-y|}{d^2} + \frac{\varepsilon^{5/2}}{d^3}\right)\right]$$
$$= -3^{3/2}\omega_2\varepsilon H(y,y)\int_{\mathbb{R}^3}\frac{\mathrm{d}x}{(1+|x|^2)^{5/2}} + o\left(\frac{\varepsilon}{d}\right).$$

If $d\to 0$, then

$$S_{a-\delta} = S - S^{-1/2}3^{1/2}\omega_2^2\varepsilon H_(y,y) + o\left(\frac{\varepsilon}{d}\right)$$

because $\int_{\mathbb{R}^3}\mathrm{d}x/(1+|x|^2)^{5/2} = \omega_2/3$ and

$$\int_\Omega a(x)PU^2_{\varepsilon,y} + \int_\Omega a(x)PU_{\varepsilon,y}w = O(\varepsilon) = o\left(\frac{\varepsilon}{d}\right).$$

Since $\sup_{y\in\Omega} dH(y,y) < 0$ (see (4)), we conclude $S_{a-\delta} > S$. A contradiction.

So we can exclude the boundary concentration: $y_0\in\Omega$. The expansion for $S_{a-\delta}$ becomes

$$S_{a-\delta} = S + S^{-1/2}\varepsilon\left[-3^{1/2}\omega_2^2H(y_0,y_0) + \int_\Omega a(x)\left(\frac{PU_{\varepsilon,y}}{\varepsilon^{1/2}}\right)^2\right.$$
$$\left. + \int_\Omega a(x)\left(\frac{PU_{\varepsilon,y}}{\varepsilon^{1/2}}\right)\left(\frac{w}{\varepsilon^{1/2}}\right)\right] + o(\varepsilon).$$

By (4) we get

$$\frac{PU_{\varepsilon,y}}{\varepsilon^{1/2}} = \frac{3^{1/4}}{(\varepsilon^2+|x-y|^2)^{1/2}} + 3^{1/4}\omega_2H(x,y) - \frac{f_{\varepsilon,y}}{\varepsilon^{1/2}} \to 3^{1/4}\omega_2G(x,y_0) \text{ as } \delta\to 0$$

in $L^s(\Omega)$ for any $s<\frac{3}{2}$ and uniformly in any compact set of $\Omega\setminus\{0\}$, where $G(x,y) := G_0(x,y)$. Moreover there holds

$$\int_\Omega a(x)\frac{3^{1/2}}{\varepsilon^2+|x-y|^2} \to \int_\Omega a(x)\frac{3^{1/2}}{|x-y_0|^2} \text{ as } \delta\to 0.$$

By (7), the functions $\tilde{w}=w/\varepsilon^{1/2}$ are uniformly bounded in $H^1_0(\Omega)$ and solve the equation

$$\begin{cases} -\Delta\tilde{w} + (a(x)-\delta)\left(\alpha\frac{PU_{\varepsilon,y}}{\varepsilon^{1/2}} + \tilde{w}\right) = \varepsilon^2\left(\alpha\frac{PU_{\varepsilon,y}}{\varepsilon^{1/2}} + \tilde{w}\right)^5 - \varepsilon^2\alpha\left(\frac{U_{\varepsilon,y}}{\varepsilon^{1/2}}\right)^5 & \text{in } \Omega \\ \tilde{w} = 0 & \text{on } \partial\Omega. \end{cases}$$

Up to a subsequence, we can assume that $\tilde{w}\rightharpoonup f$ as $\delta\to 0$ weakly in $H^1_0(\Omega)$. Hence f satisfies for any $\Phi\in C_0^\infty(\Omega\setminus\{y_0\})$

$$\int_\Omega \nabla f\nabla\Phi + \int_\Omega a(x)f\Phi = -3^{1/4}\omega_2\int_\Omega a(x)G(x,y_0)\Phi. \tag{11}$$

Since $f \in H_0^1(\Omega)$, it can be easily seen that (11) holds for any $\Phi \in H_0^1(\Omega)$ and hence $f(x) = 3^{1/4}\omega_2(H_a(x,y_0) - H(x,y_0))$. In view of the compactness of the embedding of $H_0^1(\Omega)$ into $L^s(\Omega)$ for any $1 \leqslant s < 6$, we get

$$\int_\Omega a(x)PU_{\varepsilon,y}^2 + \int_\Omega a(x)PU_{\varepsilon,y}w = 3^{1/2}\omega_2^2\varepsilon \int_\Omega a(x)G(x,y_0)^2 + 3^{1/4}\omega_2\varepsilon \int_\Omega a(x)G(x,y_0)f + o(\varepsilon).$$

Since $f \in W^{2,s} \cap C(\bar{\Omega})$ and $G(x,y_0) \in L^s$ for any $1 \leqslant s < 3$, we get that $G(x,y_0)$ is an admissible test function in (11) and then

$$3^{1/4}\omega_2 \int_\Omega a(x)G(x,y_0)^2 + \int_\Omega a(x)G(x,y_0)f = \int_\Omega \Delta f G(x,y_0) = -f(y_0) = 3^{1/4}\omega_2(H(y_0,y_0) - H_a(y_0,y_0)).$$

Finally, we get

$$S_{a-\delta} = S - 3^{1/2}\omega_2^2 S^{-1/2}\varepsilon H_a(y_0,y_0) + o(\varepsilon) < S$$

and then $H_a(y_0,y_0) \geqslant 0$. Hence (ii) $\Rightarrow$ (i).

To prove the converse, let us remark that the expansion for $S_{a-\delta} = J_{a-\delta}(u_\delta)$, $u_\delta = \alpha_\delta PU_{\varepsilon_\delta,y_\delta} + w_\delta$, is based on (7) and on

$$\int_\Omega |\nabla w_\delta|^2 + \int_\Omega a(x)w_\delta^2 - 5\int_\Omega U_{\varepsilon_\delta,y_\delta}^4 w_\delta^2 = -3^{1/4}\omega_2\varepsilon_\delta \int_\Omega a(x)G(x,y_0)f + o(\varepsilon_\delta) \tag{8'}$$

and not on the equations satisfied by u_δ and w_δ.

Let us assume $H_a(y_0,y_0) > 0$ and let us consider test functions in the form $u_\varepsilon = PU_{\varepsilon,y_0} + \varepsilon^{1/2}f$, $\varepsilon > 0$. Since $\int_\Omega U_{\varepsilon,y_0}^4 f^2 = O\left(\|f\|_\infty^2 \int_\Omega U_{\varepsilon,y_0}^4\right) \to 0$ as $\varepsilon \to 0$, we get that (7) and (8′) hold for $w = \varepsilon^{1/2}f$ and then the expansion

$$J_a(u_\varepsilon) = S - 3^{1/2}\omega_2^2 S^{-1/2}\varepsilon H_a(y_0,y_0) + o(\varepsilon)$$

follows. Hence $S_a < S$ and (i) $\Rightarrow$ (ii) is proved. □

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