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Blowing-up solutions for the Yamabe equation

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Abstract. Let (M, g) be a smooth, compact Riemannian manifold of dimension $N \geq 3$. We consider the almost critical problem

$$(P_\varepsilon) \qquad -\Delta_g u + \frac{N-2}{4(N-1)} \operatorname{Scal}_g u = u^{(N+2)/(N-2)+\varepsilon} \quad \text{in } M, \quad u > 0 \quad \text{in } M,$$

where Δ_g denotes the Laplace-Beltrami operator, Scal_g is the scalar curvature of g and $\varepsilon \in \mathbb{R}$ is a small parameter. It is known that problem (P_ε) does not have any blowing-up solutions when $\varepsilon \nearrow 0$, at least for $N \leq 24$ or in the locally conformally flat case, and this is not true anymore when $\varepsilon \searrow 0$. Indeed, we prove that, if $N \geq 7$ and the manifold is not locally conformally flat, then problem (P_ε) does have a family of solutions which blow-up at a maximum point of the function $\xi \to |\operatorname{Weyl}_g(\xi)|_g$ as $\varepsilon \searrow 0$. Here Weyl_g denotes the Weyl curvature tensor of g.

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1. Introduction

Let (M, g) be a smooth, compact Riemannian N-manifold, $N \geq 3$. We consider the almost critical problem

$$-\Delta_g u + \frac{N-2}{4(N-1)} \operatorname{Scal}_g u = \kappa u^{2^*-1+\varepsilon} \quad \text{in } M, \quad u > 0 \quad \text{in } M, \tag{1}$$

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where $\Delta_g := \mathrm{div}_g \nabla$ is the Laplace-Beltrami operator, Scal_g is the scalar curvature of (M, g), $\kappa \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ is a small parameter. Here $2^* := \frac{2N}{N-2}$ is the critical exponent for the embedding of the Riemannian Sobolev space $H_g^1(M)$ into Lebesgue space $L_g^{2^*}(M)$.

When $\varepsilon = 0$ equation (1) reads as the Yamabe problem. The constant κ can be restricted to the values -1, $+1$ or 0 depending on whether the *Yamabe invariant* of (M, g)

$$\Upsilon_g(M) = \inf_{\tilde{g} \in [g]} \left(\mathrm{Vol}_{\tilde{g}}(M)^{(2-N)/N} \int_M \mathrm{Scal}_{\tilde{g}}\, dv_{\tilde{g}} \right)$$

has negative sign, positive sign or vanishes, respectively. Here $[g] = \{\phi g : \phi \in C^\infty(M), \phi > 0\}$ is the conformal class of g and $\mathrm{Vol}_{\tilde{g}}(M)$ is the volume of the manifold $(M, \tilde{g})$. In particular, if u is a solution of the Yamabe equation $(1)_{\varepsilon=0}$, then the metric $\tilde{g} = u^{4/(N-2)} g$ is conformally equivalent to g and has constant Scalar curvature κ. The Yamabe problem, raised by H. Yamabe [23] in '60, was firstly solved by Trudinger [22] when $\Upsilon_g(M) \le 0$. In this case, the solution is unique, up to a normalization. In general, a solution of the Yamabe problem can be found by a direct constrained minimization method. As shown by Aubin [2], the inequality

$$\Upsilon_g(M) < \Upsilon_{g_0}(\mathbb{S}^N), \tag{2}$$

where $(\mathbb{S}^N, g_0)$ is the round sphere, is the key ingredient to show compactness of minimizing sequences, which is a non-trivial fact in view of the non-compactness of the Sobolev embedding $H_g^1(M) \hookrightarrow L_g^{2^*}(M)$. If (M, g) is not conformally equivalent to $(\mathbb{S}^N, g_0)$ (which has already constant Scalar curvature) with $\Upsilon_g(M) > 0$, the Yamabe equation has been solved via (2) by Aubin [2] in the non-locally conformally flat case with $N \ge 6$, by exploiting the non-vanishing of the Weyl curvature tensor Weyl_g of (M, g) in the construction of local test functions, and by Schoen [17] when either $N = 3, 4, 5$ or $(M, g) \ne (\mathbb{S}^N, g_0)$ is locally conformally flat, by exploiting the Positive Mass Theorem by Schoen–Yau [19], [20] in the construction of global test functions.

In this paper, we study the case when the manifold (M, g) has positive Yamabe invariant, i.e. $\Upsilon_g(M) > 0$, and the problem (1) is almost critical, i.e. $\varepsilon \ne 0$ is small. In particular, we are interested in the existence of blowing-up solution to (1) as $\varepsilon \to 0$. We say that a family of solutions $(u_\varepsilon)_\varepsilon$ of equation (1) blows-up at a point $\xi_0 \in M$ if there exists a family of points $(\xi_\varepsilon)_\varepsilon$ in M such that $\xi_\varepsilon \to \xi_0$ and $u_\varepsilon(\xi_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. The question on whether solutions of equations like (1) with $\varepsilon \to 0$ blow-up or not has been extensively studied in recent years. Schoen [18] proved that blow-up cannot occur when the manifold is locally conformally

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flat and not conformally equivalent to $(\mathbb{S}^N, g_0)$ provided $\varepsilon \nearrow 0$. More precisely, Schoen proved that sequences of solutions $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ of (1) with exponents $2^* - 1 + \varepsilon_k$ with $\varepsilon_k \le 0$ and $\varepsilon_k \to 0$, are pre-compact in $C^{2,\alpha}(M)$, $\alpha \in (0,1)$, and the non-locally conformally flat case was left open. Known as the Compactness conjecture, it has been finally proved by Khuri–Marques–Schoen [9] when $N \le 24$. Unexpectedly, compactness of Yamabe metrics ($\varepsilon = 0$) has revealed to be false in general in dimensions $N \ge 25$ by Brendle [4] and Brendle–Marques [5]. Previous contributions where the compactness of Yamabe metrics is proved in lower dimensions are by Li–Zhu [14] ($N = 3$), Druet [6] ($N \le 5$), Marques [15] ($N \le 7$), and Li–Zhang [11], [12], [13] ($N \le 11$).

In the present paper, we prove that if the exponent in (1) approaches the critical exponent from above, i.e. $\varepsilon \searrow 0$, then compactness is not true anymore. More precisely, we prove the following result.

Theorem 1.1. *Let (M, g) be a smooth, compact, non-locally conformally flat, Riemannian manifold with $N \ge 7$ and $\Upsilon_g(M) > 0$. Then for $\varepsilon > 0$ small, equation* (1) *has a solution u_ε such that the family $(u_\varepsilon)_\varepsilon$ blows-up, up to a sub-sequence, as $\varepsilon \to 0$ at some point ξ_0 so that $|\mathrm{Weyl}_g(\xi_0)|_g = \max_{\xi\in M} |\mathrm{Weyl}_g(\xi)|_g$.*

Theorem 1.1 is an immediate consequence of the following more general result:

Theorem 1.2. *Assume that there exists a C^1-stable critical set $\mathscr{D}$ of $\xi \to |\mathrm{Weyl}_g(\xi)|_g$ such that $\inf\{|\mathrm{Weyl}_g(\xi)|_g : \xi \in \mathscr{D}\} > 0$. Then for $\varepsilon > 0$ small, equation* (1) *has a solution u_ε such that the family $(u_\varepsilon)_\varepsilon$ blows up, up to a sub-sequence, at some $\xi_0 \in \mathscr{D}$ as $\varepsilon \to 0$.*

According to Li [10], given a C^1-function Φ on M, we say that a compact set $\mathscr{D} \subset M$ of critical points of Φ is a *C^1-stable critical set* of Φ if, for any compact neighborhood U of $\mathscr{D}$ in M and for any sequence of C^1-functions Φ_ε on M such that $\|\Phi_\varepsilon - \Phi\|_{C^1(U)} \to 0$ as $\varepsilon \to 0$, there exists $\xi_\varepsilon \in U$ critical point of Φ_ε if ε is small enough. We remark that a set of strict local maximum/minimum points or a set of non degenerate critical points are *C^1-stable*.

We prove the existence of blowing-up solutions by the well known Lyapunov-Schmidt reduction. The main point is to produce a suitable ansatz for the solutions we are looking for. This is done in Section 2. A similar idea has been already used by Esposito–Pistoia–Vétois in [7], [8]. In Section 3, we reduce the problem to a finite dimensional one, we study the critical points of the corresponding finite dimensional functional, i.e. the reduced energy, and we prove Theorem 1.2. A key step is the asymptotic expansion of the reduced energy, which is performed in Section 4.

2. Setting of the problem

2.1. Notations. Since $\Upsilon_g(M) > 0$, the conformal laplacian $\mathscr{L}_g := -\Delta_g + \beta_N \operatorname{Scal}_g$, $\beta_N := \frac{N-2}{4(N-1)}$, is coercive and we can provide the Sobolev space $H_g^1(M)$ with the inner product

$$\langle u, v\rangle = \int_M \langle \nabla u, \nabla v\rangle_g \, dv_g + \int_M \beta_N \operatorname{Scal}_g uv \, dv_g,$$

where dv_g is the volume element of the manifold. We let $\|\cdot\|$ be the norm induced by $\langle \cdot, \cdot\rangle$. Moreover, for any function u in $L^q(M)$, we denote the L^q-norm of u by $|u|_q = (\int_M |u|^q \, dv_g)^{1/q}$.

We let $i^* : L^{2N/(N+2)}(M) \to H_g^1(M)$ be the adjoint operator of the embedding $i : H_g^1(M) \hookrightarrow L^{2N/(N-2)}(M)$, i.e. for any w in $L^{2N/(N+2)}(M)$, the function $u = i^*(w)$ in $H_g^1(M)$ is the unique solution of the equation $\mathscr{L}_g u = w$ in M. By the continuity of the embedding of $H_g^1(M)$ into $L^{2N/(N-2)}(M)$, we get

$$\|i^*(w)\| \le C|w|_{2N/(N+2)} \quad \text{or equivalently} \quad \|u\| \le C|\mathscr{L}_g u|_{2N/(N+2)} \tag{3}$$

for some positive constant C which only depends on N. In order to study the supercritical case, it is also useful to recall that by standard elliptic estimates (see for example [16]) given a real number $s > \frac{2N}{N-2}$, i.e. $\frac{Ns}{N+2s} > \frac{2N}{N+2}$, for any $w \in L^{Ns/(N+2s)}(M)$ the function $i^*(w) \in L^s(M)$ and satisfies

$$|i^*(w)|_s \le C|w|_{Ns/(N+2s)} \quad \text{or equivalently} \quad |u|_s \le C|\mathscr{L}_g u|_{Ns/(N+2s)}, \tag{4}$$

for some positive constant C which only depends on N. Therefore, if ϵ is small enough, we set $s_\epsilon := 2^* + \frac{N}{2}\epsilon$ and we let $H_\epsilon := H_g^1(M) \cap L^{s_\epsilon}(M)$ be the Banach space equipped with the norm $\|u\|_\epsilon := \|u\| + |u|_{s_\epsilon}$. Taking into account that $\frac{Ns_\epsilon}{N+2s_\epsilon} = \frac{s_\epsilon}{2^*-1+\epsilon}$ and also that (3) and (4) hold, we can rewrite problem (1) as

$$u = i^*[f_\epsilon(u)], \qquad u \in H_\epsilon \tag{5}$$

where we set $f_\epsilon(u) := u_+^{p+\epsilon}$, $p := \frac{N+2}{N-2}$ and $u_+ := \max\{0, u\}$.

2.2. The bubbles. The main ingredient in the construction of the solution to problem (1) are the standard bubbles

$$U_{\mu,y}(x) := \mu^{-(N-2)/2} U\left(\frac{x-y}{\mu}\right), \qquad \mu > 0, \ y \in \mathbb{R}^N, \tag{6}$$

where

$$U(x) := \alpha_N \frac{1}{(1+|x|^2)^{(N-2)/2}}, \qquad \alpha_N := [N(N-2)]^{(N-2)/4}.$$

As it is well known (see [1], [21]), they are all the positive solutions of the equation $-\Delta u = u^p$ in $\mathbb{R}^N$.

Unfortunately, the standard bubble is not a good approximation of the solution we are looking for, so we have to improve the approximation in the following way. It is well known (see [3]) that any solutions of the linear equation $-\Delta v = pU^{p-1}v$ in $\mathbb{R}^N$, is a linear combination of the functions

$$Z^0(x) = x \cdot \nabla U(x) + \frac{N-2}{2} U(x) = \alpha_N \frac{N-2}{2} \frac{1-|x|^2}{(1+|x|^2)^{N/2}} \tag{7}$$

and

$$Z^i(x) = \partial_i U(x) = -(N-2)\alpha_N \frac{x_i}{(1+|x|^2)^{N/2}}, \qquad \text{for } i = 1, \dots, N. \tag{8}$$

Straightforward computations show that

(i) the function

$$w(x) = -\frac{\alpha_N^p(|x|^4+3)}{2N(N+2)(N-2)(1+|x|^2)^{N/2}}, \tag{9}$$

solves

$$-\Delta w - pU^{p-1}w = U - \zeta Z^0 \quad \text{in } \mathbb{R}^N, \quad \zeta := \frac{1}{\|Z^0\|^2_{\mathrm{L}^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} U(x) Z^0(x)\, dx \tag{10}$$

(ii) the function

$$v_{kl}(x) = \frac{\alpha_N^p\big(2\delta_{kl}(|x|^4+3) - (N+2)x_k x_l(|x|^2+3)\big)}{4N(N+2)(N-2)(1+|x|^2)^{N/2}}, \tag{11}$$

solves

$$\begin{aligned} &-\Delta v_{lk} - pU^{p-1}v_{lk} = x_l \partial_k U - \zeta_{lk} Z^0 \quad \text{in } \mathbb{R}^N, \\ &\zeta_{lk} := \frac{1}{\|Z^0\|^2_{\mathrm{L}^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} x_l \partial_k U(x) Z^0(x)\, dx \end{aligned} \tag{12}$$

(iii) the function

$$
\begin{aligned}
z_{abij}(x) = \alpha_N^p \big(&2(\delta_{ab}\delta_{ij} - 2\delta_{ai}\delta_{bj} - 2\delta_{aj}\delta_{bi})(|x|^4 + 3) \\
&+ (N+2)(\delta_{ai}x_b x_j + \delta_{aj}x_b x_i + \delta_{bi}x_a x_j + \delta_{bj}x_a x_i \\
&+ \delta_{ab}x_i x_j - 2\delta_{ij}x_a x_b)(|x|^2 + 3) \\
&+ 2(N+2)(N-2)x_a x_b x_i x_j\big) / \big(12N(N+2)(N-2)(1+|x|^2)^{N/2}\big), \quad (13)
\end{aligned}
$$

solves

$$
\begin{aligned}
&-\Delta z_{abij} - pU^{p-1} z_{abij} = x_a x_b \partial_{ij}^2 U - \zeta_{abij} Z^0 \quad \text{in } \mathbb{R}^N, \\
&\zeta_{abij} := \frac{1}{\|Z^0\|^2_{L^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} x_a x_b \partial_{ij}^2 U(x) Z^0(x)\, dx. \qquad (14)
\end{aligned}
$$

Here the notation δ_{st} stands for the Kronecker symbols. Defining the function V as

$$
V = -\frac{1}{3}\mathrm{R}_{iabj}(\xi) z_{abij} - \partial_l \Gamma^k_{ss}(\xi) v_{kl} - \beta_N \, \mathrm{Scal}_g(\xi) w + \zeta^* Z_0,
$$

where

$$
\zeta^* = \frac{1}{\|Z_0\|_2^2} \int_{\mathbb{R}^N} \left(\frac{1}{3}\mathrm{R}_{iabj}(\xi) z_{abij} + \partial_l \Gamma^k_{ss}(\xi) v_{kl} + \beta_N \, \mathrm{Scal}_g(\xi) w \right) Z_0 \, dx,
$$

then V is a solution to (see (10), (12) and (14))

$$
\begin{aligned}
-\Delta V - pU^{p-1} V = -\frac{1}{3} R_{iabj}(\xi) x_a x_b \partial_{ij}^2 U - \partial_l \Gamma^k_{ss}(\xi) x_l \partial_k U - \beta_N \, \mathrm{Scal}_g(\xi) U \\
+ \gamma Z^0 \quad \text{in } \mathbb{R}^N,
\end{aligned}
$$

where

$$
\gamma := \frac{1}{3} R_{abij}(\xi) \zeta_{abij} + \partial_l \Gamma^k_{a,a}(\xi) \zeta_{lk} + \beta_N \, \mathrm{Scal}_g(\xi) \zeta. \qquad (15)
$$

Remark 2.1. In [8] a similar construction is performed. Thanks to some symmetries properties, it is shown there that $\gamma = 0$ and the function V can be reduced to a simpler expression. The computations here are more direct and might be useful in other contexts where such symmetry properties might not be available. In particular, we aim to emphasize the fact that the condition $\gamma = 0$ is helpful but not really necessary in the construction.

Finally, we point out that the function

$$V_{\mu,y}(x) := \mu^{-(N-2)/2} V\left(\frac{x-y}{\mu}\right), \qquad \text{with } \mu > 0 \text{ and } y \in \mathbb{R}^N \tag{16}$$

solves

$$\begin{aligned} &\mu^2[-\Delta V_{\mu,y} - pU_{\mu,y}^{p-1} V_{\mu,y}] \\ &\quad = -\frac{1}{3} R_{iabj}(\xi)(x-y)_a(x-y)_b \partial_{ij}^2 U_{\mu,y} - \partial_l \Gamma_{ss}^k(\xi)(x-y)_l \partial_k U_{\mu,y} \\ &\qquad - \beta_N \operatorname{Scal}_g(\xi) U_{\mu,y} + \gamma Z_{\mu,y}^0 \qquad \text{in } \mathbb{R}^N, \end{aligned} \tag{17}$$

where $Z_{\mu,y}^0(x) := \mu^{-(N-2)/2} Z^0\left(\frac{x-y}{\mu}\right)$.

2.3. The ansatz. We let r_0 be a positive real number less than the injectivity radius of M, and χ be a smooth cutoff function such that $0 \le \chi \le 1$ in $\mathbb{R}$, $\chi \equiv 1$ in $[-r_0/2, r_0/2]$, and $\chi \equiv 0$ out of $[-r_0, r_0]$. For any point ξ in M and for any positive real number μ, we define the functions $\mathscr{U}_{\mu,\xi}$ and $\mathscr{V}_{\mu,\xi}$ on M by

$$\mathscr{U}_{\mu,\xi}(z) = \chi\big(d_g(z,\xi)\big) U_\mu\big(\exp_\xi^{-1}(z)\big), \qquad \mathscr{V}_{\mu,\xi}(z) = \chi\big(d_g(z,\xi)\big) V_\mu\big(\exp_\xi^{-1}(z)\big), \tag{18}$$

where d_g is the geodesic distance on M with respect to the metric g and the functions $U_\mu := U_{\mu,0}$ and $V_\mu := V_{\mu,0}$ are defined in (6) and (16), respectively.

We look for solutions of equation (1) or equivalently of (5) of the form

$$u_\varepsilon(z) = \mathscr{W}_{\mu,\xi}(z) + \phi_\varepsilon(z), \qquad \mathscr{W}_{\mu,\xi} := \mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi}. \tag{19}$$

Here the concentration point ξ belongs to M and the concentration parameter μ satisfies

$$\mu = d\sqrt[4]{\varepsilon} \qquad \text{with } d > 0. \tag{20}$$

The remainder term ϕ_ε is an higher order term which belongs to the following space.

For any point ξ in M and for any positive real number μ, we introduce the functions

$$\mathscr{Z}_{\mu,\xi}^i(z) = \chi\big(d_g(z,\xi)\big) Z_\mu^i\big(\exp_\xi^{-1}(z)\big) \qquad \text{for } i = 0, \dots, N,$$

where $Z_\mu^i(x) := \mu^{-(N-2)/2} Z^i(x/\mu)$ are defined in (7) and (8). We then define the projections $\Pi_{\mu,\xi}$ and $\Pi_{\mu,\xi}^\perp$ of the Sobolev space $H_g^1(M)$ onto the respective subspaces

$$K_{\mu,\xi} = \operatorname{Span}\{i^*(\mathscr{Z}_{\mu,\xi}^0), \dots, i^*(Z_{\mu,\xi}^N)\}$$

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and

$$K_{\mu,\xi}^{\perp} = \{\phi \in H_g^1(M) : \langle \phi, i^*(\mathscr{Z}_{\mu,\xi}^i) \rangle = 0,\ i = 0, \dots, N\}.$$

Therefore, equation (5) turns out to be equivalent to the system

$$\begin{cases} \Pi_{\mu,\xi}^{\perp}\{u_\varepsilon - i^*[f_\varepsilon(u_\varepsilon)]\} = 0 \\ \Pi_{\mu,\xi}\{u_\varepsilon - i^*[f_\varepsilon(u_\varepsilon)]\} = 0, \end{cases} \tag{21}$$

where u_ε is given in (19).

3. The finite dimensional reduction

3.1. The error estimate. Let

$$E_{\mu,\xi} := \Pi_{\mu,\xi}^{\perp}\{\mathscr{W}_{\mu,\xi} - i^*[f_\varepsilon(\mathscr{W}_{\mu,\xi})]\}. \tag{22}$$

Lemma 3.1. *Let $N \geq 7$. If μ is as in* (20), *then for any real numbers a and b satisfying $0 < a < b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any point ξ in M, and any real number d in $[a,b]$, there holds*

$$\|E_{\mu,\xi}\|_\varepsilon \leq C_{a,b}\varepsilon^{\eta/4}.$$

for $\eta > 2$.

Proof. First of all, by (3) and (4), taking into account that $\Pi_{\mu,\xi}^{\perp}[i^*(\gamma\mathscr{Z}_{\mu,\xi}^0)] = 0$, we have

$$\begin{aligned}
\|E_{\mu,\xi}\|_\varepsilon &\leq c\big(\|\mathscr{W}_{\mu,\xi} - i^*[f_\varepsilon(\mathscr{W}_{\mu,\xi}) + \gamma\mathscr{Z}_{\mu,\xi}^0]\| + |\mathscr{W}_{\mu,\xi} - i^*[f_\varepsilon(\mathscr{W}_{\mu,\xi}) + \gamma\mathscr{Z}_{\mu,\xi}^0]|_{s_\varepsilon}\big) \\
&\leq c|\mathscr{L}_g(\mathscr{W}_{\mu,\xi}) - f_\varepsilon(\mathscr{W}_{\mu,\xi}) - \gamma\mathscr{Z}_{\mu,\xi}^0|_{2N/(N+2)} \\
&\quad + c|\mathscr{L}_g(\mathscr{W}_{\mu,\xi}) - f_\varepsilon(\mathscr{W}_{\mu,\xi}) - \gamma\mathscr{Z}_{\mu,\xi}^0|_{Ns_\varepsilon/(N+2s_\varepsilon)} \\
&\leq c\big(|\mathscr{L}_g(\mathscr{W}_{\mu,\xi}) - \gamma\mathscr{Z}_{\mu,\xi}^0 - f_\varepsilon(\mathscr{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mathscr{V}_{\mu,\xi}|_{2N/(N+2)} \\
&\qquad + |f_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2\mathscr{V}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mathscr{V}_{\mu,\xi}|_{2N/(N+2)}\big) \\
&\quad + c\big(|\mathscr{L}_g(\mathscr{W}_{\mu,\xi}) - \gamma\mathscr{Z}_{\mu,\xi}^0 - f_\varepsilon(\mathscr{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mathscr{V}_{\mu,\xi}|_{Ns_\varepsilon/(N+2s_\varepsilon)} \\
&\qquad + |f_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2\mathscr{V}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mathscr{V}_{\mu,\xi}|_{Ns_\varepsilon/(N+2s_\varepsilon)}\big).
\end{aligned}$$

We have used here that $|\Pi_{\mu,\xi}^{\perp}u|_{s_\varepsilon} \leq |u|_{s_\varepsilon} + |\Pi_{\mu,\xi}u|_{s_\varepsilon} \leq |u|_{s_\varepsilon} + C\|\Pi_{\mu,\xi}u\| \leq |u|_{s_\varepsilon} + C\|u\|$, $C > 0$, for all $u \in H_g^1(M)$ since $|\cdot|_{s_\varepsilon}$ and $\|\cdot\|$ are equivalent norms on the finite-dimensional subspace $K_{\mu,\xi}$. Moreover, it is possible to show that the constant $C > 0$ can be choosen uniformly in ε, ξ and δ.

It is useful to point out that

$$|x|\,|U(x)| + |x|^2|\partial_k U(x)| + |x|^3|\partial^2_{ij}U(x)| \le c\frac{1}{(1+|x|^2)^{(N-3)/2}}, \qquad x \in \mathbb{R}^N \quad (23)$$

and, by (9), (11) and (13), also that

$$|V(x)| + |x|\,|\partial_k V(x)| + |x|^2|\partial^2_{ij}V(x)| \le c\frac{1}{(1+|x|^2)^{(N-4)/2}}, \qquad x \in \mathbb{R}^N. \quad (24)$$

for some constant c.

Now, by standard properties of the exponential map, in geodesic normal coordinates, there hold

$$-\Delta_g u = -\Delta u - (g^{ij} - \delta^{ij})\partial^2_{ij}u + g^{ij}\Gamma^k_{ij}\partial_k u, \quad (25)$$

$$g^{ij}(x) = \delta^{ij}(x) - \frac{1}{3}R_{i\alpha\beta j}(\xi)x^\alpha x^\beta + O(|x|^3), \quad (26)$$

and

$$g^{ij}(x)\Gamma^k_{ij}(x) = \partial_l\Gamma^k_{ii}(\xi)x^l + O(|x|^2) \quad (27)$$

as $x \to \xi$.

In normal coordinates using (25), (26) and (27) and by the choice of γ in (15), we get

$$\begin{aligned}
&\mathscr{L}_g(\mathscr{U}_{\mu,\xi} + \mu^2\mathscr{V}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mathscr{V}_{\mu,\xi} - \gamma\mathscr{Z}^0_{\mu,\xi} \\
&\quad = \chi\big(-\Delta_g U_\mu + \beta_N \operatorname{Scal}_g U_\mu - \mu^2\Delta_g V_\mu + \beta_N \operatorname{Scal}_g \mu^2 V_\mu \\
&\qquad\quad - f_\varepsilon(U_\mu) - \mu^2 f_\varepsilon'(U_\mu)V_\mu - \gamma Z^0_\mu\big) + r_1(x) + r_2(x) \\
&\quad = \chi\underbrace{\big(-\Delta U_\mu - f_0(U_\mu)\big)}_{=0} \\
&\qquad + \chi\underbrace{\Big[-\mu^2\big(\Delta V_\mu + f_0'(U_\mu)V_\mu\big) - \gamma Z^0_\mu + \frac{1}{3}R_{iabj}(\xi)x^a x^b\partial^2_{ij}U_\mu + \partial_l\Gamma^k_{aa}(\xi)x^l\partial_k U_\mu + \beta_N\operatorname{Scal}_g(\xi)U_\mu\Big]}_{=0 \text{ because of } (17)} \\
&\qquad - \chi\big(f_\varepsilon(U_\mu) - f_0(U_\mu)\big) - \mu^2\chi[f_\varepsilon'(U_\mu) - f_0'(U_\mu)]V_\mu \\
&\qquad + \chi O(|x|^3|\partial^2_{ij}U_\mu| + |x|^2|\partial_k U_\mu| + |x|U_\mu) \\
&\qquad + \mu^2\chi O(|x|^2|\partial^2_{ij}V_\mu| + |x|\,|\partial_k V_\mu| + V_\mu) + r_1(x) + r_2(x), \qquad (28)
\end{aligned}$$

where (setting $\chi(x) = \chi(|x|)$)

$$r_1(x) := -U_\mu \Delta_g \chi - 2\langle \nabla U_\mu, \nabla \chi \rangle_g - (\chi^{p+\varepsilon} - \chi) f_\varepsilon(U_\mu) \tag{29}$$

$$r_2(x) := -\mu^2 [V_\mu \Delta_g \chi + 2\langle \nabla V_\mu, \nabla \chi \rangle_g] - \mu^2 (\chi^{p+\varepsilon} - \chi) f_\varepsilon'(U_\mu) V_\mu. \tag{30}$$

We only estimate the $|\cdot|_{2N/(N+2)}$-norm, since the estimate of the $|\cdot|_{Ns_\varepsilon/(N+2s_\varepsilon)}$-norm follows in the same way. First, by (20) we deduce

$$\begin{aligned} &\big|\chi\big(f_\varepsilon(U_\mu) - f_0(U_\mu)\big)\big|_{2N/(N+2)} \\ &\quad \le c\big|(d^{-((N-2)/2)\varepsilon} e^{-((N-2)/8)\varepsilon \ln \varepsilon} U^\varepsilon - 1) U^p\big|_{\mathrm{L}^{2N/(N+2)}(\mathbb{R}^N)} = O(\varepsilon |\ln \varepsilon|) \end{aligned} \tag{31}$$

and

$$\begin{aligned} &\big|\chi\big(f_\varepsilon'(U_\mu) - f_0'(U_\mu)\big) V_\mu\big|_{2N/(N+2)} \\ &\quad \le |f_\varepsilon'(U_\mu) - f_0'(U_\mu)|_{N/2} |V_\mu|_{2N/(N-2)} \\ &\quad \le c\big|\big((p+\varepsilon) d^{-((N-2)/2)\varepsilon} e^{-((N-2)/8)\varepsilon \ln \varepsilon} U^\varepsilon - p\big) U^{p-1}\big|_{\mathrm{L}^{N/2}(\mathbb{R}^N)} = O(\varepsilon |\ln \varepsilon|) \end{aligned} \tag{32}$$

Moreover, by (23) we deduce

$$|O(|x|^3 |\partial_{ij}^2 U_\mu| + |x|^2 |\partial_k U_\mu| + |x| U_\mu)|_{2N/(N+2)} = \begin{cases} O(\mu^{5/2}) & \text{if } N = 7, \\ O(\mu^3 |\ln \mu|^{5/8}) & \text{if } N = 8, \\ O(\mu^3) & \text{if } N \ge 9 \end{cases} \tag{33}$$

and by (24) we deduce

$$|O(|x|^2 |\partial_{ij}^2 V_\mu| + |x|\, |\partial_k V_\mu| + V_\mu)|_{2N/(N+2)} = \begin{cases} O(\mu^{(N-6)/2}) & \text{if } 7 \le N \le 9, \\ O(\mu^2 |\ln \mu|^{3/5}) & \text{if } N = 10, \\ O(\mu^2) & \text{if } N \ge 11. \end{cases} \tag{34}$$

We also remark that $r_i(x) = 0$ if $|x| \le r_0$ for each $i = 1, 2, 3$. Therefore, by (23)–(24) we get

$$\|r_1\|_{2N/(N+2)} + \|r_2\|_{2N/(N+2)} = O(\mu^{(N-2)/2}). \tag{35}$$

Inserting (31)–(35) into (28), by the choice of μ in (20) we deduce that

$$\begin{aligned} &|\mathscr{L}_g(\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathscr{U}_{\mu,\xi}) \mathscr{V}_{\mu,\xi} - \gamma \mathscr{Z}_{\mu,\xi}^0|_{2N/(N+2)} \\ &\quad = O(\varepsilon^{5/8} |\ln \varepsilon|^{5/8}). \end{aligned} \tag{36}$$

Since $p + \varepsilon < 1$ for ε small and $N \geq 7$, we have the validity of

$$(a + b)_+^{p+\varepsilon} - a^{p+\varepsilon} - (p + \varepsilon)a^{p+\varepsilon-1}b = O(|b|^{p+\varepsilon}) \tag{37}$$

for all $a \geq 0$ and $b \in \mathbb{R}$, yielding to

$$\begin{aligned} &|f_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2\mathscr{V}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mathscr{V}_{\mu,\xi}|_{2N/(N+2)} \\ &\quad \leq c|\mu^2\mathscr{V}_{\mu,\xi}|^{p+\varepsilon}_{(2N/(N+2))(p+\varepsilon)} \\ &\quad = O(\mu^{2p-\varepsilon((N-6)/2)}). \end{aligned} \tag{38}$$

By choosing $2 < \eta < \min\{\frac{5}{2}, 2p\}$, the claim follows by collecting all the previous estimates in view of (20). □

3.2. The remainder term ϕ. For ε small, for any $\mu > 0$ and any point ξ in M, we introduce the linear map $L_{\mu,\xi} : K_{\mu,\xi}^\perp \to K_{\mu,\xi}^\perp$ defined by

$$L_{\mu,\xi}(\phi) = \Pi_{\mu,\xi}^\perp\{\phi - i^*[f_\varepsilon'(\mathscr{W}_{\mu,\xi})\phi]\}. \tag{39}$$

Lemma 3.2. *If μ is as in* (20)*, then for any real numbers a and b satisfying $0 < a < b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any point ξ in M, any real number d in $[a,b]$, and any function ϕ in $K_{\mu,\xi}^\perp$, there holds*

$$\|L_{\mu,\xi}(\phi)\|_\varepsilon \geq C_{a,b}\|\phi\|_\varepsilon.$$

Proof. We argue exactly as in Lemma 3.1 of [16]. □

Proposition 3.1. *Let $N \geq 7$. If μ is as in* (20)*, then for any real numbers a and b satisfying $0 < a < b$, there exists a positive constant $C_{a,b}$ such that for ε small, for any point ξ in M, and for any real number d in $[a,b]$, the first equation in the system* (21) *admits a unique solution $\phi_{\varepsilon,\mu,\xi}$ in $K_{\mu,\xi}^\perp$, which is continuously differentiable with respect to ξ and d, such that*

$$\|\phi_{\varepsilon,\mu,\xi}\| \leq C_{a,b}\varepsilon^{\eta/4} \tag{40}$$

for some $\eta > 2$.

Proof. We use a standard contraction mapping argument. For ε small, for any $\xi \in M$ and any $\mu > 0$ let $T_{\mu,\xi} : K_{\mu,\xi}^\perp \to K_{\mu,\xi}^\perp$ be defined by

$$T_{\mu,\xi}(\phi) := L_{\mu,\xi}^{-1}\big(N_{\mu,\xi}(\phi) - E_{\mu,\xi}\big),$$

where $L_{\mu,\xi}$ is defined in (39), $E_{\mu,\xi}$ is defined in (22) and

$$N_{\mu,\xi}(\phi) := \Pi_{\mu,\xi}^\perp\{i^*[f_\varepsilon(\mathscr{W}_{\mu,\xi} + \phi) - f_\varepsilon(\mathscr{W}_{\mu,\xi}) - f_\varepsilon'(\mathscr{W}_{\mu,\xi})\phi]\}.$$

By (37) we deduce that

$$\|N_{\mu,\xi}(\phi)\|_{\varepsilon} \le c\|\phi\|_{\varepsilon}^{p+\varepsilon}$$

in view of $\frac{2N}{N+2}(p+\varepsilon) < s_\varepsilon$. Similarly, since

$$\begin{aligned}(a+b_1)_+^{p+\varepsilon} - (a+b_2)_+^{p+\varepsilon} - (p+\varepsilon)a^{p+\varepsilon-1}(b_1-b_2)\\ = O(|b_1-b_2|^{p+\varepsilon} + |b_2|^{p+\varepsilon-1}|b_1-b_2|)\end{aligned}$$

for all $a \ge 0$ and $b_1, b_2 \in \mathbb{R}$, we get that

$$\|N_{\mu,\xi}(\phi_2) - N_{\mu,\xi}(\phi_1)\|_{\varepsilon} \le c(\|\phi_2-\phi_1\|_{\varepsilon}^{p+\varepsilon} + \|\phi_2\|_{\varepsilon}^{p+\varepsilon-1}\|\phi_2-\phi_1\|_{\varepsilon}).$$

Notice that $\mathscr{W}_{\mu,\xi} \ge 0$ by taking δ_0 sufficiently small. Using Lemmas 3.1 and 3.2, it is easy to show that, if ε is small enough, $T_{\mu,\xi}$ is a contraction mapping from the ball $\{\phi \in K_{\mu,\xi}^{\perp} : \|\phi\|_{\varepsilon} \le C\varepsilon^{\eta/4}\}$ into itself, provided C is large enough. The proof is concluded. □

3.3. The reduced problem. Let $J_\varepsilon : \mathrm{H}_\varepsilon \to \mathbb{R}$ be defined by

$$J_\varepsilon(u) := \frac{1}{2}\int_M |\nabla_g u|^2\,dv_g + \frac{1}{2}\int_M \beta_N\,\mathrm{Scal}_g\,u^2\,dv_g - \frac{1}{p+\varepsilon+1}\int_M u_+^{p+\varepsilon+1}\,dv_g.$$

Its critical points are the solutions of equation (1). We also define the reduced energy $\tilde{J}_\varepsilon : (0,+\infty)\times M \to \mathbb{R}$ by

$$\tilde{J}_\varepsilon(d,\xi) = J_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2\mathscr{V}_{\mu,\xi} + \phi_{\varepsilon,\mu,\xi}),$$

where $\mathscr{U}_{\mu,\xi}$ and $\mathscr{V}_{\mu,\xi}$ are given in (18) and where $\phi_{\varepsilon,\mu,\xi}$ is given by Proposition 3.1.

Proposition 3.2. (i) *If $(d_\varepsilon,\xi_\varepsilon) \in [a,b]\times M$ is a critical point of the function $\tilde{J}_\varepsilon$, then $\mathscr{U}_{\mu_\varepsilon,\xi_\varepsilon} + \mu_\varepsilon^2\mathscr{V}_{\mu_\varepsilon,\xi_\varepsilon} + \phi_{\varepsilon,\mu_\varepsilon,\xi_\varepsilon}$ is a solution of* (1).

(ii) *If μ is as in* (20)*, then for any real numbers a and b satisfying $0 < a < b$, there holds*

$$\tilde{J}_\varepsilon(d,\xi) = c_N + d_N\varepsilon + e_N\varepsilon\ln\varepsilon + \varepsilon\underbrace{[-a_N|\mathrm{Weyl}_g(\xi)|_g^2 d^4 + b_N\ln d]}_{\Theta(d,\xi)} + o(\varepsilon), \quad (41)$$

C^1-uniformly with respect to ξ in M and to d in $[a,b]$. Here $a_N,\dots,e_N$ are constants which only depend on N, with $a_N, b_N > 0$.

Proof. (i) follows arguing exactly as in Proposition 2.2 of [16]. The C^0-estimate in (ii) is proved in Section 4. The C^1-estimate follows using similar arguments as in Section 4 of [16]. □

3.4. Proof of Theorem 1.2. Let $\mathscr{D}$ be the C^1-stable critical set of the function $\xi \to |\mathrm{Weyl}_g(\xi)|_g$ such that $|\mathrm{Weyl}_g(\xi)|_g \neq 0$ for any $\xi \in \overline{\mathscr{D}}$. Then, for any $\xi \in \mathscr{D}$ there exists a unique $d(\xi) \in [a, b]$, for some uniform $0 < a < b$, such that $\partial_d \Theta(d, \xi) = 0$. It is not difficult to check that the set $\tilde{\mathscr{D}} := \{(d(\xi), \xi) \mid \xi \in \mathscr{D}\}$ is a C^1-stable critical set of the function Θ. Therefore, by (ii) of Proposition 3.2, if ε is small enough there exists $\xi_\varepsilon \in \mathscr{D}$ such that $\mathrm{dist}(\xi_\varepsilon, \mathscr{D}) \to 0$ as $\varepsilon \to 0$ and $(d(\xi_\varepsilon), \xi_\varepsilon)$ is a critical point of $\tilde{J}_\varepsilon$. Hence, by (i) of Proposition 3.2, we deduce that $u_\varepsilon = \mathscr{U}_{\mu_\varepsilon, \xi_\varepsilon} + \mu_\varepsilon^2 \mathscr{V}_{\mu_\varepsilon, \xi_\varepsilon} + \phi_{\varepsilon, \mu_\varepsilon, \xi_\varepsilon}$ is a solution of (1) which blows-up, up to a sub-sequence, at some $\xi_0 \in \mathscr{D}$ as $\varepsilon \to 0$. Finally, since $\mathscr{L}_g$ is coercive, the positivity of u_ε follows by the maximum principle.

4. The expansion of the reduced energy

The proof of (41) follows immediately by putting together estimates (45)–(47) and (58)–(60).

It is useful to introduce some notations. Set

$$K_N := \sqrt{\frac{4}{N(N-2)\omega_N^{2/N}}}.$$

For any positive real numbers p and q such that $p - q > 1$, we let

$$I_p^q = \int_0^{+\infty} \frac{r^q}{(1+r)^p}\, dr = 2 \int_0^{+\infty} \frac{s^{2q+1}}{(1+s^2)^p}\, ds. \tag{42}$$

In particular, there hold

$$I_{p+1}^q = \frac{p-q-1}{p} I_p^q \quad \text{and} \quad I_{p+1}^{q+1} = \frac{q+1}{p-q-1} I_{p+1}^q. \tag{43}$$

As it is easily checked, we get

$$I_N^{N/2} = \frac{N\omega_N}{2^{N-1}(N-2)\omega_{N-1}} = \frac{2K_N^{-N}}{\alpha_N^2 (N-2)^2 \omega_{N-1}}. \tag{44}$$

Step 1 We prove that

$$J_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi} + \phi_{\mu,\xi}) = J_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) + \mathrm{O}(\varepsilon^{\eta/2}). \tag{45}$$

Proof. It holds

$$\begin{aligned} J_\varepsilon(&\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi} + \phi_{\mu,\xi}) \\ &= J_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) \\ &\quad + \frac{1}{2}\int_M |\nabla_g \phi_{\mu,\xi}|^2 \, dv_g + \frac{1}{2}\int_M \beta_N \, \mathrm{Scal}_g \, \phi_{\mu,\xi}^2 \, dv_g \\ &\quad + \int_M [\mathcal{L}_g(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) - f_\varepsilon(\mathcal{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathcal{U}_{\mu,\xi})\mathcal{V}_{\mu,\xi}]\phi_{\mu,\xi} \, dv_g \\ &\quad - \int_M [f_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) - f_\varepsilon(\mathcal{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathcal{U}_{\mu,\xi})\mathcal{V}_{\mu,\xi}]\phi_{\mu,\xi} \, dv_g \\ &\quad - \int_M [F_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi} + \phi_{\mu,\xi}) - F_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) \\ &\qquad\quad - f_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi})\phi_{\mu,\xi}] \, dv_g, \end{aligned}$$

where $F_\varepsilon(u) := \frac{1}{p+1+\varepsilon} u_+^{p+1+\varepsilon}$ and $f_\varepsilon(u) = F_\varepsilon'(u)$. By (40) we get

$$\frac{1}{2}\int_M |\nabla_g \phi_{\mu,\xi}|^2 \, dv_g + \frac{1}{2}\int_M \beta_N \, \mathrm{Scal}_g \, \phi_{\mu,\xi}^2 \, dv_g = \mathrm{O}(\varepsilon^{\eta/2}).$$

If γ is defined as in (15), by (36), (40) and $\int_M \mathcal{Z}_{\mu,\xi}^0 \phi_{\mu,\xi} = 0$ we get

$$\begin{aligned} \Big| \int_M &[\mathcal{L}_g(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) - f_\varepsilon(\mathcal{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathcal{U}_{\mu,\xi})\mathcal{V}_{\mu,\xi}]\phi_{\mu,\xi} \, dv_g \Big| \\ &\le |\mathcal{L}_g(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) - f_\varepsilon(\mathcal{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathcal{U}_{\mu,\xi})\mathcal{V}_{\mu,\xi} - \gamma \mathcal{Z}_{\mu,\xi}^0|_{2N/(N+2)} \\ &\quad \times |\phi_{\mu,\xi}|_{2N/(N-2)} = \mathrm{O}(\varepsilon^{\eta/2}). \end{aligned}$$

By (38) and (40) we get

$$\begin{aligned} \Big| \int_M &[f_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) - f_\varepsilon(\mathcal{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathcal{U}_{\mu,\xi})\mathcal{V}_{\mu,\xi}]\phi_{\mu,\xi} \, dv_g \Big| \\ &\le c|f_\varepsilon(\mathcal{U}_{\mu,\xi} + \mu^2 \mathcal{V}_{\mu,\xi}) - f_\varepsilon(\mathcal{U}_{\mu,\xi}) - \mu^2 f_\varepsilon'(\mathcal{U}_{\mu,\xi})\mathcal{V}_{\mu,\xi}|_{2N/(N+2)} |\phi_{\mu,\xi}|_{2N/(N-2)} \\ &= \mathrm{O}(\varepsilon^{\eta/2}). \end{aligned}$$

Finally, by a Taylor expansion of F_ε we get

$$\int_M [F_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi} + \phi_{\mu,\xi}) - F_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi})\phi_{\mu,\xi}]\,dv_g$$
$$\le c\int_M (|\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi}|^{p+\varepsilon-1}|\phi_{\mu,\xi}| + |\phi_{\mu,\xi}|^{p+\varepsilon})|\phi_{\mu,\xi}|\,dv_g$$
$$\le c(|\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi}|_{p+1}^{p+\varepsilon-1}|\phi_{\mu,\xi}|^2_{(2(p+1))/(2-\varepsilon)} + |\phi_{\mu,\xi}|^{p+1+\varepsilon}_{p+1+\varepsilon}) = \mathrm{O}(\varepsilon^{\eta/2})$$

in view of $\frac{2(p+1)}{p-\varepsilon}$, $p+1+\varepsilon < s_\varepsilon$ for ε small and (40). Collecting all the previous estimates we get (45). □

Step 2 We prove that

$$J_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi}) = J_0(\mathscr{U}_{\mu,\xi}) - \int_M [F_\varepsilon(\mathscr{U}_{\mu,\xi}) - F_0(\mathscr{U}_{\mu,\xi})]\,dv_g$$
$$+ \int_M \Big\{\mu^2[\mathscr{L}_g \mathscr{U}_{\mu,\xi} - f_0(\mathscr{U}_{\mu,\xi})]$$
$$+ \frac{1}{2}\mu^4[\mathscr{L}_g \mathscr{V}_{\mu,\xi} - f_0'(\mathscr{U}_{\mu,\xi})\mathscr{V}_{\mu,\xi}]\Big\}\mathscr{V}_{\mu,\xi}\,dv_g$$
$$+ \mathrm{O}(\varepsilon^{7/5}). \tag{46}$$

Proof. It holds

$$J_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi}) = J_\varepsilon(\mathscr{U}_{\mu,\xi})$$
$$+ \mu^2 \int_M [\mathscr{L}_g \mathscr{U}_{\mu,\xi} - f_0(\mathscr{U}_{\mu,\xi})]\mathscr{V}_{\mu,\xi} + \mu^2 \int_M [f_0(\mathscr{U}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi})]\mathscr{V}_{\mu,\xi}$$
$$+ \frac{1}{2}\mu^4 \int_M [|\nabla_g \mathscr{V}_{\mu,\xi}|^2 - f_0'(\mathscr{U}_{\mu,\xi})\mathscr{V}^2_{\mu,\xi}]\,dv_g + \frac{1}{2}\mu^4 \int_M \beta_N \,\mathrm{Scal}_g\, \mathscr{V}^2_{\mu,\xi}\,dv_g$$
$$+ \frac{1}{2}\mu^4 \int_M [f_0'(\mathscr{U}_{\mu,\xi}) - f_\varepsilon'(\mathscr{U}_{\mu,\xi})]\mathscr{V}^2_{\mu,\xi}\,dv_g$$
$$- \int_M \Big[F_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2 \mathscr{V}_{\mu,\xi}) - F_\varepsilon(\mathscr{U}_{\mu,\xi})$$
$$- f_\varepsilon(\mathscr{U}_{\mu,\xi})\mu^2 \mathscr{V}_{\mu,\xi} - \frac{1}{2} f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mu^4 \mathscr{V}^2_{\mu,\xi}\Big]\,dv_g$$
$$= J_0(\mathscr{U}_{\mu,\xi}) - \int_M [F_\varepsilon(\mathscr{U}_{\mu,\xi}) - F_0(\mathscr{U}_{\mu,\xi})]\,dv_g$$
$$+ \int_M \Big\{\mu^2[\mathscr{L}_g \mathscr{U}_{\mu,\xi} - f_0(\mathscr{U}_{\mu,\xi})] + \frac{1}{2}\mu^4[\mathscr{L}_g \mathscr{V}_{\mu,\xi} - f_0'(\mathscr{U}_{\mu,\xi})\mathscr{V}_{\mu,\xi}]\Big\}\mathscr{V}_{\mu,\xi}\,dv_g$$
$$+ \mathrm{O}(\varepsilon^{7/5})$$

because by (31) we get

$$\mu^2\Big|\int_M [f_0(\mathscr{U}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi})]\mathscr{V}_{\mu,\xi}\,dv_g\Big| \\ \le c\mu^2|f_0(\mathscr{U}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi})|_{2N/(N+2)}|\mathscr{V}_{\mu,\xi}|_{2N/(N-2)} = \mathrm{O}(\varepsilon^{3/2}|\ln\varepsilon|)$$

and by (32) we get

$$\mu^4\Big|\int_M [f_0'(\mathscr{U}_{\mu,\xi}) - f_\varepsilon'(\mathscr{U}_{\mu,\xi})]\mathscr{V}_{\mu,\xi}^2\,dv_g\Big| \\ \le c\mu^4\big|\big(f_0'(\mathscr{U}_{\mu,\xi}) - f_\varepsilon'(\mathscr{U}_{\mu,\xi})\big)\mathscr{V}_{\mu,\xi}\big|_{2N/(N+2)}|\mathscr{V}_{\mu,\xi}|_{2N/(N-2)} = \mathrm{O}(\varepsilon^2|\ln\varepsilon|).$$

Moreover, since

$$\int_M\left[F_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2\mathscr{V}_{\mu,\xi}) - F_\varepsilon(\mathscr{U}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi})\mu^2\mathscr{V}_{\mu,\xi} - \frac{1}{2}f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mu^4\mathscr{V}_{\mu,\xi}^2\right] \\ = \int_M\int_0^1 [f_\varepsilon(\mathscr{U}_{\mu,\xi} + t\mu^2\mathscr{V}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi}) - f_\varepsilon'(\mathscr{U}_{\mu,\xi})t\mu^2\mathscr{V}_{\mu,\xi}]\mu^2\mathscr{V}_{\mu,\xi}\,dt$$

by (37) we deduce

$$\int_M\left|F_\varepsilon(\mathscr{U}_{\mu,\xi} + \mu^2\mathscr{V}_{\mu,\xi}) - F_\varepsilon(\mathscr{U}_{\mu,\xi}) - f_\varepsilon(\mathscr{U}_{\mu,\xi})\mu^2\mathscr{V}_{\mu,\xi} - \frac{1}{2}f_\varepsilon'(\mathscr{U}_{\mu,\xi})\mu^4\mathscr{V}_{\mu,\xi}^2\right|dv_g \\ \le c\int_M |\mu^2\mathscr{V}_{\mu,\xi}|^{p+1+\varepsilon}\,dv_g = O(\mu^{4N/(N-2)-((N-6)/2)\varepsilon}) = O(\varepsilon^{N/(N-2)}).$$

Collecting all the previous estimates we get (46). □

Step 3 We prove that

$$J_0(U_{\mu,\xi}) = c_1 + \big(-c_2|\mathrm{Weyl}_g(\xi)|_g^2 + c_3|\mathrm{E}_g(\xi)|_g^2 - c_4\,\mathrm{Scal}_g(\xi)^2\big)\mu^4 + O(\mu^5) \quad (47)$$

where

$$c_1 := \frac{K_N^{-N}}{N}, \qquad c_2 := \frac{K_N^{-N}}{24N(N-4)(N-6)}$$
$$c_3 := \frac{K_N^{-N}(2N-7)}{18N(N-2)(N-4)(N-6)}, \qquad c_4 := \frac{K_N^{-N}(N-2)(N-7)}{72N^2(N-1)(N-4)(N-6)}. \quad (48)$$

Here Weyl_g is the Weyl curvature of g and $\mathrm{E}_g = \mathrm{Ric}_g - \mathrm{Scal}_g\,g$ is the traceless part of the Ricci curvature of g.

Proof. There hold

$$\frac{1}{\omega_{N-1}r^{N-1}}\int_{\partial B_\xi(r)} \mathrm{Scal}_g\, d\sigma_g = \mathrm{Scal}_g(\xi) - \frac{1}{2N}\Lambda_g(\xi)r^2 + \mathrm{O}(r^4), \tag{49}$$

and

$$\frac{1}{\omega_{N-1}r^{N-1}}\int_{\partial B_\xi(r)} d\sigma_g = 1 - \frac{1}{6N}\,\mathrm{Scal}_g(\xi)r^2 + A_g(\xi)r^4 + \mathrm{O}(r^5), \tag{50}$$

as $r \to 0$, uniformly with respect to ξ, where $d\sigma_g$ is the volume element of $\partial B_\xi(r)$, ω_{n-1} is the volume of the unit $(N-1)$-sphere, and where

$$\Lambda_g(\xi) = \Delta_g\,\mathrm{Scal}_g(\xi) + \frac{1}{3}\,\mathrm{Scal}_g(\xi)^2 \tag{51}$$

and

$$A_g(\xi) = \frac{18\Delta_g\,\mathrm{Scal}_g(\xi) + 8|\mathrm{Ric}_g(\xi)|_g^2 - 3|\mathrm{Rm}_g(\xi)|_g^2 + 5\,\mathrm{Scal}_g(\xi)^2}{360N(N+2)}. \tag{52}$$

The orthogonal decomposition of Riemann curvature is given by

$$|\mathrm{Rm}_g(\xi)|_g^2 = |\mathrm{Weyl}_g(\xi)|_g^2 + \frac{4}{N-2}|\mathrm{E}_g(\xi)|_g^2 + \frac{2}{N(N-1)}\,\mathrm{Scal}_g(\xi)^2. \tag{53}$$

Moreover, we get

$$|\mathrm{Ric}_g(\xi)|_g^2 = |\mathrm{E}_g(\xi)|_g^2 + \frac{1}{N}\,\mathrm{Scal}_g(\xi)^2. \tag{54}$$

By (43) and (50), we compute

$$\begin{aligned}
&\int_M |\nabla U_{\mu,\xi}|_g^2\, dv_g \\
&\quad = \alpha_N^2(N-2)^2\int_0^{r_0/2\mu}\frac{r^2}{(1+r^2)^N}\int_{\partial B_\xi(r)} d\sigma_g\, dr + \mathrm{O}(\mu^{N-2}) \\
&\quad = \alpha_N^2(N-2)^2\omega_{N-1}\int_0^{r_0/2\mu}\frac{r^{N+1}}{(1+r^2)^N} \\
&\qquad \times\left(1 - \frac{1}{6N}\,\mathrm{Scal}_g(\xi)\mu^2r^2 + A_g(\xi)\mu^4r^4 + \mathrm{O}(\mu^5r^5)\right)dr + \mathrm{O}(\mu^{N-2})
\end{aligned}$$

$$= \frac{\alpha_N^2 (N-2)^2}{2} \omega_{N-1} \int_0^{(r_0/2\mu)^2} \frac{r^{N/2}}{(1+r)^N}$$

$$\times \left(1 - \frac{1}{6N} \operatorname{Scal}_g(\xi)\mu^2 r + A_g(\xi)\mu^4 r^2 + \mathrm{O}(\mu^5 r^{5/2})\right) dr + \mathrm{O}(\mu^{N-2})$$

$$= \frac{\alpha_N^2 (N-2)^2}{2} \omega_{N-1}$$

$$\times \left(I_N^{N/2} - \frac{1}{6N} I_N^{(N+2)/2} \operatorname{Scal}_g(\xi)\mu^2 + I_N^{(N+4)/2} A_g(\xi)\mu^4 + \mathrm{O}(\mu^5)\right)$$

$$= \frac{\alpha_N^2 (N-2)^2}{2} \omega_{N-1} I_N^{N/2}$$

$$\times \left(1 - \frac{N+2}{6N(N-4)} \operatorname{Scal}_g(\xi)\mu^2 + \frac{(N+2)(N+4)}{(N-4)(N-6)} A_g(\xi)\mu^4 + \mathrm{O}(\mu^5)\right) \qquad (55)$$

where $A_g(\xi)$ is as in (52) and $I_N^{N/2}$ is as in (44). By (43) and (49), we compute

$$\int_M \operatorname{Scal}_g U_{\mu,\xi}^2 \, dv_g = \alpha_N^2 \mu^2 \int_0^{r_0/2\mu} \frac{1}{(1+r^2)^{N-2}} \int_{\partial B_\xi(r)} \operatorname{Scal}_g \, d\sigma_g \, dr + \mathrm{O}(\mu^{N-2})$$

$$= \alpha_N^2 \omega_{N-1} \mu^2 \int_0^{r_0/2\mu} \frac{r^{N-1}}{(1+r^2)^{N-2}}$$

$$\times \left(\operatorname{Scal}_g(\xi) - \frac{1}{2N} \Lambda_g(\xi)\mu^2 r^2 + \mathrm{O}(\mu^4 r^4)\right) dr + \mathrm{O}(\mu^{N-2})$$

$$= \frac{\alpha_N^2}{2} \omega_{N-1} \mu^2 \int_0^{(r_0/2\mu)^2} \frac{r^{(N-2)/2}}{(1+r)^{N-2}}$$

$$\times \left(\operatorname{Scal}_g(\xi) - \frac{1}{2N} \Lambda_g(\xi)\mu^2 r + \mathrm{O}(\mu^4 r^2)\right) dr + \mathrm{O}(\mu^{N-2})$$

$$= \frac{\alpha_N^2}{2} \omega_{N-1} \mu^2 \left(I_{N-2}^{(N-2)/2} \operatorname{Scal}_g(\xi) - \frac{1}{2N} I_{N-2}^{N/2} \Lambda_g(\xi)\mu^2 + \mathrm{O}(\mu^5)\right)$$

$$= \frac{2\alpha_N^2 (N-1)(N-2)\omega_{N-1} I_N^{N/2}}{N(N-4)}$$

$$\times \mu^2 \left(\operatorname{Scal}_g(\xi) - \frac{1}{2(N-6)} \Lambda_g(\xi)\mu^2 + \mathrm{O}(\mu^5)\right) \qquad (56)$$

By (43) and (50), we compute

$$\begin{aligned}
\int_M U_{\mu,\xi}^{2^*}\,dv_g &= \alpha_N^{2^*}\int_0^{r_0/2\mu}\frac{1}{(1+r^2)^N}\int_{\partial B_\xi(r)}d\sigma_g\,dr + \mathrm{O}(\mu^N)\\
&= \alpha_N^{2^*}\omega_{N-1}\int_0^{r_0/2\mu}\frac{r^{N-1}}{(1+r^2)^N}\\
&\quad\times\left(1-\frac{1}{6N}\,\mathrm{Scal}_g(\xi)\mu^2r^2 + A_g(\xi)\mu^4r^4 + \mathrm{O}(\mu^5r^5)\right)dr + \mathrm{O}(\mu^N)\\
&= \frac{\alpha_N^{2^*}}{2}\omega_{N-1}\int_0^{(r_0/2\mu)^2}\frac{r^{(N-2)/2}}{(1+r)^N}\\
&\quad\times\left(1-\frac{1}{6N}\,\mathrm{Scal}_g(\xi)\mu^2r + A_g(\xi)\mu^4r^2 + \mathrm{O}(\mu^5r^{5/2})\right)dr + \mathrm{O}(\mu^N)\\
&= \frac{\alpha_N^{2^*}}{2}\omega_{N-1}\bigg(I_n^{(N-2)/2} - \frac{1}{6N}I_N^{N/2}\,\mathrm{Scal}_g(\xi)\mu^2\\
&\quad + I_N^{(N+2)/2}A_g(\xi)\mu^4 + \mathrm{O}(\mu^5)\bigg)\,dr\\
&= \frac{\alpha_N^2(N-2)^2\omega_{N-1}I_N^{N/2}}{2}\bigg(1-\frac{1}{6(N-2)}\,\mathrm{Scal}_g(\xi)\mu^2\\
&\quad + \frac{N(N+2)}{(N-2)(N-4)}A_g(\xi)\mu^4 + \mathrm{O}(\mu^5)\bigg)\,dr
\end{aligned} \tag{57}$$

as $\mu \to 0$, uniformly with respect to ξ, where $A_g(\xi)$ is as in (52) and $I_N^{N/2}$ is as in (44). Finally, estimate (47) follows from (55), (56), (57) by means of (53), (54). □

Step 4 We prove that

$$\int_M [F_\varepsilon(\mathcal{U}_{\mu,\xi}) - F_0(\mathcal{U}_{\mu,\xi})]\,dv_g = c_5\varepsilon - c_6\varepsilon\ln\varepsilon - 4c_6\varepsilon\ln d + o(\varepsilon|\ln\varepsilon|) \tag{58}$$

where

$$c_5 := \int_{\mathbb{R}^N} U^{p+1}\left[-\frac{1}{(p+1)^2} + \frac{1}{p+1}\ln U\right], \qquad c_6 := \frac{(N-2)^2}{16N}K_N^{-N}.$$

Proof. By the mean value theorem we deduce

$$\begin{aligned}
&\int_M [F_\varepsilon(\mathcal{U}_{\mu,\xi}) - F_0(\mathcal{U}_{\mu,\xi})]\,dv_g\\
&\quad = \int_M (\mathcal{U}_{\mu,\xi})^{p+1}\left[\frac{1}{p+1+\varepsilon}(\mathcal{U}_{\mu,\xi})^\varepsilon - \frac{1}{p+1}\right]dv_g
\end{aligned}$$

$$= \int_{\{|y| \le r_0/2\mu\}} U^{p+1} \left[\frac{1}{p+1+\varepsilon} e^{\varepsilon(-(N-2)/2 \ln \mu + \ln U)} - \frac{1}{p+1} \right] + O(\mu^N)$$

$$= \varepsilon \int_{\mathbb{R}^N} U^{p+1} \left[-\frac{1}{(p+1)^2} + \frac{1}{p+1} \left(-\frac{N-2}{2} \ln \mu + \ln U \right) \right]$$

$$+ O(\varepsilon^2 |\ln \varepsilon|^2) + O(\mu^N),$$

and so we can get (58) in view of $\int_{\mathbb{R}^N} U^{p+1} = K_N^{-N}$, as it follows by (43)–(44). □

Step 5 We prove that

$$\int_M \left\{ \mu^2 [\mathcal{L}_g \mathcal{U}_{\mu,\xi} - f_0(\mathcal{U}_{\mu,\xi})] + \frac{1}{2} \mu^4 [\mathcal{L}_g \mathcal{V}_{\mu,\xi} - f_0'(\mathcal{U}_{\mu,\xi}) \mathcal{V}_{\mu,\xi}] \right\} \mathcal{V}_{\mu,\xi} \, dv_g$$

$$= \frac{1}{2} \mu^4 \int_{\mathbb{R}^N} (\Delta V + p U^{p-1} V) V + O(\mu^5). \tag{59}$$

Proof. Since $\mathcal{U}_{\mu,\xi} \sim \mu^{(N-2)/2}$ and $\mu^2 \mathcal{V}_{\mu,\xi} \sim \mu^{(N-2)/2}$ away from ξ (along with a similar control on the derivatives), by (25)–(27) we get

$$\int_M \left\{ \mu^2 [\mathcal{L}_g \mathcal{U}_{\mu,\xi} - f_0(\mathcal{U}_{\mu,\xi})] + \frac{1}{2} \mu^4 [\mathcal{L}_g \mathcal{V}_{\mu,\xi} - f_0'(\mathcal{U}_{\mu,\xi}) \mathcal{V}_{\mu,\xi}] \right\} \mathcal{V}_{\mu,\xi} \, dv_g$$

$$= \int_{B(0,r_0/2)} \left\{ \mu^2 [-\Delta_g U_\mu + \beta_N \operatorname{Scal}_g U_\mu - U_\mu^p] \right.$$

$$\left. + \frac{1}{2} \mu^4 [-\Delta_g V_\mu + \beta_N \operatorname{Scal}_g V_\mu - p U_\mu^{p-1} V_\mu] \right\} V_\mu |g|^{1/2} + O(\mu^{N-2})$$

$$= \mu^2 \int_{B(0,r_0/2)} \left[\frac{1}{3} R_{iabj}(\xi) x^a x^b \partial_{ij}^2 U_\mu + \partial_l \Gamma_{ii}^k(\xi) x^l \partial_k U_\mu + \beta_N \operatorname{Scal}_g(\xi) U_\mu \right] V_\mu$$

$$+ \mu^2 \int_{B(0,r_0/2)} |V_\mu| O(|x|^3 |\partial_{ij}^2 U_\mu| + |x|^2 |\partial_k U_\mu| + |x| U_\mu)$$

$$+ \frac{1}{2} \mu^4 \int_{B(0,r_0/2)} [-\Delta V_\mu - p U_\mu^{p-1} V_\mu] V_\mu |g|^{1/2}$$

$$+ \mu^4 \int_{B(0,r_0/2)} |V_\mu| O(|x|^2 |\partial_{ij}^2 V_\mu| + |x| \, |\partial_k V_\mu| + |V_\mu|)$$

$$= \frac{\mu^4}{2} \int_{B(0,r_0/2)} [\Delta V_\mu + p U_\mu^{p-1} V_\mu] V_\mu + O(\mu^5 |\ln \mu|)$$

$$= \frac{\mu^4}{2} \int_{\mathbb{R}^2} [\Delta V + p U^{p-1} V] V + O(\mu^5 |\ln \mu|)$$

in view of (23)–(24) since V_μ does solve (17). □

Step 6 We prove that

$$\int_{\mathbb{R}^N} [-\Delta V - pU^{p-1}V]V = c_7 |\mathrm{E}_g(\xi)|_g^2 - c_8 \,\mathrm{Scal}_g(\xi)^2, \tag{60}$$

where (see (48))

$$c_7 := \frac{(2N-7)K_N^{-N}}{9N(N-2)(N-4)(N-6)} = 2c_3,$$
$$c_8 := \frac{(N-2)(N-7)K_N^{-N}}{36N^2(N-1)(N-4)(N-6)} = 2c_4.$$

Proof. We have

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta V - pU^{p-1}V)V \\
&\quad = \int_{\mathbb{R}^N} \left(\frac{1}{3}\mathrm{R}_{iabj}(\xi) x_a x_b \partial_{ij}^2 U + \partial_l \Gamma_{ss}^k(\xi) x_l \partial_k U + \frac{N-2}{4(N-1)} \,\mathrm{Scal}_g(\xi) U\right) \\
&\qquad \times \left(\frac{1}{3}\mathrm{R}_{i'a'b'j'}(\xi) z_{a'b'i'j'} + \partial_{l'} \Gamma_{s's'}^{k'}(\xi) v_{l'k'} + \frac{N-2}{4(N-1)} \,\mathrm{Scal}_g(\xi) w\right).
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta V - pU^{p-1}V)V \\
&\quad = \frac{1}{9}\mathrm{R}_{iabj}(\xi)\mathrm{R}_{i'a'b'j'}(\xi) \int_{\mathbb{R}^N} x_a x_b \partial_{ij}^2 U z_{a'b'i'j'} \\
&\qquad + \frac{1}{3}\partial_l \Gamma_{ss}^k(\xi) \mathrm{R}_{i'a'b'j'}(\xi) \int_{\mathbb{R}^N} x_l \partial_k U z_{a'b'i'j'} \\
&\qquad + \frac{N-2}{12(N-1)} \,\mathrm{Scal}_g(\xi) \mathrm{R}_{i'a'b'j'}(\xi) \int_{\mathbb{R}^N} U z_{a'b'i'j'} \\
&\qquad + \frac{1}{3}\mathrm{R}_{iabj}(\xi) \partial_{l'} \Gamma_{s's'}^{k'}(\xi) \int_{\mathbb{R}^N} x_a x_b \partial_{ij}^2 U v_{k'l'} + \partial_l \Gamma_{ss}^k(\xi) \partial_{l'} \Gamma_{s's'}^{k'}(\xi) \int_{\mathbb{R}^N} x_l \partial_k U v_{k'l'} \\
&\qquad + \frac{N-2}{4(N-1)} \,\mathrm{Scal}_g(\xi) \partial_{l'} \Gamma_{s's'}^{k'}(\xi) \int_{\mathbb{R}^N} U v_{k'l'} \\
&\qquad + \frac{N-2}{12(N-1)} \,\mathrm{Scal}_g(\xi) \mathrm{R}_{iabj}(\xi) \int_{\mathbb{R}^N} x_a x_b \partial_{ij}^2 U w \\
&\qquad + \frac{N-2}{4(N-1)} \,\mathrm{Scal}_g(\xi) \partial_l \Gamma_{ss}^k(\xi) \int_{\mathbb{R}^N} x_l \partial_k U w + \frac{(N-2)^2}{16(N-1)^2} \,\mathrm{Scal}_g(\xi)^2 \int_{\mathbb{R}^N} U w.
\end{aligned}$$

Since $\partial_l \Gamma^k_{ss}(\xi) = \frac{2}{3}\mathrm{R}_{sksl}(\xi)$ for all $k, l, s = 1, \dots, n$ and $\mathrm{R}_{iabj}(\xi) = -\mathrm{R}_{iajb}(\xi) = -\mathrm{R}_{aibj}(\xi)$ for all $a, b, i, j = 1, \dots, n$, we get

$$\begin{aligned}
&\frac{1}{9}\mathrm{R}_{iabj}(\xi)\mathrm{R}_{i'a'b'j'}(\xi)\int_{\mathbb{R}^N} x_a x_b \partial^2_{ij} U z_{a'b'i'j'} \\
&\quad + \frac{1}{3}\partial_l \Gamma^k_{ss}(\xi)\mathrm{R}_{i'a'b'j'}(\xi)\int_{\mathbb{R}^N} x_l \partial_k U z_{a'b'i'j'} \\
&\quad = -\frac{N-2}{9}\alpha_N \mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'b'j'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b z_{a'b'i'j'}}{(1+|x|^2)^{N/2}} \\
&\quad = -\frac{\alpha_N^{p+1}}{108N(N+2)}\Big[\big(2\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'a'i'}(\xi) - 4\mathrm{R}_{iabj}(\xi)\mathrm{R}_{a'a'b'b'}(\xi) \\
&\quad - 4\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'i'a'}(\xi)\big)\int_{\mathbb{R}^N}\frac{x_a x_b(|x|^4+3)}{(1+|x|^2)^N} \\
&\quad + (N+2)\mathrm{R}_{iaib}(\xi)\mathrm{R}_{a'a'b'j'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b x_{b'} x_{j'}(|x|^2+3)}{(1+|x|^2)^N} \\
&\quad + (N+2)\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'b'a'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b x_{b'} x_{i'}(|x|^2+3)}{(1+|x|^2)^N} \\
&\quad + (N+2)\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'i'j'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b x_{a'} x_{j'}(|x|^2+3)}{(1+|x|^2)^N} \\
&\quad + (N+2)\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'b'b'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b x_{i'} x_{a'}(|x|^2+3)}{(1+|x|^2)^N} \\
&\quad + (N+2)\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'a'j'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b x_{i'} x_{j'}(|x|^2+3)}{(1+|x|^2)^N} \\
&\quad - 2(N+2)\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'b'i'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b x_{a'} x_{b'}(|x|^2+3)}{(1+|x|^2)^N} \\
&\quad + 2(N+2)(N-2)\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'b'j'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b x_{a'} x_{b'} x_{i'} x_{j'}}{(1+|x|^2)^N}\Big] \\
&\quad = \frac{\alpha_N^{p+1}}{18N(N+2)}\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'i'a'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b(|x|^4+3)}{(1+|x|^2)^N} \\
&\quad - \frac{\alpha_N^{p+1}}{36N}\mathrm{R}_{iaib}(\xi)\mathrm{R}_{i'a'i'b'}(\xi)\int_{\mathbb{R}^N}\frac{x_a x_b x_{a'} x_{b'}(|x|^2+3)}{(1+|x|^2)^N} \\
&\quad = \frac{\alpha_N^{p+1}}{18N^2(N+2)} R_{iaia}(\xi) R_{i'a'i'a'}(\xi)\int_{\mathbb{R}^N}\frac{|x|^2(|x|^4+3)}{(1+|x|^2)^N}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\alpha_N^{p+1}}{36N^2(N+2)}\Big(2\sum_{a\neq b}R_{iaib}(\xi)R_{i'ai'b}(\xi)+\sum_{a\neq a'}R_{iaia}(\xi)R_{i'a'i'a'}(\xi)\\
&+3R_{iaia}(\xi)R_{i'ai'a}(\xi)\Big)\int_{\mathbb{R}^N}\frac{|x|^4(|x|^2+3)}{(1+|x|^2)^N}\\
=&\frac{\alpha_N^{p+1}}{18N^2(N+2)}\,\mathrm{Scal}_g(\xi)^2\int_{\mathbb{R}^N}\frac{|x|^2(|x|^4+3)}{(1+|x|^2)^N}\\
&-\frac{\alpha_N^{p+1}}{36N^2(N+2)}\big(2|\mathrm{Ric}_g(\xi)|_g^2+\mathrm{Scal}_g(\xi)^2\big)\int_{\mathbb{R}^N}\frac{|x|^4(|x|^2+3)}{(1+|x|^2)^N}\\
=&\frac{\alpha_N^{p+1}}{36N^3(N+2)}\,\mathrm{Scal}_g(\xi)^2\int_{\mathbb{R}^N}\frac{|x|^2\big((N-2)|x|^4-3(N+2)|x|^2+6N\big)}{(1+|x|^2)^N}\\
&-\frac{\alpha_N^{p+1}}{18N^2(N+2)}|\mathrm{E}_g(\xi)|_g^2\int_{\mathbb{R}^N}\frac{|x|^4(|x|^2+3)}{(1+|x|^2)^N}.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
&\mathrm{R}_{iabj}(\xi)\int_{\mathbb{R}^N}Uz_{abij}\\
&=\frac{\alpha_N^{p+1}}{12N(N+2)(N-2)}\Big[\big(2\mathrm{R}_{iaai}(\xi)-4\mathrm{R}_{aabb}(\xi)-4\mathrm{R}_{iaia}(\xi)\big)\int_{\mathbb{R}^N}\frac{|x|^4+3}{(1+|x|^2)^{N-1}}\\
&\quad+(N+2)\mathrm{R}_{aabj}(\xi)\int_{\mathbb{R}^N}\frac{x_bx_j(|x|^2+3)}{(1+|x|^2)^{N-1}}+(N+2)\mathrm{R}_{iaba}(\xi)\int_{\mathbb{R}^N}\frac{x_bx_i(|x|^2+3)}{(1+|x|^2)^{N-1}}\\
&\quad+(N+2)\mathrm{R}_{iaij}(\xi)\int_{\mathbb{R}^N}\frac{x_ax_j(|x|^2+3)}{(1+|x|^2)^{N-1}}+(N+2)\mathrm{R}_{iabb}(\xi)\int_{\mathbb{R}^N}\frac{x_ix_a(|x|^2+3)}{(1+|x|^2)^{N-1}}\\
&\quad+(N+2)\mathrm{R}_{iaaj}(\xi)\int_{\mathbb{R}^N}\frac{x_ix_j(|x|^2+3)}{(1+|x|^2)^{N-1}}-2(N+2)\mathrm{R}_{iabi}(\xi)\int_{\mathbb{R}^N}\frac{x_ax_b(|x|^2+3)}{(1+|x|^2)^{N-1}}\\
&\quad+2(N+2)(N-2)\mathrm{R}_{iabj}(\xi)\int_{\mathbb{R}^N}\frac{x_ax_bx_ix_j}{(1+|x|^2)^{N-1}}\Big]\\
&=-\frac{\alpha_N^{p+1}}{2N(N+2)(N-2)}\mathrm{R}_{iaia}(\xi)\int_{\mathbb{R}^N}\frac{|x|^4+3}{(1+|x|^2)^{N-1}}\\
&\quad+\frac{\alpha_N^{p+1}}{4N^2(N-2)}\mathrm{R}_{iaia}(\xi)\int_{\mathbb{R}^N}\frac{|x|^2(|x|^2+3)}{(1+|x|^2)^{N-1}}\\
&=-\frac{\alpha_N^{p+1}}{4N^2(N+2)(N-2)}\,\mathrm{Scal}_g(\xi)\int_{\mathbb{R}^N}\frac{(N-2)|x|^4-3(N+2)|x|^2+6N}{(1+|x|^2)^{N-1}}.
\end{aligned}
$$

We also compute

$$\begin{aligned}
&\partial_l \Gamma_{ss}^k(\xi) \int_{\mathbb{R}^N} U v_{kl} \\
&\quad = \frac{\alpha_N^{p+1}}{3N(N+2)(N-2)} \mathrm{R}_{sksk}(\xi) \int_{\mathbb{R}^N} \frac{|x|^4 + 3}{(1+|x|^2)^{N-1}} \\
&\qquad - \frac{\alpha_N^{p+1}}{6N^2(N-2)} \mathrm{R}_{sksk}(\xi) \int_{\mathbb{R}^N} \frac{|x|^2(|x|^2+3)}{(1+|x|^2)^{N-1}} \\
&\quad = \frac{\alpha_N^{p+1}}{6N^2(N+2)(N-2)} \operatorname{Scal}_g(\xi) \int_{\mathbb{R}^N} \frac{(N-2)|x|^4 - 3(N+2)|x|^2 + 6N}{(1+|x|^2)^{N-1}}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{3} \mathrm{R}_{iabj}(\xi) \partial_l \Gamma_{ss}^k(\xi) \int_{\mathbb{R}^N} x_a x_b \partial_{ij}^2 U v_{kl} \\
&\qquad + \partial_l \Gamma_{ss}^k(\xi) \partial_{l'} \Gamma_{s's'}^{k'}(\xi) \int_{\mathbb{R}^N} x_{l'} \partial_{k'} U v_{kl} \\
&\quad = -\frac{2(N-2)}{9} \alpha_N \mathrm{R}_{iaib}(\xi) \mathrm{R}_{sksl}(\xi) \int_{\mathbb{R}^N} \frac{x_a x_b v_{kl}}{(1+|x|^2)^{N/2}} \\
&\quad = -\frac{\alpha_N^{p+1}}{9N(N+2)} \mathrm{R}_{iaib}(\xi) \mathrm{R}_{sksk}(\xi) \int_{\mathbb{R}^N} \frac{x_a x_b (|x|^4+3)}{(1+|x|^2)^N} \\
&\qquad + \frac{\alpha_N^{p+1}}{18N} \mathrm{R}_{iaib}(\xi) \mathrm{R}_{sksl}(\xi) \int_{\mathbb{R}^N} \frac{x_a x_b x_k x_l (|x|^2+3)}{(1+|x|^2)^N} \\
&\quad = -\frac{\alpha_N^{p+1}}{9N^2(N+2)} R_{iaia}(\xi) R_{sksk}(\xi) \int_{\mathbb{R}^N} \frac{|x|^2(|x|^4+3)}{(1+|x|^2)^N} \\
&\qquad + \frac{\alpha_N^{p+1}}{18N^2(N+2)} \Big(2 \sum_{a \neq b} R_{iaib}(\xi) R_{sasb}(\xi) + \sum_{a \neq k} R_{iaia}(\xi) R_{sksk}(\xi) \\
&\qquad + 3 R_{iaia}(\xi) R_{sasa}(\xi) \Big) \int_{\mathbb{R}^N} \frac{|x|^4(|x|^2+3)}{(1+|x|^2)^N} \\
&\quad = -\frac{\alpha_N^{p+1}}{9N^2(N+2)} \operatorname{Scal}_g(\xi)^2 \int_{\mathbb{R}^N} \frac{|x|^2(|x|^4+3)}{(1+|x|^2)^N} \\
&\qquad + \frac{\alpha_N^{p+1}}{18N^2(N+2)} \big(2|\operatorname{Ric}_g(\xi)|_g^2 + \operatorname{Scal}_g(\xi)^2 \big) \int_{\mathbb{R}^N} \frac{|x|^4(|x|^2+3)}{(1+|x|^2)^N}
\end{aligned}$$

$$= -\frac{\alpha_N^{p+1}}{18N^3(N+2)} \operatorname{Scal}_g(\xi)^2 \int_{\mathbb{R}^N} \frac{|x|^2\big((N-2)|x|^4 - 3(N+2)|x|^2 + 6N\big)}{(1+|x|^2)^N}$$
$$+ \frac{\alpha_N^{p+1}}{9N^2(N+2)} |\mathrm{E}_g(\xi)|_g^2 \int_{\mathbb{R}^N} \frac{|x|^4(|x|^2+3)}{(1+|x|^2)^N}.$$

Moreover, we get

$$\frac{1}{3}\mathrm{R}_{iabj}(\xi)\int_{\mathbb{R}^N} x_a x_b \partial_{ij}^2 Uw + \partial_l \Gamma_{ss}^k(\xi)\int_{\mathbb{R}^N} x_l \partial_k Uw$$
$$= -\frac{N-2}{3N}\alpha_N \mathrm{R}_{iaia}(\xi)\int_{\mathbb{R}^N} \frac{|x|^2 w(x)}{(1+|x|^2)^{N/2}}$$
$$= \frac{\alpha_N^{p+1}}{6N^2(N+2)} \operatorname{Scal}_g(\xi)\int_{\mathbb{R}^N} \frac{|x|^2(|x|^4+3)}{(1+|x|^2)^N}$$

and

$$\int_{\mathbb{R}^N} Uw = -\frac{\alpha_N^{p+1}}{2N(N+2)(N-2)}\int_{\mathbb{R}^N} \frac{|x|^4+3}{(1+|x|^2)^{N-1}}.$$

It follows from the above estimates that

$$\int_{\mathbb{R}^N} (-\Delta V - pU^{p-1}V)V$$
$$= \frac{\alpha_N^{p+1}}{18N^2(N+2)} |\mathrm{E}_g(\xi)|_g^2 \int_{\mathbb{R}^N} \frac{|x|^4(|x|^2+3)}{(1+|x|^2)^N}$$
$$+ \frac{\alpha_N^{p+1}}{288N^3(N-1)^2} \operatorname{Scal}_g(\xi)^2 \Big[(N-2)(N-4)\int_{\mathbb{R}^N} \frac{|x|^6}{(1+|x|^2)^N}$$
$$+ 3(N^2-8N+8)\int_{\mathbb{R}^N} \frac{|x|^4}{(1+|x|^2)^N}$$
$$- 3N(7N-10)\int_{\mathbb{R}^N} \frac{|x|^2}{(1+|x|^2)^N} + 9N^2\int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)^N}\Big]$$
$$= \frac{\alpha_N^{p+1}\omega_{N-1}}{36N^2(N+2)} |\mathrm{E}_g(\xi)|_g^2 (I_N^{(N+4)/2} + 3I_N^{(N+2)/2})$$
$$+ \frac{\alpha_N^{p+1}\omega_{N-1}}{576N^3(N-1)^2} \operatorname{Scal}_g(\xi)^2 [(N-2)(N-4)I_N^{(N+4)/2}$$
$$+ 3(N^2-8N+8)I_N^{(N+2)/2} - 3N(7N-10)I_N^{N/2} + 9N^2 I_N^{(N-2)/2}]$$

$$
\begin{aligned}
&= \frac{(2N-7)\alpha_N^{p+1}\omega_{N-1}}{18N^2(N-4)(N-6)} I_N^{N/2} |\mathrm{E}_g(\xi)|_g^2 \\
&\quad - \frac{(N-2)^2(N-7)\alpha_N^{p+1}\omega_{N-1}}{72N^3(N-1)(N-4)(N-6)} I_N^{N/2} \operatorname{Scal}_g(\xi)^2 \\
&= \frac{(2N-7)K_N^{-N}}{9N(N-2)(N-4)(N-6)} |\mathrm{E}_g(\xi)|_g^2 \\
&\quad - \frac{(N-2)(N-7)K_N^{-N}}{36N^2(N-1)(N-4)(N-6)} \operatorname{Scal}_g(\xi)^2,
\end{aligned}
$$

which proves (60).

We have used the following important fact:

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left(\frac{1}{3} \mathrm{R}_{iabj}(\xi) x_a x_b \partial_{ij}^2 U + \partial_l \Gamma_{ss}^k(\xi) x_l \partial_k U + \beta_N \operatorname{Scal}_g(\xi) U \right) Z_0 \\
&\quad = -\frac{(N-2)^2}{6} \alpha_N^2 \mathrm{R}_{iaib}(\xi) \int_{\mathbb{R}^N} \frac{x_a x_b (1-|x|^2)}{(1+|x|^2)^N} \\
&\qquad + \frac{(N-2)^2}{8(N-1)} \alpha_N^2 \operatorname{Scal}_g(\xi) \int_{\mathbb{R}^N} \frac{1-|x|^2}{(1+|x|^2)^{N-1}} \\
&\quad = \frac{(N-2)^2}{24N(N-1)} \alpha_N^2 \operatorname{Scal}_g(\xi) \int_{\mathbb{R}^N} \frac{(N-4)|x|^4 - 4(N-1)|x|^2 + 3N}{(1+|x|^2)^N} \\
&\quad = \frac{(N-2)^2}{48N(N-1)} \alpha_N^2 \omega_{N-1} \operatorname{Scal}_g(\xi) \\
&\qquad \times \left((N-4) I_N^{(N+2)/2} - 4(N-1) I_N^{N/2} + 3N I_N^{(N-2)/2} \right) \\
&\quad = 0. \qquad \square
\end{aligned}
$$

References

[1] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev.* J. Differential Geometry **11** no. 4 (1976), 573–598.

[2] T. Aubin, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire.* J. Math. Pures Appl. (9) **55** (1976), 269–296.

[3] E. Bianchi, H. Egnell, *A note on the Sobolev inequality*. J. Funct. Anal. **100** (1991), 18–24.

[4] S. Brendle, *Blow-up phenomena for the Yamabe equation.* J. Amer. Math. Soc. **21**, 951–979.

[5] S. Brendle, F. C. Marques, *Blow-up phenomena for the Yamabe equation. II*. J. Differential Geom. **81** (2009), 225–250.

[6] O. Druet, *Compactness for Yamabe metrics in low dimensions.* Int. Math. Res. Not. **23** (2004), 1143–1191.

[7] P. Esposito, A. Pistoia, J. Vétois, *The effect of linear perturbations on the Yamabe problem.* Math. Ann. **358** (2014), 511–560.

[8] P. Esposito, A. Pistoia, J. Vétois, *Blow-up solutions for linear perturbations of the Yamabe equation.* Concentration Analysis and Applications to Pde: Icts Workshop, Bangalore, January 2012, Birkauser.

[9] M. A. Khuri, F. C. Marques, R. M. Schoen, *A compactness theorem for the Yamabe problem.* J. Differential Geom. **81** (2009), 143–196.

[10] Y.-Y. Li, *On a singularly perturbed elliptic equation.* Adv. Differential Equations **2** (1997), 955–980.

[11] Y.-Y. Li, L. Zhang, *A Harnack type inequality for the Yamabe equation in low dimensions.* Calc. Var. Partial Differential Equations **20** (2004), 133–151.

[12] Y.-Y. Li, L. Zhang, *Compactness of solutions to the Yamabe problem. II.* Calc. Var. Partial Differential Equations **24** (2005), 185–237.

[13] Y.-Y. Li, L. Zhang, *Compactness of solutions to the Yamabe problem. III.* J. Funct. Anal. **245** (2007), 438–474.

[14] Y.-Y. Li, M.-J. Zhu, *Yamabe type equations on three-dimensional Riemannian manifolds.* Commun. Contemp. Math. **1** (1999), 1–50.

[15] F. C. Marques, *A priori estimates for the Yamabe problem in the non-locally conformally flat case.* J. Differential Geom. **71** (2005), 315–346.

[16] A. M. Micheletti, A. Pistoia, J. Vétois, *Blow-up solutions for asymptotically critical elliptic equations.* Indiana Univ. Math. J. **58** (2009), 1719–1746.

[17] R. M. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature.* J. Differential Geom. **20** (1984), 479–495.

[18] R. M. Schoen, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics.* Topics in calculus of variations, Montecatini Terme, 1987. Lecture Notes in Math. **1365**, 120–154, Springer, 1989.

[19] R. M. Schoen, S.-T. Yau, *On the proof of the positive mass conjecture in general relativity.* Comm. Math. Phys. **65** (1979), 45–76.

[20] R. M. Schoen, S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature.* Invent. Math. **92** (1988), 47–71.

[21] G. Talenti, *Best constant in Sobolev inequality.* Ann. Mat. Pura Appl. (IV) **110** (1976), 353–372.

[22] N. S. Trudinger, *Remarks concerning the conformal deformation of Riemannian structures on compact manifolds.* Ann. Scuola Norm. Sup. Pisa (3) **22** (1968), 265–274.

[23] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds.* Osaka Math. J. **12** (1960), 21–37.

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