

Linearized theory for entire solutions of a singular Liouville equation

Manuel del Pino * Pierpaolo Esposito † Monica Musso ‡

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Abstract

We discuss invertibility properties for entire finite-energy solutions of the regularized version of a singular Liouville equation.

1 The linear theory

In this short note, we want to show the following result

Theorem 1.1. *Let $c \in \mathbb{C}$ and $N \in \mathbb{N}$. The kernel of the operator*

$$L : \phi \rightarrow L(\phi) := \Delta\phi + \frac{8(N+1)^2|z|^{2N}}{(1+|z^{N+1}-c|^2)^2}\phi$$

in $L^\infty(\mathbb{R}^2)$ has the simple form

$$\{\phi \in L^\infty(\mathbb{R}^2) : L(\phi) = 0\} = \text{Span} \left\{ \frac{1 - |z^{N+1} - c|^2}{1 + |z^{N+1} - c|^2}, \frac{\text{Re}(z^{N+1} - c)}{1 + |z^{N+1} - c|^2}, \frac{\text{Im}(z^{N+1} - c)}{1 + |z^{N+1} - c|^2} \right\},$$

where z^{N+1} denotes the complex $(N+1)$ -power.

The case $N = 0$ was already known (see [1]), based on a Fourier expansion of the function ϕ . The aim here is to adapt the argument to the more difficult case $N \in \mathbb{N}$.

Proof: Let us recall the Liouville formula: given a holomorphic function f on \mathbb{C} , the function

$$\ln \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2} \tag{1}$$

*Departamento de Ingeniería Matemática and CMM, Universidad de Chile, Casilla 170, Correo 3, Santiago, Chile. E-mail: delpino@dim.uchile.cl. Author supported by grants Fondecyt 1070389 and FONDAP (Chile)

†Dipartimento di Matematica, Università degli Studi “Roma Tre”, Largo S. Leonardo Murialdo, 1 – 00146 Roma, Italy. E-mail: esposito@mat.uniroma3.it. Author partially supported by by FIRB-IDEAS (2008), project “Geometrical aspects in PDEs”.

‡Departamento de Matematica, Pontificia Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Macul, Santiago, Chile. E-mail: mmusso@mat.puc.cl. Author supported by Fondecyt grant 1040936 (Chile), and by M.U.R.S.T., project “Metodi variazionali e topologici nello studio di fenomeni non lineari”.

solves the equation $\Delta U + e^U = 0$ in the set $\{z \in \mathbb{C} / f'(z) \neq 0\}$. If now f' has a zero at the origin of multiplicity N , the function

$$\ln \frac{8|f'(z)|^2}{(1+|f(z)|^2)^2} - \ln |z|^{2N} \quad (2)$$

solves the equation $\Delta U + |z|^{2N} e^U = 0$ in the set $\{z \in \mathbb{C} \setminus \{0\} / f'(z) \neq 0\}$. The choice $f(z) = z^{N+1}(1 + \tau z^k) - c$, $k \geq 0$, leads to a family

$$U_{\tau,k}(z) = \ln \frac{8(N+1)^2 |1 + \tau \frac{N+1+k}{N+1} z^k|^2}{(1 + |z^{N+1}(1 + \tau z^k) - c|^2)^2}, \quad \tau \in \mathbb{C},$$

of solutions of $\Delta U + |z|^{2N} e^U = 0$ in $\mathbb{C} \setminus \{z \in \mathbb{C} : 1 + \tau \frac{N+1+k}{N+1} z^k = 0\}$. The derivative of $U_{\tau,k}$ in τ at $\tau = 0$:

$$\phi_k := \partial_\tau U_{\tau,k} \Big|_{\tau=0} = z^k \left(\frac{N+1+k}{N+1} - 2 \frac{z^{N+1} \overline{z^{N+1} - c}}{1 + |z^{N+1} - c|^2} \right)$$

solves $L(\phi_k) = 0$ in \mathbb{C} , and in particular, $\phi_0 = \text{Re } \phi_0$ and $\phi_k^1 = \frac{N+1}{N+1+k} \text{Re } \phi_k$, $\phi_k^2 = \frac{N+1}{N+1+k} \text{Im } \phi_k$ are real solutions for $k \geq 1$. We want to show that every solution ϕ of $L(\phi) = 0$ is a linear combination of ϕ_0 and ϕ_k^i , $k \geq 1$ and $i = 1, 2$:

$$\phi = a_0 \phi_0 + \sum_{k \geq 1} (a_k \phi_k^1 + b_k \phi_k^2). \quad (3)$$

The key idea is that, for ρ small, the functions $\phi_0(\rho e^{i\theta})$ and $\frac{1}{\rho^k} \phi_k^i(\rho e^{i\theta})$, $k \geq 1$ and $i = 1, 2$, are very close to the real Fourier basis 1 , $\cos(k\theta)$ and $\sin(k\theta)$ with $k \geq 1$, and then form a complete set in $L^2(\partial B_\rho(0))$. Indeed, by a standard integration by parts we can compute for $k \geq 1$

$$\int_{S^1} \phi_0(\rho e^{i\theta}) d\theta = 1 + O(\rho^{N+1}), \quad \int_{S^1} \cos(k\theta) \phi_0(\rho e^{i\theta}) d\theta = O\left(\frac{\rho^{N+1}}{k+1}\right), \quad \int_{S^1} \sin(k\theta) \phi_0(\rho e^{i\theta}) d\theta = O\left(\frac{\rho^{N+1}}{k+1}\right)$$

and

$$\int_{S^1} \cos(k\theta) \phi_j^1(\rho e^{i\theta}) d\theta = \pi \delta_{kj} + O\left(\frac{\rho^{N+1}}{(k+1)(j+1)}\right), \quad \int_{S^1} \sin(k\theta) \phi_j^1(\rho e^{i\theta}) d\theta = O\left(\frac{\rho^{N+1}}{(k+1)(j+1)}\right)$$

and

$$\int_{S^1} \cos(k\theta) \phi_j^2(\rho e^{i\theta}) d\theta = O\left(\frac{\rho^{N+1}}{(k+1)(j+1)}\right), \quad \int_{S^1} \sin(k\theta) \phi_j^2(\rho e^{i\theta}) d\theta = \pi \delta_{kj} + O\left(\frac{\rho^{N+1}}{(k+1)(j+1)}\right).$$

Letting $\psi \in L^2(\partial B_\rho(0))$ in the form

$$\psi(\rho e^{i\theta}) = c_0 + \sum_{k \geq 1} (c_k \cos(k\theta) + d_k \sin(k\theta)),$$

we can compute

$$\tilde{c}_0 := \int_{S^1} \psi(\rho e^{i\theta}) \phi_0(\rho e^{i\theta}) d\theta = c_0 + \rho^{N+1} O\left(\sum_{k \geq 0} \frac{|c_k|}{k+1} + \sum_{k \geq 1} \frac{|d_k|}{k+1}\right)$$

and

$$\tilde{c}_j := \int_{S^1} \psi(\rho e^{i\theta}) \phi_j^1(\rho e^{i\theta}) d\theta = c_j + \frac{\rho^{N+1}}{j+1} O\left(\sum_{k \geq 0} \frac{|c_k|}{k+1} + \sum_{k \geq 1} \frac{|d_k|}{k+1}\right)$$

$$\tilde{d}_j := \int_{S^1} \psi(\rho e^{i\theta}) \phi_j^2(\rho e^{i\theta}) d\theta = d_j + \frac{\rho^{N+1}}{j+1} \left(\sum_{k \geq 0} \frac{|c_k|}{k+1} + \sum_{k \geq 1} \frac{|d_k|}{k+1} \right).$$

We consider the operator

$$T : (c_0, c_1, d_1, \dots) \in l_2 \rightarrow (\tilde{c}_0, \tilde{c}_1, \tilde{d}_1, \dots) \in l^2.$$

We have shown so far that

$$\|T - \text{Id}\| \leq C\rho^{N+1} \left(\sum_{j \geq 0} \frac{1}{(j+1)^2} \right).$$

In conclusion, for ρ small we have that T is an invertible operator, and by injectivity we then deduce that, if $\psi \in L^2(\partial B_\rho(0))$ is so that

$$\int_{S^1} \psi(\rho e^{i\theta}) \phi_0(\rho e^{i\theta}) d\theta = \int_{S^1} \psi(\rho e^{i\theta}) \phi_k^j(\rho e^{i\theta}) d\theta = 0 \quad \forall : k \geq 1, j = 1, 2,$$

then its Fourier coefficients c_j vanish and $\psi = 0$. This means that, for ρ small, the space $L^2(\partial B_\rho(0))$ coincides with the closure in L^2 -norm of

$$\text{Span} \{ \phi_0, \phi_k^j : k \geq 1, j = 1, 2 \}.$$

In particular, every solution $\phi \in L^\infty(\mathbb{C})$ can be written on $\partial B_\rho(0)$, ρ small, as

$$\phi(\rho e^{i\theta}) = a_0 \phi_0(\rho e^{i\theta}) + \sum_{k \geq 1} (a_k \phi_k^1(\rho e^{i\theta}) + b_k \phi_k^2(\rho e^{i\theta})),$$

for suitable a_j and b_j . By regularity theory $\phi \in C^\infty(\mathbb{C})$, and then $\phi|_{\partial B_\rho(0)} \in C^\infty(\partial B_\rho(0))$. Arguing as for the Fourier coefficients, it is easily seen that a_k and b_k tend to zero as $k \rightarrow +\infty$ faster than any power of k . In particular, the function

$$\hat{\phi}(z) = a_0 \phi_0(z) + \sum_{k \geq 1} [a_k \phi_k^1(z) + b_k \phi_k^2(z)]$$

is well defined, is in $C^\infty(\mathbb{C})$ and satisfies $L(\hat{\phi}) = 0$ in \mathbb{C} . Since $\phi = \hat{\phi}$ on $\partial B_\rho(0)$ and $\delta = \phi - \hat{\phi}$ satisfies $L(\delta) = 0$ in \mathbb{C} , an integration by parts yields to

$$\int_{B_\rho(0)} |\nabla \delta|^2 = \int_{B_\rho(0)} V \delta^2 - \int_{B_\rho(0)} L(\delta) \delta = \int_{B_\rho(0)} V \delta^2 \leq C\rho^{2N} \int_{B_\rho(0)} \delta^2,$$

where $V(z) = \frac{8(N+1)^2 |z|^{2N}}{(1+|z^{N+1}-c|^2)^2}$. As soon as $C\rho^{2N} < \lambda_1(B_\rho(0))$ (λ_1 being the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions), we get that necessarily $\delta = 0$ in $B_\rho(0) = 0$. Then, for ρ small we get that $\delta = 0$ in $B_\rho(0)$, and by the strong maximum principle $\delta = 0$ in \mathbb{C} . So we have shown that

$$\phi(z) = a_0 \phi_0(z) + \sum_{k \geq 1} [a_k \phi_k^1(z) + b_k \phi_k^2(z)]$$

in \mathbb{C} . Let us look now at the behavior of $\phi(z)$ as $|z| \rightarrow +\infty$. Since the only bounded components in $\phi(z)$ are ϕ_0 and $\phi_{N+1}^1, \phi_{N+1}^2$, we expect that $a_k = b_k = 0$ for $k \neq 0, N+1$. Also in this case we will use that the components of ϕ are very close to the Fourier basis as $|z| \rightarrow +\infty$.

Indeed, observe that

$$\frac{z^{N+1} \overline{z^{N+1} - c}}{1 + |z^{N+1} - c|^2} = 1 + O\left(\frac{1}{|z|^{N+1}}\right) \quad \text{as } |z| \rightarrow +\infty,$$

and then

$$\phi_k(z) = z^k \left(\frac{k-N-1}{N+1} + O\left(\frac{1}{|z|^{N+1}}\right) \right)$$

at infinity. So we have that

$$\begin{aligned} \phi_0(z) &= -1 + O\left(\frac{1}{|z|^{N+1}}\right), \quad \phi_k(z) = \frac{k-N-1}{N+1+k} |z|^k \cos(k\theta) \left(1 + O\left(\frac{1}{|z|^{N+1}}\right)\right) \\ \phi_k^2(z) &= \frac{k-N-1}{N+1+k} |z|^k \sin(k\theta) \left(1 + O\left(\frac{1}{k|z|^{N+1}}\right)\right). \end{aligned}$$

Let us now compute by Cauchy-Schwartz inequality

$$\begin{aligned} \frac{1}{R} \int_{\partial B_R} |\phi|^2 &= \pi \left(\sum_{k \geq 0} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k} |a_k|^2 + \sum_{k \geq 1} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k} |b_k|^2 \right) \\ &\quad + o \left(\sum_{k,j} R^{k+j} (|a_k| |a_j| + |b_k| |b_j| + |a_k| |b_j|) \right) \\ &= \pi(1 + o(1)) \left(\sum_{k \geq 0} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k} |a_k|^2 + \sum_{k \geq 1} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k} |b_k|^2 \right) \end{aligned}$$

as $R \rightarrow +\infty$. Since $\phi \in L^\infty(\mathbb{C})$, we have that $\frac{1}{R} \int_{\partial B_R} |\phi|^2$ is bounded in R , and then

$$\sum_{k \geq 0} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k} |a_k|^2 + \sum_{k \geq 1} \frac{(k-N-1)^2}{(N+1+k)^2} R^{2k} |b_k|^2$$

is bounded in R . Then $a_k = 0$ and $b_k = 0$ for $k \geq 1$ unless $k = N+1$. For a solution $\phi \in L^\infty(\mathbb{C})$ of $L(\phi) = 0$ we have then shown that

$$\phi(z) = a_0 \phi_0(z) + a_{N+1} \phi_{N+1}^1(z) + b_{N+1} \phi_{N+1}^2(z).$$

To conclude, we need simply to rewrite ϕ_0 and ϕ_{N+1} in a more expressive way. We have that

$$\phi_0(z) = 1 - 2 \frac{z^{N+1} \overline{z^{N+1} - c}}{1 + |z^{N+1} - c|^2} = \frac{1 - |z^{N+1} - c|^2}{1 + |z^{N+1} - c|^2} - 2c \frac{\overline{z^{N+1} - c}}{1 + |z^{N+1} - c|^2}$$

and

$$\phi_{N+1}(z) = 2z^{N+1} \left(1 - \frac{z^{N+1} \overline{z^{N+1} - c}}{1 + |z^{N+1} - c|^2} \right) = 2c \frac{1 - |z^{N+1} - c|^2}{1 + |z^{N+1} - c|^2} + 2 \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2} - 2c^2 \frac{\overline{z^{N+1} - c}}{1 + |z^{N+1} - c|^2}.$$

In real form we can then write that

$$\phi_0(z) = \frac{1 - |z^{N+1} - c|^2}{1 + |z^{N+1} - c|^2} - 2c_1 \operatorname{Re} \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2} - 2c_2 \operatorname{Im} \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2}$$

and

$$\phi_{N+1}^1(z) = c_1 \frac{1 - |z^{N+1} - c|^2}{1 + |z^{N+1} - c|^2} + (1 - c_1^2 + c_2^2) \operatorname{Re} \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2} - 2c_1 c_2 \operatorname{Im} \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2}$$

and

$$\phi_{N+1}^2(z) = c_2 \frac{1 - |z^{N+1} - c|^2}{1 + |z^{N+1} - c|^2} - 2c_1 c_2 \operatorname{Re} \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2} + (1 + c_1^2 - c_2^2) \operatorname{Im} \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2},$$

where $c = c_1 + ic_2$. In conclusion, the function ϕ can be written as

$$\begin{aligned} \phi = & (a_0 + a_{N+1}c_1 + b_{N+1}c_2) \frac{1 - |z^{N+1} - c|^2}{1 + |z^{N+1} - c|^2} + [-2a_0c_1 + a_{N+1}(1 - c_1^2 + c_2^2) - 2b_{N+1}c_1c_2] \operatorname{Re} \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2} \\ & + [-2a_0c_2 - 2a_{N+1}c_1c_2 + b_{N+1}(1 + c_1^2 - c_2^2)] \operatorname{Im} \frac{z^{N+1} - c}{1 + |z^{N+1} - c|^2} \end{aligned}$$

and the Theorem is established. ■

References

- [1] S. Baraket, F. Pacard, Construction of singular limits for a semilinear elliptic equation in dimension 2, *Calc. Var. P.D.E.* **6** (1998), 1-38.