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BLOWUP SOLUTIONS FOR A LIOUVILLE EQUATION WITH SINGULAR DATA*

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Abstract. We study the existence of multiple blowup solutions for a semilinear elliptic equation with homogeneous Dirichlet boundary condition, exponential nonlinearity, and a singular source term given by Dirac masses. In particular, we extend the result of Baraket and Pacard [*Calc. Var. Partial Differential Equations*, 6 (1998), pp. 1–38] by allowing the presence, in the equation, of a weight function possibly vanishing in some points.

Key words. Liouville equation, singular data, exponential nonlinearity, blowup solutions

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1. Main results and examples. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded open set. We are concerned with the existence of solutions in the distributional sense for the problem

$$(1) \qquad \begin{cases} -\triangle u = \rho^2 e^u - 4\pi \sum_{i=1}^N \alpha_i \delta_{p_i} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with the property that $\rho^2 e^u$ "concentrates" when the parameter $\rho \to 0$. Here $\{\alpha_1, \dots, \alpha_N\}$ are positive numbers, δ_p defines the Dirac mass at p, and $\Gamma := \{p_1, \dots, p_N\} \subset \Omega$ is the set of singular sources in (1).

Problem (1) with $\Gamma = \emptyset$ has been largely studied in connection with many physical models such as thermionic emission [21], the theory of the isothermal gas sphere [14], gas combustion [25], and in the context of statistical mechanics in [11], [12], and [23]. The asymptotic analysis for blowup solutions to problem (1) as $\rho \to 0$ is contained in [36] (see also [27]) and alternatively it can be obtained as a by-product of the general blowup analysis of [8]: it leads in the limit to a quantization property of the energy $\rho^2 \int_\Omega e^u$ in terms of the number of blowup points and to a characterization of the location of the blowup points. For the converse question, namely, the construction of solutions to (1) which do blow up at the "admissible" points as $\rho \to 0$, the first result is due to Weston [38] who constructed a sequence of solutions on simply connected domains "concentrating" on a single blowup point according to [36] (see also [26] for more general nonlinearities). The general case of the existence of multiple blowup solutions has been treated only in the beautiful paper [4]. Subsequently Chen and Lin in [17] have given an alternative proof in the special case of an annulus. Thus, perturbative problems with exponential nonlinearities in dimension two seem to be very difficult to handle. So far only a few results have been derived that cover some special cases (beside [4] and its extensions [2], [3], and [5], see also [13] and [29]) in contrast to the vast literature available in higher dimensions; see, for example, [1], [15], [28], and [32].

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Motivated by some singular elliptic equations arising in the study of Chern–Simons vortex theory (we refer the reader to [39] and the references therein), we are interested in analyzing (1) with $\Gamma \neq \emptyset$. We mention that some of the progress made about condensates in Chern–Simons models is contained in [10], [30], [33], [34], [35], and [37] to quote a few.

Let $G(z, z')$ denote the Green's function of $-\triangle$ on Ω with Dirichlet boundary condition, namely,

$$\begin{cases} -\triangle_z G(z, z') = \delta_{z'} & \text{in } \Omega, \\ G(z, z') = 0 & \text{on } \partial\Omega, \end{cases}$$

and let $H(z, z') = \frac{1}{2\pi} \ln |z - z'| + G(z, z')$ be the regular part of $G(z, z')$. Problem (1) is equivalent to solving for $v = u + 4\pi \sum_{i=1}^N \alpha_i G(z, p_i)$, the regular part of u, the equation

$$\begin{cases} -\triangle v = \rho^2 |z - p_1|^{2\alpha_1} \cdots |z - p_N|^{2\alpha_N} e^{-4\pi \sum_{i=1}^N \alpha_i H(z,p_i)} e^v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Thus, we may consider the following general model problem:

$$(Q)_\rho \qquad \begin{cases} -\triangle v = \rho^2 |z - p_1|^{2\alpha_1} \cdots |z - p_N|^{2\alpha_N} f(z) e^v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Gamma = \{p_1, \ldots, p_N\} \subset \Omega$ and $\{\alpha_1, \ldots, \alpha_N\}$ are positive numbers, $f : \Omega \to \mathbb{R}$ is a smooth function such that $f(p_i) > 0$ for any $i = 1, \ldots, N$. An extension to the singular case of the blowup analysis in [8] is due to [7] (see also [6]). It permits us to perform an asymptotic analysis in the spirit of [36] (see [20] for a proof). To this purpose, set $\Gamma = \{p_1, \ldots, p_N\}$ and $\Omega' = \Omega \cap \{f > 0\}$, and for given $m \in \mathbb{N}$ and $s \in \{1, \ldots, N\}$ define

$$\begin{aligned} \tilde{\mathcal{F}}(z_1, \ldots, z_m) = &\sum_{i=1}^m H(z_i, z_i) + \sum_{i \neq j} G(z_i, z_j) \\ &+ \frac{1}{4\pi} \sum_{i=1}^m \ln \left(|z_i - p_1|^{2\alpha_1} \cdots |z_i - p_N|^{2\alpha_N} f(z_i) \right) \end{aligned}$$

which is well defined in $(\Omega' \setminus \Gamma)^m$ for $z_i \neq z_j$ whenever $i \neq j$, and let

$$\mathcal{G}(z_1, \ldots, z_m, \omega_1, \ldots, \omega_s) = \frac{1}{4\pi} \left(\sum_{i=1}^m \sum_{j=1}^s 8\pi(1 + \alpha_j) G(z_i, \omega_j) \right)$$

be well defined for $z_i \neq \omega_j$, with $z_i \in \Omega$, $\omega_j \in \mathbb{C}$, $i = 1, \ldots, m$, $j = 1, \ldots, s$.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded open set and let f be a smooth positive function. Let v_ρ be a sequence of solutions of $(Q)_\rho$ such that* $\sup_\rho T_\rho < +\infty$, $T_\rho = \rho^2 \int_\Omega |z - p_1|^{2\alpha_1} \cdots |z - p_N|^{2\alpha_N} f(z) e^{v_\rho}$. *If $T_\rho \to 0$ as $\rho \to 0$, then $v_\rho \to 0$ in $C^{2,\beta}(\Omega)$ and, for ρ small, v_ρ coincides with the unique minimal solution of $(Q)_\rho$. If $T_\rho \to L \neq 0$, then (up to a subsequence) there exists a nonempty finite set $S = \{q_1, \ldots, q_K\} \subset \Omega$ (blowup set) such that $\rho^2 |z - p_1|^{2\alpha_1} \cdots |z - p_N|^{2\alpha_N} f(z) e^{v_\rho} \rightharpoonup \sum_{i=1}^K b_i \delta_{q_i}$ in the sense of measures and $v_\rho \to \sum_{i=1}^K b_i G(z, q_i)$ in $C^{2,\beta}_{loc}(\Omega \setminus S)$ for some $\beta \in (0, 1)$, with $b_i = 8\pi$ if $q_i \notin \Gamma$, or $b_i = 8\pi(1 + \alpha_j)$ if $q_i = p_j$ for some $j = 1, \ldots, N$.*

Moreover, if $S \cap \Gamma = \emptyset$, then $(q_1, \dots, q_K)$ is a critical point for the function $\tilde{\mathcal{F}}$; if $S \cap \Gamma = \{p_{j_1}, \dots, p_{j_s}\}$ and $S \setminus \Gamma = \{q_{i_1}, \dots, q_{i_m}\}$ with $m + s = K$, then $(q_{i_1}, \dots, q_{i_m})$ is a critical point for the function $\tilde{\mathcal{F}} + \mathcal{G}(\cdot, p_{j_1}, \dots, p_{j_s})$.

For the existence of a minimal solution of $(Q)_\rho$, we refer the reader to [19]. As a vice versa of Theorem 1.1, we establish the following result.

THEOREM 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded open set, f be a smooth function, and $\{\alpha_1, \dots, \alpha_N\} \subset (0, +\infty) \setminus \mathbb{N}$ be real numbers. We have*

(a) *let $S = \{p_{j_1}, \dots, p_{j_s}\} \subset \Gamma$, then there exist $\rho_0 > 0$ small and a family $\{v_\rho\}_{0<\rho<\rho_0}$ of solutions for equation $(Q)_\rho$ such that $\rho^2|z - p_1|^{2\alpha_1} \cdots |z - p_N|^{2\alpha_N} f(z)e^{v_\rho} \rightharpoonup \sum_{i=1}^{s} 8\pi(1 + \alpha_{j_i})\delta_{p_{j_i}}$ in the sense of measures and $v_\rho \to \sum_{i=1}^{s} 8\pi(1 + \alpha_{j_i})G(z, p_{j_i})$ in $C^{2,\beta}_{loc}(\Omega \setminus S)$ for some $\beta \in (0, 1)$;*

(b) *let $S = \{q_1, \dots, q_m\} \subset \Omega' \setminus \Gamma$ and $(q_1, \dots, q_m)$ be a nondegenerate critical point of $\tilde{\mathcal{F}}$ such that $\Delta \ln f(q_1) = \cdots = \Delta \ln f(q_m) = 0$, then there exist $\rho_0 > 0$ small and a family $\{v_\rho\}_{0<\rho<\rho_0}$ of solutions for $(Q)_\rho$ such that $\rho^2|z-p_1|^{2\alpha_1} \cdots |z-p_N|^{2\alpha_N} f(z)e^{v_\rho} \rightharpoonup \sum_{i=1}^{m} 8\pi\delta_{q_i}$ in the sense of measures and $v_\rho \to \sum_{i=1}^{m} 8\pi G(z, q_i)$ in $C^{2,\beta}_{loc}(\Omega \setminus S)$ for some $\beta \in (0, 1)$;*

(c) *let S be such that $S \cap \Gamma = \{p_{j_1}, \dots, p_{j_s}\}$, $S \setminus \Gamma = \{q_1, \dots, q_m\}$, and $(q_1, \dots, q_m)$ is a critical point of $\tilde{\mathcal{F}} + \mathcal{G}(\cdot, p_{j_1}, \dots, p_{j_s})$ such that $\Delta \ln f(q_1) = \cdots = \Delta \ln f(q_m) = 0$, then there exist $\rho_0 > 0$ small and a family $\{v_\rho\}_{0<\rho<\rho_0}$ of solutions for $(Q)_\rho$ such that $\rho^2|z-p_1|^{2\alpha_1} \cdots |z-p_N|^{2\alpha_N} f(z)e^{v_\rho} \rightharpoonup \sum_{k=1}^{s} 8\pi(1+\alpha_{j_k})\delta_{p_{j_k}} + \sum_{j=1}^{m} 8\pi\delta_{q_j}$ in the sense of measures and $v_\rho \to \sum_{k=1}^{s} 8\pi(1 + \alpha_{j_k})G(z, p_{j_k}) + \sum_{j=1}^{m} 8\pi G(z, q_j)$ in $C^{2,\beta}_{loc}(\Omega \setminus S)$ for some $\beta \in (0, 1)$.*

Let us point out that the assumption $\Delta \ln f(q_i) = 0$ for any $q_i \in S \setminus \Gamma$ is always fulfilled by the original problem (2), so in some sense it seems a "natural" assumption from a physical point of view. In case $\Gamma = \emptyset$, part (b) in Theorem 1.2 gives a direct extension of the result in [4], which has largely motivated our approach. More precisely, it states the following.

COROLLARY 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded open set, f be a smooth function, and $S = \{q_1, \dots, q_m\} \subset \Omega'$ be a nonempty set. Assume that $(q_1, \dots, q_m)$ is a nondegenerate critical point of $\tilde{\mathcal{F}}(z_1, \dots, z_m) = \sum_{i=1}^{m} H(z_i, z_i) + \sum_{i \neq j} G(z_i, z_j) + \frac{1}{4\pi} \sum_{i=1}^{m} \ln f(z_i)$ in $(\Omega')^m$ such that $\Delta \ln f(q_1) = \cdots = \Delta \ln f(q_m) = 0$. There exist $\rho_0 > 0$ small and a family $\{v_\rho\}_{0<\rho<\rho_0}$ of solutions for the equation*

$$\begin{cases} -\triangle v = \rho^2 f(z)e^v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $\rho^2 f(z)e^{v_\rho} \rightharpoonup \sum_{i=1}^{m} 8\pi\delta_{q_i}$ in the sense of measures and $v_\rho \to \sum_{i=1}^{m} 8\pi G(z, q_i)$ in $C^{2,\beta}_{loc}(\Omega \setminus S)$ for some $\beta \in (0, 1)$.

Thus, from Corollary 1.3 the result in [4] is recovered by taking $f = \text{const} > 0$.

To avoid technicalities, we derive the proof of Theorem 1.2 only in the following significant cases: (a) holds with $S = \{p\}$ and $p \in \Gamma$, (b) holds with $S = \{q\}$ and $q \notin \Gamma$, and (c) holds with $S = \{p, q\}$, $p \in \Gamma$, and $q \notin \Gamma$. Our approach generalizes to any number of "peaks," the technical details are worked out in [20]. So we restrict our attention to the problem

$$(P)_\rho \qquad \begin{cases} -\triangle v = \rho^2 |z - p|^{2\alpha} f(z)e^v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha \in (0, +\infty) \setminus \mathbb{N}$ is a real number and $f : \Omega \to \mathbb{R}$ is a smooth function not necessarily positive. We will prove the following result.

THEOREM 1.4. *Under the above assumptions we have*

(a) *if* $p \in \Omega$ *with* $f(p) > 0$, *there exist* $\rho_0 > 0$ *small and a family* $\{v_\rho\}_{0<\rho<\rho_0}$ *of solutions for* $(P)_\rho$ *such that* $\rho^2|z-p|^{2\alpha} f(z)e^{v_\rho} \rightharpoonup 8\pi(1+\alpha)\delta_p$ *in the sense of measures and* $v_\rho \to 8\pi(1+\alpha)G(z,p)$ *in* $C^{2,\beta}_{loc}(\Omega \setminus \{p\})$ *for some* $\beta \in (0,1)$;

(b) *if* $q \in \Omega' \setminus \{p\}$ *is a nondegenerate critical point of* $\mathcal{F}(z) = H(z,z) + \frac{1}{4\pi}\ln[|z-p|^{2\alpha} f(z)]$ *in* $\Omega' \setminus \{p\}$ *such that* $\Delta \ln f(q) = 0$, *then there exist* $\rho_0 > 0$ *small and a family* $\{v_\rho\}_{0<\rho<\rho_0}$ *of solutions for* $(P)_\rho$ *such that* $\rho^2|z-p|^{2\alpha} f(z)e^{v_\rho} \rightharpoonup 8\pi\delta_q$ *in the sense of measures and* $v_\rho \to 8\pi G(z,q)$ *in* $C^{2,\beta}_{loc}(\Omega \setminus \{q\})$ *for some* $\beta \in (0,1)$;

(c) *if* $p \in \Omega$ *with* $f(p) > 0$ *and* $q \neq p$ *is a nondegenerate critical point of* $\mathcal{F}(z) = \tilde{\mathcal{F}}(z) + \mathcal{G}(z,p) = H(z,z) + \frac{1}{4\pi}\ln(|z-p|^{2\alpha} f(z)) + 2(1+\alpha)G(z,p)$ *in* $\Omega' \setminus \{p\}$ *such that* $\Delta \ln f(q) = 0$, *then there exist* $\rho_0 > 0$ *small and a family* $\{v_\rho\}_{0<\rho<\rho_0}$ *of solutions for* $(P)_\rho$ *such that* $\rho^2|z-p|^{2\alpha} f(z)e^{v_\rho} \rightharpoonup 8\pi(1+\alpha)\delta_p + 8\pi\delta_q$ *in the sense of measures and* $v_\rho \to 8\pi(1+\alpha)G(z,p) + 8\pi G(z,q)$ *in* $C^{2,\beta}_{loc}(\Omega \setminus \{p,q\})$ *for some* $\beta \in (0,1)$.

We now discuss some applications of the results above. As it is well known, for $\alpha \geq 0$, the problem

$$\begin{cases} -\triangle v = \lambda \frac{|z-p|^{2\alpha} f(z) e^v}{\int_\Omega |z-p|^{2\alpha} f(z) e^v} & \text{in } B(0,1), \\ v = 0 & \text{on } \partial B(0,1) \end{cases} \tag{3}$$

with $p = 0$ and $f(z) = 1$ possesses a radial solution for $0 < \lambda < 8\pi(\alpha+1)$ and, as a consequence of a Pohozaev identity, has no solution for $\lambda \geq 8\pi(\alpha+1)$. By means of Theorem 1.4, we can show that such a threshold for existence of (3) is no longer valid if we perturb (3) either by replacing $f = 1$ with a suitable nonconstant function or by moving p close to $\partial B(0,1)$. In fact we will be able to produce solutions v_ρ for (3) with $\lambda_\rho = \rho^2 \int_\Omega |z-p|^{2\alpha} f(z)e^{v_\rho} \to 8\pi(\alpha+1) + 8\pi > 8\pi(\alpha+1)$ concentrating on two points. According to Theorem 1.4, for this purpose we need to exhibit a nondegenerate critical point q for $\mathcal{F}(z) = H(z,z) - \frac{2+\alpha}{2\pi}\ln|z-p| + 2(1+\alpha)H(z,p) + \frac{1}{4\pi}\ln f(z)$ such that $\Delta \ln f(q) = 0$. Let us recall that $H(z,p) = \frac{1}{4\pi}\ln\left(|p|^2|z|^2 - 2\langle p,z\rangle + 1\right)$ and $H(z,z) = \frac{1}{2\pi}\ln\left(1-|z|^2\right)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^2$. Hence we obtain for $\mathcal{F}(z)$ the expression

$$\mathcal{F}(z) = \frac{1}{2\pi}\ln\left(1-|z|^2\right) - \frac{2+\alpha}{2\pi}\ln|z-p| + \frac{1+\alpha}{2\pi}\ln\left(|p|^2|z|^2 - 2\langle p,z\rangle + 1\right) + \frac{1}{4\pi}\ln f(z).$$

Example 1.5. We study now the case $p = 0$. For fixed $q \in B(0,1) \setminus \{0\}$, we can define a function $f(z)$ such that in a small neighborhood of q it takes the form

$$f(z) = \exp\left((z_1-q_1)^2 - c_q(z_2-q_2)^2 - 2\ln\left(1-|z|^2\right) + 2(2+\alpha)\ln|z|\right),$$

where $c_q = 1 + \frac{4}{(1-|q|^2)^2} > 0$. For such a function $f(z)$, the function $\mathcal{F}(z)$ near q takes the form $\mathcal{F}(z) = \frac{1}{4\pi}\left[(z_1-q_1)^2 - c_q(z_2-q_2)^2\right]$ and hence q is a nondegenerate critical point of $\mathcal{F}(z)$ such that $\Delta \ln f(q) = 2 - 2c_q + \frac{8}{(1-|q|^2)^2} = 0$. Moreover, if we choose q such that $|q| = r_\alpha$, with $r_\alpha \in (0,1)$ satisfying $r_\alpha^{2+\alpha} + r_\alpha^2 - 1 = 0$, then such an f may be constructed as a small perturbation of the constant function 1. In fact, for ϵ small we can just take f_ϵ of the form

$$f_\epsilon(z) = \left(1 - \chi\left(\frac{z-q}{\epsilon}\right)\right) + \chi\left(\frac{z-q}{\epsilon}\right) \times \exp\left(\epsilon(z_1-q_1)^2 - c_\epsilon(z_2-q_2)^2 - 2\ln\left(1-|z|^2\right) + 2(2+\alpha)\ln|z|\right),$$

where $c_\epsilon = \epsilon + \frac{4}{(1-|q|^2)^2}$ and $0 \le \chi \le 1$ is a smooth cut-off function such that $\chi = 1$ in $B(0,1)$ and $\chi = 0$ in $\mathbb{R}^2 \setminus B(0,2)$.

Example 1.6. We study now the case $f(z) = 1$. The function $\mathcal{F}(z)$ becomes

$$\mathcal{F}(z) = \frac{1}{2\pi} \ln\left(1 - |z|^2\right) - \frac{2+\alpha}{2\pi} \ln|z-p| + \frac{1+\alpha}{2\pi} \ln\left(|p|^2|z|^2 - 2\langle p, z\rangle + 1\right).$$

Let us remark that, according to the nonexistence result stated above, for $p = 0$ the function $\mathcal{F}(z) = \frac{1}{2\pi}\ln\left(1-|z|^2\right) - \frac{2+\alpha}{2\pi}\ln|z|$ has no critical points in $B(0,1)\setminus\{0\}$ and this remains true for p close to zero. On the other hand, we can take $p \in B(0,1)$ such that $p \to e \in \partial B(0,1)$ along a straight line. We consider a point $q = se$ for $s \in (-1,1)$. We have that

$$\nabla\mathcal{F}(q) = \left(\frac{(\alpha+2)s+\alpha}{2\pi(s^2-1)} + o(1)\right) e \quad \text{ as } p \to e$$

for $|s-1|$ bounded away from zero. Let $s_0 = -\frac{\alpha}{\alpha+2}$, for p close to e we find a point s_p such that $\nabla\mathcal{F}(s_p e) = 0$ and $s_p \to s_0$ as $p \to e$. We evaluate now the determinant of $D^2\mathcal{F}(s_p e)$:

$$\det D^2\mathcal{F}(s_p e) = \frac{(\alpha+2)^6}{64\pi^2(\alpha+1)^3} + o(1) \quad \text{ as } p \to e.$$

Hence $q_p = s_p e$ is a nondegenerate critical point of $\mathcal{F}(z)$ for p close to e such that $q_p \to -\frac{\alpha}{\alpha+2} e$ as $p \to e$.

As in [4], Theorem 1.4 is based on the construction of a suitable family of approximate solutions $v(\rho,\lambda,a)$ for problem $(P)_\rho$, with (λ, a) a suitable set of parameters, such that the linearized operator about $v(\rho,\lambda,a)$ is invertible. Thus, for ρ small a fixed point argument will provide a solution v_ρ close in some sense to $v(\rho,\lambda,a)$ with the required asymptotic properties.

2. Construction of approximating solutions. As far as part (a) in Theorem 1.4 is concerned, in view of the expected asymptotic behavior, the approximating function $v(\rho,0,0)$ will be constructed by gluing in a small neighborhood of p the limit function $8\pi(1+\alpha)G(z,p)$ with a suitable local solution of $-\Delta v = \rho^2|z-p|^{2\alpha}f(p)e^v$. Using the scale invariance $v(z) \to v_t(z) = v(tz) + 2(\alpha+1)\ln t$, $t > 0$, valid for the solutions of the equation

$$-\Delta v = \rho^2 |z|^{2\alpha} e^v, \tag{4}$$

we can construct local solutions which are very concentrated near p in such a way that the gluing with $8\pi(1+\alpha)G(z,p)$ is sufficiently accurate. This is possible in view of the fact that $8\pi(1+\alpha)G(z,p) \to +\infty$ as $z \to p$. For part (b) in Theorem 1.4, we glue in a small neighborhood of q the limit function $8\pi G(z,q)$ with a suitable local solution of $-\Delta v = \rho^2|q-p|^{2\alpha}f(q)e^v$. The scale invariance involved here is $v(z) \to v_t(z) = v(tz) + 2\ln t$, $t > 0$, valid for solutions of

$$-\Delta v = \rho^2 e^v. \tag{5}$$

Finally, for part (c) in Theorem 1.4 we combine the two previous constructions by gluing the limit function $8\pi(1+\alpha)G(z,p) + 8\pi G(z,q)$ with a local solution of $-\Delta v = \rho^2|z-p|^{2\alpha}f(p)e^v$ near p and with a local solution of $-\Delta v = \rho^2|q-p|^{2\alpha}f(q)e^v$ near q.

To this purpose, we recall some known facts. The solutions of $-\Delta u = e^u$ in $\mathbb{R}^2$ have been completely classified by Liouville in [24] and in complex notations they satisfy the so-called Liouville formula

$$\ln \frac{8|F'(z)|^2}{(1+|F(z)|^2)^2} \tag{6}$$

for some meromorphic function F with $F'(z) \neq 0$ whenever defined.

This representation formula generalizes to solutions in the punctured plane $\mathbb{C}\setminus\{0\}$, as proved in [18], by choosing some multivalued meromorphic function $F : \mathbb{C} \to \mathbb{C}$, locally univalent in $\mathbb{C} \setminus \{0\}$, satisfying

$$\text{either} \qquad F(z) = G(z)z^\gamma,\ \gamma \in \mathbb{R}, \qquad \text{or} \qquad F(z) = \Phi(\sqrt{z}),$$

where G and Φ are single-valued holomorphic functions away from the origin and where $\Phi(z)\Phi(-z) = 1$.

A complete classification for solutions of

$$\begin{cases} -\Delta u = e^u & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < +\infty \end{cases} \tag{7}$$

can be performed either by the Liouville formula or via the moving plane method (see [16]) and it leads to the only possible choice of $F(z) = az + b$, with $a, b \in \mathbb{C}$. The complete classification for solutions of

$$\begin{cases} -\Delta u = e^u - 4\pi\alpha\delta_{p=0} & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^u < +\infty \end{cases} \tag{8}$$

is due to [31] and it corresponds to the choice $F(z) = az^{\alpha+1} + b$, with $a, b \in \mathbb{C}$ and $b = 0$ if $\alpha \notin \mathbb{N}$. By choosing $F(z) = \frac{1}{\tau\rho} z(1+\gamma z^2)$, $\tau > 0$, $\gamma \in \mathbb{C}$ such that $|\gamma| < \frac{1}{3}$, and $F(z) = \frac{1}{\tau\rho} z^{\alpha+1}$, we can provide, respectively, solutions for (7) and (8) in $B(0,1)$. By taking the regular part of this functions and adding a term $2\ln\frac{1}{\rho}$, we obtain a large class of solutions for (4) and (5) in $B(0,1)$, respectively, in the form

$$v_{\rho,\tau} = \ln \frac{8(\alpha+1)^2\tau^2}{(\tau^2\rho^2 + |z|^{2(\alpha+1)})^2}, \qquad v_{\rho,\tau,\gamma} = \ln \frac{8\tau^2|1+3\gamma z^2|^2}{(\tau^2\rho^2 + |z|^2|1+\gamma z^2|^2)^2}. \tag{9}$$

Let $h(z)$ be some smooth function such that $h(0) > 0$. The function $v_{\rho,\tau} - \ln h(0)$ satisfies the equation $-\Delta\left(v_{\rho,\tau} - \ln h(0)\right) = \rho^2 h(0)|z|^{2\alpha} e^{v_{\rho,\tau} - \ln h(0)}$ in $B(0,1)$. Similarly $v_{\rho,\tau,0} - \ln h(0)$ is a solution for $-\Delta\left(v_{\rho,\tau,0} - \ln h(0)\right) = \rho^2 h(0) e^{v_{\rho,\tau,0} - \ln h(0)}$ in $B(0,1)$. For $\rho > 0$ small, they can be viewed as approximating solutions when we replace $h(0)$ by $h(z)$: such an approximation, however, may not be accurate enough to carry out our fixed point argument. In fact, we will need to define the local approximating solution $U_{\rho,\tau}$ as the difference between, respectively, $v_{\rho,\tau}$, $v_{\rho,\tau,0}$ and a Taylor expansion of $\ln h(z)$ at $z = 0$, taking into account two basic facts:

(a) $U_{\rho,\tau}$ must be a "good" local approximating solution;

(b) translating $U_{\rho,\tau}$ at some point $q \in S$, the difference between this local function and the related limit function as $\rho \to 0$ must be small in a small annulus centered at q.

In case $\alpha > 0$, $v_{\rho,\tau} - \ln h(0)$ is satisfactory for (a). For (b), if $p \in S \cap \Gamma$, we choose some $\tau > 0$ such that the Taylor expansion corresponding to the difference function in a small annulus centered in p contains powers of $z-p$ of degree 1. In case $\alpha=0$ the situation is more delicate as there is more degeneracy. Assuming $\Delta \ln h(0)=0$, for (a) we need to take the local function $U_{\rho,\tau}$ of the form $v_{\rho,\tau,\gamma}-\ln h(0)-2\bar z\cdot\partial_z \ln h(0)-\bar z^2\cdot\partial_{zz}\ln h(0)$. While for (b) we need the difference function to be an infinitesimal term of order 3 as $z\to q$. This condition will be attained by specifying $\tau>0$ and γ suitably and by the condition $\partial_z\mathcal{F}(q)=0$. The invertibility of $D^2\mathcal{F}(q)$ will guarantee the invertibility of the linearized operator around such an approximating solution at $\rho=0$.

Summarizing, an appropriate approximating solution for our problem near a blowup point should look like

$$U_{\rho,\tau}(z)=\begin{cases} v_{\rho,\tau}(z)-\ln h(0) & \text{if } \alpha>0,\\ v_{\rho,\tau,\gamma}(z)-\ln h(0)-2\bar z\cdot\partial_z\ln h(0)-\bar z^2\cdot\partial_{zz}\ln h(0) & \text{if } \alpha=0\end{cases}$$

with τ and γ suitably chosen. Introduce the differential operators $\partial_z=\frac{1}{2}(\partial_1-i\partial_2)$, $\partial_{\bar z}=\frac12(\partial_1+i\partial_2)$, and the notation $2z\cdot z'=z\overline{z'}+\bar z z'=2\mathrm{Re}\,(z\overline{z'})$. Thus $\Delta=4\partial_z\partial_{\bar z}$ and the Taylor expansion in 0 for any smooth function $h:\Omega\to\mathbb{R}$ takes the form

$$h(z)=h(0)+2\bar z\cdot\partial_z h(0)+\bar z^2\cdot\partial_{zz}h(0)+\frac{|z|^2}{4}\Delta h(0)+O(|z|^3).$$

Hence $U_{\rho,\tau}$ is a solution in $B(0,1)$ of

$$-\Delta U_{\rho,\tau}=\begin{cases}\rho^2|z|^{2\alpha}h(0)e^{U_{\rho,\tau}} & \text{if } \alpha>0,\\ \rho^2 e^{\ln h(0)+2\bar z\cdot\partial_z\ln h(0)+\bar z^2\cdot\partial_{zz}\ln h(0)}e^{U_{\rho,\tau}} & \text{if } \alpha=0,\end{cases}\tag{10}$$

and we see that the right-hand side (RHS) of (10) may be expressed as follows:

$$\text{RHS }=\begin{cases}\rho^2|z|^{2\alpha}h(z)e^{U_{\rho,\tau}}+O\left(\rho^2|z|^{2\alpha+1}e^{U_{\rho,\tau}}\right) & \text{if } \alpha>0,\\ \rho^2h(z)e^{U_{\rho,\tau}}+O\left(\rho^2|z|^3e^{U_{\rho,\tau}}\right) & \text{if } \alpha=0\end{cases}$$

provided that when $\alpha=0$ we also satisfy $\Delta\ln h(0)=0$.

By the assumptions in Theorem 1.4, we may translate the function $U_{\rho,\tau}(z)$ around the points p and q by defining

$$\begin{cases} U^1_\rho(z)=v_{\rho,\tau_1}(z-p)-\ln f(p),\\ U^2_\rho(z)=v_{\rho,\tau_2,\gamma}(z-q)-\ln\left(|z-p|^{2\alpha}f\right)(q)-2\overline{z-q}\cdot\partial_z\ln\left(|z-p|^{2\alpha}f\right)(q)\\ \qquad\qquad -\overline{z-q}^2\cdot\partial_{zz}\ln\left(|z-p|^{2\alpha}f\right)(q)\end{cases}$$

with τ_1, τ_2, and γ to be specified below. Thus, we have

$$\Delta U^i_\rho(z)+\rho^2|z-p|^{2\alpha}f(z)e^{U^i_\rho(z)}=\begin{cases} O\left(\rho^2|z-p|^{2\alpha+1}e^{U^1_\rho(z)}\right) & \text{in } B(p,1),\\ O\left(\rho^2|z-q|^3e^{U^2_\rho(z)}\right) & \text{in } B(q,1).\end{cases}\tag{11}$$

Note that the following expansions hold as $\rho\to0$:

$$v_{\rho,\tau}(z-p)=\ln 8(1+\alpha)^2\tau^2-4(1+\alpha)\ln|z-p|+O\left(\frac{\tau^2\rho^2}{|z-p|^{2(\alpha+1)}}\right),$$
$$v_{\rho,\tau,\gamma}(z-q)=\ln 8\tau^2-4\ln|z-q|+2\overline{z-q}^2\cdot\gamma+O\left(|z-q|^4+\frac{\tau^2\rho^2}{|z-q|^2}\right).$$

Let us define the limit function $L(z)$ as

$$L(z) = \begin{cases} 8\pi(1+\alpha)G(z,p) & \text{if } S = \{p\}, \\ 8\pi G(z,q) & \text{if } S = \{q\}, \\ 8\pi(1+\alpha)G(z,p) + 8\pi G(z,q) & \text{if } S = \{p,q\}. \end{cases}$$

Hence in $|z-p| < 1$ we get

$$U_\rho^1(z) - L(z) = \ln 8(\alpha+1)^2\tau_1^2 - \mathcal{F}_1(p) + O\left(\frac{\tau_1^2\rho^2}{|z-p|^{2(1+\alpha)}} + |z-p|\right),$$

while for $|z-q| < 1$,

$$\begin{aligned} U_\rho^2(z) - L(z) &= \ln 8\tau_2^2 - \mathcal{F}_2(q) - 2\overline{z-q}\cdot\partial_z\mathcal{F}_2(q) \\ &-\overline{z-q}^2\cdot(\partial_{zz}\mathcal{F}_2(q) - 2\gamma) + O\left(\frac{\tau_2^2\rho^2}{|z-q|^2} + |z-q|^3\right), \end{aligned}$$

where

$$\mathcal{F}_1(z) = \begin{cases} 8\pi(1+\alpha)H(z,p) + \ln f(z) & \text{if } S = \{p\}, \\ 8\pi(1+\alpha)H(z,p) + 8\pi G(z,q) + \ln f(z) & \text{if } S = \{p,q\} \end{cases}$$

and

$$\mathcal{F}_2(z) = \begin{cases} 8\pi H(z,q) + \ln\left(|z-p|^{2\alpha}f(z)\right) & \text{if } S = \{q\}, \\ 8\pi H(z,q) + \ln\left(|z-p|^{2\alpha}f(z)\right) + 8\pi(1+\alpha)G(z,p) & \text{if } S = \{p,q\}. \end{cases}$$

Let us remark that by assumption $\partial_z\mathcal{F}_2(q) = 0$. Now we specify the values for τ_1, τ_2, and γ to be fixed as follows:

$$\tau_1 = \frac{e^{\frac{1}{2}\mathcal{F}_1(p)}}{\sqrt{8}(1+\alpha)}, \qquad \tau_2 = \frac{e^{\frac{1}{2}\mathcal{F}_2(q)}}{\sqrt{8}}, \qquad \gamma = \frac{1}{2}\partial_{zz}\mathcal{F}_2(q).$$

In such a way we obtain

$$U_\rho^1(z) - L(z) = O\left(\frac{\tau_1^2\rho^2}{|z-p|^{2(1+\alpha)}} + |z-p|\right) \quad \text{in } |z-p| < 1 \tag{12}$$

and

$$U_\rho^2(z) - L(z) = O\left(\frac{\tau_2^2\rho^2}{|z-q|^2} + |z-q|^3\right) \quad \text{in } |z-q| < 1. \tag{13}$$

By scaling the variables, we can always assume that $\overline{B(p,2)}\cap\overline{B(q,2)} = \emptyset$, $\overline{B(p,2)} \subset \Omega$, $\overline{B(q,2)} \subset \Omega$, and $|\gamma| < \frac{1}{3}$. For $i = 1,2$ let $r_i = r_i(\rho)$ be a positive smooth function such that $\frac{\rho^2}{r_1^{4\alpha+5}} = O(1)$ as $\rho \to 0$ and $\frac{\rho^2}{r_2^5} = O(1)$ as $\rho \to 0$. Let χ be a radial smooth function such that $0 \le \chi \le 1$, $\chi = 1$ in $B(0,1)$, and $\chi = 0$ in $\mathbb{R}^2 \setminus B(0,2)$.

To obtain part (a) in Theorem 1.4, for $\lambda_1 \in \mathbb{R}$, $|\lambda_1| < \frac{1}{2}\tau_1$, we consider the approximating function

$$\begin{aligned} v(\rho,\lambda_1)(z) &= \left(1 - \chi\left(\frac{z-p}{r_1}\right)\right) 8\pi(1+\alpha)G(z,p) \\ &+ \chi\left(\frac{z-p}{r_1}\right)(v_{\rho,\tau_1+\lambda_1}(z-p) - \ln f(p)). \end{aligned}$$

So $v(\rho,0)=U^1_\rho$ in $|z-p|<r_1$. For part (b) in Theorem 1.4, we need a three-parameter family of approximating functions and, for $(\lambda_2,a)\in\mathbb{R}\times\mathbb{C}$, $|\lambda_2|<\frac{1}{2}\tau_2$, $|a|<\frac{1}{2}$, and $g(z)=|z-p|^{2\alpha}f(z)$, we consider

$$\begin{aligned} v(\rho,\lambda_2,a)(z) \;&= \left(1-\chi\left(\frac{z-q-a}{r_2}\right)\right) 8\pi G(z,q+a)\\ &\quad +\chi\left(\frac{z-q-a}{r_2}\right)(v_{\rho,\tau_2+\lambda_2,\gamma}(z-q-a)-P_a(z)), \end{aligned}$$

where $P_a(z)=\ln g(q+a)+2\overline{z-q-a}\cdot\partial_z\ln g(q+a)+\overline{z-q-a}^2\cdot\partial_{zz}\ln g(q+a)$. So $v(\rho,0,0)=U^2_\rho$ in $|z-q|<r_2$. Finally, for part (c) in Theorem 1.4, we need a four-parameter family of approximating functions and, for $(\lambda,a)\in\mathbb{R}^2\times\mathbb{C}$, $\lambda=(\lambda_1,\lambda_2)$, $|\lambda|<\frac{1}{2}\min\{\tau_1,\tau_2\}$, $|a|<\frac{1}{2}$, and $g(z)=|z-p|^{2\alpha}f(z)$, we take

$$\begin{aligned} v(\rho,\lambda,a)(z) \;&= \left(1-\chi\left(\frac{z-p}{r_1}\right)\right)(8\pi(1+\alpha)G(z,p)+8\pi G(z,q+a))\\ &\quad +\chi\left(\frac{z-p}{r_1}\right)(v_{\rho,\tau_1+\lambda_1}(z-p)-\ln f(p)) \quad \text{in } B(p,1), \end{aligned}$$

$$\begin{aligned} v(\rho,\lambda,a)(z) \;&= \left(1-\chi\left(\frac{z-q-a}{r_2}\right)\right)(8\pi(1+\alpha)G(z,p)+8\pi G(z,q+a))\\ &\quad +\chi\left(\frac{z-q-a}{r_2}\right)(v_{\rho,\tau_2+\lambda_2,\gamma}(z-q-a)-P_a(z)) \quad \text{in } B(q,1), \end{aligned}$$

and

$$v(\rho,\lambda,a)(z)=8\pi(1+\alpha)G(z,p)+8\pi G(z,q+a) \qquad \text{in } \Omega\setminus(B(p,1)\cup B(q,1)).$$

To unify notation, from now on we will use the convention that

- $\lambda_2=0$, $a=0$ if $S=\{p\}$,
- $\lambda_1=0$ if $S=\{q\}$,
- every expression containing p (or q) does really exist only if $p\in S$ (or $q\in S$).

We remark that in such a way the last definition of $v(\rho,\lambda,a)$ contains the previous ones and (λ,a) always lie in $\mathbb{R}^2\times\mathbb{C}$.

3. A fixed point argument. In this section we obtain the desired existence result by means of a fixed point argument. To this end we have postponed the proof of the most technical aspects necessary to such an approach in the next two sections.

For $a\in\mathbb{C}$, $|a|<\frac{1}{2}$, it is possible to construct a diffeomorphism $\Psi(a,\cdot):\Omega\to\Omega$, smoothly depending on a, such that $\Psi(0,\cdot)=\mathrm{Id}$, $\Psi(a,z)=z-a$ for all $z\in B(q,\frac{3}{2})$, and $\Psi(a,\cdot)=\mathrm{Id}$ for all $z\in\Omega\setminus B(q,2)$. We can suppose that all derivatives of $\Psi(a,z)$ in a, $\bar a$, z, $\bar z$ up to order 3 are bounded in Ω.

We define now suitable function spaces of weighted Hölder type appropriate for our problem, which were introduced for the first time by Caffarelli, Hardt, and Simon in [9].

DEFINITION 3.1. *For any $\nu\in\mathbb{R}$, $k\in\mathbb{N}$, $\beta\in[0,1]$, define the space*

$$C^{k,\beta}_\nu(B(0,1)):=\{w\in C^{k,\beta}(B(0,1)\setminus\{0\},\mathbb{R}) \,:\, \|w\|_{k,\beta,\nu}<+\infty\},$$

where

$$\|w\|_{k,\beta,\nu} := \sup_{r\le 1} r^{-\nu}\left\{\sup_{\{z:\,\frac{r}{2}<|z|<r\}}\left(\sum_{j=0}^{k} r^j|\nabla^j w(z)|\right)\right.$$
$$\left. + r^{k+\beta}\sup_{\{x,y:\,|x|,|y|\in(\frac{r}{2},r)\}}\left(\frac{|\nabla^k w(x)-\nabla^k w(y)|}{|x-y|^\beta}\right)\right\}.$$

Let $\nu_1 \in (0,1)$ and $\nu_2 \in (1,2)$ be two real numbers. Set $\tilde\Omega = \Omega\setminus B$, $B = B(p,1)\cup B(q,1)$, and define

$$X = \{w \in C^{2,\beta}(\Omega\setminus S,\mathbb{R}):\ w\equiv 0 \text{ on } \partial\Omega\,,\ \|w\|_X < +\infty\},$$

where $\|w\|_X = \|w\|_{2,\beta,\tilde\Omega} + \|w\|_{2,\beta,\nu_1,B(p,1)} + \|w\|_{2,\beta,\nu_2,B(q,1)}$, and

$$Y = \{w\in C^{0,\beta}(\Omega\setminus S,\mathbb{R}):\ \|w\|_Y < +\infty\},$$

where $\|w\|_Y = \|w\|_{0,\beta,\tilde\Omega} + \|w\|_{0,\beta,\nu_1-2,B(p,1)} + \|w\|_{0,\beta,\nu_2-2,B(q,1)}$.

We can replace the norm in X with an equivalent one (for ρ fixed) of the form

$$\|w\|_{X'} = \|w\|_{2,\beta,\tilde\Omega} + r_1^{\nu_1}\|w\|_{2,\beta,\nu_1,B(p,1)} + r_2^{\nu_2}\|w\|_{2,\beta,\nu_2,B(q,1)}$$

and we will refer to the space X, endowed with the norm $\|\cdot\|_{X'}$, as X'.

Finally, we define

$$\mathcal{E} = \{(w,\lambda,a) : w\in X,\ \lambda\in\mathbb{R}^2,\ a\in\mathbb{C}\}$$

with the norm $\|(w,\lambda,a)\|_{\mathcal{E}} = \|w\|_X + |\lambda| + |a|$, and $\mathcal{E}'$ as the space $\mathcal{E}$ endowed with the equivalent norm $\|(w,\lambda,a)\|_{\mathcal{E}'} = \|w\|_{X'} + |\lambda| + |a|$.

We can produce a solution $v(\rho,\lambda,a) + w\circ\Psi(a,\cdot)$, $(w,\lambda,a)\in\mathcal{E}'$, for problem $(P)_\rho$ if (w,λ,a) is a zero for the nonlinear map

$$\begin{aligned} N : \mathcal{E}' &\to Y\\ (w,\lambda,a) &\to N(w,\lambda,a) = \Delta\left[v(\rho,\lambda,a) + w\circ\Psi(a,\cdot)\right]\circ\Psi(a,\cdot)^{-1}\\ &\quad + \rho^2 g\circ\Psi(a,\cdot)^{-1} e^{v(\rho,\lambda,a)\circ\Psi(a,\cdot)^{-1}+w}, \end{aligned}$$

where $g(z) = |z-p|^{2\alpha} f(z)$. Define $L_{(0,\lambda,a)} : \mathcal{E}' \to Y$ as the linearized operator of N at $(0,\lambda,a)$. Hence,

$$\begin{aligned} L_{(0,\lambda,a)}(h,\sigma,b) &= \Delta\left(h\circ\Psi(a,\cdot)\right)\circ\Psi(a,\cdot)^{-1} + \rho^2 g\circ\Psi(a,\cdot)^{-1} e^{v(\rho,\lambda,a)\circ\Psi(a,\cdot)^{-1}} h\\ &\quad + \sum_i \sigma_i\left[\Delta\partial_{\lambda_i} v(\rho,\lambda,a) + \rho^2 g(z) e^{v(\rho,\lambda,a)}\partial_{\lambda_i} v(\rho,\lambda,a)\right]\circ\Psi(a,\cdot)^{-1}\\ &\quad + 2b\cdot\left[\Delta\partial_{\overline{a}} v(\rho,\lambda,a) + \rho^2 g(z) e^{v(\rho,\lambda,a)}\partial_{\overline{a}} v(\rho,\lambda,a)\right]\circ\Psi(a,\cdot)^{-1}\\ &\quad + 2\partial_{\bar z}\left[\Delta v(\rho,\lambda,a) + \rho^2 g(z) e^{v(\rho,\lambda,a)}\right]|_{\Psi(a,\cdot)^{-1}}\cdot\left[\left(b\partial_a + \bar b\partial_{\overline{a}}\right)\Psi(a,\cdot)^{-1}\right]. \end{aligned}$$

In Theorem 4.13 below, we show that the map $L_{(0,0,0)} : \mathcal{E}' \to Y$ is uniformly invertible for ρ small. We can decompose

$$\begin{aligned} &N(w,\lambda,a) - N(0,0,0) - L_{(0,0,0)}(w,\lambda,a)\\ &\quad = \left[N(w,\lambda,a) - N(0,\lambda,a) - L_{(0,\lambda,a)}(w,0,0)\right] + \left(L_{(0,\lambda,a)} - L_{(0,0,0)}\right)(w,0,0)\\ &\qquad + \left[N(0,\lambda,a) - N(0,0,0) - L_{(0,0,0)}(0,\lambda,a)\right]. \end{aligned}$$

In the following three steps we estimate in Y each term above. For simplicity, we show only how to derive the estimates for the L^∞ part in $\|\cdot\|_Y$ since the estimates of the Hölder term can be established in a similar way: all along the paper we will use implicitly this fact to simplify all the computations.

Step 1. Let

$$\begin{aligned} f_1(w,\lambda,a) &= N(w,\lambda,a) - N(0,\lambda,a) - L_{(0,\lambda,a)}(w,0,0) \\ &= \rho^2 g \circ \Psi(a,\cdot)^{-1} e^{v(\rho,\lambda,a)\circ\Psi(a,\cdot)^{-1}} (e^w - 1 - w). \end{aligned}$$

Since

$$\Psi(a,\cdot)^{-1}: \begin{array}{ll} B(q,1) & \to B(q+a,1) \\ z & \to z+a, \end{array}$$

we obtain the bounds

(14)

$$|\rho^2 g\circ\Psi(a,\cdot)^{-1}e^{v(\rho,\lambda,a)\circ\Psi(a,\cdot)^{-1}}| = \begin{cases} O\left(\frac{\rho^2|z-p|^{2\alpha}}{(\rho^2+|z-p|^{2\alpha+2})^2}\right) & \text{in } B(p,r_1), \\ O\left(\frac{\rho^2}{|z-p|^{2\alpha+4}}\right) & \text{in } B(p,1)\setminus B(p,r_1), \\ O\left(\frac{\rho^2}{(\rho^2+|z-q|^2)^2}\right) & \text{in } B(q,r_2), \\ O\left(\frac{\rho^2}{|z-q|^4}\right) & \text{in } B(q,1)\setminus B(q,r_2), \\ O(\rho^2) & \text{in } \tilde\Omega, \end{cases}$$

(15)

$$|\partial_{\overline{a}}\, \rho^2 g\circ\Psi(a,\cdot)^{-1}e^{v(\rho,\lambda,a)\circ\Psi(a,\cdot)^{-1}}| = \begin{cases} O\left(\frac{\rho^2}{(\rho^2+|z-q|^2)^2}\right) & \text{in } B(q,r_2), \\ O\left(\frac{\rho^2}{|z-q|^4}\right) & \text{in } B(q,1)\setminus B(q,r_2), \\ O(\rho^2) & \text{in } \Omega\setminus B(q,1), \end{cases}$$

(16)

$$|\partial_{\lambda_i}\, \rho^2 g\circ\Psi(a,\cdot)^{-1}e^{v(\rho,\lambda,a)\circ\Psi(a,\cdot)^{-1}}| = \begin{cases} O\left(\frac{\rho^2|z-p|^{2\alpha}}{(\rho^2+|z-p|^{2\alpha+2})^2}\right) & \text{in } B(p,2r_1) \text{ if } i=1, \\ O\left(\frac{\rho^2}{(\rho^2+|z-q|^2)^2}\right) & \text{in } B(q,2r_2) \text{ if } i=2, \\ 0 & \text{elsewhere.} \end{cases}$$

Hence, we can derive

$$\begin{aligned} &\|f_1(w_1,\lambda_1,a_1) - f_1(w_2,\lambda_2,a_2)\|_{0,\beta,\tilde\Omega} \\ &\quad = O\left[\rho^2(\|w_1\|_{2,\beta,\tilde\Omega} + \|w_2\|_{2,\beta,\tilde\Omega})\|(w_1,\lambda_1,a_1)-(w_2,\lambda_2,a_2)\|_{\mathcal{E}}\right], \\ &\|f_1(w_1,\lambda_1,a_1) - f_1(w_2,\lambda_2,a_2)\|_{0,\beta,\nu_1-2,B(p,1)} \\ &\quad = O\Big[\rho^{\frac{\nu_1}{\alpha+1}}(\|w_1-w_2\|_{2,\beta,\nu_1,B(p,1)} \\ &\qquad + |\lambda_1-\lambda_2| + |a_1-a_2|)\left(\|w_1\|_{2,\beta,\nu_1,B(p,1)} + \|w_2\|_{2,\beta,\nu_1,B(p,1)}\right)\Big], \end{aligned}$$

and

$$\begin{aligned}&\|f_1(w_1,\lambda_1,a_1)-f_1(w_2,\lambda_2,a_2)\|_{0,\beta,\nu_2-2,B(q,1)}\\&\quad=O\left[\rho^{\nu_2}\left(\|w_1-w_2\|_{2,\beta,\nu_2,B(q,1)}\right.\right.\\&\qquad\quad\left.\left.+|\lambda_1-\lambda_2|+|a_1-a_2|\right)\left(\|w_1\|_{2,\beta,\nu_2,B(q,1)}+\|w_2\|_{2,\beta,\nu_2,B(q,1)}\right)\right].\end{aligned}$$

Since $\frac{\rho^2}{r_1^{4\alpha+5}}+\frac{\rho^2}{r_2^5}=O(1)$, finally we get

$$\begin{aligned}\|f_1(w_1,\lambda_1,a_1)-f_1(w_2,\lambda_2,a_2)\|_Y=O\,[&\|(w_1,\lambda_1,a_1)-(w_2,\lambda_2,a_2)\|_{\mathcal{E}'}\\&\times(\|(w_1,\lambda_1,a_1)\|_{\mathcal{E}'}+\|(w_2,\lambda_2,a_2)\|_{\mathcal{E}'})]\,.\end{aligned}$$

Step 2. Define

$$f_2(w,\lambda,a)=\left(L_{(0,\lambda,a)}-L_{(0,0,0)}\right)(w,0,0)=f_2^1(w,a)+f_2^2(w,\lambda,a),$$

where

$$f_2^1(w,a)=\Delta\left[w\circ\Psi(a,\cdot)\right]\circ\Psi(a,\cdot)^{-1}-\Delta w$$

and

$$f_2^2(w,\lambda,a)=\rho^2 g\circ\Psi(a,\cdot)^{-1}e^{v(\rho,\lambda,a)\circ\Psi(a,\cdot)^{-1}}w-\rho^2 g e^{v(\rho,0,0)}w.$$

Using the identities

$$\begin{aligned}\partial_{\bar z}(w\circ\Psi)&=(\partial_z w\circ\Psi)\,\partial_{\bar z}\Psi+(\partial_{\bar z}w\circ\Psi)\,\partial_{\bar z}\bar\Psi,\\\Delta(w\circ\Psi)&=(\Delta w\circ\Psi)\left[|\partial_z\Psi|^2+|\partial_{\bar z}\Psi|^2\right]+8\mathrm{Re}\,\left[(\partial_{zz}w\circ\Psi)\,\partial_z\Psi\partial_{\bar z}\Psi\right]\\&\quad+8\mathrm{Re}\,\left[(\partial_z w\circ\Psi)\,\partial_{z\bar z}\Psi\right],\end{aligned}$$

we obtain

$$\begin{aligned}f_2^1(w_1,a_1)-f_2^1(w_2,a_2)&=\Delta(w_1-w_2)\left(|\partial_z\Psi(a_1,\cdot)|^2+|\partial_{\bar z}\Psi(a_1,\cdot)|^2-1\right)|_{\Psi(a_1,\cdot)^{-1}}\\&\quad+8\mathrm{Re}\left(\partial_{zz}(w_1-w_2)\partial_z\Psi(a_1,\cdot)\,|_{\Psi(a_1,\cdot)^{-1}}\,\partial_{\bar z}\Psi(a_1,\cdot)\,|_{\Psi(a_1,\cdot)^{-1}}\right)\\&\quad+8\mathrm{Re}\left(\partial_z(w_1-w_2)\partial_{z\bar z}\Psi(a_1,\cdot)\,|_{\Psi(a_1,\cdot)^{-1}}\right)\\&\quad+\Delta\left[w_2\circ\Psi(a_1,\cdot)\right]\circ\Psi(a_1,\cdot)^{-1}-\Delta\left[w_2\circ\Psi(a_2,\cdot)\right]\circ\Psi(a_2,\cdot)^{-1}.\end{aligned}$$

Therefore

$$\begin{aligned}f_2^1(w_1,a_1)-f_2^1(w_2,a_2)=O\,&\left(|D^2(w_1-w_2)||a_1|+|\nabla(w_1-w_2)||a_1|\right.\\&\left.+|D^2w_2||a_1-a_2|+|\nabla w_2||a_1-a_2|\right)\end{aligned}$$

in $B(q,2)\setminus B(q,1)$ and $f_2^1(w_1,a_1)-f_2^1(w_2,a_2)=0$ outside this region. Hence

$$\begin{aligned}\|f_2^1(w_1,a_1)-f_2^1(w_2,a_2)\|_Y=O\,[&\|(w_1,\lambda_1,a_1)-(w_2,\lambda_2,a_2)\|_{\mathcal{E}'}\\&\times(\|(w_1,\lambda_1,a_1)\|_{\mathcal{E}'}+\|(w_2,\lambda_2,a_2)\|_{\mathcal{E}'})]\,.\end{aligned}$$

By (14), (15), and (16), for f_2^2 we get

$$\begin{aligned}&\|f_2^2(w_1,\lambda_1,a_1)-f_2^2(w_2,\lambda_2,a_2)\|_Y\\&\quad=O\,[(\|w_1\|_X+\|w_2\|_X)\,\|(w_1,\lambda_1,a_1)-(w_2,\lambda_2,a_2)\|_{\mathcal{E}'}\\&\qquad+\|w_1-w_2\|_X\,(\|(w_1,\lambda_1,a_1)\|_{\mathcal{E}'}+\|(w_2,\lambda_2,a_2)\|_{\mathcal{E}'})]\,.\end{aligned}$$

Step 3. Set $f_3(\lambda,a)=N(0,\lambda,a)-N(0,0,0)-L_{(0,0,0)}(0,\lambda,a)$. Let us write explicitly $N(0,\lambda,a)$ in $B(p,1)$,

$$\begin{aligned}N(0,\lambda,a)(z)=&\frac{1}{r_1^2}\Delta\chi\left(\frac{z-p}{r_1}\right)\Delta^1(z)+\frac{8}{r_1}\partial_{\bar z}\chi\left(\frac{z-p}{r_1}\right)\cdot\partial_{\bar z}\Delta^1(z)\\&-\chi\left(\frac{z-p}{r_1}\right)\rho^2|z-p|^{2\alpha}e^{v_{\rho,\tau_1+\lambda_1}(z-p)}\\&+\rho^2|z-p|^{2\alpha}f(z)e^{8\pi(1+\alpha)G(z,p)+8\pi G(z,q+a)+\chi(\frac{z-p}{r_1})\Delta^1(z)}\end{aligned}$$

and in $B(q,1)$

$$\begin{aligned}N(0,\lambda,a)(z)=&\frac{1}{r_2^2}\Delta\chi\left(\frac{z-q}{r_2}\right)\Delta^2(z)+\frac{8}{r_2}\partial_{\bar z}\chi\left(\frac{z-q}{r_2}\right)\cdot\partial_{\bar z}\Delta^2(z)\\&-\chi\left(\frac{z-q}{r_2}\right)\rho^2e^{v_{\rho,\tau_2+\lambda_2,\gamma}(z-q)}\\&+\rho^2g(z+a)e^{8\pi(1+\alpha)G(z+a,p)+8\pi G(z+a,q+a)+\chi(\frac{z-q}{r_2})\Delta^2(z)},\end{aligned}$$

where $g(z)=|z-p|^{2\alpha}f(z)$ and

$$\begin{aligned}\Delta^1(z):=&\,v_{\rho,\tau_1+\lambda_1}(z-p)-\ln f(p)-8\pi(1+\alpha)G(z,p)-8\pi G(z,q+a)\text{ in }B(p,1),\\\Delta^2(z):=&\,v_{\rho,\tau_2+\lambda_2,\gamma}(z-q)-P_a(z+a)-8\pi(1+\alpha)G(z+a,p)\\&-8\pi G(z+a,q+a)\text{ in }B(q,1).\end{aligned}$$

In $B(p,1)$ we get

$$\begin{aligned}\partial_{\lambda_s\lambda_k}N(0,\lambda,a)(z)=&\frac{1}{r_1^2}\Delta\chi\left(\frac{z-p}{r_1}\right)v^1_{sk}(z)\delta_{s1}\delta_{k1}+\frac{8}{r_1}\partial_{\bar z}\chi\left(\frac{z-p}{r_1}\right)\cdot\partial_{\bar z}v^1_{sk}(z)\delta_{s1}\delta_{k1}\\&-\chi\left(\frac{z-p}{r_1}\right)\rho^2|z-p|^{2\alpha}e^{v_{\rho,\tau_1+\lambda_1}(z-p)}\left(v^1_s(z)v^1_k(z)+v^1_{sk}(z)\right)\delta_{s1}\delta_{k1}\\&+\rho^2|z-p|^{2\alpha}f(z)e^{8\pi(1+\alpha)G(z,p)+8\pi G(z,q+a)+\chi(\frac{z-p}{r_1})\Delta^1(z)}\\&\times\chi\left(\frac{z-p}{r_1}\right)\left(v^1_{sk}(z)+\chi\left(\frac{z-p}{r_1}\right)v^1_s(z)v^1_k(z)\right)\delta_{s1}\delta_{k1},\end{aligned}$$

$$\begin{aligned}\partial_{a\lambda_k}N(0,\lambda,a)(z)=&\,\rho^2|z-p|^{2\alpha}f(z)e^{v_{\rho,\tau_1+\lambda_1}(z-p)-\ln f(p)+(\chi(\frac{z-p}{r_1})-1)\Delta^1(z)}\\&\times\chi\left(\frac{z-p}{r_1}\right)\left(\chi\left(\frac{z-p}{r_1}\right)-1\right)v^1_k(z)\partial_a\Delta^1(z)\delta_{k1},\end{aligned}$$

and

$$\begin{aligned}\partial_{aa}N(0,\lambda,a)(z)=&\frac{1}{r_1^2}\Delta\chi\left(\frac{z-p}{r_1}\right)\partial_{aa}\Delta^1(z)+\frac{8}{r_1}\partial_{\bar z}\chi\left(\frac{z-p}{r_1}\right)\cdot\partial_{aa\bar z}\Delta^1(z)\\&+\rho^2|z-p|^{2\alpha}f(z)e^{v_{\rho,\tau_1+\lambda_1}(z-p)-\ln f(p)+(\chi(\frac{z-p}{r_1})-1)\Delta^1(z)}\\&\times\left\{\left(\chi\left(\frac{z-p}{r_1}\right)-1\right)\partial_{aa}\Delta^1(z)+\left(\chi\left(\frac{z-p}{r_1}\right)-1\right)^2\partial_a\Delta^1(z)\partial_a\Delta^1(z)\right\},\end{aligned}$$

where $v^1_j(z):=\partial_{\lambda_j}v_{\rho,\tau_1+\lambda_1}(z-p)$ and $v^1_{jm}(z):=\partial_{\lambda_j\lambda_m}v_{\rho,\tau_1+\lambda_1}(z-p)$ for any j,m.

Similarly, in $B(q,1)$ we get

$$\begin{aligned}\partial_{\lambda_s\lambda_k}N(0,\lambda,a)(z) &= \frac{1}{r_2^2}\Delta\chi\left(\frac{z-q}{r_2}\right)v_{sk}^2(z)\delta_{s2}\delta_{k2} + \frac{8}{r_2}\partial_{\bar z}\chi\left(\frac{z-q}{r_2}\right)\cdot\partial_{\bar z}v_{sk}^2(z)\delta_{s2}\delta_{k2}\\ &\quad -\chi\left(\frac{z-q}{r_2}\right)\rho^2 e^{v_{\rho,\tau_2+\lambda_2,\gamma}(z-q)}\left(v_s^2(z)v_k^2(z)+v_{sk}^2(z)\right)\delta_{s2}\delta_{k2}\\ &\quad +\rho^2 g(z+a)e^{8\pi(1+\alpha)G(z+a,p)+8\pi G(z+a,q+a)+\chi(\frac{z-q}{r_2})\Delta^2(z)}\\ &\quad \times\chi\left(\frac{z-q}{r_2}\right)\left(v_{sk}^2(z)+\chi\left(\frac{z-q}{r_2}\right)v_s^2(z)v_k^2(z)\right)\delta_{s2}\delta_{k2},\end{aligned}$$

$$\begin{aligned}\partial_{a\lambda_k}N(0,\lambda,a)(z) &= \rho^2 e^{v_{\rho,\tau_2+\lambda_2,\gamma}(z-q)+(\chi(\frac{z-q}{r_2})-1)\Delta^2(z)}\chi\left(\frac{z-q}{r_2}\right)v_k^2(z)\delta_{k2}\\ &\quad\times\Bigg[\left(\chi\left(\frac{z-q}{r_2}\right)-1\right)\partial_a\Delta^2(z)g(z+a)e^{-P_a(z+a)}\\ &\qquad +\partial_a\left(g(z+a)e^{-P_a(z+a)}\right)\Bigg],\end{aligned}$$

and

$$\begin{aligned}\partial_{aa}N(0,\lambda,a)(z) &= \frac{1}{r_2^2}\Delta\chi\left(\frac{z-q}{r_2}\right)\partial_{aa}\Delta^2(z) + \frac{8}{r_2}\partial_{\bar z}\chi\left(\frac{z-q}{r_2}\right)\cdot\partial_{aa\bar z}\Delta^2(z)\\ &\quad +\rho^2 e^{v_{\rho,\tau_2+\lambda_2,\gamma}(z-q)+(\chi(\frac{z-q}{r_2})-1)\Delta^2(z)}\Bigg\{\partial_{aa}\left(g(z+a)e^{-P_a(z+a)}\right)\\ &\quad +\left(\chi\left(\frac{z-q}{r_2}\right)-1\right)\Big[2\partial_a\Delta^2(z)\partial_a\Big(g(z+a)e^{-P_a(z+a)}\Big)\\ &\qquad\qquad +\partial_{aa}\Delta^2(z)g(z+a)e^{-P_a(z+a)}\Big]\\ &\quad +\left(\chi\left(\frac{z-q}{r_2}\right)-1\right)^2(\partial_a\Delta^2(z))^2 g(z+a)e^{-P_a(z+a)}\Bigg\},\end{aligned}$$

where $v_j^2(z) := \partial_{\lambda_j}v_{\rho,\tau_2+\lambda_2,\gamma}(z-q)$ and $v_{jm}^2(z) := \partial_{\lambda_j\lambda_m}v_{\rho,\tau_2+\lambda_2,\gamma}(z-q)$ for any j,m.

We have that

$$|\partial_\tau v_{\rho,\tau,\lambda}(z)| + |\partial_{\tau\tau}v_{\rho,\tau,\lambda}(z)| + |z||\nabla\partial_{\tau\tau}v_{\rho,\tau,\lambda}(z)| = O(1)$$

for $\lambda\in\{0,\gamma\}$, $f(z)e^{-\ln f(p)} = 1+O(|z-p|)$, $g(z+a)e^{-P_a(z+a)} = 1+O(|z-q|^2)$, and

$$|\partial_a\Delta^1(z)| + |\partial_{aa}\Delta^1(z)| + |z-p||\partial_{aa\bar z}\Delta^1(z)| = O(1) \quad \text{in } B(p,1),$$

$$|\partial_a\Delta^2(z)| + |\partial_{aa}\Delta^2(z)| + |z-q||\partial_{aa\bar z}\Delta^2(z)| = O(1) \quad \text{in } B(q,1),$$

$$\left|\partial_a\left(g(z+a)e^{-P_a(z+a)}\right)\right| + \left|\partial_{aa}\left(g(z+a)e^{-P_a(z+a)}\right)\right| = O(|z-q|^2).$$

So we can derive $\|\partial^2 N(0,\lambda,a)\|_{0,\beta,\nu_1-2,B(p,\frac12)} = O(r_1^{-\nu_1})$, $\|\partial^2 N(0,\lambda,a)\|_{0,\beta,\nu_2-2,B(q,\frac12)} = O(r_2^{-\nu_2})$, where ∂^2 denotes some second-order derivative of $N(0,\lambda,a)$ in the variables λ and a. Since $\|\partial^2 N(0,\lambda,a)\|_{0,\beta} = O(\rho^2)$ in $\tilde\Omega$, we conclude that $\|\partial^2 N(0,\lambda,a)\|_Y = O(\sum_{i=1}^2 r_i^{-\nu_i})$. Finally, we obtain

$$\|f_3(\lambda_1,a_1)-f_3(\lambda_2,a_2)\|_Y = O\left(\left(\sum_{i=1}^2 r_i^{-\nu_i}\right)\|(w_1,\lambda_1,a_1)-(w_2,\lambda_2,a_2)\|_{\mathcal{E}'}^2\right).$$

Step 4. We define

$$\begin{aligned} K : \mathcal{E}' &\to \mathcal{E}' \\ (w, \lambda, a) &\to -L^{-1}_{(0,0,0)} \left[N(0,0,0) + \left(N(w,\lambda,a) - N(0,0,0) - L_{(0,0,0)}(w,\lambda,a) \right) \right]. \end{aligned}$$

Let us remark that (w, λ, a) is a zero for $N \Leftrightarrow (w, \lambda, a)$ is a fixed point for K. Summarizing the previous steps and by means of the uniform estimates derived in Theorem 4.13 below, we have

$$\begin{aligned} &\|K(w_1, \lambda_1, a_1) - K(w_2, \lambda_2, a_2)\|_{\mathcal{E}'} \\ &\leq C_0 \left(\sum_{i=1}^{2} r_i^{-\nu_i} \right) \left(\|(w_1, \lambda_1, a_1)\|_{\mathcal{E}'} + \|(w_2, \lambda_2, a_2)\|_{\mathcal{E}'} \right) \\ &\times \|(w_1, \lambda_1, a_1) - (w_2, \lambda_2, a_2)\|_{\mathcal{E}'} \end{aligned}$$

for some constant $C_0 > 0$, where we have taken into account that

$$\|w\|_X \leq \left(\sum_{i=1}^{2} r_i^{-\nu_i} \right) \cdot \|(w, \lambda, a)\|_{\mathcal{E}'}.$$

We can choose $\nu_1 \in (0,1)$ and $\nu_2 \in (1,2)$ in such a way that $(\nu_1, 1 - \nu_1) \cap (\nu_2 - 1, 2 - \nu_2) \neq \emptyset$ and let us fix some $\delta > 0$ in this set. Define $\sigma = \frac{4\alpha+5}{2\nu_1} + 1$, $r_i = \rho^{\frac{1}{\sigma \nu_i}}$, and note that $N(0,0,0) = \eta$ where η is the error term defined and estimated in section 5. In fact, from the technical estimates contained in sections 4 and 5 we see that $\|L^{-1}_{(0,0,0)} \eta\|_{\mathcal{E}'} = O(r_1^{1-\delta} + r_2^{2-\delta})$ (see (37) below), and we get

$$\|K(w, \lambda, a)\|_{\mathcal{E}'} \leq C_1 \left[\left(\sum_{i=1}^{2} r_i^{-\nu_i} \right) \|(w, \lambda, a)\|^2_{\mathcal{E}'} + r_1^{1-\delta} + r_2^{2-\delta} \right]$$

for some constant $C_1 > 0$, where we have used the fact that $\|K(w,\lambda,a) - K(0,0,0)\|_{\mathcal{E}'} \leq C_0 (\sum_{i=1}^{2} r_i^{-\nu_i}) \|(w, \lambda, a)\|^2_{\mathcal{E}'}$. Thus, the suitable choice of r_1, r_2, as expressed by property (38) below, allows us to conclude that for ρ small the map K is a contraction of the space

$$\mathcal{E}' \cap \left\{ (w, \lambda, a) : \|(w, \lambda, a)\|_{\mathcal{E}'} \leq 2C_1 \left(r_1^{1-\delta} + r_2^{2-\delta} \right) \right\}$$

into itself. So there exists a unique fixed point $(w^\rho, \lambda^\rho, a^\rho)$ of the map K for $0 < \rho < \rho_0$, $\rho_0 > 0$ small, such that

$$\|w^\rho\|_{2,\beta,\tilde{\Omega}} + r_1^{\nu_1} \|w^\rho\|_{2,\beta,\nu_1,B(p,1)} + r_2^{\nu_2} \|w^\rho\|_{2,\beta,\nu_2,B(q,1)} + |\lambda^\rho| + |a^\rho| \leq 2C_1 \left(r_1^{1-\delta} + r_2^{2-\delta} \right).$$

Hence $v_\rho = v(\rho, \lambda^\rho, a^\rho) + w^\rho \circ \Psi(a^\rho, \cdot)$ is the solution we are looking for in Theorem 1.4. It admits the desired properties in view of the definition of $v(\rho, \lambda, a)$, the fact that $\frac{\rho^2}{r_1^{2\alpha+4}} + \frac{\rho^2}{r_2^4} \to 0$ as $\rho \to 0$ and $w^\rho \to 0$ uniformly in Ω and in $C^{2,\beta}_{loc}(\Omega \setminus S)$, as follows by (38).

4. Invertibility of the linearized operator $L_{(0,0,0)}$.

4.1. Some local operator. The radial case. We are interested in studying the linearized operator of the equation

$$-\Delta v = \rho^2 |z|^{2\alpha} e^v \quad \text{ in } B(0,1)\,,\ \ \alpha \geq 0 \tag{17}$$

about the radial solutions $v_{\rho,\tau}$ defined in (9) in case either $\alpha = 0$ or $\alpha \notin \mathbb{N}$. We define the linearized operator about $v_{\rho,\tau}$ by setting

$$L_{\rho,\tau} w = \Delta w + \rho^2 |z|^{2\alpha} e^{v_{\rho,\tau}} w$$

and we investigate the invertibility of $L_{\rho,\tau}$ under Dirichlet boundary condition. Inspired by the work of Caffarelli, Hardt, and Simon in [9] also used in [4], we have the following result.

PROPOSITION 4.1. *Let $\alpha \notin \mathbb{N}$. For all $\nu \in (0,1)$ and $\tau > 0$, there exist $\rho_0 > 0$, a continuous linear form $H^0_{\rho,\tau} : C^{0,\beta}_{\nu-2}(B(0,1)) \to \mathbb{R}$, and a linear operator $G_{\rho,\tau}$: $C^{0,\beta}_{\nu-2}(B(0,1)) \to C^{2,\beta}_{\nu}(B(0,1))$, uniformly bounded for $0 < \rho < \rho_0$, such that for all $\rho \in (0,\rho_0)$ and for all $f \in C^{0,\beta}_{\nu-2}(B(0,1))$ there exists a unique bounded solution w of*

$$\begin{cases} L_{\rho,\tau} w = f & \text{in } B(0,1), \\ w = 0 & \text{on } \partial B(0,1) \end{cases} \tag{18}$$

which can be uniquely decomposed as follows:

$$w(z) = G_{\rho,\tau}(f)(z) + H^0_{\rho,\tau}(f) \frac{\tau^2\rho^2 - |z|^{2(\alpha+1)}}{\tau^2\rho^2 + |z|^{2(\alpha+1)}}.$$

Moreover, $H^0_{\rho,\tau}(f) = 0$ for any f such that $\int_0^{2\pi} f(re^{i\theta})d\theta = 0$ for all $r \in (0,1]$.

PROPOSITION 4.2. *Let $\alpha = 0$, $\nu \in (1,2)$, and $\tau > 0$. There exist $\rho_0 > 0$, two continuous linear forms $H^0_{\rho,\tau} : C^{0,\beta}_{\nu-2}(B(0,1)) \to \mathbb{R}$, $H^1_{\rho,\tau} : C^{0,\beta}_{\nu-2}(B(0,1)) \to \mathbb{C}$, and a linear operator $G_{\rho,\tau} : C^{0,\beta}_{\nu-2}(B(0,1)) \to C^{2,\beta}_{\nu}(B(0,1))$, uniformly bounded for $0 < \rho < \rho_0$, such that for all $\rho \in (0,\rho_0)$ and for all $f \in C^{0,\beta}_{\nu-2}(B(0,1))$ there exists a unique bounded solution w of*

$$\begin{cases} L_{\rho,\tau} w = f & \text{in } B(0,1), \\ w = 0 & \text{on } \partial B(0,1) \end{cases}$$

which can be uniquely decomposed as follows:

$$w(z) = G_{\rho,\tau}(f)(z) + H^0_{\rho,\tau}(f) \frac{\tau^2\rho^2 - |z|^2}{\tau^2\rho^2 + |z|^2} + 2H^1_{\rho,\tau}(f) \cdot \frac{z}{\tau^2\rho^2 + |z|^2}.$$

Moreover, $H^0_{\rho,\tau}(f) = 0$, $H^1_{\rho,\tau}(f) = 0$ for any f such that $\int_0^{2\pi} f(re^{i\theta})d\theta = 0$ and $\int_0^{2\pi} f(re^{i\theta})e^{-i\theta}d\theta = 0$ for all $r \in (0,1]$.

By the Liouville formula (6), we get that, for any $j \in \mathbb{Z}$ and $|a| < \frac{\alpha+1}{|j|+\alpha+1}$,

$$\ln \frac{8(\alpha+1)^2\tau^2 |1 + \frac{j+\alpha+1}{\alpha+1} a z^j|^2}{(\tau^2\rho^2 + |z|^{2(\alpha+1)}|1 + az^j|^2)^2}$$

solves (17). Hence by taking its derivative with respect to a, evaluated at $a = 0$, we obtain a solution of $L_{\rho,\tau} w = 0$ in the form

$$\frac{1}{\alpha+1} \frac{(j+\alpha+1)\tau^2\rho^2 + (j-\alpha-1)|z|^{2(\alpha+1)}}{\tau^2\rho^2 + |z|^{2(\alpha+1)}} z^j, \quad j \in \mathbb{Z}.$$

Consequently,

$$a_j(r) := \frac{(j+\alpha+1)\tau^2\rho^2 + (j-\alpha-1)r^{2(\alpha+1)}}{\tau^2\rho^2 + r^{2(\alpha+1)}} r^j, \quad j \in \mathbb{Z}$$

is a solution for the ordinary differential equation

$$\ddot{a}_j + \frac{1}{r}\dot{a}_j - \frac{j^2}{r^2} a_j + \frac{8(\alpha+1)^2\tau^2\rho^2 r^{2\alpha}}{(\tau^2\rho^2 + r^{2(\alpha+1)})^2} a_j = 0 \quad \text{ in } (0,1).$$

Let us remark that for $j > 0$, $\{a_j(r), a_{-j}(r)\}$ is a set of linearly independent solutions for the same homogeneous equation. Hence any other solution is obtained as a linear combination of $a_j(r)$ and $a_{-j}(r)$. For $j = 0$, another independent solution can be explicitly found and it behaves like $\ln r$ as $r \to 0$. Since $\ln r$ and $a_{-j}(r)$, $j > 0$, are not bounded in a neighborhood of $r = 0$ and $a_j(1) \neq 0$, $j \geq 0$, by means of Fourier decomposition, it is easy to derive the following lemma.

LEMMA 4.3. *Let w be a bounded solution of*

$$\begin{cases} L_{\rho,\tau} w = 0 & \text{in } B(0,1), \\ w = 0 & \text{on } \partial B(0,1). \end{cases}$$

Then $w = 0$.

We decompose w and f into Fourier series:

$$w(z) = w_0(r) + 2\sum_{j=1}^{+\infty} w_j(r) \cdot e^{-ij\theta}, \quad f(z) = f_0(r) + 2\sum_{j=1}^{+\infty} f_j(r) \cdot e^{-ij\theta}.$$

So problem (18) becomes equivalent to

$$\text{(Pj)} \qquad \begin{cases} \ddot{w}_j + \frac{1}{r}\dot{w}_j - \frac{j^2}{r^2} w_j + \frac{8(\alpha+1)^2\tau^2\rho^2 r^{2\alpha}}{(\tau^2\rho^2 + r^{2(\alpha+1)})^2} w_j = f_j, & \text{in } (0,1), \\ w_j(1) = 0 \end{cases}$$

for $j \in \mathbb{N}$. Set $j_\alpha = \min\{j \in \mathbb{N} : j > \alpha + 1\}$ and $m_\alpha = \max\{j \in \mathbb{N} : j < \alpha + 1\}$.

Step 1. By the variation of constants formula, for $j \geq j_\alpha$ and $\nu > -j$,

$$w_j(r) = \left(\int_1^r \frac{ds}{s a_j^2(s)} \int_0^s t a_j(t) f_j(t) dt \right) a_j(r), \quad r > 0$$

defines a solution of (Pj). Since $0 < j - \alpha - 1 \leq \frac{(j+\alpha+1)\tau^2\rho^2 + (j-\alpha-1)r^{2(\alpha+1)}}{\tau^2\rho^2 + r^{2(\alpha+1)}} \leq j + \alpha + 1$, we have that for $-j < \nu < j$ $r^{-\nu}|w_j(r)| \leq \frac{(j+\alpha+1)^2}{(j-\alpha-1)^2} \frac{1}{j^2 - \nu^2} \|f_j\|_{0,\beta,\nu-2}$ and, by classical rescaled Schauder estimates (see [22]), we find

$$\|w_j\|_{2,\beta,\nu} \leq C \frac{(j+\alpha+1)^2}{(j-\alpha-1)^2} \frac{1}{j^2 - \nu^2} \|f_j\|_{0,\beta,\nu-2}$$

for suitable $C > 0$. Finally, for $-j_\alpha < \nu < j_\alpha$, we can define $h(z) = 2\sum_{j=j_\alpha}^{+\infty} w_j(r) \cdot e^{-ij\theta}$ and, since $\|f_j\|_{0,\beta,\nu-2} \le \|f\|_{0,\beta,\nu-2}$, there holds the estimate

$$\sum_{j=j_\alpha}^{+\infty} \|w_j\|_{2,\beta,\nu} \le C \left(\sum_{j=j_\alpha}^{+\infty} \frac{(j+\alpha+1)^2}{(j-\alpha-1)^2} \frac{1}{j^2-\nu^2} \right) \|f\|_{0,\beta,\nu-2}.$$

So $h(z)$ is a well-defined function in $C^{2,\beta}_\nu(B(0,1))$ and satisfies

$$\begin{cases} L_{\rho,\tau} h = 2\sum_{j=j_\alpha}^{+\infty} f_j(r) \cdot e^{-ij\theta} & \text{in } B(0,1), \\ h = 0 & \text{on } \partial B(0,1) \end{cases}$$

together with the estimate $\|h\|_{2,\beta,\nu} \le C\|f\|_{0,\beta,\nu-2}$ for $-j_\alpha < \nu < j_\alpha$.

Step 2. For $0 < j \le m_\alpha$, $\nu > -j$, and $r > \bar r := (\frac{j+\alpha+1}{\alpha+1-j}\tau^2\rho^2)^{\frac{1}{2\alpha+2}}$, it is possible to define

$$\tilde w_j(r) = \left(\int_1^r \frac{ds}{s a_j^2(s)} \int_0^s t a_j(t) f_j(t) dt \right) a_j(r).$$

Note that $a_j(\bar r) = 0$ and hence $\tilde w_j(r)$ is not well defined up to $\bar r$. To be able to obtain an extension of $\tilde w_j$ for $r \le \bar r$, define $\psi_j(s,\rho) = (s-\bar r)^2 \frac{1}{s a_j^2(s)} \int_0^s t a_j(t) f_j(t) dt$ and set

$$w_j(r) = a_j(r) \left[\int_1^r \frac{\psi_j(s,\rho) - \psi_j(\bar r,\rho)}{(s-\bar r)^2} ds - \frac{1-r}{(1-\bar r)(r-\bar r)} \psi_j(\bar r,\rho) \right]. \tag{19}$$

The function w_j is well defined also for $r \le \bar r$ and gives an extension of $\tilde w_j$. We will refer to the first and second terms in the expression of $w_j(r)$ above as $w_j^1(r)$ and $w_j^2(r)$, respectively. Since for $0 < r < \bar r - \delta$, $\delta > 0$, we have

$$\begin{aligned} w_j(r) = a_j(r) \Bigg[& \int_1^{\bar r - \delta} \frac{\psi_j(s,\rho) - \psi_j(\bar r,\rho)}{(s-\bar r)^2} ds \\ & + \int_{\bar r-\delta}^r \frac{ds}{s a_j(s)^2} \int_0^s t a_j(t) f_j(t) dt + \left(\frac{1}{1-\bar r} + \frac{1}{\delta} \right) \psi_j(\bar r,\rho) \Bigg], \end{aligned}$$

the function $w_j(r)$ does solve (Pj) for $r > 0$. Note that for $\nu < j$, we have that $\sup_{r\in(\bar r,1)} r^{-\nu}|w_j(r)| \le C\|f_j\|_{0,\beta,\nu-2}$. In fact, for $r \ge 2\bar r$ we find $|w_j(r)| = |\tilde w_j(r)| \le C\|f_j\|_{0,\beta,\nu-2} r^\nu$ as $\frac{1}{a_j^2(s)} = O(\frac{1}{s^{2j}})$ for $s \ge r$. While for $\bar r \le r \le 2\bar r$ there holds

$$\begin{aligned} |w_j(r)| = |\tilde w_j(r)| &\le C\|f_j\|_{0,\beta,\nu-2} \left(\frac{r}{\bar r} - 1 \right) \bar r^j \int_r^1 \frac{(\tau^2\rho^2 + s^{2(\alpha+1)})^2 s^{\nu-j-1} ds}{[(j+\alpha+1)\tau^2\rho^2 + (j-\alpha-1)s^{2(\alpha+1)}]^2} \\ &\le C\|f_j\|_{0,\beta,\nu-2} \left(\frac{r}{\bar r} - 1 \right) \bar r^\nu \left[\bar r \int_r^{2\bar r} \frac{ds}{(s-\bar r)^2} + 1 \right] \le C\|f_j\|_{0,\beta,\nu-2} \bar r^\nu. \end{aligned}$$

Since $|\psi_j(s,\rho)| \le C\|f_j\|_{0,\beta,\nu-2} \bar r^2 s^{\nu-j-1}$ for $s \le \bar r$, then $|\psi_j(\bar r,\rho)| \le C\|f_j\|_{0,\beta,\nu-2} \bar r^{\nu-j+1}$ and in turn for $s \le \frac{\bar r}{2}$

$$\left| \frac{\psi_j(s,\rho) - \psi_j(\bar r,\rho)}{(s-\bar r)^2} \right| \le C\|f_j\|_{0,\beta,\nu-2} \left(s^{\nu-j-1} + \bar r^{\nu-j-1} \right). \tag{20}$$

For $s \in [\frac{\bar r}{2}, 2\bar r] \setminus \{\bar r\}$, we decompose

$$\frac{\psi_j(s,\rho)-\psi_j(\bar r,\rho)}{(s-\bar r)^2} = \left[\frac{1}{s a_j^2(s)} - \frac{\bar r^{-(2j-1)}(s-\bar r)^{-2}}{[(\alpha+1)^2-j^2]^2}\right]\int_0^{\bar r} t a_j(t) f_j(t)dt + \frac{1}{s a_j^2(s)}\int_{\bar r}^s t a_j(t) f_j(t) dt$$

and hence, using a homogeneity argument, we get

$$\begin{aligned}&\left|\frac{\psi_j(s,\rho)-\psi_j(\bar r,\rho)}{(s-\bar r)^2}\right|\\ \le\quad & C\|f_j\|_{0,\beta,\nu-2}\bar r^{\nu-j-1}\left\{\frac{[1+z^{2(\alpha+1)}]^2}{(1-z^{2(\alpha+1)})^2 z^{2j+1}}\left|\int_1^z \frac{1-t^{2(\alpha+1)}}{1+t^{2(\alpha+1)}} t^{\nu+j-1}dt\right|\right.\\ &\left.+\left|\frac{[(\alpha+1-j)+(\alpha+1+j)z^{2(\alpha+1)}]^2}{(1-z^{2(\alpha+1)})^2 z^{2j+1}} - \frac{1}{(z-1)^2}\right|\right\}\Bigg|_{z=\frac{s}{\bar r}\in[\frac12,2]\setminus\{1\}}.\end{aligned}$$

Consequently, for $\frac{\bar r}{2} \le s \le 2\bar r$ we obtain

$$\left|\frac{\psi_j(s,\rho)-\psi_j(\bar r,\rho)}{(s-\bar r)^2}\right| \le C\|f_j\|_{0,\beta,\nu-2}s^{\nu-j-1}. \tag{21}$$

Finally, it is easy to see that for $r \le 2\bar r$ and for w_j^2 there holds

$$\sup_{r\in(0,2\bar r)} r^{-\nu}|w_j^2(r)| \le C\|f_j\|_{0,\beta,\nu-2}\sup_{0\le t\le 2} t^{-\nu+j}\frac{|1-t^{2(\alpha+1)}|}{|1-t|} \le C\|f_j\|_{0,\beta,\nu-2}$$

in view of the estimate available for $\psi_j(\bar r,\rho)$ and $\nu < j$. Since $|w_j^1(2\bar r)| \le |w_j(2\bar r)| + |w_j^2(2\bar r)| \le C\|f_j\|_{0,\beta,\nu-2}\bar r^{\nu}$, then $|\int_1^{2\bar r}\frac{\psi_j(s,\rho)-\psi_j(\bar r,\rho)}{(s-\bar r)^2}ds| \le C\|f_j\|_{0,\beta,\nu-2}\bar r^{\nu-j}$. So we derive, by splitting the integral in (19) as $\int_1^r = \int_1^{2\bar r} + \int_{2\bar r}^r$ and using (20) and (21), the estimate $\sup_{r\in(0,\bar r)} r^{-\nu}|w_j^1(r)| \le C\|f_j\|_{0,\beta,\nu-2}$ for $\nu < j$. Finally, the estimate $\sup_{r\in(0,1)} r^{-\nu}|w_j(r)| \le C\|f_j\|_{0,\beta,\nu-2}$ does hold and, using classical rescaled Schauder estimates, we get the existence of some constant $C > 0$ such that $\|w_j\|_{2,\beta,\nu} \le C\|f\|_{0,\beta,\nu-2}$ for $0 < j \le m_\alpha$ and $-j < \nu < j$.

Step 3. Analogously, for $j = 0$ and $\nu > 0$, it is possible to consider, for $0 < r < \bar r := (\tau\rho)^{\frac{1}{\alpha+1}}$,

$$\tilde w_0(r) = \left(\int_0^r \frac{ds}{s a_0^2(s)}\int_0^s t a_0(t) f_0(t)dt\right) a_0(r).$$

By defining $\psi_0(s,\rho) = (s-\bar r)^2\frac{1}{s a_0^2(s)}\int_0^s t a_0(t) f_0(t)dt$, we can extend $\tilde w_0$ for $r \ge \bar r$ by considering

$$\hat w_0(r) = a_0(r)\left[\int_0^r \frac{\psi_0(s,\rho)-\psi_0(\bar r,\rho)}{(s-\bar r)^2}ds + \frac{r}{\bar r(\bar r - r)}\psi_0(\bar r,\rho)\right]$$

which defines a solution for $(P0)$, with $\hat w_0(1) \ne 0$ in general. We have the estimate $\sup_{r\in(0,\bar r)} r^{-\nu}|\hat w_0(r)| \le C\|f_0\|_{0,\beta,\nu-2}$. In fact, for $r \le \frac{\bar r}{2}$ we see that $|\hat w_0(r)| = |\tilde w_0(r)| \le C\|f_0\|_{0,\beta,\nu-2} r^{\nu}$ since $\frac{1}{a_0^2(s)} = O(1)$ for $s \le r$. While for $\frac{\bar r}{2} \le r \le \bar r$ there holds

$$|\hat w_0(r)| = |\tilde w_0(r)| \le C\|f_0\|_{0,\beta,\nu-2}\left(1-\frac{r}{\bar r}\right)\bar r^{\nu+1}\int_0^r \frac{ds}{(s-\bar r)^2} \le C\|f_0\|_{0,\beta,\nu-2}\bar r^{\nu}.$$

Furthermore, since $|\psi_0(s,\rho)| \leq C\|f_0\|_{0,\beta,\nu-2}\bar{r}^2 s^{\nu-1}$ for $s \leq \bar{r}$, as above for $s \leq \frac{\bar{r}}{2}$ we obtain

$$\left|\frac{\psi_0(s,\rho)-\psi_0(\bar{r},\rho)}{(s-\bar{r})^2}\right| \leq C\|f_0\|_{0,\beta,\nu-2}\left(s^{\nu-1}+\bar{r}^{\nu-1}\right). \tag{22}$$

On the other hand, for $s \geq \frac{\bar{r}}{2}$ we have

$$\left|\frac{\psi_0(s,\rho)-\psi_0(\bar{r},\rho)}{(s-\bar{r})^2}\right| \leq C\|f_0\|_{0,\beta,\nu-2}s^{\nu-1}. \tag{23}$$

In fact, (23) follows as in (21) when $s \in [\frac{\bar{r}}{2}, 2\bar{r}] \setminus \{\bar{r}\}$. While for $s \geq 2\bar{r}$, we have

$$\begin{aligned}\left|\frac{\psi_0(s,\rho)-\psi_0(\bar{r},\rho)}{(s-\bar{r})^2}\right| &= \left[\frac{1}{sa_0^2(s)} - \frac{\bar{r}}{(\alpha+1)^4(s-\bar{r})^2}\right]\int_0^{\bar{r}} ta_0(t)f_0(t)dt \\ &\quad + \frac{1}{sa_0^2(s)}\int_{\bar{r}}^s ta_0(t)f_0(t)dt \\ &\leq C\|f_0\|_{0,\beta,\nu-2}\left(\frac{\bar{r}^\nu}{s} + \frac{s^{\nu-1}}{(\frac{s}{\bar{r}})^{\nu-1}(\frac{s}{\bar{r}}-1)^2} + s^{\nu-1}\right).\end{aligned}$$

Finally, since $\nu > 0$ it is easy to see that $\sup_{r\in(\bar{r},1)} r^{-\nu}|a_0(r)\psi_0(\bar{r},\rho)\frac{r}{\bar{r}(r-\bar{r})}| \leq C\|f_0\|_{0,\beta,\nu-2}$. While by (22) and (23) for $r \geq \bar{r}$ we get $|a_0(r)\int_0^r \frac{\psi_0(s,\rho)-\psi_0(\bar{r},\rho)}{(s-\bar{r})^2}ds| \leq C\|f_0\|_{0,\beta,\nu-2}r^\nu$. Hence $\sup_{r\in(0,1)} r^{-\nu}|\hat{w}_0(r)| \leq C\|f_0\|_{0,\beta,\nu-2}$. Consequently, by classical rescaled Schauder estimates, we find a suitable constant $C > 0$ such that $\|\hat{w}_0\|_{2,\beta,\nu} \leq C\|f\|_{0,\beta,\nu-2}$. We set now

$$w_0(r) = \hat{w}_0(r) + H^0_{\rho,\tau}(f)\frac{\tau^2\rho^2 - r^{2(\alpha+1)}}{\tau^2\rho^2 + r^{2(\alpha+1)}}, \tag{24}$$

where $H^0_{\rho,\tau}(f) \in \mathbb{R}$ is such that $w_0(1) = 0$. Hence $w_0(r)$ is a solution for $(P0)$ and for $\nu > 0$,

$$|H^0_{\rho,\tau}(f)| \leq C|\hat{w}_0(1)| \leq C\|f\|_{0,\beta,\nu-2}.$$

Notice that for $\alpha > 0$, $\alpha \notin \mathbb{N}$, Steps 1–3 lead to the proof of Proposition 4.1 by choosing $\nu \in (0,1)$ and $G_{\rho,\tau}(f) = \hat{w}_0(r) + 2\sum_{j=1}^{+\infty} w_j(r)\cdot e^{-ij\theta}$.

Step 4. To obtain Proposition 4.2, it remains for problem $(P1)$ to be considered with $\alpha = 0$ while for the validity of Steps 1–3 we must specify $\nu \in (0,2)$. To account also for $(P1)$ we further specify $1 < \nu < 2$. Then it is possible to define

$$\begin{aligned}\hat{w}_1(r) &= \left(\int_0^r \frac{ds}{sa_1^2(s)}\int_0^s ta_1(t)f_1(t)dt\right)a_1(r) \\ &= \frac{r}{\tau^2\rho^2+r^2}\int_0^r \frac{(\tau^2\rho^2+s^2)^2}{s^3}ds\int_0^s \frac{t^2}{\tau^2\rho^2+t^2}f_1(t)dt.\end{aligned}$$

To estimate $\|\hat{w}_1(r)\|_{2,\beta,\nu}$, introduce $z = \frac{r}{\tau\rho}$ and observe that

$$\begin{aligned}\sup_{r\in(0,1)} r^{-\nu}|\hat{w}_1(r)| &\leq \|f_1\|_{0,\beta,\nu-2}\sup_{z\in(0,(\tau\rho)^{-1})}\frac{z^{1-\nu}}{1+z^2}\int_0^z \frac{(1+s^2)^2}{s^3}ds\int_0^s \frac{t^\nu}{1+t^2}dt \\ &\leq C\|f\|_{0,\beta,\nu-2}.\end{aligned}$$

Set

$$w_1(r) = \hat{w}_1(r) + \overline{H^1_{\rho,\tau}(f)} \frac{r}{\tau^2\rho^2 + r^2}, \tag{25}$$

where $H^1_{\rho,\tau}(f) \in \mathbb{C}$ is such that $w_1(1) = 0$. Hence $w_1(r)$ is a solution for $(P1)$ and

$$|H^1_{\rho,\tau}(f)| \le C|\hat{w}_1(1)| \le C\|f\|_{0,\beta,\nu-2}.$$

Therefore, for $\alpha = 0$ Proposition 4.2 also follows with

$$G_{\rho,\tau}(f) = \hat{w}_0(r) + 2\hat{w}_1(r) \cdot e^{-i\theta} + 2\sum_{j=2}^{+\infty} w_j(r) \cdot e^{-ij\theta}$$

whenever $\nu \in (1, 2)$. Finally, using Lemma 4.3 we can deduce the uniqueness of $w_j(r)$. The uniqueness of the decomposition (24) follows by evaluating $w_0(r)$ at $r = 0$. Similarly, if $\alpha = 0$ the uniqueness of the decomposition (25) follows by evaluating $\frac{w_1(r)}{r}$ at $r = 0$. Hence Propositions 4.1 and 4.2 are completely established.

Remark 4.4. The function $G_{\rho,\tau}(f)$ is the unique solution in $C^{2,\beta}_{\nu}(B(0,1))$ for $L_{\rho,\tau}w = f$ in $B(0,1)$ such that

$$G_{\rho,\tau}(f)|_{\partial B(0,1)} = \begin{cases} \hat{w}_0(1) & \text{if } \alpha \notin \mathbb{N}, \\ \hat{w}_0(1) + 2\hat{w}_1(1) \cdot e^{-i\theta} & \text{if } \alpha = 0. \end{cases}$$

4.2. Some local operator. The nonradial case. In case $\alpha = 0$, we discuss now the invertibility of the operator

$$L_{\rho,\tau,\gamma}w = \Delta w + \rho^2 e^{v_{\rho,\tau,\gamma}} w$$

under Dirichlet boundary condition. The following result holds.

PROPOSITION 4.5. *Let $\alpha = 0$. For all $\nu \in (1, 2)$ and $\gamma \in \mathbb{C}$, $|\gamma| < \frac{1}{3}$, $\tau > 0$, there exist $\rho_0 > 0$, two continuous linear forms $H^0_{\rho,\tau,\gamma} : C^{0,\beta}_{\nu-2}(B(0,1)) \to \mathbb{R}$, $H^1_{\rho,\tau,\gamma} : C^{0,\beta}_{\nu-2}(B(0,1)) \to \mathbb{C}$, and a linear operator $G_{\rho,\tau,\gamma} : C^{0,\beta}_{\nu-2}(B(0,1)) \to C^{2,\beta}_{\nu}(B(0,1))$, uniformly bounded for $0 < \rho < \rho_0$, such that for all $\rho \in (0, \rho_0)$ and for all $f \in C^{0,\beta}_{\nu-2}(B(0,1))$ there exists a unique bounded solution w of*

$$\begin{cases} L_{\rho,\tau,\gamma}w = f & \text{in } B(0,1), \\ w = 0 & \text{on } \partial B(0,1) \end{cases}$$

which can be uniquely decomposed as

$$w(z) = G_{\rho,\tau,\gamma}(f)(z) + H^0_{\rho,\tau,\gamma}(f)\partial_\tau v_{\rho,\tau,\gamma} + 2H^1_{\rho,\tau,\gamma}(f) \cdot \partial_{\bar{z}} v_{\rho,\tau,\gamma}.$$

Moreover, the following estimates hold:

$$\|G_{\rho,\tau,\gamma}(f)\|_{2,\beta,\nu} \le C\left(\|G_{\rho,\tau}(f)\|_{2,\beta,\nu} + \rho^2|H^0_{\rho,\tau}(f)| + |H^1_{\rho,\tau}(f)|\right), \tag{26}$$

$$|H^0_{\rho,\tau,\gamma}(f)| \le C\left(\rho^2\|G_{\rho,\tau}(f)\|_{2,\beta,\nu} + |H^0_{\rho,\tau}(f)| + \rho^2|H^1_{\rho,\tau}(f)|\right), \tag{27}$$

$$|H^1_{\rho,\tau,\gamma}(f)| \le C\left(\rho^2\|G_{\rho,\tau}(f)\|_{2,\beta,\nu} + \rho^2|H^0_{\rho,\tau}(f)| + |H^1_{\rho,\tau}(f)|\right), \tag{28}$$

$$\|\partial_r G_{\rho,\tau,\gamma}(f)|_{\partial B(0,1)}\|_{1,\beta} \le C\left(\|\partial_r G_{\rho,\tau}(f)|_{\partial B(0,1)}\|_{1,\beta} + \rho^2\|G_{\rho,\tau}(f)\|_{2,\beta,\nu} + \rho^2|H^0_{\rho,\tau}(f)| + |H^1_{\rho,\tau}(f)|\right) \tag{29}$$

for some constant $C>0$.

Proof. In case $\alpha=0$, we compute

$$\begin{aligned}
\lim_{\rho\to 0}\partial_\tau v_{\rho,\tau,\gamma}(z) &= \lim_{\rho\to 0}\left(\frac{2}{\tau}\frac{|z|^2|1+\gamma z^2|^2-\tau^2\rho^2}{|z|^2|1+\gamma z^2|^2+\tau^2\rho^2}\right)=\frac{2}{\tau},\\
\lim_{\rho\to 0}\partial_{\bar z} v_{\rho,\tau,\gamma}(z) &= \lim_{\rho\to 0}\left(\frac{6\bar\gamma\bar z}{1+3\bar\gamma\bar z^2}-2\frac{z(1+\gamma z^2)(1+3\bar\gamma\bar z^2)}{|z|^2|1+\gamma z^2|^2+\tau^2\rho^2}\right)\\
&= 6\bar\gamma\bar z\left(\sum_{k=0}^{+\infty}(-1)^k3^k\bar\gamma^k\bar z^{2k}\right)-2(1+3\bar\gamma\bar z^2)\left(\sum_{k=0}^{+\infty}(-1)^k\bar\gamma^k\bar z^{2k-1}\right)\\
&= -\frac{2}{\bar z}+2\bar\gamma\bar z+\eta^\perp(z)
\end{aligned}$$

uniformly on compact sets in $B(0,1)\setminus\{0\}$, where $\eta^\perp(z)=2\sum_{k=0}^{+\infty}(-1)^{k+1}(3^{k+2}-2)\bar\gamma^{k+2}\bar z^{2k+3}$ is orthogonal to $\{1,e^{\pm i\theta}\}$ (in the sense $\eta^\perp(re^{i\theta})$ is orthogonal to 1 and $e^{\pm i\theta}$ in $L^2([0,2\pi])$ for any $r\in(0,1]$) and it is a harmonic function. Set $\mathrm{Span}\{1,e^{-i\theta}\}=\{a_0+2a_1\cdot e^{-i\theta}:\ a_0\in\mathbb{R},\ a_1\in\mathbb{C}\}$ and define π as the orthogonal projection over $\mathrm{Span}\{1,e^{-i\theta}\}$. Define the mapping $\psi_\rho:(h_0,h_1)\in\mathbb{R}\times\mathbb{C}\to(\psi^1_\rho(h_0,h_1),\psi^2_\rho(h_0,h_1))\in\mathbb{R}\times\mathbb{C}$ by setting

$$\psi^1_\rho(h_0,h_1)+2\psi^2_\rho(h_0,h_1)\cdot e^{-i\theta}=\pi\left(h_0\partial_\tau v_{\rho,\tau,\gamma}(e^{i\theta})+2h_1\cdot\partial_{\bar z}v_{\rho,\tau,\gamma}(e^{i\theta})\right).$$

Note that $\psi_\rho\to\psi_0$ as $\rho\to 0$ in the operatorial norm, where $\psi_0(h_0,h_1)=(\frac{2}{\tau}h_0,-2\bar h_1+2\gamma h_1)$ is an invertible operator for $|\gamma|<1$ with inverse $\psi_0^{-1}(\tilde h_0,\tilde h_1)=(\frac{\tau}{2}\tilde h_0,-\frac{1}{2(1-|\gamma|^2)}\cdot(\bar\gamma\tilde h_1+\bar{\tilde h}_1))$. Hence, for ρ small ψ_ρ is invertible with uniformly bounded inverse. Let $f\in C^{0,\beta}_{\nu-2}(B(0,1))$ be a given function and $\nu\in(1,2)$. By Remark 4.4, there exists a unique $w_0\in C^{2,\beta}_\nu(B(0,1))$ such that

$$\begin{cases} L_{\rho,\tau}w_0=f & \text{in } B(0,1),\\ w_0\,|_{\partial B(0,1)}=\tilde h_0+2\tilde h_1\cdot e^{-i\theta}\in\mathrm{Span}\,\{1,e^{-i\theta}\}. \end{cases}$$

Let $(h_0,h_1)=\psi_\rho^{-1}(\tilde h_0,\tilde h_1)$. We define on $\partial B(0,1)$

$$\phi(\theta):=h_0\partial_\tau v_{\rho,\tau,\gamma}(e^{i\theta})+2h_1\cdot\partial_{\bar z}v_{\rho,\tau,\gamma}(e^{i\theta})-\tilde h_0-2\tilde h_1\cdot e^{-i\theta}$$

in such a way that $\pi\phi=0$. We extend ϕ in $B(0,1)$ as $\tilde\phi(z)=\sigma(r)\phi(\theta)$, where $0\le\sigma\le 1$ is a smooth function with $\sigma\equiv 1$ in $[\frac12,1]$ and $\sigma\equiv 0$ in $[0,\frac14]$. Since $L_{\rho,\tau}\tilde\phi\in\mathrm{Span}\,\{1,e^{-i\theta}\}^\perp$, by Proposition 4.2 we get $H^0_{\rho,\tau}(-L_{\rho,\tau}\tilde\phi)=0$, $H^1_{\rho,\tau}(-L_{\rho,\tau}\tilde\phi)=0$ and hence $w_1:=G_{\rho,\tau}(-L_{\rho,\tau}\tilde\phi)$ vanishes on $\partial B(0,1)$. The function $w_2:=w_0+\tilde\phi+w_1\in C^{2,\beta}_\nu(B(0,1))$ solves

$$\begin{cases} L_{\rho,\tau}w_2=f & \text{in } B(0,1),\\ w_2\,|_{\partial B(0,1)}=h_0\partial_\tau v_{\rho,\tau,\gamma}(e^{i\theta})+2h_1\cdot\partial_{\bar z}v_{\rho,\tau,\gamma}(e^{i\theta}) \end{cases}$$

with $\|w_2\|_{2,\beta,\nu}\le C\|f\|_{0,\beta,\nu-2}$. Moreover, w_2 is the unique solution in $C^{2,\beta}_\nu(B(0,1))$ for the problem. If w_2' is a solution in $C^{2,\beta}_\nu(B(0,1))$ with

$$w_2'\,|_{\partial B(0,1)}=h_0'\partial_\tau v_{\rho,\tau,\gamma}(e^{i\theta})+2h_1'\cdot\partial_{\bar z}v_{\rho,\tau,\gamma}(e^{i\theta})=\tilde h_0'+2\tilde h_1'\cdot e^{-i\theta}+\phi',$$

then by the uniqueness part in Proposition 4.2 we derive $w_0 = w_2' - \tilde{\phi}' - w_1'$, $\tilde{h}_0' = \tilde{h}_0$, $\tilde{h}_1' = \tilde{h}_1$, where $w_1' = G_{\rho,\tau}(-L_{\rho,\tau}\tilde{\phi}')$ and $\tilde{\phi}'$ extends ϕ' as before. Since ψ_ρ is injective, then $h_0' = h_0$, $h_1' = h_1$, $\phi' = \phi$ and hence $\tilde{\phi}' + w_1' = \tilde{\phi} + w_1$ and $w_2' = w_2$.

Then $L_{\rho,\tau}$, as an operator between

$$\{w \in C^{2,\beta}_\nu\left(B(0,1)\right) : w\,|_{\partial B(0,1)} = h_0 \partial_\tau v_{\rho,\tau,\gamma}(e^{i\theta}) + 2h_1 \cdot \partial_{\bar{z}} v_{\rho,\tau,\gamma}(e^{i\theta}),\, (h_0,h_1) \in \mathbb{R}\times\mathbb{C}\}$$

and $C^{0,\beta}_{\nu-2}\left(B(0,1)\right)$, is an isomorphism with inverse uniformly bounded with respect to $\|\cdot\|_{0,\beta,\nu-2}$ and $\|\cdot\|_{2,\beta,\nu}$. We will denote this inverse operator as $L^{-1}_{\rho,\tau}$. Moreover, we have the estimate $|h_0(f)| + |h_1(f)| \le C\|f\|_{0,\beta,\nu-2}$. We use now a perturbation argument to prove Proposition 4.5. Since for $z, x, y \in B(0,1)$ we have

$$|v_{\rho,\tau,\gamma} - v_{\rho,\tau}|(z) = \left|\ln \frac{(\tau^2\rho^2 + |z|^2)^2 |1 + 3\gamma z^2|^2}{(\tau^2\rho^2 + |z|^2|1+\gamma z^2|^2)^2}\right| \le C|z|^2,$$

$$\begin{aligned}\left|\frac{\left(v_{\rho,\tau,\gamma} - v_{\rho,\tau}\right)(x) - \left(v_{\rho,\tau,\gamma} - v_{\rho,\tau}\right)(y)}{|x-y|^\beta}\right| &\le 2|\partial_z(v_{\rho,\tau,\gamma} - v_{\rho,\tau})(\xi)||x-y|^{1-\beta} \\ &\le C\left(\max\{|x|,|y|\}\right)^{2-\beta}\end{aligned}$$

for some point ξ on the segment joining x and y, we get that $\|v_{\rho,\tau,\gamma} - v_{\rho,\tau}\|_{0,\beta,2} \le C$. Hence, for $w \in C^{2,\beta}_\nu\left(B(0,1)\right)$ we have the estimate

$$\|\left(L_{\rho,\tau,\gamma} - L_{\rho,\tau}\right) w\|_{0,\beta,\nu-2} = \rho^2 \|\left(e^{v_{\rho,\tau,\gamma}} - e^{v_{\rho,\tau}}\right) w\|_{0,\beta,\nu-2} \le C\rho^2 \|w\|_{2,\beta,\nu}. \tag{30}$$

A solution for the problem

$$\begin{cases} L_{\rho,\tau,\gamma} w = f & \text{in } B(0,1), \\ w\,|_{\partial B(0,1)} = -H^0_{\rho,\tau,\gamma}(f)\partial_\tau v_{\rho,\tau,\gamma}(e^{i\theta}) - 2H^1_{\rho,\tau,\gamma}(f)\cdot\partial_{\bar{z}} v_{\rho,\tau,\gamma}(e^{i\theta}) \end{cases}$$

corresponds to a fixed point for the map $w \to L^{-1}_{\rho,\tau} f + L^{-1}_{\rho,\tau}(L_{\rho,\tau} - L_{\rho,\tau,\gamma})w$. By (30) we deduce that this map is a contraction. So it has a unique fixed point $w = G_{\rho,\tau,\gamma}(f)$ which satisfies $\|G_{\rho,\tau,\gamma}(f)\|_{2,\beta,\nu} \le C\|L^{-1}_{\rho,\tau}(f)\|_{2,\beta,\nu}$. At this point, we deduce (26)–(29): since

$$|\partial_\tau v_{\rho,\tau,\gamma}(e^{i\theta}) - \frac{2}{\tau}| + |\partial_{\bar{z}} v_{\rho,\tau,\gamma}(e^{i\theta}) - \left(-2e^{i\theta} + 2\bar{\gamma}e^{-i\theta} + \eta^\perp(e^{i\theta})\right)| \le C\rho^2,$$

there holds the estimate $\|\psi_\rho - \psi_0\| + \|\psi_\rho^{-1} - \psi_0^{-1}\| \le C\rho^2$. Therefore

$$\begin{aligned}(h_0,h_1) &= \psi_0^{-1}(\tilde{h}_0,\tilde{h}_1) + O\left(\rho^2|H^0_{\rho,\tau}(f)| + \rho^2|H^1_{\rho,\tau}(f)|\right) \\ &= \left(O(|H^0_{\rho,\tau}(f)| + \rho^2|H^1_{\rho,\tau}(f)|), O(\rho^2|H^0_{\rho,\tau}(f)| + |H^1_{\rho,\tau}(f)|)\right)\end{aligned} \tag{31}$$

as $\tilde{h}_0 = -H^0_{\rho,\tau}(f)\frac{\tau^2\rho^2-1}{\tau^2\rho^2+1}$ and $\tilde{h}_1 = -\overline{H^1_{\rho,\tau}(f)}\frac{1}{\tau^2\rho^2+1}$. On $\partial B(0,1)$ there holds

$$\begin{aligned}\phi(\theta) =& \frac{2}{\tau}h_0 + 2(-2\bar{h}_1 + 2\gamma h_1)\cdot e^{-i\theta} - \tilde{h}_0 - 2\tilde{h}_1 \cdot e^{-i\theta} + 2h_1\cdot\eta^\perp(e^{i\theta}) \\ &+ O\left(\rho^2|h_0| + \rho^2|h_1|\right) = 2h_1\cdot\eta^\perp(e^{i\theta}) + O\left(\rho^2|H^0_{\rho,\tau}(f)| + \rho^2|H^1_{\rho,\tau}(f)|\right) \\ =& O\left(\rho^2|H^0_{\rho,\tau}(f)| + |H^1_{\rho,\tau}(f)|\right)\end{aligned}$$

and we deduce

$$\|\tilde{\phi}\|_{2,\beta,\nu} + \|w_1\|_{2,\beta,\nu} = O\left(\rho^2|H^0_{\rho,\tau}(f)| + |H^1_{\rho,\tau}(f)|\right). \tag{32}$$

Since $G_{\rho,\tau,\gamma}(f) = L_{\rho,\tau}^{-1} f + L_{\rho,\tau}^{-1}(L_{\rho,\tau} - L_{\rho,\tau,\gamma})G_{\rho,\tau,\gamma}(f)$ with $L_{\rho,\tau}^{-1} f = G_{\rho,\tau}(f) + \tilde{\phi} + w_1$, we get (26) as follows:

$$\begin{aligned}\|G_{\rho,\tau,\gamma}(f)\|_{2,\beta,\nu} &\le C\left(\|G_{\rho,\tau}(f)\|_{2,\beta,\nu} + \|\tilde{\phi}\|_{2,\beta,\nu} + \|w_1\|_{2,\beta,\nu}\right)\\ &\le C\left(\|G_{\rho,\tau}(f)\|_{2,\beta,\nu} + \rho^2 |H^0_{\rho,\tau}(f)| + |H^1_{\rho,\tau}(f)|\right)\end{aligned}$$

and in turn by (32) and (26) we obtain (29).

Letting $S = f + (L_{\rho,\tau} - L_{\rho,\tau,\gamma})G_{\rho,\tau,\gamma}(f)$, by (31) and (26) we find

$$\begin{aligned}|H^0_{\rho,\tau,\gamma}(f)| = |h_0(S)| = O\left(\rho^2 \|G_{\rho,\tau}(f)\|_{2,\beta,\nu} + |H^0_{\rho,\tau}(f)| + \rho^2 |H^1_{\rho,\tau}(f)|\right),\\ |H^1_{\rho,\tau,\gamma}(f)| = |h_1(S)| = O\left(\rho^2 \|G_{\rho,\tau}(f)\|_{2,\beta,\nu} + \rho^2 |H^0_{\rho,\tau}(f)| + |H^1_{\rho,\tau}(f)|\right),\end{aligned}$$

and the proof of Proposition 4.5 is completed. □

4.3. Some global operator. Let $\alpha \in (0, +\infty) \setminus \mathbb{N}$ be a fixed number. Let χ be a radial smooth function such that $0 \le \chi \le 1$, $\chi = 1$ in $B(0,1)$, $\chi = 0$ in $\mathbb{R}^2 \setminus B(0,2)$. In Theorem 1.4 we are interested in dealing with three possible cases:

(a) the concentration set S is a single point which is a singular source, that is, $S = \{p\}$, and in this case we consider the associated potential as given by $V_\rho(z) = \rho^2 \chi(z-p)|z-p|^{2\alpha} e^{v_{\rho,\tau_1}(z-p)}$ where $\tau_1 > 0$ is defined in section 2;

(b) the concentration set S is a single point which is not a singular source, that is, $S = \{q\}$ with $q \ne p$, and the associated potential considered in this case is $V_\rho(z) = \rho^2 \chi(z-q) e^{v_{\rho,\tau_2,\gamma}(z-q)}$ where $\tau_2 > 0$ and γ are defined in section 2;

(c) the concentration set $S = \{p, q\}$ and the associated potential is $V_\rho(z) = \rho^2 \chi(z-p)|z-p|^{2\alpha} e^{v_{\rho,\tau_1}(z-p)} + \rho^2 \chi(z-q) e^{v_{\rho,\tau_2,\gamma}(z-q)}$ where $\tau_1, \tau_2 > 0$ and γ are defined in section 2.

We are assuming that $\overline{B(p,2)} \cap \overline{B(q,2)} = \emptyset$, $\overline{B(p,2)} \subset \Omega$, and $\overline{B(q,2)} \subset \Omega$. Set $B = B(p,1) \cup B(q,1)$ and $\tilde{\Omega} = \Omega \setminus B$.

We introduce the operator $\mathcal{L}_\rho = \Delta + V_\rho$ where the potential V_ρ is defined above according to the cases (a), (b), and (c) we wish to deal with. We investigate the invertibility of $\mathcal{L}_\rho$ between X and Y (see section 3 for the definition of X and Y). We will prove the following result.

THEOREM 4.6. *There exist $\rho_0 > 0$ small, continuous linear forms $\mathcal{H}^0_{\rho,1}, \mathcal{H}^0_{\rho,2} : Y \to \mathbb{R}$ and $\mathcal{H}^1_{\rho,2} : Y \to \mathbb{C}$, a linear operator $\mathcal{G}_\rho : Y \to, X$, uniformly bounded for $\rho \in (0, \rho_0)$, such that for all $f \in Y$ and $\rho \in (0, \rho_0)$ there exists a unique solution $w(z)$ of*

$$\begin{cases}\mathcal{L}_\rho w = f & \text{in } \Omega,\\ w = 0 & \text{on } \partial\Omega\end{cases}$$

which can be decomposed in a unique way in the form

$$w(z) = \mathcal{G}_\rho(f)(z) + \chi(z-p)\mathcal{H}^0_{\rho,1} \partial_\tau v_{\rho,\tau_1}(z-p)$$

in case (a)*, in the form*

$$w(z) = \mathcal{G}_\rho(f)(z) + \chi(z-q)\mathcal{H}^0_{\rho,2} \partial_\tau v_{\rho,\tau_2,\gamma}(z-q) + 2\chi(z-q)\mathcal{H}^1_{\rho,2} \cdot \partial_{\bar{z}} v_{\rho,\tau_2,\gamma}(z-q)$$

in case (b)*, and in the form*

$$\begin{aligned}w(z) = \mathcal{G}_\rho(f)(z) &+ \chi(z-p)\mathcal{H}^0_{\rho,1} \partial_\tau v_{\rho,\tau_1}(z-p)\\ &+ \chi(z-q)\mathcal{H}^0_{\rho,2} \partial_\tau v_{\rho,\tau_2,\gamma}(z-q) + 2\chi(z-q)\mathcal{H}^1_{\rho,2} \cdot \partial_{\bar{z}} v_{\rho,\tau_2,\gamma}(z-q)\end{aligned}$$

in case (c).

We collect some preliminary results which will be crucial to the proof of Theorem 4.6. Since $|V_\rho| \le C\rho^2$ in $\tilde\Omega$, from classical elliptic theory we have the following lemma.

LEMMA 4.7. *There exists $\rho_0 > 0$ small such that for all $f \in C^{0,\beta}(\tilde\Omega)$ there exists a unique solution $w \in C^{2,\beta}(\tilde\Omega)$ for the problem*

$$\begin{cases} \mathcal{L}_\rho w = f & \text{in } \tilde\Omega, \\ w = 0 & \text{on } \partial\tilde\Omega. \end{cases}$$

Moreover, $\|w\|_{2,\beta,\tilde\Omega} \le C\|f\|_{0,\beta,\tilde\Omega}$.

We introduce now the exterior Dirichlet to Neumann map. Let $\Phi \in C^{2,\beta}(\partial B)$; we can extend Φ inside $\tilde\Omega$ in such a way that $\tilde\Phi \in C^{2,\beta}(\tilde\Omega)$, $\tilde\Phi = 0$ on $\partial\Omega$, and $\|\tilde\Phi\|_{2,\beta,\tilde\Omega} \le C\|\Phi\|_{2,\beta,\partial B}$.

By Lemma 4.7 we can find a solution $\bar w$ for

$$\begin{cases} \mathcal{L}_\rho \bar w = -\mathcal{L}_\rho \tilde\Phi & \text{in } \tilde\Omega, \\ \bar w = 0 & \text{on } \partial\tilde\Omega \end{cases}$$

and hence $w_\Phi = \bar w + \tilde\Phi$ solves

$$\begin{cases} \mathcal{L}_\rho w_\Phi = 0 & \text{in } \tilde\Omega, \\ w_\Phi = 0 & \text{on } \partial\Omega, \\ w_\Phi = \Phi & \text{on } \partial B \end{cases}$$

with $\|w_\Phi\|_{2,\beta,\tilde\Omega} \le C\|\Phi\|_{C^{2,\beta}(\partial B)}$.

Define

$$S_\rho : C^{2,\beta}(\partial B) \to C^{1,\beta}(\partial B)$$
$$\Phi \to S_\rho(\Phi) = \frac{\partial w_\Phi}{\partial n} \,|_{\partial B},$$

where n is the unit inward normal on ∂B to $\tilde\Omega$. If $\tilde w$ denotes the solution of

$$\begin{cases} \Delta \tilde w = -\Delta \tilde\Phi & \text{in } \tilde\Omega, \\ \tilde w = 0 & \text{on } \partial\tilde\Omega, \end{cases}$$

then

$$\begin{cases} \Delta(\bar w - \tilde w) = -V_\rho w_\Phi & \text{in } \tilde\Omega, \\ \bar w - \tilde w = 0 & \text{on } \partial\tilde\Omega \end{cases}$$

and so, by classical Schauder estimates, $\|\bar w - \tilde w\|_{2,\beta,\tilde\Omega} \le C\rho^2\|\Phi\|_{C^{2,\beta}(\partial B)}$. Hence, if S_0 denotes the Dirichlet to Neumann map corresponding to Δ on $\tilde\Omega$, we have that $S_\rho = S_0 + O(\rho^2)$. Summarizing, we have the following lemma.

LEMMA 4.8. *There exists $\rho_0 > 0$ small such that for $\rho \in (0, \rho_0)$ the map S_ρ is well defined and $S_\rho \to S_0$ as $\rho \to 0$ in the operatorial norm.*

We introduce now the interior Dirichlet to Neumann map. Let $\Phi \in C^{2,\beta}(\partial B)$, which we extend as $\tilde\Phi$ in B in such a way that $\|\tilde\Phi\|_{2,\beta,\nu_1,B(p,1)} + \|\tilde\Phi\|_{2,\beta,\nu_2,B(q,1)} \le$

$C\|\Phi\|_{2,\beta,\partial B}$. By Propositions 4.1 and 4.5, we see that there exists a unique solution $\bar{v}$ of

$$\begin{cases} \mathcal{L}_\rho \bar{v} = -\mathcal{L}_\rho \tilde{\Phi} & \text{in } B, \\ \bar{v} = 0 & \text{on } \partial B \end{cases}$$

and hence $v_\Phi = \bar{v} + \tilde{\Phi}$ uniquely solves

$$\begin{cases} \mathcal{L}_\rho v_\Phi = 0 & \text{in } B, \\ v_\Phi = \Phi & \text{on } \partial B \end{cases}$$

with $\|v_\Phi \mid_{B(p,1)} \|_{\mathcal{E}_1} + \|v_\Phi \mid_{B(q,1)} \|_{\mathcal{E}_2} \le C\|\Phi\|_{2,\beta,\partial B}$.

The space

$$\mathcal{E}_1 = \{w = h + \lambda \partial_\tau v_{\rho,\tau_1}(z-p) :\ h \in C^{2,\beta}_{\nu_1}(B(p,1)),\ \lambda \in \mathbb{R}\}$$

is endowed with the norm $\|w\|_{\mathcal{E}_1} = \|h\|_{2,\beta,\nu_1,B(p,1)} + |\lambda|$, and the space

$$\mathcal{E}_2 = \{w = h + \lambda\partial_\tau v_{\rho,\tau_2,\gamma}(z-q) + 2a\cdot\partial_{\bar z} v_{\rho,\tau_2,\gamma}(z-q) : h \in C^{2,\beta}_{\nu_2}(B(q,1)),\ \lambda \in \mathbb{R}, a \in \mathbb{C}\}$$

with the norm $\|w\|_{\mathcal{E}_2} = \|h\|_{2,\beta,\nu_2,B(q,1)} + |\lambda| + |a|$.

Define

$$\begin{aligned} T^1_\rho : C^{2,\beta}(\partial B(p,1)) &\to C^{1,\beta}(\partial B(p,1)) \\ \phi_1 &\to T^1_\rho(\phi_1) = \partial_{r_1} v_\Phi \mid_{\partial B(p,1)}, \\ T^2_\rho : C^{2,\beta}(\partial B(q,1)) &\to C^{1,\beta}(\partial B(q,1)) \\ \phi_2 &\to T^2_\rho(\phi_2) = \partial_{r_2} v_\Phi \mid_{\partial B(q,1)}, \end{aligned}$$

where $\Phi = (\phi_1, \phi_2)$, $r_1 = |z-p|$, and $r_2 = |z-q|$. T^i_ρ is a uniformly bounded operator such that the following lemma holds.

LEMMA 4.9. *$T^i_\rho \to T^i_0$ as $\rho \to 0$ in the operatorial norm, where*

$$T^1_0 \phi_1 = 2\sum_{n=1}^{+\infty} n a_n \cdot e^{-in\theta}$$

with $\phi_1 = a_0 + 2\sum_{n=1}^{+\infty} a_n \cdot e^{-in\theta}$, while

$$T^2_0 \phi_2 = -2a_1 \cdot (e^{i\theta} + \bar{\gamma} e^{-i\theta}) + 2\sum_{n=2}^{+\infty} n a_n \cdot e^{-in\theta},$$

where $\phi_2 = a_0 + 2a_1 \cdot (e^{i\theta} - \bar{\gamma} e^{-i\theta}) + 2\sum_{n=2}^{+\infty} a_n \cdot e^{-in\theta}$. The variable θ denotes the angular variable of $\frac{z-p}{|z-p|}$ and $\frac{z-q}{|z-q|}$, respectively.

Remark 4.10. (1) The map $a_1 \in \mathbb{C} \to \bar{a}_1 - \gamma a_1 \in \mathbb{C}$ is invertible; see the discussion for the invertibility of ψ_0. Since $a_1 \cdot (e^{i\theta} - \bar{\gamma} e^{-i\theta}) = (\bar{a}_1 - \gamma a_1)\cdot e^{-i\theta}$, the statement of Lemma 4.9 makes good sense.

(2) The operator T^2_0 is the interior Dirichlet to Neumann map associated with Δ on $B(q,1)\setminus\{q\}$ with a first-order singularity in q, since $w = a_0 + 2a_1\cdot(\frac{z-q}{|z-q|^2} - \bar{\gamma}\overline{z-q}) + 2\sum_{n=2}^{+\infty} a_n \cdot \overline{z-q}^n$ is a harmonic extension of ϕ_2 in $B(q,1)\setminus\{q\}$ with $\partial_{r_2} w \mid_{\partial B(q,1)} = T^2_0\phi_2$.

Proof. Assume for simplicity that $p=0$. Write

$$\phi_1(\theta)=a_0+2\sum_{n=1}^{+\infty}a_n\cdot e^{-in\theta}=\frac{\tau_1(1+\tau_1^2\rho^2)}{2(1-\tau_1^2\rho^2)}a_0\partial_\tau v_{\rho,\tau_1}(e^{i\theta})+2\sum_{n=1}^{+\infty}a_n\cdot e^{-in\theta}.$$

Then $w=\frac{\tau_1(1+\tau_1^2\rho^2)}{2(1-\tau_1^2\rho^2)}a_0\partial_\tau v_{\rho,\tau_1}(z)+2\sum_{n=1}^{+\infty}r^n a_n\cdot e^{-in\theta}+w_1$ solves

$$\begin{cases} L_{\rho,\tau_1}w=0 & \text{in } B(p,1),\\ w=\phi_1 & \text{on } \partial B(p,1)\end{cases}$$

if and only if w_1 solves

$$\begin{cases} L_{\rho,\tau_1}w_1=f_1:=-\rho^2|z-p|^{2\alpha}e^{v_{\rho,\tau_1}}(2\sum_{n=1}^{+\infty}r^n a_n\cdot e^{-in\theta}) & \text{in } B(p,1),\\ w_1=0 & \text{on } \partial B(p,1).\end{cases}$$

The well-known estimate $\|\sum_{n=1}^{+\infty}r^n a_n\cdot e^{-in\theta}\|_{2,\beta,1}\le C\|\phi_1\|_{2,\beta}$ implies $\|f_1\|_{0,\beta,\nu_1-2}\le C\rho^{\frac{1-\nu_1}{\alpha+1}}\|\phi_1\|_{2,\beta}$. Since $\int_0^{2\pi}f_1(re^{i\theta})d\theta=0$ for all $r\in(0,1]$, by Proposition 4.1 $w_1(z)=G_{\rho,\tau_1}(f_1)(z)$ with $\|G_{\rho,\tau_1}(f_1)\|_{2,\beta,\nu_1}\le C\rho^{\frac{1-\nu_1}{\alpha+1}}\|\phi_1\|_{2,\beta}$. Therefore $\|\partial_r w_1\mid_{\partial B(p,1)}\|_{1,\beta}\le C\rho^{\frac{1-\nu_1}{\alpha+1}}\|\phi_1\|_{2,\beta}$ and hence

$$\begin{aligned} T_\rho^1\phi_1 &= \frac{\tau_1(1+\tau_1^2\rho^2)}{2(1-\tau_1^2\rho^2)}a_0\partial_r\partial_\tau v_{\rho,\tau_1}(z)\mid_{\partial B(p,1)}+2\sum_{n=1}^{+\infty}na_n\cdot e^{-in\theta}+O\left(\rho^{\frac{1-\nu_1}{\alpha+1}}\|\phi_1\|_{2,\beta}\right)\\ &=\frac{4(\alpha+1)\tau_1^2\rho^2}{(1+\tau_1^2\rho^2)(1-\tau_1^2\rho^2)}a_0+2\sum_{n=1}^{+\infty}na_n\cdot e^{-in\theta}+O\left(\rho^{\frac{1-\nu_1}{\alpha+1}}\|\phi_1\|_{2,\beta}\right)\\ &=T_0^1\phi_1+O\left(\rho^{\frac{1-\nu_1}{\alpha+1}}\|\phi_1\|_{2,\beta}\right).\end{aligned}$$

Assuming for simplicity that $q=0$, for ϕ_2 as above we can write

$$\begin{aligned}\phi_2(\theta)&=\frac{\tau_2}{2}a_0\partial_\tau v_{\rho,\tau_2,\gamma}(e^{i\theta})+\frac{2\tau_2^2\rho^2}{\tau_2^2\rho^2+|1+\gamma e^{2i\theta}|^2}a_0-a_1\cdot\partial_{\bar z}v_{\rho,\tau_2,\gamma}(e^{i\theta})\\ &\quad+\frac{2\tau_2^2\rho^2}{\tau_2^2\rho^2+|1+\gamma e^{2i\theta}|^2}a_1\cdot\frac{1+3\bar\gamma e^{-2i\theta}}{1+\bar\gamma e^{-2i\theta}}e^{i\theta}+a_1\cdot\eta^\perp(e^{i\theta})+2\sum_{n=2}^{+\infty}a_n\cdot e^{-in\theta}.\end{aligned}$$

Let $h(z):=\frac{2\tau_2^2\rho^2|z|^2}{\tau_2^2\rho^2+|1+\gamma z^2|^2}a_0+\frac{2\tau_2^2\rho^2|z|^2}{\tau_2^2\rho^2+|1+\gamma z^2|^2}a_1\cdot\frac{1+3\bar\gamma\bar z^2}{1+\bar\gamma\bar z^2}z$, then

$$w(z)=\frac{\tau_2}{2}a_0\partial_\tau v_{\rho,\tau_2,\gamma}(z)-a_1\cdot\partial_{\bar z}v_{\rho,\tau_2,\gamma}(z)+h(z)+a_1\cdot\eta^\perp(z)+2\sum_{n=2}^{+\infty}a_n\cdot\bar z^n+w_1$$

solves

$$\begin{cases} L_{\rho,\tau_2,\gamma}w=0 & \text{in } B(q,1),\\ w=\phi_2 & \text{on } \partial B(q,1)\end{cases}$$

if and only if w_1 solves

$$\begin{cases} L_{\rho,\tau_2,\gamma}w_1=f_2:=-\rho^2e^{v_{\rho,\tau_2,\gamma}}(2\sum_{n=1}^{+\infty}a_n\cdot\bar z^n+a_1\cdot\eta^\perp(z))-L_{\rho,\tau_2,\gamma}h & \text{in } B(q,1),\\ w_1=0 & \text{on } \partial B(q,1).\end{cases}$$

Since $\|2\sum_{n=2}^{+\infty} a_n \cdot \bar{z}^n + a_1 \cdot \eta^\perp(z)\|_{2,\beta,2} \le C\|\phi_2\|_{2,\beta}$, we get $\|f_2\|_{0,\beta,\nu_2-2} \le C\rho^{2-\nu_2}\|\phi_2\|_{2,\beta}$. By Proposition 4.5,

$$w_1(z) = G_{\rho,\tau_2,\gamma}(f_2)(z) + H^0_{\rho,\tau_2,\gamma}(f_2)\partial_\tau v_{\rho,\tau_2,\gamma}(z) + 2H^1_{\rho,\tau_2,\gamma}(f_2)\cdot \partial_{\bar z} v_{\rho,\tau_2,\gamma}(z)$$

with $\|G_{\rho,\tau_2,\gamma}(f_2)\|_{2,\beta,\nu_2} + |H^0_{\rho,\tau_2,\gamma}(f_2)| + |H^1_{\rho,\tau_2,\gamma}(f_2)| \le C\rho^{2-\nu_2}\|\phi_2\|_{2,\beta}$.

Therefore $\|\partial_r w_1 \mid_{\partial B(q,1)} \|_{1,\beta} \le C\rho^{2-\nu_2}\|\phi_2\|_{2,\beta}$, and so

$$\begin{aligned} T^2_\rho \phi_2 = {} & \frac{\tau_2}{2} a_0 \partial_r \partial_\tau v_{\rho,\tau_2,\gamma} \mid_{\partial B(q,1)} - a_1 \cdot \partial_r \partial_{\bar z} v_{\rho,\tau_2,\gamma} \mid_{\partial B(q,1)} + a_1 \cdot \partial_r \eta^\perp \mid_{\partial B(q,1)} \\ & + 2\sum_{n=2}^{+\infty} n a_n \cdot e^{-in\theta} + O\left(\rho^{2-\nu_2}\|\phi_2\|_{2,\beta}\right). \end{aligned}$$

By direct computation, we find $\partial_r\partial_\tau v_{\rho,\tau_2,\gamma} \mid_{\partial B(q,1)} = O(\rho^2)$ and

$$\begin{aligned} \partial_r \partial_{\bar z} v_{\rho,\tau_2,\gamma} \mid_{\partial B(q,1)} &= \partial_r \left(-\frac{2}{r}e^{i\theta} + 2\bar\gamma r e^{-i\theta} + \eta^\perp(re^{i\theta})\right) \mid_{r=1} + O(\rho^2) \\ &= 2(e^{i\theta} + \bar\gamma e^{-i\theta}) + \partial_r \eta^\perp(re^{i\theta}) \mid_{r=1} + O(\rho^2). \end{aligned}$$

Consequently, $T^2_\rho\phi_2 = T^2_0\phi_2 + O(\rho^{2-\nu_2}\|\phi_2\|_{2,\beta})$ and the proof of the lemma is completed. ∎

Define

$$\begin{aligned} T_\rho : {} & C^{2,\beta}(\partial B) \to C^{1,\beta}(\partial B) \\ & \Phi = (\phi_1,\phi_2) \to T_\rho\Phi = (T^1_\rho\phi_1, T^2_\rho\phi_2) \end{aligned}$$

and similarly the operator T_0. We want to prove the following lemma.

Lemma 4.11. *There exists $\rho_0 > 0$ small such that the operator $S_\rho - T_\rho$ is invertible with uniformly bounded inverse for $\rho \in (0,\rho_0)$.*

Proof. Since $S_\rho - T_\rho \to S_0 - T_0$ as $\rho \to 0$ in the operatorial norm, we want to prove that $S_0 - T_0$ is invertible. By an idea of R. Mazzeo used in [4], we claim that it is enough to prove that $S_0 - T_0$ is injective. Regarding $S_0 - T_0$ as an operator from $H^1(\partial B)$ into $L^2(\partial B)$, it is a self-adjoint first-order pseudodifferential operator. Since S_0 and T_0 are elliptic with principal symbols $-|\xi|$ and $|\xi|$, respectively, the difference $S_0 - T_0$ is also elliptic and semibounded. Hence, $S_0 - T_0$ has a discrete spectrum and the invertibility reduces to prove injectivity. The invertibility in Hölder spaces then will follow by classical regularity theory. Let $\Phi \in H^1(\partial B)$ such that $(S_0 - T_0)\Phi = 0 \in L^2(\partial B)$. In view of (2) in Remark 4.10, by Lemmas 4.8 and 4.9 there exists a solution w_0 for the problem

$$\begin{cases} \Delta w_0 = 0 & \text{in } \Omega \setminus S, \\ w_0 = 0 & \text{on } \partial\Omega, \\ w_0 = \Phi & \text{on } \partial B, \end{cases}$$

such that

$$w(z) = \begin{cases} s_1 + 2q_1 \cdot \overline{z-p} + O(|z-p|^2) & \text{as } z \to p, \\ s_2 + 2q_2 \cdot \left(\frac{z-q}{|z-q|^2} - \bar\gamma \overline{z-q}\right) + O(|z-q|^2) & \text{as } z \to q \end{cases}$$

for some $s_i \in \mathbb{R}$ and $q_i \in \mathbb{C}$. The assumption $(S_0 - T_0)\Phi = 0$ ensures that we are gluing harmonic functions in $\tilde\Omega$ and B which coincide with their normal derivative on ∂B. In this way the resulting function is harmonic in $\Omega \setminus S$.

In case $S = \{p\}$, w_0 is bounded near p and it extends to a harmonic function in Ω with homogeneous Dirichlet boundary condition, hence $w_0 = 0$, $\Phi = 0$, and the injectivity of $S_0 - T_0$ is proved. In the remaining cases S contains the point $q \neq p$, the solution w_0 must be equal to $8\pi q_2 \cdot \partial_{\bar{z}'} G(z,q)$ because their difference is a harmonic function in $\Omega \setminus S$ with removable singularities. Moreover, there holds $2\partial_z \left(q_2 \cdot \partial_{\bar{z}'} H(z,q)\right)|_{z=q} = -\frac{q_2\gamma}{4\pi}$ which can be rewritten as follows:

$$q_2 \partial_{zz'} H(q,q) + \bar{q}_2 \partial_{z\bar{z}'} H(q,q) = -\frac{q_2 \gamma}{4\pi}. \tag{33}$$

Let us recall that, if $S = \{q\}$, $\mathcal{F}(z) = H(z,z) + \frac{1}{4\pi}\ln\left(|z-p|^{2\alpha} f(z)\right)$ and $\gamma = 4\pi\partial_{zz} H(q,q) + \frac{1}{2}\left[\partial_{zz}\left(|z-p|^{2\alpha} f(z)\right)\right](q)$; while if $S = \{p,q\}$, $\mathcal{F}(z) = H(z,z) + \frac{1}{4\pi}\ln(|z-p|^{2\alpha} f(z)) + 2(1+\alpha)G(z,p)$ and $\gamma = 4\pi\partial_{zz} H(q,q) + \frac{1}{2}\left[\partial_{zz}\left(|z-p|^{2\alpha} f(z)\right)\right](q) + 4\pi(1+\alpha)\partial_{zz} G(q,p)$. Hence (33) is equivalent to $D^2\mathcal{F}(q)(\frac{q_2}{\bar{q}_2}) = 0$ when we assume further that $\Delta \ln f(q) = 0$. The assumption that q is a nondegenerate critical point for $\mathcal{F}(z)$ provides $q_2 = 0$. Then w_0 is not singular in the points of S and as before $w_0 = 0$, $\Phi = 0$, and the injectivity of $S_0 - T_0$ follows. $\square$

We are now in position to give the proof of Theorem 4.6.

Proof of Theorem 4.6. By Lemma 4.7 and Propositions 4.1 and 4.5, for any $f \in Y$ we can find $w_{\text{ext}} \in C^{2,\beta}(\tilde{\Omega})$ and $w_{\text{int}}, i \in \mathcal{E}_i$, $i = 1,2$, which solve

$$\begin{cases} \mathcal{L}_\rho w_{\text{ext}} = f & \text{in } \tilde{\Omega}, \\ w_{\text{ext}} = 0 & \text{on } \partial\tilde{\Omega}, \end{cases} \quad \begin{cases} \mathcal{L}_\rho w_{\text{int}}, 1 = f & \text{in } B(p,1), \\ w_{\text{int}}, 1 = 0 & \text{on } \partial B(p,1), \end{cases} \quad \begin{cases} \mathcal{L}_\rho w_{\text{int}}, 2 = f & \text{in } B(q,1), \\ w_{\text{int}}, 2 = 0 & \text{on } \partial B(q,1). \end{cases}$$

Moreover, $\|w_{\text{ext}}\|_{2,\beta,\tilde{\Omega}} + \sum_i \|w_{\text{int}}, i\|_{\mathcal{E}_i} \leq C\|f\|_Y$. By Lemma 4.11, we find $\Phi \in C^{2,\beta} \cdot (\partial B)$ such that

$$(S_\rho - T_\rho)\Phi = \left(-\partial_{r_1}(w_{\text{ext}} - w_{\text{int}}, 1)\,|_{\partial B(p,1)}, -\partial_{r_2}(w_{\text{ext}} - w_{\text{int}}, 2)\,|_{\partial B(q,1)}\right)$$

with $\|\Phi\|_{C^{2,\beta}(\partial B)} \leq C\|f\|_Y$. At this point, we define $w_{\text{ker}} \in C(\Omega \setminus S)$ by solving

$$\begin{cases} \mathcal{L}_\rho w_{\text{ker}} = 0 & \text{in } \Omega \setminus \partial B, \\ w_{\text{ker}} = 0 & \text{on } \partial\Omega, \\ w_{\text{ker}} = \Phi & \text{on } \partial B. \end{cases}$$

Define

$$w(z) = \begin{cases} w_{\text{ext}}(z) + w_{\text{ker}}(z) & \text{in } \tilde{\Omega}, \\ w_{\text{int}}, 1(z) + w_{\text{ker}}(z) & \text{in } B(p,1), \\ w_{\text{int}}, 2(z) + w_{\text{ker}}(z) & \text{in } B(q,1). \end{cases}$$

Since the external and internal normal derivative of $w(z)$ on ∂B coincide, we conclude that $w(z)$ is a solution for the problem

$$\begin{cases} \mathcal{L}_\rho w = f & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \\ w \in C^{2,\beta}(\Omega \setminus S). \end{cases} \tag{34}$$

It remains to discuss the uniqueness of w: let w' be another solution of (34). Set $\Phi' = (w'\,|_{\partial B(p,1)}, w'\,|_{\partial B(q,1)})$. Then $(S_\rho - T_\rho)\Phi' = (S_\rho - T_\rho)\Phi = 0$ and, by injectivity of $S_\rho - T_\rho$, we deduce $\Phi' = \Phi$ and so $w' = w$. $\square$

4.4. The linearized operator. Now we want to pass the information on the invertibility of $\mathcal{L}_\rho$ to $\Lambda_\rho = \Delta + W_\rho$, where $W_\rho(z) = \rho^2|z-p|^{2\alpha} f(z) e^{v(\rho,0,0)(z)}$, and in turn to $L_{(0,0,0)}$. To an element $(h, \lambda, a) \in \mathcal{E}$ we associate in a canonical way the function

$$w(z) = h(z) + \chi(z-p)\lambda_1 \partial_\tau v_{\rho,\tau_1}(z-p) + \chi(z-q)\lambda_2 \partial_\tau v_{\rho,\tau_2,\gamma}(z-q) \\ + \chi(z-q) 2a \cdot \partial_{\bar{z}} v_{\rho,\tau_2,\gamma}(z-q)$$

(with the understanding that $\lambda_1 = 0$ if $p \notin S$ and $\lambda_2 = 0$, $a = 0$ if $q \notin S$) and we want to evaluate the difference $\Lambda_\rho - \mathcal{L}_\rho$ on $w(z)$. We have

$$\|(\Lambda_\rho - \mathcal{L}_\rho)w\|_Y \le C\rho^2(\|h\|_{2,\beta,\tilde{\Omega}} + |\lambda| + |a|) \\ + \|\rho^2|z-p|^{2\alpha}(f(z)e^{v(\rho,0,0)(z)} - e^{v_{\rho,\tau_1}(z-p)})w\|_{0,\beta,\nu_1-2,B(p,1)} \\ + \|\rho^2(|z-p|^{2\alpha} f(z) e^{v(\rho,0,0)(z)} - e^{v_{\rho,\tau_2,\gamma}(z-q)})w\|_{0,\beta,\nu_2-2,B(q,1)}.$$

Therefore,

$$(\Lambda_\rho - \mathcal{L}_\rho)(z) = \begin{cases} O\left(\rho^2|z-p|^{2\alpha+1} e^{v_{\rho,\tau_1}(z-p)}\right) & \text{in } B(p,1), \\ O\left(\rho^2|z-q|^3 e^{v_{\rho,\tau_2,\gamma}(z-q)}\right) & \text{in } B(q,1) \end{cases}$$

in view of (11), (12), and (13). Since $|\partial_\tau v_{\rho,\tau,\lambda}(z)| + |z||\partial_{\bar{z}} v_{\rho,\tau,\lambda}(z)| = O(1)$ in $B(0,1)$ when $\lambda \in \{0, \gamma\}$, we deduce $\|(\Lambda_\rho - \mathcal{L}_\rho)w\|_{0,\beta,\nu_1-2,B(p,1)} + \|(\Lambda_\rho - \mathcal{L}_\rho)w\|_{0,\beta,\nu_2-2,B(q,1)} = O\left(r^s\|(h,\lambda,a)\|_{\mathcal{E}'}\right)$, where $r := \max\{r_1, r_2\}$ and $s = \min\{1-\nu_1, 2-\nu_2\} > 0$. This implies $\|(\Lambda_\rho - \mathcal{L}_\rho)w\|_Y \le Cr^s\|(h,\lambda,a)\|_{\mathcal{E}'}$. Note that Theorem 4.6 can be restated as follows: for any $f \in Y$ there exists $(h_0, \lambda_0, a_0) = \mathcal{L}_\rho^{-1} f \in \mathcal{E}$ such that

$$w_0(z) = h_0(z) + \chi(z-p)(\lambda_0)_1 \partial_\tau v_{\rho,\tau_1}(z-p) + \chi(z-q)(\lambda_0)_2 \partial_\tau v_{\rho,\tau_2,\gamma}(z-q) \\ + 2\chi(z-q) a_0 \cdot \partial_{\bar{z}} v_{\rho,\tau_2,\gamma}(z-q)$$

is a solution for $\mathcal{L}_\rho w_0 = f$ in Ω and $\|(h_0, \lambda_0, a_0)\|_{\mathcal{E}} \le C\|f\|_Y$, provided $\rho > 0$ is small enough.

On the other hand, for given $(h, \lambda, a) \in \mathcal{E}$ the associated $w(z)$ solves $\Lambda_\rho w = f$ in Ω if and only if it corresponds to a fixed point for the map

$$\mathcal{E} \to \mathcal{E} \\ (h, \lambda, a) \to \mathcal{L}_\rho^{-1} f - \mathcal{L}_\rho^{-1}(\Lambda_\rho - \mathcal{L}_\rho) w.$$

Since

$$\|\mathcal{L}_\rho^{-1}(\Lambda_\rho - \mathcal{L}_\rho) w\|_{\mathcal{E}} \le C\|(\Lambda_\rho - \mathcal{L}_\rho)w)\|_Y \le Cr^s\|(h,\lambda,a)\|_{\mathcal{E}},$$

there exists ρ_0 small such that for $0 < \rho < \rho_0$ such a map defines a contraction. Thus, for any $f \in Y$ there exists a unique $(h, \lambda, a) \in \mathcal{E}$ solving $\Lambda_\rho w = f$ in Ω with $\|(h,\lambda,a)\|_{\mathcal{E}} \le C\|\mathcal{L}_\rho^{-1} f\|_{\mathcal{E}}$.

We rewrite the solution $w(z)$ in the form

$$w(z) = h'(z) + \sum_i \lambda_i \partial_{\lambda_i} v(\rho,0,0)(z) + 2(-a) \cdot \partial_{\overline{a}} v(\rho,0,0)(z)$$

with

$$\begin{aligned} h'(z) = h(z) &+ \chi(z-p)\lambda_1 \left(\partial_\tau v_{\rho,\tau_1}(z-p) - \partial_{\lambda_1} v(\rho,0,0)(z)\right) \\ &+ \chi(z-q)\lambda_2 \left(\partial_\tau v_{\rho,\tau_2,\gamma}(z-q) - \partial_{\lambda_2} v(\rho,0,0)(z)\right) \\ &+ 2\left(1-\chi(z-q)\right) a \cdot \partial_{\overline{a}} v(\rho,0,0)(z) \\ &+ 2\chi(z-q)\, a \cdot \left(\partial_{\bar z} v_{\rho,\tau_2,\gamma}(z-q) + \partial_{\overline{a}} v(\rho,0,0)(z)\right), \end{aligned}$$

where we have taken into account that $(1-\chi(z-p))\,\partial_{\lambda_1} v(\rho,0,0)(z)$ and $(1-\chi(z-q))\partial_{\lambda_2} v(\rho,0,0)(z)$ are identically zero. Let us compute the derivatives of $v(\rho,\lambda,a)$:

$$\begin{aligned} \partial_{\lambda_1} v(\rho,0,0)(z) &= \chi\left(\frac{z-p}{r_1}\right)\partial_\tau v_{\rho,\tau_1}(z-p), \\ \partial_{\lambda_2} v(\rho,0,0)(z) &= \chi\left(\frac{z-q}{r_2}\right)\partial_\tau v_{\rho,\tau_2,\gamma}(z-q), \end{aligned}$$

and

$$\partial_{\overline{a}} v(\rho,0,0)(z) = \begin{cases} \left(1-\chi(\frac{z-p}{r_1})\right) 8\pi \partial_{\bar z'} G(z,q) & \text{in } B(p,1), \\ -\partial_{\bar z} v_{\rho,\tau_2,\gamma}(z-q) - \partial_{\overline{a}} P_0(z) & \\ +\frac{1}{r_2}\partial_{\bar z}\chi(\frac{z-q}{r_2})\left(8\pi(1+\alpha)G(z,p) + 8\pi G(z,q) - U_\rho^2(z)\right) & \\ +\left(1-\chi(\frac{z-q}{r_2})\right)\left(8\pi\partial_{\bar z'} G(z,q)\right. & \\ \qquad \left. +\partial_{\bar z} v_{\rho,\tau_2,\gamma}(z-q) + \partial_{\overline{a}} P_0(z)\right) & \text{in } B(q,1), \\ 8\pi \partial_{\bar z'} G(z,q) \quad \text{in } \tilde\Omega. & \end{cases}$$

Using again that $\Delta \ln f(q) = 0$, we get $\partial_{\overline{a}} P_0(z) = O(|z-q|^2)$, and so

$$\begin{aligned} \|\chi(z-p)\left(\partial_\tau v_{\rho,\tau_1}(z-p) - \partial_{\lambda_1} v(\rho,0,0)(z)\right)\|_{2,\beta,\nu_1,B(p,1)} &\le C(r_1)^{-\nu_1}, \\ \|\chi(z-q)\left(\partial_\tau v_{\rho,\tau_2,\gamma}(z-q) - \partial_{\lambda_2} v(\rho,0,0)(z)\right)\|_{2,\beta,\nu_2,B(q,1)} &\le C(r_2)^{-\nu_2}, \\ \|\chi(z-q)\left(\partial_{\bar z} v_{\rho,\tau_2,\gamma}(z-q) + \partial_{\overline{a}} v(\rho,0,0)(z)\right)\|_{2,\beta,\nu_2,B(q,1)} &\le C(r_2)^{-\nu_2}, \end{aligned}$$

the last estimate being valid in view of the fact that

$$8\pi\partial_{\bar z'} G(z,q) + \partial_{\bar z} v_{\rho,\tau_2,\gamma}(z-q) = 2\frac{z-q}{|z-q|^2} + O(1) - 2\frac{z-q}{|z-q|^2} + O\left(\frac{\rho^2}{|z-q|^3}\right) = O(1)$$

for any $z \in B(q,1) \setminus B(q,r_2)$. Hence $\|h'\|_{X'} \le C\|\mathcal{L}_\rho^{-1}(f)\|_{\mathcal{E}}$ for some uniform constant $C > 0$. Thus, we have proved the following result.

THEOREM 4.12. *There exists $\rho_0 > 0$ small such that for any $\rho \in (0,\rho_0)$, we have that for any $f \in Y$ there exists a unique solution $(h,\lambda,a) \in \mathcal{E}'$ satisfying*

$$\begin{cases} \Lambda_\rho w = f \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega, \\ w(z) = h(z) + \sum_i \lambda_i \partial_{\lambda_i} v(\rho,0,0)(z) + 2a \cdot \partial_{\overline{a}} v(\rho,0,0)(z) \end{cases}$$

with $\|(h,\lambda,a)\|_{\mathcal{E}'} \le C\|\mathcal{L}_\rho^{-1}(f)\|_{\mathcal{E}}$ for some uniform constant $C > 0$.

Let us recall now the definition of $L_{(0,0,0)} : \mathcal{E}' \to Y$ (see section 3): for any $(h,\sigma,b) \in \mathcal{E}'$ we set

$$\begin{aligned} L_{(0,0,0)}(h,\sigma,b) = \Lambda_\rho\Big(h + \sum_i \sigma_i \partial_{\lambda_i} v(\rho,0,0) + 2b \cdot \partial_{\overline{a}} v(\rho,0,0)\Big) \\ + 2\partial_{\bar z}\big[\triangle v(\rho,0,0) + \rho^2|z-p|^{2\alpha} f(z) e^{v(\rho,0,0)}\big] \\ \cdot \big[\big(b\partial_a + \bar b \partial_{\overline{a}}\big)\Psi(0,\cdot)^{-1}\big]. \end{aligned}$$

We have to estimate in Y the term

$$\partial_{\bar z}\big[\Delta v(\rho,0,0) + \rho^2|z-p|^{2\alpha} f(z) e^{v(\rho,0,0)}\big] \cdot \big[\big(b\partial_a + \bar b \partial_{\overline{a}}\big)\Psi(0,\cdot)^{-1}\big].$$

Since $\Psi(a,z) \equiv z$ for $z \in \Omega \setminus B(q,2)$, we have that $\big(b\partial_a + \bar b\partial_{\overline{a}}\big)\Psi(0,\cdot)^{-1} \equiv 0$ in $\Omega \setminus B(q,2)$. In view of (39) and (40) we get

$$\Big\|\partial_{\bar z}\big[\Delta v(\rho,0,0) + \rho^2|z-p|^{2\alpha} f(z) e^{v(\rho,0,0)}\big] \cdot \big[\big(b\partial_a + \bar b \partial_{\overline{a}}\big)\Psi(0,\cdot)^{-1}\big]\Big\|_Y = o(1)|b|,$$

and so using a perturbation argument by Theorem 4.12 we derive the following result.

THEOREM 4.13. *There exists $\rho_0 > 0$ small such that for any $\rho \in (0,\rho_0)$ and $f \in Y$ there exists a unique solution $(h,\sigma,b) \in \mathcal{E}'$ satisfying*

$$\begin{cases} L_{(0,0,0)} w = f \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega, \\ w(z) = h(z) + \sum_i \sigma_i \partial_{\lambda_i} v(\rho,0,0)(z) + 2b \cdot \partial_{\overline{a}} v(\rho,0,0)(z) \end{cases}$$

such that $\|L^{-1}_{(0,0,0)}(f)\|_{\mathcal{E}'} \le C\|\mathcal{L}_\rho^{-1}(f)\|_{\mathcal{E}}$ for some uniform constant $C > 0$.

5. Some estimates. In order to apply a fixed point argument to K, we used in a crucial way the fact that $K : \mathcal{E}' \to \mathcal{E}'$ maps a suitable small ball into itself; see Step 4 of section 3. To obtain such information we need the estimate contained in (37) below. For this end, first we estimate the preimage through $\mathcal{L}_\rho$ of the error term $\eta = \Delta v(\rho,0,0) + \rho^2|z-p|^{2\alpha} f(z) e^{v(\rho,0,0)}$. In $\tilde\Omega = \Omega \setminus B$, $v(\rho,0,0)$ is a harmonic function and hence

$$|\eta(z)| = |\rho^2|z-p|^{2\alpha} f(z) \exp\left(8\pi(1+\alpha)G(z,p) + 8\pi G(z,q)\right)| = O(\rho^2)$$

in $\tilde\Omega$. In $B(p,1)$ we have that

$$\begin{aligned}\eta(z) = &\frac{1}{r_1^2}\Delta\chi\left(\frac{z-p}{r_1}\right)\left[v_{\rho,\tau_1}(z-p)-\ln f(p)-8\pi(1+\alpha)G(z,p)-8\pi G(z,q)\right]\\ &+\frac{8}{r_1}\partial_{\bar z}\chi\left(\frac{z-p}{r_1}\right)\cdot\partial_{\bar z}\left[v_{\rho,\tau_1}(z-p)-\ln f(p)-8\pi(1+\alpha)G(z,p)-8\pi G(z,q)\right]\\ &+\rho^2|z-p|^{2\alpha}e^{v_{\rho,\tau_1}(z-p)}\Bigg\{-\chi\left(\frac{z-p}{r_1}\right)+f(z)e^{-\ln f(p)}\\ &\times\exp\Bigg[\left(1-\chi\left(\frac{z-p}{r_1}\right)\right)(8\pi(1+\alpha)G(z,p)+8\pi G(z,q)\\ &-v_{\rho,\tau_1}(z-p)+\ln f(p))\Bigg]\Bigg\}\end{aligned}$$

and in $B(q,1)$

$$\begin{aligned}\eta(z) = &\frac{1}{r_2^2}\Delta\chi\left(\frac{z-q}{r_2}\right)\left[v_{\rho,\tau_2,\gamma}(z-q)-P_0(z)-8\pi(1+\alpha)G(z,p)-8\pi G(z,q)\right]\\ &+\frac{8}{r_2}\partial_{\bar z}\chi\left(\frac{z-q}{r_2}\right)\cdot\partial_{\bar z}\left[v_{\rho,\tau_2,\gamma}(z-q)-P_0(z)-8\pi(1+\alpha)G(z,p)-8\pi G(z,q)\right]\\ &+\rho^2e^{v_{\rho,\tau_2,\gamma}(z-q)}\Bigg\{-\chi\left(\frac{z-q}{r_2}\right)+|z-p|^{2\alpha}f(z)e^{-P_0(z)}\\ &\times\exp\Bigg[\left(1-\chi\left(\frac{z-q}{r_2}\right)\right)(8\pi(1+\alpha)G(z,p)+8\pi G(z,q)\\ &-v_{\rho,\tau_2,\gamma}(z-q)+P_0(z))\Bigg]\Bigg\},\end{aligned}$$

where $P_a(z)$ is defined in section 2. In $B(q,1)$ there holds

$$\begin{aligned}&\partial_{\bar z}\left(v_{\rho,\tau_2,\gamma}(z-q)-P_0(z)-8\pi(1+\alpha)G(z,p)-8\pi G(z,q)\right)=-\partial_{\bar z}\mathcal{F}_2(q)\\ &-\overline{z-q}\left(\partial_{\bar z\bar z}\mathcal{F}_2(q)-2\overline{\gamma}\right)+O\left(|z-q|^2+\frac{\tau_2^2\rho^2}{|z-q|^3}\right)=O\left(|z-q|^2+\frac{\tau_2^2\rho^2}{|z-q|^3}\right)\end{aligned}\tag{35}$$

in view of the fact that $\partial_z\mathcal{F}_2(q)=0$ and $\gamma=\frac{1}{2}\partial_{zz}\mathcal{F}_2(q)$ (see section 2 for the definitions of $\mathcal{F}_2(z)$ and γ). Similarly, in $B(p,1)$ we get

$$\partial_{\bar z}\left(v_{\rho,\tau_1}(z-p)-\ln f(p)-8\pi(1+\alpha)G(z,p)-8\pi G(z,q)\right)=O\left(1+\frac{\tau_1^2\rho^2}{|z-p|^{2\alpha+3}}\right).$$

As far as second derivatives are concerned, in $B(q,1)$ we have the estimate

$$\partial^2_{\bar z\bar z}\left(v_{\rho,\tau_2,\gamma}(z-q)-P_0(z)-8\pi(1+\alpha)G(z,p)-8\pi G(z,q)\right)=O\left(|z-q|+\frac{\tau_2^2\rho^2}{|z-q|^4}\right).\tag{36}$$

Since $\frac{\rho^2}{r_1^{2\alpha+5}}+\frac{\rho^2}{r_2^5}=O(1)$, recalling (11), (12), and (13), for any $\nu\in(0,2)$ we get the estimates

$$\|\eta\|_{0,\beta,\nu-2,B(p,1)}=O\left(r_1^{1-\nu}\right),\qquad \|\eta\|_{0,\beta,\nu-2,B(q,1)}=O\left(r_2^{3-\nu}\right).$$

Fix $0<\delta<1$ to be specified below. Following the notations of section 4.1, we find

$$|H^0_{\rho,\tau_1}(\eta\ |_{B(p,1)})|+\|\partial_{r_1}G_{\rho,\tau_1}(\eta\ |_{B(p,1)})\|_{1,\beta,\partial B(p,1)}=O(r_1^{1-\delta})$$

and

$$|H^0_{\rho,\tau_2}(\eta\ |_{B(q,1)})|+|H^1_{\rho,\tau_2}(\eta\ |_{B(q,1)})|+\|\partial_{r_2}G_{\rho,\tau_2}(\eta\ |_{B(q,1)})\|_{1,\beta,\partial B(q,1)}=O(r_2^{3-\delta}).$$

Moreover, choosing $\nu=\nu_1$ in $B(p,1)$ and $\nu=\nu_2$ in $B(q,1)$ we get

$$\|G_{\rho,\tau_1}(\eta\ |_{B(p,1)})\|_{2,\beta,\nu_1,B(p,1)}=O(r_1^{1-\nu_1})\ ,\ \ \|G_{\rho,\tau_2,\gamma}(\eta\ |_{B(q,1)})\|_{2,\beta,\nu_2,B(q,1)}=O(r_2^{3-\nu_2}).$$

By Proposition 4.5 we have

$$|H^0_{\rho,\tau_2,\gamma}(\eta\ |_{B(q,1)})|+|H^1_{\rho,\tau_2,\gamma}(\eta\ |_{B(q,1)})|+\|\partial_{r_2}G_{\rho,\tau_2,\gamma}(\eta\ |_{B(q,1)})\|_{1,\beta,\partial B(q,1)}=O(r_2^{3-\delta}),$$

$$\|G_{\rho,\tau_2,\gamma}(\eta\ |_{B(q,1)})\|_{2,\beta,\nu_2,B(q,1)}=O(\|\eta\|_{0,\beta,\nu_2-2,B(q,1)})=O(r_2^{3-\nu_2}).$$

Hence, following the notation and the construction of section 4.3, we obtain $\|w_{\rm ext}\|_{2,\beta,\tilde\Omega}\le C\|\eta\|_{0,\beta,\tilde\Omega}=O(\rho^2)$, $\|\partial_{r_1}w_{{\rm int},1}\|_{1,\beta,\partial B(p,1)}=O(r_1^{1-\delta})$, and $\|\partial_{r_2}w_{{\rm int},2}\|_{1,\beta,\partial B(q,1)}=O(r_2^{3-\delta})$.

Let $\Phi=-(S_\rho-T_\rho)^{-1}\left(\partial_{r_1}(w_{\rm ext}-w_{{\rm int},1})\ |_{\partial B(p,1)},\partial_{r_2}(w_{\rm ext}-w_{{\rm int},2})\ |_{\partial B(q,1)}\right)$. So $\|\Phi\|_{C^{2,\beta}(\partial B)}=O(r_1^{1-\delta}+r_2^{3-\delta})$. Consequently, $\|w_{\rm ker}\|_{2,\beta,\tilde\Omega}+\|w_{\rm ker}\ |_{B(p,1)}\ \|_{\mathcal{E}_1}+\|w_{\rm ker}\ |_{B(q,1)}\ \|_{\mathcal{E}_2}=O(r_1^{1-\delta}+r_2^{3-\delta})$ which in turn implies $\|\mathcal{L}_\rho^{-1}\eta\|_{\mathcal{E}}=O(r_1^{1-\delta}+r_1^{1-\nu_1}+r_2^{3-\delta}+r_2^{3-\nu_2})$. We can choose $\nu_1\in(0,1)$ and $\nu_2\in(1,2)$ in such a way that $(\nu_1,1-\nu_1)\cap(\nu_2-1,2-\nu_2)\neq\emptyset$ and we can suppose that δ is fixed to belong in this set. Hence, by Theorem 4.13 we get

$$\|L^{-1}_{(0,0,0)}\eta\|_{\mathcal{E}'}=O\left(r_1^{1-\delta}+r_2^{2-\delta}\right). \tag{37}$$

Let us define $\sigma=\frac{4\alpha+5}{2\nu_1}+1$ and choose $r_i=\rho^{\frac{1}{\sigma\nu_i}}$. In this way, we have $\frac{\rho^2}{r_1^{4\alpha+5}}=\rho^{2(1-\frac{4\alpha+5}{2\sigma\nu_1})}\to0$ as $\rho\to0$, $\frac{\rho^2}{r_2^5}=\rho^{2(1-\frac{5}{2\sigma\nu_2})}\to0$ as $\rho\to0$, and

$$\left(\sum_{i=1}^2 r_i^{-\nu_i}\right)\left(r_1^{1-\delta}+r_2^{2-\delta}\right)=2\left(\rho^{\frac{1-\delta-\nu_1}{\sigma\nu_1}}+\rho^{\frac{2-\delta-\nu_2}{\sigma\nu_2}}\right)\to0 \tag{38}$$

as $\rho\to0$.

Now, taking the derivative with respect to $\bar z$ in the expression for $\eta=\Delta v(\rho,0,0)+\rho^2|z-p|^{2\alpha}f(z)e^{v(\rho,0,0)}$ and using (13), (35), and (36), we can conclude

$$\left\|\partial_{\bar z}\left(\Delta v(\rho,0,0)+\rho^2|z-p|^{2\alpha}f(z)e^{v(\rho,0,0)}\right)\right\|_{0,\beta,B(q,2)\setminus B(q,1)}=O(\rho^2) \tag{39}$$

and

$$\left\|\partial_{\bar z}\left(\Delta v(\rho,0,0)+\rho^2|z-p|^{2\alpha}f(z)e^{v(\rho,0,0)}\right)\right\|_{0,\beta,\nu_2-2,B(q,1)}=O\left(r_2^{2-\nu_2}\right), \tag{40}$$

where we use in a crucial way the fact that $|z-p|^{2\alpha}f(z)e^{-P_0(z)}-1=O(|z-q|^3)$.

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