SOME REMARKS CONCERNING SYMMETRY-BREAKING FOR THE GINZBURG-LANDAU EQUATION

PIERPAOLO ESPOSITO

Abstract. The correlation term, introduced in [13] to describe the interaction between very far apart vortices, governs symmetry-breaking for the Ginzburg-Landau equation in $\mathbb{R}^2$ or bounded domains. It is a homogeneous function of degree (-2), and then for $\frac{2\pi}{N}-$symmetric vortex configurations can be expressed in terms of the so-called correlation coefficient. Ovchinnikov and Sigal [13] have computed it in few cases and conjectured its value to be an integer multiple of $\frac{\pi}{4}$. We will disprove this conjecture by showing that the correlation coefficient always vanishes, and will discuss some of its consequences.

Keywords: Ginzburg-Landau equation, Symmetry-breaking, correlation term

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1. Introduction

The Ginzburg-Landau theory is a very popular model in super-conductivity [6]. Stationary states are described by complex-valued solutions u of the planar equation

$$-\Delta u = k^2 u(1-|u|^2),$$

where $k>0$ is the Ginzburg-Landau parameter. The condensate wave function u describes the superconductive regime in the sample by simply interpreting $|u|^2$ as the density of Cooper electrons pairs. The zeroes of u, where the normal state is restored, are called vortices. The parameter k depends on the physical properties of the material and distinguishes between Type I superconductors $k<\frac{1}{\sqrt{2}}$ (in this normalization of constants) and Type II superconductors $k>\frac{1}{\sqrt{2}}$.

In the entire plane $\mathbb{R}^2$ the parameter k does not play any role, as we can reduce to the case $k=1$ by simply changing u into $u(\frac{x}{k})$. Supplemented by the correct asymptotic behavior at infinity, the Ginzburg-Landau equation now reads as

$$\begin{cases} -\Delta U = U(1-|U|^2) & \text{in } \mathbb{R}^2 \\ |U| \to 1 \text{ as } |x|\to\infty. \end{cases} \tag{1.1}$$

The condition $|U|\to 1$ as $|x|\to\infty$ allows to define the (topological) degree $\deg U$ of U as the winding number of U at ∞:

$$\deg U = \frac{1}{2\pi}\int_{|x|=R} d(arg\, U),$$

where $R>0$ is chosen large so that $|U|\geq \frac{1}{2}$ in $\mathbb{R}^2\setminus B_R(0)$. Given $n\in\mathbb{Z}$, the only known solution of (1.1) with $\deg U = n$ is the "radially symmetric" one $U_n(x)=S_n(|x|)(\frac{x}{|x|})^n$ (in complex notations with $x\in\mathbb{C}$), where S_n is the solution of the following ODE:

$$\begin{cases} \ddot{S}_n + \frac{1}{r}\dot{S}_n - \frac{n^2}{r^2}S_n + S_n(1-S_n^2) = 0 & \text{in } (0,+\infty) \\ S_n(0)=0\,, \quad \lim\limits_{r\to+\infty} S_n = 1. \end{cases}$$

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Existence and uniqueness of S_n is shown in [4,7]. Moreover, the solution U_n is stable for $|n| \leq 1$ and unstable for $|n| > 1$ [11]. When $n = \pm 1$, the solution $U_{\pm 1}$ is unique, modulo translations and rotations, in the class of functions U with deg $U = \pm 1$ and $\int_{\mathbb{R}^2} (|U|^2 - 1)^2 dx < +\infty$ [10].

One of the open problems (Problem 1)– that Brezis-Merle-Rivière raise out in [3]– concerns the existence of solutions U of (1.1) with deg $U = n$, $|n| > 1$, which are not "radially symmetric" around any point. So far there is no rigorous answer, but a strategy to find them has been proposed in [12]. Formally, a solution U of (1.1) is a critical point of the functional

$$\mathcal{E}(\Psi) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \Psi|^2 dx + \frac{1}{4} \int_{\mathbb{R}^2} (|\Psi|^2 - 1)^2 dx.$$

Since $\mathcal{E}(\Psi) = +\infty$ for any C^1-map Ψ so that $|\Psi| \to 1$ as $|x| \to +\infty$ and deg $(\Psi) \neq 0$, Ovchinnikov and Sigal [11] have proposed to correct $\mathcal{E}$ into

$$\mathcal{E}_{\text{ren}}(\Psi) = \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla \Psi|^2 - \frac{(\text{deg } \Psi)^2}{|x|^2} \chi + \frac{1}{4} (|\Psi|^2 - 1)^2 \right) dx,$$

where χ is a smooth cut-off function with $\chi = 0$ when $|x| \leq R$ and $\chi = 1$ when $|x| \geq R + R^{-1}$, and $R >> 1$ is given. Given a vortex configuration $(\underline{a}, \underline{n}) = (a_1, \dots, a_K, n_1, \dots, n_K)$, a C^1-map Ψ so that $|\Psi| \to 1$ as $|x| \to +\infty$ has vortex configuration $(\underline{a}, \underline{n})$ if $a_1, \dots, a_K$ are the only zeroes of Ψ with local indices $n_1, \dots, n_K$, denoted for short as conf $\Psi = (\underline{a}, \underline{n})$. Given $\underline{n}_0$, Ovchinnikov and Sigal [12] introduce the "intervortex energy" E given by

$$E(\underline{a}) = \inf \{ \mathcal{E}_{\text{ren}}(\Psi) : \text{ conf } \Psi = (\underline{a}, \underline{n}_0) \},$$

and conjecture that $\underline{a}_0$ is a critical point of E if and only if there is a minimizer U for $E(\underline{a}_0)$, yielding to a solution of (1.1) with conf $U = (\underline{a}_0, \underline{n}_0)$ which is not "radially symmetric" around any point by construction. Letting $d_{\underline{a}} = \min_{i \neq j} |a_i - a_j|$, the following asymptotic expression is established [12]:

$$E(\underline{a}) = \sum_{j=1}^{K} \mathcal{E}_{\text{ren}}(U_{n_i}) + H(\frac{\underline{a}}{R}) + \text{Rem} \tag{1.2}$$

with Rem $= O(d_{\underline{a}}^{-1})$ as $d_{\underline{a}} \to +\infty$, where $H(\underline{a}) = -\pi \sum_{i \neq j} n_i n_j \ln |a_i - a_j|$ is the energy of the vortex pairs interactions. When $\nabla H(\underline{a}) = 0$, the estimate in (1.2) improves up to Rem $= O(d_{\underline{a}}^{-2})$.

When $\nabla H(\underline{a}) = 0$ (a so-called forceless vortex configuration), by choosing refined test functions the asymptotic expression (1.2) is improved [13] in the form of the following upper bound:

$$E(\underline{a}) \leq \sum_{j=1}^{K} \mathcal{E}_{\text{ren}}(U_{n_i}) + H(\frac{\underline{a}}{R}) - A(\underline{a}) + \text{Rem} \tag{1.3}$$

with Rem $= O(d_{\underline{a}}^{-2} + R^{-2})$ as $d_{\underline{a}} \to +\infty$, where the correlation term $A(\underline{a})$ is a homogeneous function of degree (-2) given as

$$A(\underline{a}) = \frac{1}{4} \int_{\mathbb{R}^2} \left[|\sum_{j=1}^{K} \nabla \varphi_j|^4 - \sum_{j=1}^{K} |\nabla \varphi_j|^4 \right],$$

with $\varphi_j(x) = n_j \theta(x - a_j)$, $j = 1, \dots, K$, and $\theta(x)$ the polar angle of $x \in \mathbb{R}^2$.

To push further the analysis, in [13] the attention is restricted to symmetric vortex configurations in order to reduce the number of independent variables in $E(\underline{a})$. In particular, the simplest $\frac{2\pi}{N}-$symmetric vortex configurations $(\underline{a}, \underline{n})$ (which are invariant under $\frac{2\pi}{N}-$rotations

and reflections w.r.t. the real axis) have the form: $a_0 = 0$, $a_1, \dots, a_N$ are the vertices of a regular $N-$polygon with $a_1 = 1$ and $n_1 = \dots = n_N = m$. We impose also the forceless condition $\nabla H(\underline{a}) = 0$, which simply reads as $n_0 = -\frac{N-1}{2}m$.
Since $|a_1| = \dots = |a_N|$, the only variable is the size $a = |a_1|$ of the polygon, and then the intervortex energy will be in the form $E(a)$. Since $A(\underline{a})$ is homogeneous of degree -2, we have that $A(\underline{a}) = \frac{A_0}{a^2}$, where

$$A_0 := A(1, e^{\frac{2\pi i}{N}}, \dots, e^{\frac{2\pi i(N-1)}{N}}) \tag{1.4}$$

is the correlation coefficient for given $n_0 = -\frac{N-1}{2}m$ and $n_1 = \dots = n_N = m$. In [13] the existence of c.p.'s of $E(a)$ is shown for the cases $(N, m) = (2, 2)$ and $(N, m) = (4, 2)$ by comparing $E(a)$ for a small and large, and using the positive sign of A_0 (the correlation coefficient has value 8π and 80π, respectively). It is also conjectured [13] that the correlation coefficient has values which are integer multiples of $\frac{\pi}{4}$. With a long but tricky computation, in the next section we will disprove such a conjecture by showing

Theorem 1.1. *The correlation coefficient in (1.4) always vanishes: $A_0 = 0$, for all $N \geq 2$ and $m \in \mathbb{Z}$.*

Beside the role of A_0 in symmetry-breaking phenomena for (1.1) in $\mathbb{R}^2$, as already discussed, we will also explain its connection with the Ginzburg-Landau equation

$$\begin{cases} -\Delta u = k^2 u(1 - |u|^2) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

on a bounded domain Ω for strongly Type II superconductors $k \to +\infty$, where $g : \partial\Omega \to S^1$ is a smooth map.

The energy functional for (1.5)

$$E_k(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{k^2}{4}\int_\Omega (1 - |u|^2)^2$$

has always a minimizer $\bar{u}_k$ in the space $H = \{u \in H^1(\Omega, \mathbb{C}) : \ u = g \text{ on } \partial\Omega\}$. When $d = \deg g \neq 0$, by [2,15,16] we know that on simply connected domains $\bar{u}_k$ has exactly $|d|$ simple zeroes $a_1, \dots, a_{|d|}$ for k large, where $(a_1, \dots, a_{|d|})$ is a critical point for a suitable "renormalized energy" $W(a_1, \dots, a_{|d|})$. The symmetry-breaking phenomenon here takes place, driven by an external mechanism like the boundary condition that forces the confinement of vortices in some equilibrium configuration. A similar result does hold [2] on star-shaped domains for any solutions sequence u_k of (1.5). Near any vortex a_i, the function $u(\frac{x}{k} + a_i)$ behaves like $U_{n_i}(x)$.

Once the asymptotic behavior is well understood, a natural question concerns the construction of such solutions for any given c.p. $(a_1, \dots, a_K)$ of W, and a positive answer has been given by a heat-flow approach [8,9], by topological methods [1] and by perturbative methods [5,14] in case $n_1 = \dots = n_K = \pm 1$. In [14], page 12, it is presented as an open problem to know whether or not there are solutions having vortices collapsing as $k \to \infty$, the simplest situation being problem (1.5) on the unit ball B with boundary value $g_0 = \frac{x^2}{|x|^2}$:

$$\begin{cases} -\Delta u = k^2 u(1 - |u|^2) & \text{in } B \\ u = g_0 & \text{on } \partial B. \end{cases} \tag{1.6}$$

It is conjectured the existence of solutions to (1.6) having a vortex of degree -1 at the origin $a_0 = 0$ and three vortices of degrees $+1$ at the vertices la_j, $a_j = e^{\frac{2\pi i}{3}(j-1)}$ for $j = 1, 2, 3$, of a small ($l << 1$) equilateral triangle centered at 0. This vortex configuration is $\frac{2\pi}{3}-$symmetric, forceless and has "renormalized energy"

$$W(l) = -6\pi \ln 3 - 6\pi \ln(1 - l^6) + O(l^9)\,, \quad l > 0. \tag{1.7}$$

In collaboration with J. Wei, we were working on this problem. Inspired by [5], we were aiming to use a reduction argument of Lyapunov-Schmidt type, starting from the approximating solutions U_k for (1.6) given by

$$U_k(x) = e^{i\varphi_k(x)} U_{-1}(kx) \prod_{j=1}^{3} U_1\left(k(x - le^{\frac{2\pi i}{3}(j-1)})\right)$$

with $l \to 0$ and $lk \to +\infty$, where the function φ_k is an harmonic function so that $U_k\mid_{\partial B} = g_0$. The interaction due to the collapsing of three vortices onto 0 gives at main order a term $(lk)^{-2}$ with the plus sign, i.e. for some $J_0 > 0$ there holds the energy expansion

$$\begin{aligned} E_k(U_k) &= 4\pi \ln k + I + \frac{1}{2} W(l) + J_0 (lk)^{-2} + o((lk)^{-2}) \\ &= 4\pi \ln k + I - 3\pi \ln 3 + 3\pi l^6 + J_0 (lk)^{-2} + o\left(l^6 + (lk)^{-2}\right), \end{aligned} \tag{1.8}$$

in view of (1.7). The aim is to construct a solution u_k in the form $U_k[\eta(1+\psi) + (1-\eta)e^{\psi}]$, where $\psi = \psi(k)$ is a remainder term small in a weighted $L^\infty(B)-$norm and $l = l(k)$ as $k \to +\infty$. The function η is a smooth cut-off function with $\eta = 1$ in $\cup_{j=0}^{3} B_{1/k}(la_j)$ and $\eta = 0$ in $B \setminus \cup_{j=0}^{3} B_{2/k}(la_j)$. The function $\psi = \psi(k)$ is found thanks to the solvability theory (up to a finite-dimensional kernel) of the linearized operator for (1.6) at U_k as $l \to 0$ and $lk \to +\infty$, and by the Lyapunov-Schmidt reduction the existence of $l(k)$ follows as a c.p. of

$$\tilde{E}_k := E_k\left(U_k[\eta(1 + \psi(k)) + (1-\eta)e^{\psi(k)}]\right).$$

If U_k is sufficiently good as an approximating solution of (1.6), we have that $\tilde{E}_k = E_k(U_k) + o((lk)^{-2})$. Since $3\pi l^6 + J_0(lk)^{-2}$ has always a minimum point of order $k^{-\frac{1}{4}}$ as $k \to +\infty$, by (1.8) we get the existence of $l = l(k)$ in view of the persistence of minimum points under small perturbations.

Unfortunately, this is not the case. Pushing further the analysis, we were able to identify the leading term $\psi_0 = \psi_0(k)$ of $\psi = \psi(k)$, and compute its contribution into the energy expansion, yielding to a correction in the form:

$$\tilde{E}_k = 4\pi \ln k + I + \frac{1}{2} W(l) + J_1 (lk)^{-2} + o((lk)^{-2}). \tag{1.9}$$

By (1.7) and (1.9) a c.p. $l(k)$ of $\tilde{E}_k$ always exists provided $J_1 > 0$. First numerically, and then rigorously, we were disappointed to find that $J_1 = 0$.

Later on, we realized that $-J_1$ is exactly the correlation coefficient A_0 in (1.4) (with $N = 3$ and $m = 1$) introduced by Ovchinnikov and Sigal [13]. If u is a solution of (1.6) with vortices $a_0 = 0$ and la_j, $a_j = e^{\frac{2\pi i}{3}(j-1)}$ for $j = 1, 2, 3$, with $n_0 = -1$ and $n_1 = n_2 = n_3 = 1$, then the function $U(x) = u(\frac{x}{k})$ does solve

$$\begin{cases} -\Delta U = U(1 - |U|^2) & \text{in } B_k \\ U = g_0 & \text{on } \partial B_k \end{cases} \tag{1.10}$$

with vortices a_0 and lka_j of vorticities $n_0 = -1$, $n_1 = n_2 = n_3 = 1$. Since (1.1) and (1.10) formally coincide when $k = +\infty$, it is natural to find a correlation term in the energy expansion

$\tilde{E}_k$ in the form $-\frac{A_0}{a^2} = J_1(lk)^{-2}$, where $a = lk$ is the modulus of the lka_j's for $j = 1, 2, 3$. Even more and not surprisingly, the function $\tilde{U}_k(\frac{x}{k})$, where

$$U_k[\eta(1 + \psi_0(k)) + (1 - \eta)e^{\psi_0(k)}]$$

is a very good approximating solution for (1.6) which improves the approximation rate of U_k, does coincide with the refined test functions used by Ovchinnikov and Sigal [13] to get the improved upper bound (1.3).

In conclusion, the vanishing of the correlation coefficient A_0 does not support any conjecture concerning symmetry-breaking phenomena for (1.1) or the existence of collapsing vortices for (1.6) when $k \to +\infty$. Higher-order expansions would be needed in their study.

2. The correlation coefficient

Let $N \geq 2$. Let $a_j = e^{\frac{2\pi i (j-1)}{N}}$, $j = 1, \dots, N$, be the $N-$roots of unity, and set $n_j = m \in \mathbb{Z}$ for all $j = 1, \dots, N$, $a_0 = 0$ and $n_0 = -\frac{N-1}{2} m$. We aim to compute the correlation coefficient $A_0 = A_0(m)$ given in (1.4). Since (in complex notation) $\nabla\theta(x) = |x|^{-2}(-x_2, x_1)$ has the same modulus as $\frac{1}{x} = \frac{\bar{x}}{|x|^2}$, the correlation coefficient takes the form

$$A_0 = \frac{1}{4} \int_{\mathbb{R}^2} \left[|\sum_{j=0}^N \frac{n_j}{x - a_j}|^4 - \sum_{j=0}^N |\frac{n_j}{x - a_j}|^4 \right]. \tag{2.1}$$

Since the integer m comes out as m^4 from the expression (2.1), we have that $A_0(m) = m^4 A_0(1)$. Hereafter, we will assume $m = 1$ and simply denote $A_0(1)$ as A_0.

Let us first notice that A_0 is not well-defined without further specifications, because the integral function in (2.1) is not integrable near the points a_j, $j = 0, \dots, N$. Recall that the $N-$roots of unity $a_1, \dots, a_N$ do satisfy the following symmetry properties:

$$\sum_{j=1}^N a_j^l = 0 \qquad \forall\, |l| \leq N,\, l \neq 0, \tag{2.2}$$

as it can be easily deduced by the relation $x^N - 1 = \prod_{j=1}^N (x - a_j)$. A first application of (2.2) is the validity of

$$\sum_{j=1}^N \frac{1}{x - a_j} = \sum_{j=1}^N \frac{x^{N-1} + a_j x^{N-2} + \dots + a_j^{N-1}}{x^N - 1} = \frac{N x^{N-1}}{x^N - 1}, \tag{2.3}$$

which implies that the integral function in (2.1) near 0 has the form

$$|\sum_{j=0}^N \frac{n_j}{x - a_j}|^4 - \sum_{j=0}^N |\frac{n_j}{x - a_j}|^4 = -\frac{N(N-1)^3}{2} \mathrm{Re}\left(\frac{x^N}{(x^N - 1)|x|^4}\right) + O(1) \tag{2.4}$$

and is not integrable at 0 when $N = 2$. Similarly, setting $\alpha_k(x) = -\frac{N-1}{2x} + \sum_{\substack{j=1\\j\neq k}}^{N} \frac{1}{x-a_j}$ for $k = 1, \dots, N$, near a_k we have that

$$|\sum_{j=0}^{N} \frac{n_j}{x-a_j}|^4 - \sum_{j=0}^{N} |\frac{n_j}{x-a_j}|^4 = \frac{4}{|x-a_k|^4}\mathrm{Re}[(x-a_k)\alpha_k(x)] + \frac{2}{|x-a_k|^2}|\alpha_k(x)|^2 \tag{2.5}$$
$$+\left(2\mathrm{Re}\frac{(x-a_k)\alpha_k(x)}{|x-a_k|^2} + |\alpha_k(x)|^2\right)^2 - \frac{(N-1)^4}{16|x|^4} - \sum_{\substack{j=1\\j\neq k}}^{N} \frac{1}{|x-a_j|^4}.$$

The function α_k can not be computed explicitly, but we know that

$$\begin{aligned}\alpha_k(a_k) &= -\frac{N-1}{2a_k} + \sum_{\substack{j=1\\j\neq k}}^{N} \frac{1}{a_k-a_j} = a_k^{N-1}\left(-\frac{N-1}{2} + \sum_{j=2}^{N} \frac{1}{1-a_j}\right) \\ &= a_k^{N-1}\left(-\frac{N-1}{2} + \sum_{j=2}^{N} \frac{1-\cos\frac{2\pi(j-1)}{N} + i\sin\frac{2\pi(j-1)}{N}}{2(1-\cos\frac{2\pi(j-1)}{N})}\right) \\ &= i a_k^{N-1} \sum_{j=2}^{N} \frac{\sin\frac{2\pi(j-1)}{N}}{2(1-\cos\frac{2\pi(j-1)}{N})} = 0\end{aligned} \tag{2.6}$$

in view of $\{a_j a_k^{N-1} : j = 1, \dots, N,\ j \neq k\} = \{a_2, \dots, a_N\}$ and the symmetry of $\{a_1, \dots, a_N\}$ under reflections w.r.t. the real axis. By inserting (2.6) into (2.5) we deduce that the integral function in (2.1) near a_k has the form

$$|\sum_{j=0}^{N} \frac{n_j}{x-a_j}|^4 - \sum_{j=0}^{N} |\frac{n_j}{x-a_j}|^4 = \frac{4}{|x-a_k|^4}\mathrm{Re}[\alpha_k'(a_k)(x-a_k)^2] + O(\frac{1}{|x-a_k|}) \tag{2.7}$$

and is not integrable at a_k when $\alpha_k'(a_k) \neq 0$. Since the (possible) singular term in (2.4), (2.7) has vanishing integrals on circles, the meaning of A_0 is in terms of a principal value:

$$A_0 = \frac{1}{4}\lim_{\epsilon\to 0} \int_{\mathbb{R}^2 \setminus \cup_{k=0}^{N} B_\epsilon(a_k)} \left[|\sum_{j=0}^{N} \frac{n_j}{x-a_j}|^4 - \sum_{j=0}^{N} |\frac{n_j}{x-a_j}|^4\right]. \tag{2.8}$$

We would like to compute A_0 in polar coordinates, even tough the set $\mathbb{R}^2 \setminus \cup_{k=0}^{N} B_\epsilon(a_k)$ is not radially symmetric. The key idea is to make the integral function in (2.8) integrable near any a_j, $j = 1, \dots, N$, by adding suitable singular terms, in such a way that the integral in (2.8) will have to be computed just on the radially symmetric set $\mathbb{R}^2 \setminus B_\epsilon(a_0)$. To this aim, it is crucial to compute $\alpha_k'(a_k)$. Arguing as before, we get that

$$\begin{aligned}\alpha_k'(a_k) &= \frac{N-1}{2a_k^2} - \sum_{\substack{j=1\\j\neq k}}^{N} \frac{1}{(a_k-a_j)^2} = a_k^{N-2}\left(\frac{N-1}{2} - \sum_{j=2}^{N} \frac{1}{(1-a_j)^2}\right) \\ &= a_k^{N-2}\left(\frac{N-1}{2} - \sum_{j=2}^{N} \frac{(1-\cos\frac{2\pi(j-1)}{N})^2 - \sin^2\frac{2\pi(j-1)}{N}}{4(1-\cos\frac{2\pi(j-1)}{N})^2}\right) \\ &= a_k^{N-2} \sum_{j=2}^{N} \frac{1}{2(1-\cos\frac{2\pi(j-1)}{N})} = a_k^{N-2} \sum_{j=2}^{N} \frac{1}{|1-a_j|^2}.\end{aligned} \tag{2.9}$$

Since there holds $\sum_{j=1}^{N-1} a_k^j = \sum_{j=2}^{N} a_j = -1$ for all $k = 2, \dots, N$ in view of (2.2), we have that

$$\prod_{j=2}^{N}(z-a_j) = \frac{z^N-1}{z-1} = \sum_{p=0}^{N-1} z^p\,, \quad \prod_{\substack{j=2\\ j\neq k}}^{N}(z-a_j) = \frac{\sum_{p=0}^{N-1} z^p}{z-a_k} = \sum_{p=0}^{N-2} z^p \sum_{l=0}^{N-2-p} a_k^l,$$

and then

$$\prod_{j=2}^{N}(1-a_j) = N\,, \quad \prod_{\substack{j=2\\ j\neq k}}^{N}(1-a_j) = \sum_{l=0}^{N-2}(N-l-1)a_k^l. \tag{2.10}$$

By (2.10) we get that

$$\begin{aligned}
\beta_N \ &:= \ \sum_{j=2}^{N} \frac{4}{|1-a_j|^2} = \sum_{j=2}^{N} \frac{4}{N^2} \prod_{\substack{k=2\\ k\neq j}}^{N} |1-a_k|^2 = \sum_{j=2}^{N} \frac{4}{N^2} \sum_{l,p=0}^{N-2} (N-l-1)(N-p-1)a_j^{l-p} \\
&= \ 4\frac{N-1}{N^2}\sum_{l=1}^{N-1} l^2 - \frac{4}{N^2}\sum_{\substack{l,p=1\\ l\neq p}}^{N-1} lp = \frac{4}{N}\sum_{l=1}^{N-1} l^2 - \frac{4}{N^2}(\sum_{l=1}^{N-1} l)^2 = \frac{2(N-1)(2N-1)}{3} - (N-1)^2 \\
&= \ \frac{N^2-1}{3}
\end{aligned}$$

in view of (2.2). Since by (2.9) $\alpha_k'(a_k) = \frac{\beta_N}{4} a_k^{N-2}$, by (2.7) we have that

$$|\sum_{j=0}^{N} \frac{n_j}{x-a_j}|^4 - \sum_{j=0}^{N} |\frac{n_j}{x-a_j}|^4 - \sum_{j=1}^{N} \text{Re}[\frac{\beta_N a_j^2}{(x-a_j)^2(1+|x-a_j|^2)}] \in L^1(\mathbb{R}^2 \setminus \{0\}).$$

Since

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^2 \setminus \cup_{k=0}^{N} B_\epsilon(a_k)} \frac{a_j^2}{(x-a_j)^2(1+|x-a_j|^2)} = \lim_{\epsilon\to 0} \int_{\mathbb{R}^2 \setminus B_\epsilon(a_j)} \frac{a_j^2}{(x-a_j)^2(1+|x-a_j|^2)} = 0,$$

we can re-write A_0 as

$$\begin{aligned}
A_0 \ &= \ \frac{1}{4}\lim_{\epsilon\to 0}\int_{\mathbb{R}^2\setminus B_\epsilon(0)} \Bigg[|\frac{(N+1)x^N+(N-1)}{2x(x^N-1)}|^4 - \frac{(N-1)^4}{16|x|^4} - \sum_{j=1}^{N} \frac{1}{|x-a_j|^4} \\
&\quad - \sum_{j=1}^{N} \text{Re}[\frac{\beta_N a_j^2}{(x-a_j)^2(1+|x-a_j|^2)}] \Bigg] \\
&= \ \frac{1}{4}\lim_{\epsilon\to 0}\int_{\mathbb{R}^2\setminus(B_\epsilon(0)\cup\{1-\epsilon\le|x|\le\frac{1}{1-\epsilon}\})} \Bigg[|\frac{(N+1)x^N+(N-1)}{2x(x^N-1)}|^4 - \frac{(N-1)^4}{16|x|^4} - \sum_{j=1}^{N} \frac{1}{|x-a_j|^4} \Bigg] \\
&\quad -\frac{1}{4}\text{Re}\Bigg[\lim_{\epsilon\to 0}\int_{\mathbb{R}^2\setminus(B_\epsilon(0)\cup\{1-\epsilon\le|x|\le\frac{1}{1-\epsilon}\})} \sum_{j=1}^{N} \frac{\beta_N a_j^2}{(x-a_j)^2(1+|x-a_j|^2)} \Bigg] =: \frac{1}{4}\text{I} - \frac{1}{4}\text{II}
\end{aligned} \tag{2.11}$$

in view of (2.3).

As far as I, let us write the following Taylor expansions: for $|x|<1$ there hold

$$\begin{aligned}\frac{((N+1)x^N+(N-1))^2}{(1-x^N)^2} &= \left((N-1)^2+2(N^2-1)x^N+(N+1)^2x^{2N}\right)\sum_{k\geq 0}(k+1)x^{kN}\\ &= (N-1)^2+\sum_{k\geq 1}4N(kN-1)x^{kN}=\sum_{k\geq 0}c_kx^{kN} \qquad (2.12)\end{aligned}$$

and

$$\begin{aligned}\frac{((N-1)x^N+(N+1))^2}{(1-x^N)^2} &= \left((N+1)^2+2(N^2-1)x^N+(N-1)^2x^{2N}\right)\sum_{k\geq 0}(k+1)x^{kN}\\ &= (N+1)^2+\sum_{k\geq 1}4N(kN+1)x^{kN}=\sum_{k\geq 0}d_kx^{kN}, \qquad (2.13)\end{aligned}$$

where $c_k=\max\{4N(kN-1),(N-1)^2\}$ and $d_k=\max\{4N(kN+1),(N+1)^2\}$. Letting $\epsilon>0$ small, by (2.12)-(2.13) we have that in polar coordinates (w.r.t. to the origin) I writes as

$$\begin{aligned}\mathrm{I} &= \int_\epsilon^{1-\epsilon}\rho d\rho\int_0^{2\pi}d\theta\left[\frac{1}{16\rho^4}|\sum_{k\geq 0}c_k\rho^{kN}e^{ikN\theta}|^2-\frac{(N-1)^4}{16\rho^4}-\sum_{j=1}^N|\sum_{k\geq 0}(k+1)a_j^{k(N-1)}\rho^ke^{ik\theta}|^2\right]\\ &+\int_{\frac{1}{1-\epsilon}}^\infty\rho d\rho\int_0^{2\pi}d\theta\left[\frac{1}{16\rho^4}|\sum_{k\geq 0}d_k\rho^{-kN}e^{-ikN\theta}|^2-\frac{(N-1)^4}{16\rho^4}-\frac{1}{\rho^4}\sum_{j=1}^N|\sum_{k\geq 0}(k+1)a_j^k\rho^{-k}e^{-ik\theta}|^2\right]\\ &+o_\epsilon(1)\end{aligned}$$

with $o_\epsilon(1)\to 0$ as $\epsilon\to 0$, in view of

$$|x-a_j|^{-4}=|a_j^{N-1}x-1|^{-4}=|\sum_{k\geq 0}(k+1)a_j^{k(N-1)}x^k|^2,\quad |1-a_jx|^{-4}=|\sum_{k\geq 0}(k+1)a_j^kx^k|^2$$

for $|x|<1$. By the Parseval's Theorem we get that

$$\begin{aligned}\mathrm{I} &= 2\pi\int_\epsilon^{1-\epsilon}\left[\frac{1}{16}\sum_{k\geq 1}|c_k|^2\rho^{2kN-3}-N\sum_{k\geq 0}(k+1)^2\rho^{2k+1}\right]d\rho\\ &+2\pi\int_{\frac{1}{1-\epsilon}}^\infty\left[\frac{1}{16}\sum_{k\geq 1}|d_k|^2\rho^{-2KN-3}+\frac{(N+1)^4-(N-1)^4}{16\rho^3}-N\sum_{k\geq 0}(k+1)^2\rho^{-2k-3}\right]d\rho+o_\epsilon(1)\\ &= 2\pi N\int_0^{1-\epsilon}\left[N\sum_{k\geq 0}(kN+N-1)^2\rho^{2kN+2N-3}-\sum_{k\geq 0}(k+1)^2\rho^{2k+1}\right]d\rho\\ &+2\pi N\int_{\frac{1}{1-\epsilon}}^\infty\left[N\sum_{k\geq 0}(kN+N+1)^2\rho^{-2kN-2N-3}-\sum_{k\geq 0}(k+1)^2\rho^{-2k-3}\right]d\rho+N(N^2+1)\frac{\pi}{2}\\ &+o_\epsilon(1)=2\pi N\int_0^{1-\epsilon}\left[N\sum_{k\geq 0}(kN+N-1)^2\rho^{2kN+2N-3}+N\sum_{k\geq 0}(kN+N+1)^2\rho^{2kN+2N+1}\right.\\ &\left.-2\sum_{k\geq 0}(k+1)^2\rho^{2k+1}\right]d\rho+N(N^2+1)\frac{\pi}{2}+o_\epsilon(1)\end{aligned}$$

as $\epsilon \to 0$. We compute now the integrals and let $\epsilon \to 0$ to end up with

$$\begin{aligned} \mathrm{I} &= 2\pi N \left[\frac{N}{2} \sum_{k\geq 0} (kN+N-1)\rho^{2kN+2N-2} + \frac{N}{2} \sum_{k\geq 0} (kN+N+1)\rho^{2kN+2N+2} - \sum_{k\geq 0} (k+1)\rho^{2k+2} \right] \Big|_0^1 \\ &\quad + N(N^2+1)\frac{\pi}{2}. \end{aligned}$$

Denoting the function inside brackets as $f(\rho)$, we need now to determine the explicit expression of $f(\rho)$ for $\rho < 1$:

$$\begin{aligned} f(\rho) &= \frac{N^2}{2}\rho^{2N-2}(1+\rho^4)\sum_{k\geq 0}(k+1)(\rho^{2N})^k - \frac{N}{2}\rho^{2N-2}(1-\rho^4)\sum_{k\geq 0}(\rho^{2N})^k - \rho^2 \sum_{k\geq 0}(k+1)(\rho^2)^k \\ &= \frac{N^2}{2}\rho^{2N-2}\frac{1+\rho^4}{(1-\rho^{2N})^2} - \frac{N}{2}\rho^{2N-2}\frac{1-\rho^4}{1-\rho^{2N}} - \frac{\rho^2}{(1-\rho^2)^2} \\ &= \frac{1}{2}\frac{N^2\rho^{2N-2}(1+\rho^4) - N\rho^{2N-2}(1-\rho^4)(1-\rho^{2N}) - 2\rho^2(\sum_{j=0}^{N-1}\rho^{2j})^2}{(1-\rho^{2N})^2}, \end{aligned}$$

and then by the l'Hôpital's rule we get that

$$\begin{aligned} 4N^2 f(1) &= 2\lim_{\rho\to 1} \frac{N(N-1)\rho^{N-1} + N(N+1)\rho^{N+1} - 2\rho(\sum_{j=0}^{N-1}\rho^j)^2 + N\rho^{2N-1} - N\rho^{2N+1}}{(1-\rho)^2} \\ &= \lim_{\rho\to 1} \frac{-N^2(N-2)\rho^{N-2} - N^2(N+2)\rho^N + 2(\sum_{j=0}^{N-1}\rho^j)^2 + 4\rho(\sum_{j=0}^{N-1}\rho^j)(\sum_{j=0}^{N-2}(j+1)\rho^j)}{1-\rho} \\ &\quad + N\lim_{\rho\to 1}\frac{(2N+1)\rho^{2N} - (2N-1)\rho^{2N-2} - \rho^{N-2} - \rho^N}{1-\rho} = -\frac{N^2(N^2+5)}{3}. \end{aligned}$$

In conclusion, for I we get the value

$$\mathrm{I} = \frac{\pi}{3}N(N^2-1). \tag{2.14}$$

Remark 2.1. *In* [13] *the value of A_0 was computed neglecting the term II in* (2.11)*. By* (2.14) *notice that $\frac{m^4}{4}I = \frac{\pi}{12}m^4N(N^2-1)$ does coincide with 8π when $(N,m) = (2,2)$ and 80π when $(N,m) = (4,2)$, in agreement with the computations in* [13].

As far as II, let us compute in polar coordinates the value of

$$\lim_{\epsilon\to 0}\sum_{j=1}^N \int_{\mathbb{R}^2\setminus(B_\epsilon(0)\cup\{1-\epsilon\leq|x|\leq\frac{1}{1-\epsilon}\})} \frac{a_j^2}{(x-a_j)^2(1+|x-a_j|^2)} = \lim_{\epsilon\to 0}\int_{(0,1-\epsilon)\cup(\frac{1}{1-\epsilon},+\infty)} \rho\Gamma(\rho)d\rho,$$

where the function Γ is defined in the following way:

$$\begin{aligned} \Gamma(\rho) &= \sum_{j=1}^N \int_0^{2\pi} \frac{a_j^2}{(\rho e^{i\theta}-a_j)^2(2+\rho^2 - a_j\rho e^{-i\theta} - a_j^{N-1}\rho e^{i\theta})}d\theta \\ &= \frac{i}{\rho}\sum_{j=1}^N a_j^3 \int_\gamma \frac{dw}{(\rho w - a_j)^2(w^2 - \frac{2+\rho^2}{\rho}a_j w + a_j^2)}, \end{aligned}$$

with γ the counterclockwise unit circle around the origin. Since

$$w^2 - \frac{2+\rho^2}{\rho} a_j w + a_j^2 = \left(w - \frac{2+\rho^2}{2\rho} a_j \right)^2 + a_j^2 \left(1 - (\frac{2+\rho^2}{2\rho})^2 \right),$$

observe that $w^2 - \frac{2+\rho^2}{\rho} a_j w + a_j^2$ vanishes at $\rho_\pm a_j$, with

$$\rho_\pm = \frac{2+\rho^2}{2\rho} \pm \sqrt{(\frac{2+\rho^2}{2\rho})^2 - 1}$$

satisfying $\rho_- < 1 < \rho_+$ in view of $\frac{2+\rho^2}{2\rho} \geq \sqrt{2}$. Since

$$(\frac{1}{w^2 - \frac{2+\rho^2}{\rho} a_j w + a_j^2})'(\frac{a_j}{\rho}) = a_j^{N-3} \rho^5,$$

by the Cauchy's residue Theorem the function $\Gamma(\rho)$ can now be computed explicitly as

$$\begin{aligned} \Gamma(\rho) &= \frac{i}{\rho^3} \sum_{j=1}^N a_j^3 \int_\gamma \frac{dw}{(w - \frac{a_j}{\rho})^2 (w - \rho_- a_j)(w - \rho_+ a_j)} \\ &= 2\pi N \begin{cases} (\rho\rho_- - 1)^{-2} (\rho\rho_+ - \rho\rho_-)^{-1} & \text{if } \rho < 1 \\ (\rho\rho_- - 1)^{-2} (\rho\rho_+ - \rho\rho_-)^{-1} - \rho^2 & \text{if } \rho > 1. \end{cases} \end{aligned}$$

Since we have that

$$(\rho\rho_- - 1)^2 = \frac{1}{4}(\rho^2 - \sqrt{\rho^4+4})^2 = \frac{1}{2}(\rho^4 + 2 - \rho^2\sqrt{\rho^4+4}), \qquad \rho\rho_+ - \rho\rho_- = \sqrt{\rho^4+4},$$

we get that

$$(\rho\rho_- - 1)^{-2} (\rho\rho_+ - \rho\rho_-)^{-1} = \frac{2}{(\rho^4+2)\sqrt{\rho^4+4} - \rho^2(\rho^4+4)} = \frac{\rho^4+2}{2\sqrt{\rho^4+4}} + \frac{\rho^2}{2},$$

and the expression of $\Gamma(\rho)$ now follows in the form

$$\Gamma(\rho) = \pi N \frac{\rho^4+2}{\sqrt{\rho^4+4}} - \pi N \rho^2 + \begin{cases} 2\pi N \rho^2 & \text{if } \rho < 1 \\ 0 & \text{if } \rho > 1. \end{cases} \tag{2.15}$$

Note that

$$\rho \left(\frac{\rho^4+2}{\sqrt{\rho^4+4}} - \rho^2 \right) = \frac{4\rho}{(\rho^4+2)\sqrt{\rho^4+4} + \rho^2(\rho^4+4)}$$

is integrable in $(0, \infty)$, and we have that

$$\begin{aligned} \int_0^\infty \rho(\frac{\rho^4+2}{\sqrt{\rho^4+4}} - \rho^2) d\rho &= \lim_{M\to+\infty} \frac{1}{2} \int_0^M (\frac{s^2+2}{\sqrt{s^2+4}} - s) ds \qquad (2.16) \\ = \lim_{M\to+\infty} \left[\frac{s}{4}\sqrt{s^2+4} \,\Big|_0^M - \frac{M^2}{4} \right] &= \lim_{M\to+\infty} \frac{M}{4}(\sqrt{M^2+4} - M) = \frac{1}{2}. \end{aligned}$$

Thanks to (2.15)-(2.16) we can compute

$$\lim_{\epsilon\to 0} \int_{(0,1-\epsilon)\cup(\frac{1}{1-\epsilon},+\infty)} \rho\Gamma(\rho) d\rho = \int_0^{+\infty} \rho\Gamma(\rho) d\rho = \pi N,$$

and for II we get the value

$$\text{II} = \frac{\pi}{3} N(N^2 - 1). \tag{2.17}$$

Finally, inserting (2.14) and (2.17) into (2.11) we get that the correlation coefficient vanishes: $A_0 = 0$. Then, there holds $A_0(m) = 0$ for all $m \in \mathbb{Z}$, as claimed.

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Pierpaolo Esposito, Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S. Leonardo Murialdo 1, 00146 Roma, Italy

E-mail address: esposito@mat.uniroma3.it