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Linear instability of entire solutions for a class of non-autonomous elliptic equations

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We study the effect of the potential $|y|^\alpha$ on the stability of entire solutions for elliptic equations on $\mathbb{R}^N$, $N \geqslant 2$, with exponential or smoooth/singular polynomial nonlinearities. Instability properties are crucial in order to establish regularity of the extremal solution to some related Dirichlet nonlinear eigenvalue problem on bounded domains. As a by-product of our results, we will improve the known results about the regularity of such solutions.

1. Introduction

We are concerned with the study of

$$-\Delta U = |y|^\alpha \operatorname{sgn}(\dot{F}(U))F(U) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\alpha \geqslant 0$, $N \geqslant 2$ and $F(s) = \mathrm{e}^s, s^p$ with $p < 0$ or $p > 1$.

When $\alpha = 0$, classification results for (1.1) are available in the literature. In [19] Gidas and Spruck proved that the only non-negative solution of such an equation with a subcritical nonlinearity $F(s) = s^p$, $1 < p < (N+2)/(N-2)$ and $N \geqslant 3$, is the trivial one (see also [1] for bounded changing-sign solutions with finite Morse index). For the critical exponent $p = (N+2)/(N-2)$, $N \geqslant 3$, the problem admits exactly a three-parameter family of solutions as shown in the celebrated papers [8, 20] (see also [10]). In dimension $N = 2$, a similar classification is available in [10] for the exponential nonlinearity $F(s) = \mathrm{e}^s$ under the finite-energy condition:

$$\int_{\mathbb{R}^2} \mathrm{e}^U < +\infty.$$

In all these situations, the solutions are radial about some point in $\mathbb{R}^N$. For singular polynomial nonlinearities $f(s) = s^p$, $p < 0$, only partial results are available. In [22] it is shown that any positive solution u of (1.1) is a radial function, provided that u satisfies a growth assumption modelled on $|y|^{2/(1-p)}$ at infinity. Only when $N = 2$, every solution which is symmetric in both variables and arises from a limiting procedure (in a sense which we will explain later) is radially symmetric, as shown in [21].

When $\alpha > 0$, as far we know, quite a few things are known. In dimension $N = 2$ a complete classification has been proved by Prajapat and Tarantello [25]: when

$\frac{1}{2}\alpha \notin \mathbb{N}$, all the solutions are radial around the origin and are 'dilations' of the same function; when $\frac{1}{2}\alpha \in \mathbb{N}$, there is a three-parameter family of solutions and most of them are not symmetric around any point of $\mathbb{R}^2$.

We focus now on stability properties. Given a solution U of (1.1), we define the 'first eigenvalue' of the linearized operator in the following way:

$$\mu_1(U) = \inf\left\{ \int |\nabla\phi|^2 - \int |y|^\alpha |\dot{F}(U)|\phi^2 : \phi \in C_0^\infty(\mathbb{R}^N),\ \int \phi^2 = 1 \right\}.$$

We will say that U is a semi-stable (respectively, stable) solution if $\mu_1(U) \geqslant 0$ (respectively, $\mu_1(U) > 0$). An unstable solution U corresponds to the opposite situation, $\mu_1(U) < 0$.

In the case when $\alpha = 0$ and $F(s) = s^p$ with $p > 1$, for any

$$2 \leqslant N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}},$$

Farina [17] extends the Liouville-type results of [1] (which were established in the subcritical case) to possibly unbounded and changing-sign solutions which are semi-stable outside a compact set (see also [15]): such a class includes, in particular, semi-stable and finite Morse index solutions. In a different direction, in [6] Cabré and Capella show that, for general smooth nonlinearities $F(U)$ (convex and increasing), any bounded, radial solution of (1.1) with $\alpha = 0$ is unstable when $N \leqslant 10$. The result is sharp because problem (1.1) for some $F(s)$ admits a bounded, radial solution which is semi-stable when $N \geqslant 11$.

As for singular polynomial nonlinearities, Esposito *et al*. show in [14] that for $F(U) = 1/U^2$ all the (possibly non-radial) solutions bounded away from zero are unstable when either $2 \leqslant N \leqslant 7$ or $N \geqslant 8$ and

$$\alpha > \frac{3N - 14 - 4\sqrt{6}}{4 + 2\sqrt{6}},$$

exhibiting an effect of the potential $|y|^\alpha$ on the stability. In this situation the result is sharp.

In the spirit (and as a continuation) of [14], we focus our attention on the simplest situation of semi-stable solutions and extend our previous result to a class of more general nonlinearities.

Theorem 1.1. *Let $F(s) = \mathrm{e}^s, s^p$, with $p < 0$ or $p > 1 + \frac{2}{\sqrt{3}}$. Let U be a non-trivial solution of*

$$\left.\begin{aligned} -\Delta U &= |y|^\alpha \operatorname{sgn}(\dot{F}(U))F(U) \quad &&\text{in } \mathbb{R}^N, \\ F(U(y)) &\leqslant F(U(0)) = 1 &&\text{in } \mathbb{R}^N. \end{aligned}\right\} \tag{1.2}$$

Then, U is unstable provided that

(a) *either $2 \leqslant N < 10$ or $N \geqslant 10$ and $\alpha > \frac{1}{4}(N - 10)$, when $F(s) = \mathrm{e}^s$,*

(b) *either*

$$2 \leqslant N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$$

or

$$N \geqslant 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}, \qquad \alpha > \frac{|p|(N-2)(p-1)}{2p(|p| + \sqrt{p(p-1)})} - 2,$$

when $F(s) = s^p$.

The critical dimension which appears in theorem 1.1 is sharp, as we will see later by exhibiting well-known counterexamples. As far as semi-stable solutions are concerned, for $F(s) = s^p$, $p > 1$, our result extends what was known in [1, 17] for $\alpha = 0$ to the case $\alpha \geqslant 0$ under the technical assumption $p > 1 + \frac{2}{\sqrt{3}}$. For $1 < p \leqslant 1 + \frac{2}{\sqrt{3}}$ we are able to establish a similar statement as in theorem 1.1: however, it would not be sharp and we prefer to omit it. Our first aim is to cover the exponential and the singular situation $F(s) = s^p$, $p < 0$, because, as we recently discovered, for exponential nonlinearities such an instability property was not even known for $\alpha = 0$. For $N = 3$, let us quote a recent non-existence result of finite Morse index solutions by Dancer [12]. A special emphasis is given to the presence of $|y|^\alpha$ in (1.2) and our second aim is to investigate the dependence on α of stability properties.

The instability of solutions to (1.2) is related to various nonlinear eigenvalue problems. Let us take a nonlinearity $f(s)$ in the class e^s, $(1-s)^p$ with $p < 0$, $(1+s)^p$ with $p > 1 + \frac{2}{\sqrt{3}}$. Let us consider the problem

$$\left.\begin{aligned} -\Delta u &= \lambda g(x) f(u) \quad \text{in } \Omega, \\ u &= 0 \quad\quad\quad\quad\;\; \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.3}$$

where $g(x)$ is a non-negative Hölder function in Ω and $\lambda \geqslant 0$. According to the literature (see, for example, [11] and [18, 23] for $(1-s)^p$, $p < 0$), it is possible to define an extremal value in the following way:

$$\lambda^* = \sup\{\lambda > 0 : (1.3) \text{ has a classical solution}\} \in (0, +\infty),$$

such that, for any $\lambda \in (0, \lambda^*)$, problem (1.3) has a unique minimal solution u_λ (it is the pointwise smallest positive solution of (1.3)). The solution u_λ is completely characterized as the unique stable solution:

$$\mu_1(u_\lambda) = \inf\left\{\int |\nabla\phi|^2 - \lambda \int g(x)\dot{f}(u_\lambda)\phi^2 : \phi \in C_0^\infty(\Omega),\ \int_\Omega \phi^2 = 1\right\} > 0$$

(see [4] for a survey on the subject and an exhaustive list of related references). As λ approaches λ^*, the family $\{u_\lambda\}$ can be compact:

$$\sup_{\lambda \in (0,\lambda^*)} \|f(u_\lambda)\|_\infty < +\infty,$$

or $f(u_\lambda)$ can develop a blow-up phenomenon. The extremal function

$$u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$$

is always a weak solution of (1.3) with $\lambda = \lambda^*$; in the compact situation, u^* is actually a classical solution of (1.3). Hence, to establish the regularity of u^*, we have to study and exclude the blow-up of $f(u_\lambda)$. As $\lambda \to \lambda^*$, a suitable rescaling of u_λ converges to a solution of the corresponding limiting problem on $\mathbb{R}^N$ in the form (1.2) for some $\alpha \geqslant 0$, semi-stable by the stability of u_λ.

Now let $g(x) = |x|^\alpha h(x)$, $h \geqslant C > 0$. If the blow-up occurs 'essentially' at the origin, the limiting equation (1.2) presents a potential $|y|^\alpha$ and theorem 1.1 allows us to exclude blow-up of $f(u_\lambda)$ for suitable values of α. This occurs on the unit ball B with a radial potential $g(x)$ because in this case the minimal solution u_λ achieves the maximum value exactly at the origin (u_λ is radial and radially decreasing).

THEOREM 1.2. *Let* $f(s) = \mathrm{e}^s$, $(1-s)^p$ *with* $p < 0$, $(1+s)^p$ *with* $p > 1 + \frac{2}{\sqrt{3}}$. *Assume that the potential* $g(x)$ *has the form*

$$g(x) = |x|^\alpha h(|x|), \quad h \geqslant C > 0 \text{ on } B.$$

Let u_λ *be the minimal solution of (1.3) for* $\lambda \in (0, \lambda^*)$. *Then*

$$\sup_{\lambda \in (0,\lambda^*)} \|f(u_\lambda)\|_\infty < \infty$$

and $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ *is a classical solution of (1.3) at* λ^*, *provided that*

(a) *either* $2 \leqslant N < 10$ *or* $N \geqslant 10$ *and* $\alpha > \frac{1}{4}(N-10)$, *when* $f(s) = \mathrm{e}^s$,

(b) *either*

$$2 \leqslant N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$$

or

$$N \geqslant 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}, \qquad \alpha > \frac{|p|(N-2)(p-1)}{2p(|p| + \sqrt{p(p-1)})} - 2,$$

when $f(s) = (1-s)^p/(1+s)^p$.

Theorem 1.2 is shown for $p = -2$ in [14]. Eigenvalue problems with singular nonlinearities as $(1-u)^{-2}$ show special features (as shown in [13, 14]): not only is the minimal branch compact in low dimensions but the unstable branches are also compact as far as their Morse indices remain bounded (in [14] solutions of Morse index 1 were considered). In [9] we consider the behaviour of stable and unstable branches for singular nonlinearities in a larger class than $(1-s)^p$, $p < 0$. We also include the m-Laplace operator, $m > 1$, in that study.

We also mention here the recent developments by Cabré and Capella [5, 7]: for quite general nonlinearities, the extremal function u^* is a classical solution on the ball for any $N \leqslant 9$ and on a general domain for any $N \leqslant 4$ (see also a former result of Nedev [24]).

On a general domain, the form of $g(x) = |x|^\alpha h(x)$ does not help because the blow-up can occur outside the origin and the limiting problem would have a constant positive potential. Hence, the dimensions for compactness to hold correspond to the worst situation $\alpha = 0$; for the sake of completeness, let us state the following result (it is already known, see [11, 18, 23]).

THEOREM 1.3. *Let* $f(s) = \mathrm{e}^s$, $(1-s)^p$ *with* $p < 0$, $(1+s)^p$ *with* $p > 1 + \frac{2}{\sqrt{3}}$. *Let* u_λ *be the minimal solution of (1.3) for* $\lambda \in (0, \lambda^*)$. *Then*

$$\sup_{\lambda \in (0,\lambda^*)} \|f(u_\lambda)\|_\infty < \infty$$

and $u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda$ *is a classical solution of (1.3) at* λ^*, *provided that* $2 \leqslant N < 10$ *when* $f(s) = \mathrm{e}^s$ *or*

$$2 \leqslant N < 2 + \frac{4p}{p-1} + 4\sqrt{\frac{p}{p-1}}$$

when $f(s) = (1-s)^p/(1+s)^p$.

REMARK 1.4. Theorems 1.2 and 1.3 turn into a regularity property for semi-stable solutions of Dirichlet elliptic problems. Let u be a semi-stable $H_0^1(\Omega)$-weak solution of (1.3) with $\lambda = 1$. Then, denoting by λ^* and u^* the extremal value and extremal solution, respectively, of the nonlinear eigenvalue problem (1.3), it is possible to show that $\lambda^* \geqslant 1$ (see [2, 3]). Since $f(u)$ is convex, when $\lambda^* > 1$, u coincides with the minimal solution $u_\lambda|_{\lambda=1}$, which is a classical solution, while, if $\lambda^* = 1$, then $u = u^*$ and, by theorems 1.2 and 1.3, it is still a smooth function. We refer the reader to the appendix in [14] for details when $f(u) = (1-u)^{-2}$.

The paper is organized in the following way. In § 2, we derive weighted $L^q(\mathbb{R}^N)$-bounds for semi-stable solutions of (1.2) which yield to the proof of theorem 1.1. In § 3, we describe the blow-up procedure to relate (1.3) to (1.2) in the case of non-compactness: theorem 1.2 then follows easily. We also provide some counterexamples to show the sharpness of the results.

Theorem 1.3 is based on the same techniques and requires us only to show (perhaps in an easy way, as done in [13, 14]) that blow-up 'essentially' does not occur on $\partial\Omega$. Nonetheless, we do not provide the details of its proof.

While this paper was under review, Farina informed us that, for the exponential nonlinearity $F(s) = \mathrm{e}^s$ and $\alpha = 0$, he had obtained in [16] a result similar to our theorem 1.1: all the solutions are linearly unstable as long as $N \leqslant 9$. In the exponential case the question of whether or not there exist solutions which are semi-stable outside a compact set of $\mathbb{R}^N$ is still open: a non-existence result would allow us to prove compactness along any unstable branch with uniformly bounded Morse indices as already shown in [13] for $F(U) = 1/U^2$.

2. Instability of entire solutions

Let $F(s) = \mathrm{e}^s, s^p$ with $p < 0$ or $p > 1 + \frac{2}{\sqrt{3}}$. Let U be a solution of

$$\left.\begin{aligned} -\Delta U &= |y|^\alpha \operatorname{sgn}(\dot{F}(U)) F(U) \quad && \text{in } \mathbb{R}^N, \\ F(U(y)) &\leqslant F(U(0)) = 1 && \text{in } \mathbb{R}^N. \end{aligned}\right\} \tag{2.1}$$

For semi-stable solutions of (2.1) the following useful weighted integral estimates on $F(U)$ hold.

THEOREM 2.1. *Assume that U is a semi-stable solution of (2.1). Denoting by $[\cdot]$ the integer part, let*

$$\bar q = \begin{cases} 4 \quad \textit{if } F(s) = \mathrm{e}^s, \\ \dfrac{p-1}{p} \quad \textit{if } F(s) = s^p,\ 0 > p \geqslant 1 - \frac{2}{\sqrt{3}}, \\ \dfrac{p-1}{p}\left[\dfrac{2p + 2\sqrt{p(p-1)}}{p-1}\right] + \dfrac{p-1}{p} \\ \qquad \textit{if } F(s) = s^p,\ |p-1| > \frac{2}{\sqrt{3}},\ \dfrac{2p+2\sqrt{p(p-1)}}{p-1} \notin \mathbb{N}, \\ 2 + 2\sqrt{\dfrac{p-1}{p}} \\ \qquad \textit{if } F(s) = s^p,\ |p-1| > \frac{2}{\sqrt{3}},\ \dfrac{2p+2\sqrt{p(p-1)}}{p-1} \in \mathbb{N}, \end{cases}$$

and

$$\bar\beta = \begin{cases} \frac{1}{2}N - 2(2+\alpha) & \textit{if } F(s) = \mathrm{e}^s, \\ \frac{1}{2}N - \dfrac{p}{p-1}(1 + \frac{1}{2}\alpha)\bar q & \textit{if } F(s) = s^p. \end{cases}$$

Then, for any $q \geqslant \bar q$ and $\beta > \bar\beta$, we have that

$$\int \frac{F^q(U)}{(1+|y|^2)^\beta} < \infty. \tag{2.2}$$

Proof. By the semi-stability assumption on U, the following inequality holds:

$$\int |\nabla\phi|^2 - \int |y|^\alpha |\dot F(U)| \phi^2 \geqslant 0 \quad \forall \phi \in C_0^\infty(\mathbb{R}^N). \tag{2.3}$$

Given $\delta > 0$, for the test function

$$\phi = \frac{1}{(1+|y|^2)^{(N-2)/4+(\delta/2)}}$$

we can find a sequence $\phi_n \in C_0^\infty(\mathbb{R}^N)$ so that $\phi_n \to \phi$ almost everywhere, $\nabla\phi_n \to \nabla\phi$ in $L^2(\mathbb{R}^N)$. Applying (2.3) to ϕ_n and taking the limit as $n \to \infty$, by Fatou's theorem we obtain

$$\int \frac{|y|^\alpha}{(1+|y|^2)^{((N-2)/2)+\delta}} |\dot F(U)| \leqslant C \int \frac{1}{(1+|y|^2)^{(N/2)+\delta}} < +\infty \tag{2.4}$$

for any $\delta > 0$.

Define

$$0 < \Lambda = \begin{cases} 1 & \text{if } F(s) = \mathrm{e}^s, \\ \dfrac{p-1}{p} & \text{if } F(s) = s^p, \end{cases} \qquad 0 \neq \gamma = \begin{cases} 1 & \text{if } F(s) = \mathrm{e}^s, \\ p & \text{if } F(s) = s^p, \end{cases} \tag{2.5}$$

in such a way the following relation holds:

$$\dot F(U) = \gamma F^\Lambda(U). \tag{2.6}$$

By a Moser-type iteration scheme based on (2.3), we will show that, for any $2-2\sqrt{\varLambda}<q<2+2\sqrt{\varLambda}$, $q\neq 0$, and β,

$$\int \frac{F^{q+\varLambda}(U)}{(1+|y|^2)^{\beta-1-(\alpha/2)}} \leqslant C_q\left(1+\int \frac{F^{q}(U)}{(1+|y|^2)^{\beta}}\right) \tag{2.7}$$

(provided the second integral is finite).

Estimates (2.4), (2.7) now provide the validity of (2.2). Indeed, in view of (2.6), estimate (2.4) can be rewritten as

$$\int \frac{F^{q_1}(U)}{(1+|y|^2)^{\beta_1+\delta}} < +\infty,$$

where $\beta_1=\frac{1}{2}N-(1+\frac{1}{2}\alpha)$ and $q_1=\varLambda$. Note that, by means of (2.5), the assumption $p>1+\frac{2}{\sqrt{3}}$ may be expressed in terms of $\varLambda$ as $\varLambda>4-2\sqrt{3}$ or, equivalently, as $q_1=\varLambda>2-2\sqrt{\varLambda}$.

Let $\beta_i=\frac{1}{2}N-i(1+\frac{1}{2}\alpha)$ and $q_i=i\varLambda$. The index $\bar{\imath}$, defined by

$$\bar{\imath}=\begin{cases} 1 & \text{if } \varLambda\geqslant 2+2\sqrt{\varLambda},\\ \left[\dfrac{2+2\sqrt{\varLambda}}{\varLambda}\right]+1 & \text{if } 1<\dfrac{2+2\sqrt{\varLambda}}{\varLambda}\notin\mathbb{N},\\ \dfrac{2+2\sqrt{\varLambda}}{\varLambda} & \text{if } 1<\dfrac{2+2\sqrt{\varLambda}}{\varLambda}\in\mathbb{N}, \end{cases} \tag{2.8}$$

is the smallest positive integer such that $q_{\bar{\imath}}\geqslant 2+2\sqrt{\varLambda}$. Now applying (2.7) for $i=1,\dots,\bar{\imath}-1$ only in the case when $\bar{\imath}\geqslant 2$, we get that

$$\int \frac{F^{q_{\bar{\imath}}}(U)}{(1+|y|^2)^{\beta_{\bar{\imath}}+\delta}} < +\infty. \tag{2.9}$$

Observe that $\varLambda<2+2\sqrt{\varLambda}$ holds exactly when $\varLambda<4+2\sqrt{3}$, which is expressed in terms of p as $p\notin[1-\frac{2}{\sqrt{3}},0)$ by means of (2.5). Moreover, the relations

$$\bar{q}=q_{\bar{\imath}}:=\bar{\imath}\varLambda\geqslant 2+2\sqrt{\varLambda}, \qquad \bar{\beta}=\beta_{\bar{\imath}}:=\tfrac{1}{2}N-\bar{\imath}(1+\tfrac{1}{2}\alpha) \tag{2.10}$$

do hold, following definitions (2.5) of $\varLambda$ and (2.8) of $\bar{\imath}$. Therefore, estimate (2.9) for any $\delta>0$ clearly implies the validity of (2.2).

In order to complete the proof, we need to show the validity of (2.7). Given $R>0$, consider a smooth radial cut-off function η so that

$$0\leqslant\eta\leqslant 1,\quad \eta=1 \text{ on } B_R(0),\quad \eta=0 \text{ on } \mathbb{R}^N\setminus B_{2R}(0),\quad R|\nabla\eta|+R^2|\Delta\eta|\leqslant 2. \tag{2.11}$$

Note that $F(U(y))>0$ in $\mathbb{R}^N$, unless $F(s)=s^p$, $p>0$ and $U=0$ (by the Hopf lemma). Now multiply (2.1) by

$$\gamma\frac{\eta^2}{(1+|y|^2)^{\beta-1}}F^{q-1+\varLambda}(U),\quad q\neq 0,$$

and, by (2.3), (2.6), an integration by parts yields

$$\begin{aligned}
\gamma \int & \frac{\eta^2|y|^\alpha}{(1+|y|^2)^{\beta-1}} \operatorname{sgn}(\dot F(U)) F^{q+\Lambda}(U) \\
&= \gamma(q-1+\Lambda) \int \frac{\eta^2}{(1+|y|^2)^{\beta-1}} F^{q-2+\Lambda}(U) \dot F(U) |\nabla U|^2 \\
&\quad + \gamma \int F^{q-1+\Lambda}(U) \nabla \Big(\frac{\eta^2}{(1+|y|^2)^{\beta-1}} \Big) \nabla U \\
&= (q-1+\Lambda) \int \frac{\eta^2}{(1+|y|^2)^{\beta-1}} F^{q-2}(U) (\dot F(U))^2 |\nabla U|^2 \\
&\quad + \int F^{q-1}(U) \dot F(U) \nabla \Big(\frac{\eta^2}{(1+|y|^2)^{\beta-1}} \Big) \nabla U \\
&= \frac{4(q-1+\Lambda)}{q^2} \int \Big| \nabla \frac{\eta F^{q/2}(U)}{(1+|y|^2)^{(\beta-1)/2}} \Big|^2 \\
&\quad - \frac{4(q-1+\Lambda)}{q^2} \int F^q(U) \Big| \nabla \frac{\eta}{(1+|y|^2)^{(\beta-1)/2}} \Big|^2 \\
&\quad + \frac{2-q-2\Lambda}{q^2} \int \nabla (F^q(U)) \nabla \Big(\frac{\eta^2}{(1+|y|^2)^{\beta-1}} \Big) \\
&\geqslant \frac{4(q-1+\Lambda)}{q^2} \int \frac{\eta^2 |y|^\alpha}{(1+|y|^2)^{\beta-1}} |\dot F(U)| F^q(U) \\
&\quad - \frac{4(q-1+\Lambda)}{q^2} \int F^q(U) \Big| \nabla \frac{\eta}{(1+|y|^2)^{(\beta-1)/2}} \Big|^2 \\
&\quad - \frac{2-q-2\Lambda}{q^2} \int F^q(U) \Delta \Big(\frac{\eta^2}{(1+|y|^2)^{\beta-1}} \Big).
\end{aligned}$$

By (2.11) it is straightforward to see that

$$\Big| \nabla \frac{\eta}{(1+|y|^2)^{(\beta-1)/2}} \Big|^2 + \Big| \Delta \frac{\eta^2}{(1+|y|^2)^{\beta-1}} \Big| \leqslant \frac{C}{(1+|y|^2)^\beta}$$

for some constant C independent on $R > 0$. Since (2.6) implies $\gamma \operatorname{sgn}(\dot F(U)) = |\gamma| > 0$ and $|\dot F(U)| = |\gamma| F^\Lambda(U)$, we finally obtain

$$\Big(\frac{4(q-1+\Lambda)}{q^2} - 1 \Big) \int \frac{\eta^2 |y|^\alpha}{(1+|y|^2)^{\beta-1}} F^{q+\Lambda}(U) \leqslant C \int \frac{F^q(U)}{(1+|y|^2)^\beta},$$

where C does not depend on $R > 0$. For any $2 - 2\sqrt{\Lambda} < q < 2 + 2\sqrt{\Lambda}$ it holds that $(4(q-1+\Lambda)/q^2) - 1 > 0$ and then we get

$$\int \frac{\eta^2 |y|^\alpha}{(1+|y|^2)^{\beta-1}} F^{q+\Lambda}(U) \leqslant C \int \frac{F^q(U)}{(1+|y|^2)^\beta}$$

for any $2 - 2\sqrt{\Lambda} < q < 2 + 2\sqrt{\Lambda}$, $q \neq 0$, where C does not depend on $R > 0$. Taking the limit as $R \to +\infty$, since $F(U(y))$ is locally bounded, we easily obtain the validity of (2.7) for any $2 - 2\sqrt{\Lambda} < q < 2 + 2\sqrt{\Lambda}$, $q \neq 0$, and the proof is completed. □

As a by-product of theorem 2.1, the following corollary holds.

COROLLARY 2.2. *Let U be a semi-stable solution of (2.1). Define*

$$\bar{N} = \begin{cases} 10 & \text{if } F(s) = \mathrm{e}^s, \\ 2 + \dfrac{4p}{p-1} + 4\sqrt{\dfrac{p}{p-1}} & \text{if } F(s) = s^p \end{cases}$$

and

$$\bar{\alpha} = \begin{cases} \dfrac{N-10}{4} & \text{if } F(s) = \mathrm{e}^s, \\ \dfrac{|p|(N-2)(p-1)}{2p(|p| + \sqrt{p(p-1)})} - 2 & \text{if } F(s) = s^p. \end{cases}$$

If either $N < \bar{N}$ or $N \geqslant \bar{N}$ and $\alpha > \bar{\alpha}$, then it holds that

$$\int \frac{F^q(U)}{1+|y|^2} < \infty \tag{2.12}$$

for any

$$\frac{\Lambda(N-2)}{2+\alpha} < q < 2 + 2\sqrt{\Lambda},$$

where Λ is defined in (2.5).

Proof. First, note that $\bar{\alpha} \geqslant 0$ only when $N \geqslant \bar{N}$. Hence, the inequality $\alpha > \bar{\alpha}$ is automatically satisfied when $N < \bar{N}$ and is equivalent to the condition that

$$\frac{\Lambda(N-2)}{2+\alpha} < 2 + 2\sqrt{\Lambda}.$$

Let us now fix some

$$q \in \left(\frac{\Lambda(N-2)}{2+\alpha}, 2 + 2\sqrt{\Lambda} \right).$$

By (2.10), the requirement

$$\frac{\Lambda(N-2)}{2+\alpha} < q < 2 + 2\sqrt{\Lambda}$$

implies that

$$\frac{(N-2)\bar{q}}{N-2\bar{\beta}} < q < \bar{q},$$

or, equivalently,

$$\frac{\bar{q} - \bar{\beta} q}{\bar{q} - q} > \frac{N}{2}.$$

Therefore, for $\delta > 0$ small we have

$$\int \frac{1}{(1+|y|^2)^{(\bar{q}-(\bar{\beta}+\delta)q)/(\bar{q}-q)}} < \infty. \tag{2.13}$$

In order to prove (2.12), by the Hölder inequality we obtain

$$\begin{aligned}\int \frac{F^q(U)}{1+|y|^2} &= \int \frac{F^q(U)}{(1+|y|^2)^{(\bar\beta+\delta)q/\bar q}} \frac{1}{(1+|y|^2)^{1-(\bar\beta+\delta)q/\bar q}} \\ &\leqslant \Big(\int \frac{F^{\bar q}(U)}{(1+|y|^2)^{\bar\beta+\delta}}\Big)^{q/\bar q} \Big(\int \frac{1}{(1+|y|^2)^{(\bar q-(\bar\beta+\delta)q)/(\bar q-q)}}\Big)^{(\bar q-q)/\bar q} < +\infty\end{aligned}$$

by means of (2.2) in theorem 2.1. □

Proof of theorem 1.1. By contradiction, assume that $\mu_1(U) \geqslant 0$. The function U is then a semi-stable solution of (2.1) and corollary 2.2 implies that (2.12) holds for any

$$\frac{\Lambda(N-2)}{2+\alpha} < q < 2+2\sqrt{\Lambda},$$

where Λ is defined in (2.5). Since $1-\Lambda < 2+2\sqrt{\Lambda}$, let us fix some q for which (2.12) is available and

$$1-\frac{q^2}{4(q-1+\Lambda)} > 0.$$

Let η be a cut-off function satisfying (2.11). Using equation (2.1) and relation (2.6) we compute

$$\begin{aligned}&\int |\nabla \eta F^{q/2}(U)|^2 - \int |y|^\alpha |\dot F(U)| (\eta F^{q/2}(U))^2 \\ &= \tfrac14 q^2 \int \eta^2 F^{q-2}(U)(\dot F(U))^2 |\nabla U|^2 + \int |\nabla\eta|^2 F^q(U) \\ &\quad + \tfrac12 q \int F^{q-1}(U)\dot F(U)\nabla U \nabla(\eta^2) - \int \eta^2 |y|^\alpha |\dot F(U)| F^q(U) \\ &= \tfrac14 \gamma q^2 \int \eta^2 F^{q-2+\Lambda}(U)\dot F(U)|\nabla U|^2 + \int |\nabla\eta|^2 F^q(U) \\ &\quad + \tfrac12 q \int F^{q-1}(U)\dot F(U)\nabla U\nabla(\eta^2) - \int \eta^2|y|^\alpha \dot F(U)|F^q(U) \\ &= \gamma \frac{q^2}{4(q-1+\Lambda)} \int \nabla U \nabla(\eta^2 F^{q-1+\Lambda}(U)) + \int |\nabla\eta|^2 F^q(U) \\ &\quad - \int \eta^2 |y|^\alpha |\dot F(U)| F^q(U) \\ &\quad + \Big(\tfrac12 q - \frac{q^2}{4(q-1+\Lambda)}\Big)\int F^{q-1}(U)\dot F(U)\nabla U\nabla(\eta^2) \\ &= -\Big(1-\frac{q^2}{4(q-1+\Lambda)}\Big)\int \eta^2|y|^\alpha|\dot F(U)|F^q(U) + \int |\nabla\eta|^2 F^q(U) \\ &\quad + \Big(\frac12 - \frac{q}{4(q-1+\Lambda)}\Big)\int \nabla(F^q(U))\nabla(\eta^2)\end{aligned}$$

$$= -\left(1 - \frac{q^2}{4(q-1+\Lambda)}\right) \int \eta^2 |y|^\alpha |\dot{F}(U)| F^q(U) + \int |\nabla\eta|^2 F^q(U)$$
$$+ \left(\frac{q}{4(q-1+\Lambda)} - \frac{1}{2}\right) F^q(U)\Delta(\eta^2).$$

Since

$$1 - \frac{q^2}{4(q-1+\Lambda)} > 0,$$

we get

$$\int |\nabla\eta F^{q/2}(U)|^2 - \int |y|^\alpha |\dot{F}(U)| (\eta F^{q/2}(U))^2$$
$$\leqslant -\left(1 - \frac{q^2}{4(q-1+\Lambda)}\right) \int_{B_1(0)} |y|^\alpha |\dot{F}(U)| F^q(U) + O\left(\int_{|y|\geqslant R} \frac{F^q(U)}{1+|y|^2}\right)$$
$$\to -\left(1 - \frac{q^2}{4(q-1+\Lambda)}\right) \int_{B_1(0)} |y|^\alpha |\dot{F}(U)| F^q(U) < 0 \quad \text{as } R \to +\infty$$

for the non-trivial solution U. This is in contradiction to $\mu_1(U) \geqslant 0$. □

3. Compactness of the minimal branch on the ball

First, we prove theorem 1.2, as follows.

Proof of theorem 1.2. We argue by contradiction. Assume the existence of a sequence $\lambda_n \uparrow \lambda^*$ and associated solution $u_n := u_{\lambda_n}$ of (1.3) on the unit ball B so that

$$\|f(u_n)\|_\infty \to +\infty \quad \text{as } n \to +\infty. \tag{3.1}$$

Recall that u_n is a radial and radially decreasing function. Let us now discuss all the possible cases.

If $f(s) = \mathrm{e}^s$, (3.1) implies that $u_n(0) = \|u_n\|_\infty \to +\infty$ as $n \to +\infty$. Let

$$\varepsilon_n = \exp\left(-\frac{u_n(0)}{\alpha+2}\right)$$

and $U_n(y) = u_n(\varepsilon_n y) + (2+\alpha)\ln\varepsilon_n$, $y \in B_n := B_{1/\varepsilon_n}(0)$. Then, $\varepsilon_n \to 0$, $B_n \to \mathbb{R}^N$ as $n \to +\infty$, and U_n solves

$$-\Delta U_n = \lambda_n |y|^\alpha h(\varepsilon_n |y|)\mathrm{e}^{U_n} \quad \text{in } B_n,$$
$$U_n(y) \leqslant U_n(0) = 0.$$

Since U_n satisfies $U_n(0) \leqslant U_n(0) = 0$ and has a uniformly bounded Laplacian, we can find a subsequence of U_n (still denoted by U_n) so that $U_n \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, where U is a solution of (1.2) with $F(s) = \mathrm{e}^s$ (up to reabsorption of the positive coefficient $\lambda^* h(0)$). Indeed, let us fix some ball $B_R(0)$ of large radius R. We can decompose U_n as $U_n = U_n^1 + U_n^2$, where $U_n^2 = 0$ on $\partial B_R(0)$ with $\Delta U_n^2 = \Delta U_n$ uniformly bounded. By elliptic regularity theory, U_n^2 is uniformly bounded in $C^{1,\gamma}(B_R(0))$, $\gamma > 0$. Since U_n is negative and U_n^2 is uniformly bounded on $B_R(0)$, $U_n^1 = U_n - U_n^2$ is a harmonic function which is also one-side uniformly bounded too. By the mean-value theorem, since $U_n(0) = 0$, U_n^1 (and then U_n) is uniformly bounded in $C^{1,\gamma}(B_{R/2}(0))$, $\gamma > 0$, for any $R > 0$. By the Ascoli–Arzelà theorem and a diagonal process, U_n has a converging subsequence in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$.

If $f(s) = (1+s)^p$, $p > 1$, then (3.1) implies that $u_n(0) = \|u_n\|_\infty \to +\infty$ as $n \to +\infty$. Let $\varepsilon_n = u_n(0)^{-(p-1)/(2+\alpha)}$ and

$$U_n(y) = \varepsilon_n^{(2+\alpha)/(p-1)} u_n(\varepsilon_n y), \quad y \in B_n := B_{1/\varepsilon_n}(0).$$

Then $\varepsilon_n \to 0$, $B_n \to \mathbb{R}^N$ as $n \to +\infty$, and U_n solves

$$\begin{gathered} -\Delta U_n = \lambda_n |y|^\alpha h(\varepsilon_n |y|)(\varepsilon_n^{(2+\alpha)/(p-1)} + U_n)^p \quad \text{in } B_n, \\ U_n(y) \leqslant U_n(0) = 1. \end{gathered}$$

Also in this case, U_n is negative with $U_n(0) = 1$ and has a uniformly bounded Laplacian. Then, a subsequence of U_n (still denoted by U_n) exists such that $U_n \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, where U is a solution of (1.2) with $F(s) = s^p$ (up to a positive coefficient).

If $f(s) = (1-s)^p$, $p < 0$, (3.1) implies that $u_n(0) = \|u_n\|_\infty \to 1^-$ as $n \to +\infty$. Let $\varepsilon_n = (1 - u_n(0))^{(1-p)/(2+\alpha)}$ and

$$U_n(y) = \varepsilon_n^{-(2+\alpha)/(1-p)} (1 - u_n(\varepsilon_n y)), \quad y \in B_n := B_{1/\varepsilon_n}(0).$$

Then, $\varepsilon_n \to 0$, $B_n \to \mathbb{R}^N$ as $n \to +\infty$, and U_n solves

$$\begin{gathered} \Delta U_n = \lambda_n |y|^\alpha h(\varepsilon_n |y|) U_n^p \quad \text{in } B_n, \\ U_n(y) \geqslant U_n(0) = 1. \end{gathered}$$

The family U_n satisfies $U_n \geqslant U_n(0) = 1$ and has a uniformly bounded Laplacian. Up to a subsequence, $U_n \to U$ in $C^1_{\mathrm{loc}}(\mathbb{R}^N)$, where U is a solution of (1.2) with $F(s) = s^p$ (up to a positive coefficient).

Any function $\phi \in C_0^\infty(\mathbb{R}^N)$ so that

$$\int |\nabla \phi|^2 - \lambda^* h(0) \int |y|^\alpha |\dot{F}(U)| \phi^2 < 0$$

could be rescaled back to a function $\phi_n \in C_0^\infty(B_{R\varepsilon_n}(0))$, for some $R > 0$, so that

$$\int_\Omega |\nabla \phi_n|^2 - \lambda_n \int_\Omega g(x) \dot{f}(u_n) \phi_n^2 \to \int |\nabla \phi|^2 - \lambda^* h(0) \int |y|^\alpha |\dot{F}(U)| \phi^2 < 0$$

as $n \to +\infty$, in contradiction to the semi-stability of u_n. Hence, $\mu_1(U) \geqslant 0$. Theorem 1.1 excludes the existence of such a solution when either $N < \bar{N}$ or $N \geqslant \bar{N}$ and $\alpha > \bar{\alpha}$, and then the blow-up assumption (3.1) leads to a contradiction in such cases (for the definitions of $\bar{N}$ and $\bar{\alpha}$, see the statement of corollary 2.2). □

To describe the counterexamples, we want to compute u^* and λ^* explicitly on the unit ball B with $g(x) = |x|^\alpha$ and $N \geqslant \bar{N}$, $0 \leqslant \alpha \leqslant \bar{\alpha}$. This will provide an example of an extremal function u^* so that $\|f(u^*)\|_\infty = \infty$, which is not a classical solution. Therefore, theorem 1.2 cannot be improved. The limiting profile U around zero of the minimal branch u_λ as $\lambda \to \lambda^*$ (which is non-compact in this case) provides an example of a radial, semi-stable, non-trivial solution of (1.2). Hence, theorem 1.1 is sharp too.

Our examples are based on the following useful characterization of the extremal solution (see [2,3] for $f(s) = \mathrm{e}^s, (1+s)^p$ with $p > 1$, and [13,14] for $f(s) = (1-s)^p$ with $p < 0$).

Theorem 3.1. *Let $g(x)$ be a non-negative Hölder function. Let u be a $H_0^1(\Omega)$-weak solution of (1.3) so that $\|f(u)\|_\infty = +\infty$. Then the following assertions are equivalent:*

(i) *u satisfies*

$$\int_\Omega |\nabla\phi|^2 - \lambda \int_\Omega g(x)\dot{f}(u)\phi^2 \geqslant 0 \quad \forall \phi \in H_0^1(\Omega); \tag{3.2}$$

(ii) $\lambda = \lambda^*$ *and* $u = u^*$.

Theorem 3.1 and the Hardy inequality

$$\int_\Omega |\nabla\phi|^2 \geqslant \frac{(N-2)^2}{4} \int_\Omega \frac{\phi^2}{|x|^2} \quad \forall \phi \in H_0^1(\Omega)$$

provide the counterexamples. When $f(s) = \mathrm{e}^s$, for $N > 2$ the function $u_0(x) = (2+\alpha)\ln 1/|x|$ is a singular $H_0^1(B)$-weak solution of $-\Delta u_0 = \lambda_0 |x|^\alpha \mathrm{e}^{u_0}$, $\lambda_0 = (2+\alpha)(N-2)$. When $N \geqslant \bar{N}$ and $0 \leqslant \alpha \leqslant \bar{\alpha}$, by the Hardy inequality we can prove that (3.2) holds for $\lambda = \lambda_0$ and $u = u_0$. By theorem 3.1, we obtain $u_0 = u^*$ and $\lambda_0 = \lambda^*$.

When $N \geqslant \bar{N}$ and $0 \leqslant \alpha \leqslant \bar{\alpha}$, for $f(s) = (1+s)^p$, $p > 1$, and $f(s) = (1-s)^p$, $p < 0$, it is possible to see in the same way that

$$\lambda^* = \frac{2+\alpha}{p-1}\left(N - 2 - \frac{2+\alpha}{p-1}\right), \quad u^* = |x|^{-(2+\alpha)/(p-1)} - 1,$$

and

$$\lambda^* = \frac{2+\alpha}{1-p}\left(N - 2 + \frac{2+\alpha}{1-p}\right), \quad u^* = 1 - |x|^{(2+\alpha)/(1-p)}$$

are the extremal value and solution of (1.3) on B, with $g(x) = |x|^\alpha$, respectively. Observe that, in the case $p > 1$, we have

$$\frac{2+\alpha}{p-1} < \frac{N-2}{2},$$

which guarantees that $u^* = |x|^{-(2+\alpha)/(p-1)} - 1 \in H_0^1(B)$.

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