Singular mean field equations on compact Riemann surfaces

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\textbf{ABSTRACT}

For a general class of elliptic PDE’s in mean field form on compact Riemann surfaces with exponential nonlinearity, we address the question of the existence of solutions with concentrated nonlinear term, which, in view of the applications, are physically of definite interest. In the model, we also include the possible presence of singular sources in the form of Dirac masses, which makes the problem more difficult to attack.

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\textbf{1. Introduction}

Let us consider the problem

\begin{equation}
-\Delta_g u = \lambda \left( \frac{k e^u}{\int_S k e^u dv_g} - 1 \right)
\end{equation}

on a compact, orientable Riemann surface \((S, g)\), where \(\lambda > 0\), \(k\) is a smooth function and \(|S|\) is the area of \(S\). Here, \(\Delta_g\) is the Laplace–Beltrami operator and \(dv_g\) is the area element in \((S, g)\).

Eq. (1) and its variants arise in many different contexts. The Nirenberg problem concerns the existence on \(S^2\) of metrics – conformal to the standard round metric \(g_0\) – with Gaussian curvature \(k\), and corresponds to Eq. (1) with \(\lambda = 8\pi\). Indeed, a solution \(u\) of (1) on \((S^2, g_0)\) with \(\lambda = 8\pi\) provides a metric \(\frac{4\pi e^u}{\int_{S^2} e^u dv_{g_0}} g_0\), conformal to \(g_0\), with Gaussian curvature \(k\). For a general compact Riemann surface, the prescribed Gaussian curvature problem is referred to as the Kazdan–Warner problem.

Since there are plenty of results in the literature, let us just quote the ones due to Kazdan and Warner [1], Chang and Yang [2] and Chang, Gursky, Yang [3]. For bounded domains of \(\mathbb{R}^2\), a variant of (1) with Dirichlet boundary condition arises in fluid mechanics as the equation for the stream function of a turbulent Euler flow with vortices of same orientation. By a statistical mechanics approach a rigorous derivation of it can be given as the mean-field limit of Onsager’s vortex theory, as shown by Kiessling [4,5] and Caglioti, Lions, Marchioro, Pulvirenti [6,7], and it is referred to as the “mean field equation”. In all these contexts, the function \(k\) is typically positive.

Notice that (1) contains also the singular mean-field equation

\begin{equation}
-\Delta_g v = \lambda \left( \frac{he^v}{\int_S he^v dv_g} - 1 \right) + \frac{4\pi N}{|S|} - 4\pi \sum_{j=1}^{l} n_j \delta_{p_j} \text{ in } S
\end{equation}
as a special case, where \( h > 0, p_j \in S, j = 1, \ldots, l \), are distinct points, \( n_i > 0 \) and \( N = \sum_{j=1}^{l} n_i \). Indeed, introducing the Green function \( G(x, p) \) with pole at \( p \in S \) as the solution of

\[
\begin{cases}
-\Delta G(\cdot, p) = \frac{1}{|S|} & \text{in } S \\
\int_{S} G(x, p)d\nu_k = 0,
\end{cases}
\]

(3)

the function \( u(x) = v(x) + 4\pi \sum_{j=1}^{l} n_i G(x, p_j) \) does solve (1) with \( k(x) = h(x)e^{-4\pi \sum_{j=1}^{l} n_i G(x, p_j)} \). Here, the function \( k \) is no longer positive, but is still nonnegative with zero set \( \{p_1, \ldots, p_l\} \). On a flat torus \( T \), singular mean-field equations with integer multiplicities \( \{n_1, \ldots, n_l\} \subset \mathbb{N} \) arise in the study of the asymptotics for non-topological (stationary) condensates in the relativistic abelian Chern–Simons–Higgs model as the Chern–Simons parameter tends to zero, as shown by Nolasco and Tarantello [8]. In the context of Euler flows, the presence of singular sources models the interaction of the fluid running on the given surface \( S \) with sinks of given vorticities and opposite orientation w.r.t. all the vortices present in the fluid.

Observe that (1) admits a variational structure, in the sense that weak solutions for (1) are critical points of the following energy functional

\[
J_{\lambda}(u) = \frac{1}{2} \int_{S} |\nabla u|_g^2 dv_g - \lambda \log \left( \int_{S} k e^u dv_g \right), \quad u \in \bar{H},
\]

(4)

where \( \bar{H} = \{ u \in H^1(S) : \int_{S} u dv_g = 0 \} \). For \( \lambda < 8\pi \), \( J_{\lambda} \) is bounded from below and the infimum of \( J_{\lambda} \) is achieved by the well-known Moser–Trudinger inequality.

Let us focus first on the regular case \( k > 0 \). For \( k = 1 \) Struwe and Tarantello [9] were able to obtain non-trivial solutions of (1) for \( 8\pi < \lambda < 8\pi m \) on the square flat torus \( T \). In the case of compact Riemann surfaces with genus \( g \geq 1 \), the existence of solutions for (1) with \( 8\pi < \lambda < 16\pi \) was shown by Ding, Jost, Li, Wang [10] still by a variational approach. The case \( S = S^2 \) of zero genus was considered by Lin [11] who proved nonvanishing of the Leray–Schauder degree \( d_\lambda \) associated to (1) for \( 8\pi < \lambda < 16\pi \) and \( d_\lambda = 0 \) for \( 16\pi < \lambda < 24\pi \).

Since the solutions set of (1) is bounded in \( C^{2,\alpha}(S), \alpha \in (0, 1) \), as long as \( \lambda \) is far from the critical parameter’s range \( 8\pi \mathbb{N} \), the degree \( d_\lambda \) is well-defined and constant for all \( \lambda \in (8\pi m, 8\pi (m + 1)), m \in \mathbb{N} \). As observed by Y.Y. Li [12], its value should just depend on \( m \) and the topology of \( S \). The program for computing \( d_\lambda \), initiated in [12], was completely settled by Chen and Lin [13] showing that

\[
d_\lambda = \left( \frac{m - \chi(S)}{m} \right),
\]

where \( \chi(S) = 2 - 2g \) is the Euler characteristic of \( S \) (see also the variational approach later developed by Malchiodi [14]). For \( S \neq S^2 \), the degree \( d_\lambda \) is always non-trivial yielding to a solution of (1) for all \( \lambda \notin 8\pi \mathbb{N} \). While, as already partially proved by Lin [11], for \( S^2 \) there holds \( d_\lambda = 0 \) for all \( \lambda > 16\pi \) with \( \lambda \notin 8\pi \mathbb{N} \), and no existence statements can be deduced. A complete positive answer to the existence issue for (1) has been provided by Djadli [15] for all \( \lambda \notin 8\pi \mathbb{N} \) by means of a variational approach of min–max type, inspired by the result of Djadli and Malchiodi [16] concerning the fourth-order Paneitz operator in conformal geometry. Multiplicity results have been provided by De Marchis [17,18].

Solutions of (1) are no longer a pre-compact set when \( \lambda \to 8\pi \mathbb{N} \): blow-up in the \( L^\infty \)-norm along with the concentration of the measure \( \lambda \frac{ke^u}{\int_{S} k e^u dv_g} \) as a sum of Dirac masses possibly arise for sequences of solutions as \( \lambda \to 8\pi \mathbb{N} \). Since \( d_\lambda \) can change just when \( \lambda \) crosses the values \( 8\pi m, m \in \mathbb{N} \), it is crucial to have a precise asymptotic knowledge of blow–up solutions \( u_i \), and uniquely characterize them as \( \lambda \to 8\pi m \). The most refined asymptotic analysis is given by Chen and Lin [19]: in particular, as \( \lambda \to 8\pi mu_\lambda \) has \( m \) well-separated maximum points (up to a subsequence) which converge to a critical point in \( S^m \setminus \Delta \) of

\[
\varphi_m(\xi) = \frac{1}{4\pi} \sum_{j=1}^{m} \log k(\xi_j) + \sum_{j=1}^{m} H(\xi_j, \xi_j) + \sum_{\substack{l \neq j \\mid \xi_l = \xi_j}} G(\xi_l, \xi_j),
\]

(5)

where \( H(x, \xi) \) is the regular part of \( G(x, \xi) \) and \( \Delta = \{ \xi \in S^m : \xi_i = \xi_j \text{ for } i \neq j \} \) is the diagonal set in \( S^m \). Let us notice that a critical point \( \xi \) of \( \varphi_m \) does satisfy

\[
\nabla \left[ \log k(x) + 8\pi H(x, \xi) + 8\pi \sum_{j \neq i} G(x, \xi_j) \right]_{x=\xi_i} = 0
\]

for all \( i = 1, \ldots, m \). In [13] blow-up solutions are constructed and their contribution to the degree is computed, so to determine (by local uniqueness of blow-up solutions) the jump in the values of \( d_\lambda \) across \( \lambda = 8\pi m \). Since the degree \( d_\lambda \) does not depend on \( k \), it is possible to choose a positive function \( k \) so that all the c.p.’s of \( \varphi_m \) are non-degenerate, and then in [13] the authors simply address the existence of blow-up sequences of solutions for (1) which concentrate at non-degenerate c.p.’s of \( \varphi_m \) as \( \lambda \to 8\pi m \).

The aim of the present paper is twofold. On one hand, we are interested in the construction of blow-up solutions with a general potential \( k \) for which the corresponding \( \varphi_m \) can possibly have degenerate but “stable” c.p.’s. On the other hand, we are interested in the singular mean-field equation or, equivalently, in (1) with a non-negative potential \( k \) which vanishes somewhere.
Let us focus now on the singular case. The first asymptotic analysis has been carried out by Bartolucci and Tarantello [20], with an application in the electroweak theory following [10]. The asymptotic analysis has been refined later in [21,22], with the on-going project by Chen and Lin [23] of computing the Leray–Schauder degree \( d_\lambda, \lambda \notin \Lambda \), where

\[
\Lambda = 8\pi \mathbb{N} + \left\{ 8\pi \sum_{i \leq j} (1 + n_j) : j \in \{1, \ldots, l\} \right\}
\]

is the corresponding critical set of parameters where compactness might fail, see [20]. For \( n_j \geq 1 \) the degree \( d_\lambda \) has been computed by Chen, Lin and Wang [24] for \( \lambda \in (8\pi, 16\pi) \), revealing the special role played by \( S^2 \), the sphere being the only surface for which the degree can vanish (precisely, it vanishes only for \( l = 1 \)). The critical regime \( \lambda = 8\pi \) has been considered in [25,26] for a general surface. However, as we will explain below, the problem on the torus with total multiplicity \( N = 2 \) becomes more degenerate. In this case, existence/non-existence issues have been discussed in [24] for a rectangular torus (along with the computation of \( d_{8\pi} \)) and in [27] for the general case, physically relevant issues in connection with non-topological 2-condensates in the Chern–Simons–Higgs model [8]. Existence results have been recently obtained by means of a variational approach of min–max type, inspired by [10,16], confirming the special role of \( S^2 \) (see also the discussion in [28,29]).

For \( \lambda \notin \Lambda \), the singular problem (2) is solvable for \( S \neq S^2 \) [30] (see also [31] for an application in the electroweak theory). The case of the sphere has been first considered by Malchiodi and Ruiz [32] for \( n_j \leq 1 \) and \( \lambda \in (8\pi, 16\pi) \setminus \Lambda \): the crucial assumption to have existence for (2) is that \( \# J \neq 1 \), where \( J = \{j = 1, \ldots, l : \lambda < 8\pi (1 + n_j)\} \). The result has been extended by Bartolucci and Malchiodi [33] to general \( n_j \)'s and \( \lambda \) under the condition \( l \geq 2 \) and \( \lambda < 8\pi \min \{1 + n_j : j = 1, \ldots, l\} \), corresponding to the situation \( \# J = l \).

In some of the above-mentioned papers, the regular/singular mean field equation has been also considered on a bounded domain \( \Omega \subset \mathbb{R}^2 \) with homogeneous Dirichlet b.c. Since \( \int_{\tilde{S}} k e^{u_k} d\nu_k \rightarrow +\infty \) along any non-compact sequence of solutions \( u_k \) for (1), through the setting \( \rho = \frac{\lambda}{\int_{\tilde{S}} k e^{u_k} d\nu_k} \) problem (1) is naturally related (but not equivalent) to \( -\Delta u = \rho \left( k e^u - \frac{1}{|\tilde{S}|} \int_{\tilde{S}} k e^u d\nu_k \right) \) with \( \rho \rightarrow 0^+ \), which has been recently studied by the second author in [34]. Blow-up solutions for the corresponding Dirichlet problem

\[
\begin{align*}
-\Delta u &= \rho k e^u & \text{in } \Omega \\
0 &= u & \text{on } \partial \Omega
\end{align*}
\]

on a bounded domain \( \Omega \subset \mathbb{R}^2 \) have been constructed at c.p.'s of \( \varphi_m \) which are non-degenerate [35] or, more generally, "stable" [36,37]. A "stable" critical value for \( \varphi_m \) has been constructed by del Pino, Kowalczyk and Musso [36] for the regular problem on a non-simply connected domain and for the singular problem with \( l = 1 \). The latter case has been extended to the flat torus [34], and a similar result is still in order for multiple singular sources as shown by D'Aprile [38], under suitable relations between \( m \) and the \( n_j \)'s.

Setting

\[
\rho_j(x) = k(x) e^{8\pi H(x, \xi_j) + 8\pi \sum_{\ell \neq j} G(x, \xi_{\ell})},
\]

for \( \xi \in S^m \setminus \Delta \) we introduce the notation

\[
A(\xi) = 4\pi \sum_{j=1}^m \left[ \Delta_{\tilde{S}} \rho_j(\xi_j) - 2K(\xi) \rho_j(\xi_j) \right],
\]

where \( K \) is the Gaussian curvature of \( (S, g) \). Letting \( \tilde{S} = \{k > 0\} \), our first main result is

**Theorem 1.1.** Let \( D \subset \subset \tilde{S}^m \setminus \Delta \) be a stable critical set of \( \varphi_m \). Assume that \( A(\xi) > 0 \) (\(< 0 \) resp.) for all \( \xi \in D \). Then, for all \( \lambda \) in a small right (left resp.) neighborhood of \( 8\pi m \) there is a solution \( u_k \) of (1) so that (along sub-sequences)

\[
\frac{\lambda k e^{u_k}}{\int_{\tilde{S}} k e^{u_k} d\nu_k} - 8\pi \sum_{j=1}^m \delta_{q_j} \to +\infty \text{ in } C_{\text{loc}}(S \setminus \{q_1, \ldots, q_m\})
\]

as \( \lambda \rightarrow 8\pi m \) in the sense of measures in \( S \), for some \( q = (q_1, \ldots, q_m) \in D \).

Along with (8) notice that there always hold that \( u_k - \log \int_{\tilde{S}} k e^{u_k} d\nu_k \to -\infty \) in \( C_{\text{loc}}(S \setminus \{q_1, \ldots, q_m\}) \) and

\[
\sup_{U_j} \left( u_j - \log \int_{\tilde{S}} k e^{u_j} d\nu_j \right) \to +\infty
\]

as \( \lambda \rightarrow 8\pi m \), for any neighborhood \( U_j \) of \( q_j \) in \( S \), \( j = 1, \ldots, m \). The notion of stability we are using here is the one introduced in [39]:

**Definition 1.2.** A critical set \( D \subset \subset \tilde{S}^m \setminus \Delta \) of \( \varphi_m \) is stable if for any closed neighborhood \( U \) of \( D \) in \( \tilde{S}^m \setminus \Delta \) there exists \( \delta > 0 \) such that, if \( ||G - \varphi_m||_{C(U)} \leq \delta \), then \( G \) has at least one critical point in \( U \). In particular, the minimal/maximal set of \( \varphi_m \) is stable (if \( \varphi_m \) is not constant) as well as any isolated c.p. of \( \varphi_m \) with non-trivial local degree.
Since \( A(\xi) \) can be re-written as
\[
A(\xi) = 4\pi \sum_{j=1}^{m} \rho_j(\xi_j) [\Delta g \log \rho_j(\xi_j) + |\nabla \log \rho_j(\xi_j)|_g^2 - 2K(\xi_j)]
\]
\[
= 4\pi \sum_{j=1}^{m} \rho_j(\xi_j) \left[ \Delta_g \log k(\xi_j) + \frac{8\pi m}{|S|} + |\nabla \log \rho_j(\xi_j)|_g^2 - 2K(\xi_j) \right],
\]
for a c.p. of \( \varphi_m \) we have that
\[
A(\xi) = 4\pi \sum_{j=1}^{m} \rho_j(\xi_j) \left[ \Delta_g \log k(\xi_j) + \frac{8\pi m}{|S|} - 2K(\xi_j) \right]
\]
in view of \( \nabla \rho_j(\xi_j) = 0 \) for all \( j = 1, \ldots, m \). Since for \( k > 0 \) the function \( \varphi_m \) always attains its minimum value in \( S^m \setminus \Delta \) and the minimal set is clearly stable, as a first by-product we have (see also [13]):

**Corollary 1.3.** Assume \( k > 0 \). Let \( m \in \mathbb{N} \) be so that either \( 1 \leq m < \inf_{S^2} \{ 2K - \Delta_g \log k \} \) or \( m > \sup_{S^2} \{ 2K - \Delta_g \log k \} \). Then there exist solutions \( u_\lambda \) of (1) which concentrate at \( m \) points \( q_1, \ldots, q_m \) in the sense (8) as \( \lambda \to 8\pi m \), where \( q = (q_1, \ldots, q_m) \) is a minimum point of \( \varphi_m \) in \( S^m \setminus \Delta \).

When the surface \((S, g)\) has constant Gaussian curvature, by the Gauss–Bonnet formula we have that \( K = \frac{2\pi \chi(S)}{|S|} \). For \( k = 1 \), Corollary 1.3 then provides the existence of blow-up solutions \( u_\lambda \) concentrating at \( m \) points as \( \lambda \to 8\pi m \) for all \( m \geq 2 \), where \( \lambda \) belongs to a small right neighborhood of \( 8\pi m \). The case \( m = 1 \) is problematic since \( \varphi_1 \) is a constant function.

Concerning the singular problem (2), in general the function \( \varphi_m \) has neither maximum nor minimum points, and it is then natural to search for saddle critical points. The min–max scheme introduced in [38] works in the Euclidean context as well as in the case of a surface [40]. In particular, on \( S^2 \) the function \( \varphi_m \) has a “stable” critical value of min–max type as soon as \( l \geq 2 \) and
\[
8\pi m < 8\pi \pi N + 8\pi (1 + n_1) \quad \forall j = 1, \ldots, l, \ #j \geq 2,
\]
where \( j = \{ j = 1, \ldots, l : 8\pi m < 8\pi (1 + n_1) \} \). In the construction, each singular source \( p_i \) has to be coupled with some \( p_j \neq p_i \) in order to deform \( S^2 \setminus \{ p_i, p_j \} \) onto a circle running around \( p_i \), and the condition \( l \geq 2 \) is crucial. Notice that the min–max scheme provides a critical point \( q \) of \( \varphi_m \) so that \( [q] \) is a stable critical set according to Definition 1.2. Moreover, since \( #j \geq 2 \) yields to \( 2m < 2 + N \), for \( k(x) = e^{-4\pi \sum_{j=1}^{l} n_j \chi(x,p_j)} \) we have that
\[
A(q) = \frac{16\pi^2}{|S^2|} \sum_{j=1}^{m} \rho_j(q_j)[-N + 2m - 2] < 0
\]
in view of \( K = \frac{4\pi}{|S^2|} \). As a second by-product of Theorem 1.1, we have:

**Corollary 1.4.** Let \( h = 1 \) and \( l \geq 2 \). Assume that \( S \) is topologically a sphere and that \( m \) satisfies (10). Then, for all \( \lambda \) in a small left neighborhood of \( 8\pi m \) there is a solution \( u_\lambda \) of (2) which concentrates at \( m \) points \( q_1, \ldots, q_m \) in the sense (8) as \( \lambda \to 8\pi m \).

Theorems 1.1 and 1.3 are the perturbative counter-parts of global existence results already available in the literature, obtained via degree theory or a variational approach. However, the behavior of such solutions as \( \lambda \to 8\pi m \) is not known whereas the ones we construct exhibit blow-up phenomena, a property that has a definite interest in its own. More important, Corollary 1.4 gives completely new results for the case of \( S^2 \), by showing that in a perturbative regime the condition \( #j \geq 2 \) is sufficient for the existence in the general case, beyond the results in [33]. Moreover, in [40] the cases \( #j = 1 \) are also treated.

There are cases for which \( A(\xi) \) can vanish. By invariance under rotations, it is easily seen that on \( S^2 \) the function \( H(\xi, \hat{\xi}) \) is constant, and then the c.p.‘s of \( \varphi_1 \) and \( k \) do coincide. Since in particular \( \nabla H(x, \hat{x})|_{x=\xi} = 0 \), by (9) for \( S = S^2 \) and \( m = 1 \) the coefficient \( A(\xi) \) writes as
\[
A(\xi) = 4\pi k(\xi) e^{8\pi H(\xi, \hat{\xi})} [\Delta_g \log k(\xi) + |\nabla \log k(\xi)|_g^2] = 4\pi^2 e^{8\pi H(\xi, \hat{\xi})} \Delta_g k(\xi),
\]
and might vanish at some c.p. of \( k \). Another typical example is the singular mean-field equation (2) on the flat torus \( T \) with \( h = 1 \) and even total multiplicity \( N \): since \( k = e^{-4\pi \sum_{j=1}^{l} n_j \chi(x,p_j)} \) and \( K \equiv 0 \), by (9) the coefficient \( A(\xi) \) writes for \( m = \frac{N}{2} \) as
\[
A(\xi) = 4\pi \sum_{j=1}^{m} \rho_j(\xi_j) |\nabla \log \rho_j(\xi_j)|_g^2 = (4\pi)^3 \sum_{j=1}^{m} \rho_j(\xi_j) |\nabla \varphi_m(\xi_j)|_g^2 \geq 0,
\]
and vanishes exactly at the c.p.’s of \( \varphi_m \). In all these situations, a more refined analysis is necessary.
Introduce the following quantity

\[ B(\xi) = -2\pi \sum_{j=1}^{m} \left[ \Delta_{g} \rho_{j}(\xi) - 2K(\xi) \rho_{j}(\xi) \right] \log \rho_{j}(\xi) - \frac{A(\xi)}{2} \]

\[ + \lim_{r \to 0} \left[ 8 \int_{S_{r} + m}^{\infty} \frac{8\pi}{r^{2}} \sum_{j=1}^{m} \rho_{j}(\xi) - \frac{8\pi}{r^{2}} \sum_{j=1}^{m} \rho_{j}(\xi) - A(\xi) \log \frac{1}{r} \right], \tag{12} \]

where \( B(\xi) \) denotes the pre-image of \( B_{0}(0) \) through the isothermal coordinate system at \( \xi \). The quantity \( B(\xi) \) has been first used and derived by Chang, Chen and Lin [41] in the study of the mean field equation on bounded domains (see also [24,42] for the case of the torus). We have the following general result, of which Theorem 1.1 is just a special case:

**Theorem 1.5.** Let \( D \subset \subset \bar{S}^{m} \setminus \Delta \) be a stable critical set of \( \varphi_{m} \). Assume that

either \( A(\xi) > 0 \) (\( < 0 \) resp.) or \( A(\xi) = 0 \), \( B(\xi) > 0 \) (\( < 0 \) resp.)

hold in a closed neighborhood \( U \) of \( D \) in \( \bar{S}^{m} \setminus \Delta \). Then, for all \( \lambda \) in a small right (left resp.) neighborhood of \( 8\pi m \) there is a solution \( u_{\lambda_{j}} \) of (1) which concentrates (along sub-sequences) at \( m \) points \( q_{1}, \ldots, q_{m} \) in the sense (8) as \( \lambda \to 8\pi m \), for some \( q \in D \).

To deal with a stable critical set \( D \) in the sense above, we need to require condition (13) on a neighborhood of \( D \). In case we strengthen the stability assumption, we can relax the assumption (13) to hold just on \( D \). As an instructive example, in Remark 4.5(i) we present the case of a non-degenerate local minimum/maximum point.

We can now discuss the two previous examples for which the coefficient \( A(\xi) \) vanishes. For \( S = S^{2} \) and \( m = 1 \), there holds \( \varphi_{1} = \frac{1}{4\pi} \log k + \text{const.} \) since \( H(\xi, \xi) = \text{const.} \). In view of (11), assume that \( \Delta_{g}k \geq 0 \) in a small neighborhood \( U \) of the minimal set \( D = \{ \xi \in S : \varphi_{1}(\xi) = \min \varphi_{1} \} \) so that \( A(\xi) \geq 0 \) in \( U \). We just need to show that \( B(\xi) > 0 \) in \( U \) so to use Theorem 1.5 with \( D \), which is clearly a stable critical set of \( \varphi_{1} \) as soon as \( k \) is not a constant function. Up to take \( U \) smaller, it is clearly enough to show that \( B(\xi) > 0 \) for all \( \xi \in D \) with \( A(\xi) = 0 \). Up to a rotation, we can assume that \( \xi \) is the south pole \( P \) of \( S^{2} \). The stereographic projection \( \pi : (x, y, z) \to \left( \frac{2x}{1-z^{2}}, \frac{2y}{1-z^{2}} \right) \) through the north pole is an isometry between \( (S^{2} \setminus \text{[north pole]}, g_{0}) \) and \( (\mathbb{R}^{2}, \frac{16}{(4 + u^{2} + v^{2})^{2}} \delta_{\text{eucl}}) \). Since it is easily seen that \( \text{dist}(\pi^{-1}(u, v), P) = |(u, v)| \)

\[ G(\pi^{-1}(u, v), P) = -\frac{1}{2\pi} \log |(u, v)| + \frac{1}{4\pi} \log(4 + u^{2} + v^{2}) + c_{0}, \]

in the coordinate system \( \pi \) in terms of \( \tilde{k}(u, v) = k(\pi^{-1}(u, v)) \) we can write that

\[ B(\tilde{P}) = 128e^{\omega_{0}} \lim_{r \to 0} \int_{\mathbb{R}^{2} \setminus B_{r}(0)} \tilde{k}(u, v) - k(0, 0) \frac{dudv}{(u^{2} + v^{2})^{2}} > 0 \]

in view of \( k \geq k(P) \), \( k \neq k(P) \). Similarly, we can treat the case in which \( \Delta_{g}k \leq 0 \) does hold in a small neighborhood \( U \) of the maximal set \( D \).

In the case of the flat torus \( T \) with \( N \) even, \( m = \frac{N}{2} \) and \( k = e^{\omega_{0}} , u_{0} = -4\pi \sum_{j=1}^{l} n_{j}G(x, p_{j}) \), at a c.p. \( \xi \) of \( \varphi_{m} \) the coefficient

\[ B(\xi) = \lim_{r \to 0} \left[ \frac{8}{T} \int_{T \setminus B_{r}(\xi)} e^{u_{0} + 8\pi H(x, \xi)} - \frac{8\pi}{r^{2}} \sum_{j=1}^{m} \rho_{j}(\xi) \right], \]

can be re-written in the following way:

- if \( N = 2 \), \( m = 1 \)
  \[ B(\xi) = 8e^{\omega_{0}} + 8\pi H(x, \xi) \left[ \frac{1}{|x - \xi|^{4}} \right], \]

where the integral on \( T \) is conditionally convergent in view of \( \nabla(u_{0}(x) + 8\pi H(x, \xi))|_{x=\xi} = 0 \) and \( \Delta(u_{0}(x) + 8\pi H(x, \xi)) \equiv 0 \);

- if \( N \geq 4 \) even, \( m = \frac{N}{2} \)
  \[ B(\xi) = 8 \sum_{j=1}^{m} \left[ \frac{\rho_{j}(\xi) - \rho_{j}(\xi)}{|x - \xi|^{4}} \right], \]

where \( T \) has been split into disjoint sets \( T_{1}, \ldots, T_{m} \) so that \( B_{i}(\xi) \subset T_{j} \) for \( r \) small and all \( j \).

When \( T \) is a rectangle, \( l = 1 \) and \( n_{1} = 2 \), the constant \( B(\xi) \) has been used by Chen, Lin and Wang [24] in the computation of the degree \( d_{8\pi} \). The function \( \varphi_{1} = \frac{\omega_{0}}{4\pi} + \text{const.} \) has exactly three non-degenerate critical points \( \xi_{1}, \xi_{2} \) (saddle points) and \( \xi_{3} \) (maximum point) with \( B(\xi_{i}) \). \( B(\xi_{1}) > 0 \) and \( B(\xi_{2}) < 0 \). By Theorem 1.5 and Remark 4.5-(i) we deduce the existence of

- two distinct families of solutions, for \( \lambda \) in a small right neighborhood of \( 8\pi \), concentrating at \( \xi_{1} \) and \( \xi_{2} \) as \( \lambda \to 8\pi \);

- one family of solutions, for \( \lambda \) in a small left neighborhood of \( 8\pi \), concentrating at \( \xi_{3} \) as \( \lambda \to 8\pi \).
Moreover, $B(\xi)$ has been recently used in the construction of non-topological condensates for the relativistic abelian Chern–Simons–Higgs model as the Chern–Simons parameter tends to zero, see [42]. Unfortunately, when $N \geq 4$ there are no examples where the sign of $B(\xi)$ can be determined. To explain more clearly such a connection, recall that in the relativistic abelian Chern–Simon–Higgs model the $N$ vortex-condensates are gauge-periodic stationary matter configurations with finite-energy that, in the self-dual regime, express in terms of solutions for

$$
-\Delta w = \frac{1}{\epsilon^2} e^{w} (1 - e^{-w}) - 4\pi \sum_{j=1}^{l} n_j \delta_{p_j} \tag{14}
$$

in a flat torus $T$. We refer to [43] for a complete account on the model and to [44] for the analytical results concerning it. The quantity $2\epsilon > 0$ is the Chern–Simons parameter, $p_j \in S$, $j = 1, \ldots, l$, are distinct points and $n_j \in \mathbb{N}$. Physically, $\epsilon$ is very small and two classes of solutions are relevant: either $e^{w} \to 1$ as $\epsilon \to 0^+$ (“topological” type) or $e^{w} \to 0$ as $\epsilon \to 0^+$ (“non-topological” type). Topological solutions were first found by Caffarelli and Yang [45]. However, non-topological condensates represent the main feature of the Chern–Simons–Higgs model which were absent in the classical (Maxwell–Higgs) vortex theory, whose existence was established by Tarantello [46]. Through the change $w \to w - u_0$, $u_0 = -4\pi \sum_{j=1}^{l} n_j G(x, p_j)$, the self-dual equation (14) reads equivalently as

$$
-\Delta w = \frac{1}{\epsilon^2} ke^{w} (1 - ke^{-w}) - 4\pi N \frac{1}{|T|} \tag{15}
$$

with $k = e^{u_0}$. Setting $c = \frac{1}{|T|} \int_{T} w dx$ and $u = w - c \in \tilde{H}$, an integration of (15) provides a relation between $c$ and $u$ (see [46]):

$$
e^c \int_{T} ke^{w} dx - e^{2c} \int_{T} k^2 e^{2u} dx = 4\pi Ne^2.
$$

Hence, necessarily

$$u \in \mathcal{A}_e = \left\{ u \in \tilde{H} \mid \left( \int_{T} ke^{u} \right)^2 - 16\pi Ne^2 \int_{T} k^2 e^{2u} \geq 0 \right\}
$$

and then $c = c_\pm(u)$ with

$$e^{c_\pm(u)} = \frac{8\pi Ne^2}{\int_{T} ke^{u} \mp \sqrt{\left( \int_{T} ke^{u} \right)^2 - 16\pi Ne^2 \int_{T} k^2 e^{2u}}}.
$$

For solutions of “non-topological” type it is natural to choose $c_-(u)$, and then Eq. (15) reads in terms of $u \in \mathcal{A}_e$ as

$$
-\Delta u = 4\pi N \left( \frac{ke^u}{\int_{T} ke^u} - \frac{1}{|T|} \right) + \frac{64\pi^2 N^2 \epsilon^2}{\left( \int_{T} ke^u + \sqrt{\left( \int_{T} ke^u \right)^2 - 16\pi Ne^2 \int_{T} k^2 e^{2u}} \right)^2} \left( \frac{ke^u}{\int_{T} ke^u} - \frac{k^2 e^{2u}}{\int_{T} k^2 e^{2u}} \right). \tag{16}
$$

When $N$ is even and $m = \frac{N}{4}$, Eq. (16) is a perturbation of (1) $\lambda = 8\pi m$ as $\epsilon \to 0^+$. The parallel becomes clear if we re-consider (1) itself as a perturbation of (1) $\lambda = 8\pi m$ as $\lambda \to 8\pi m$. As far as (1) is concerned, the sign of the perturbation can be chosen since it depends on $\lambda - 8\pi m$. For (16) the sign of the perturbation is given and is like the case $\lambda < 8\pi m$ in which we need to require (13) with the negative sign $< 0$. Even if we always have the wrong sign $A(\xi) \geq 0$, the coefficient $A(\xi)$ behaves like $|\nabla \phi_m(\xi)|_G^2 := \sum_{j=1}^{m} |\nabla \phi_m(\xi)|_G^2$ and, near a critical set $\mathcal{D}$ of $\phi_m$, is very small. The condition $B(\xi) < 0$ on $\mathcal{D}$ will then be enough, as stated in the following:

**Theorem 1.6.** Assume $N$ even. Let $\mathcal{D} \subset\subset (T \setminus \{p_1, \ldots, p_l\})^m \setminus \Delta$ be a stable critical set of

$$
\phi_m(\xi) = \frac{1}{4\pi} \sum_{j=1}^{m} u_0(p_j) + \sum_{l,j} G(p_j, \xi), \quad \text{Assume that } B(\xi) < 0 \text{ does not hold in } \mathcal{D}. \text{ Then, for all } \epsilon \text{ small there is a solution } u_\epsilon \text{ of (14) which concentrates at } m \text{ points } q_1, \ldots, q_m \text{ with } q = (q_1, \ldots, q_m) \in \mathcal{D}, \text{ as } \epsilon \to 0 \text{ in the sense of measures:}
$$

$$
\frac{1}{\epsilon^2} e^{u_\epsilon} (1 - e^{-u_\epsilon}) \to 8\pi \sum_{j=1}^{m} \delta_{q_j}.
$$

Correspondingly, there exist non-topological $N$ vortex-condensates of gauge potential $A_\epsilon$ and Higgs field $\phi_\epsilon$ for which the magnetic field $(F_{12})_\epsilon$ is very concentrated at the $m$ points $q_1, \ldots, q_m$ (external to the so-called vortex-set $\{p_1, \ldots, p_l\}$) as $\epsilon \to 0$.

**Theorem 1.6** slightly improves the result in [42] (see [47–49] for concentration at the vortices) where they just deal with isolated c.p.’s of $\phi_m$ with non-trivial local degree. Our “stability” assumption is more general as already explained in **Definition 1.2.** When $m = 1\varphi_1$ has always the maximal set as a “stable” critical set, while for $m > 1$ it is not clear whether there
exists or not a configuration \( p_1, \ldots, p_l \) such that \( \varphi_m \) has a nontrivial "stable" critical set. However, for \( \ell = 1, n_1 = N = 2m \) and any \( p_j \in T \) then there exists a nontrivial critical level of \( \varphi_m \), as shown in [34], so that the corresponding critical set is "stable". Unfortunately, even at an isolated critical point, it is very difficult to determine the negative sign of the quantity \( B \) (apart from the case \( \ell = m = 1, n_1 = 2, T \) a rectangle).

2. Approximation of the solution

To construct approximating solutions of (1), the main idea is to use as "basic cells" the functions

\[
u_{\delta, \xi}(x) = u_0\left(\frac{|x - \xi|}{\delta}\right) - 2\log\delta \quad \delta > 0, \xi \in \mathbb{R}^2,
\]

where

\[
u_0(r) = \log\frac{8}{(1 + r^2)^2}.
\]

They are all the solutions of

\[
\begin{cases}
\Delta u + e^u = 0 & \text{in } \mathbb{R}^2 \\
\int_{\mathbb{R}^2} e^u < \infty,
\end{cases}
\]

and do satisfy the following concentration property:

\[e^{\nu_{\delta, \xi}} \to 8\pi\delta \chi \text{ in measure sense as } \delta \to 0.\]

We will use now isothermal coordinates to pull-back \( u_{\delta, \xi} \) in \( S \).

Let us recall that every Riemann surface \((S, g)\) is locally conformally flat, and the local coordinates in which \( g \) is conformal to the Euclidean metric are referred to as isothermal coordinates (see for example the simple existence proof provided by Chern [50]). For every \( \xi \in S \) it amounts to find a local chart \( y_\xi \), with \( y_\xi(\xi) = 0 \), from a neighborhood of \( \xi \) onto \( B_{2r_0}(0) \) (the choice of \( r_0 \) is independent of \( \xi \)) in which \( g = e^{\hat{\varphi}_\xi}(y_\xi(x)) \, dx \), with \( \hat{\varphi}_\xi \in C^\infty(B_{2r_0}(0), \mathbb{R}) \). In particular, \( \hat{\varphi}_\xi \) relates with the Gaussian curvature \( K \) of \((S, g)\) through the relation:

\[
\Delta \hat{\varphi}_\xi(y) = -2K(y_\xi^{-1}(y_\xi(x)))e^{\hat{\varphi}_\xi}(y) \quad \text{for } y \in B_{2r_0}(0).
\]

(17)

We can also assume that \( y_\xi, \hat{\varphi}_\xi \) depends smoothly on \( \xi \) and that \( \hat{\varphi}_\xi(0) = 0, \nabla \hat{\varphi}_\xi(0) = 0 \).

We now pull-back \( u_{\delta, \xi} \) in \( \xi \in S \), for \( \delta > 0 \), by simply setting

\[
u_{\delta, \xi}(x) = u_{\delta, 0}(y_\xi(x)) = \log\frac{8\delta^2}{(\delta^2 + |y_\xi(x)|)^2}.
\]

for \( x \in y_\xi^{-1}(B_{2r_0}(0)) \). Letting \( \chi \in C^\infty_0(B_{2r_0}(0)) \) be a radial cut-off function so that \( 0 \leq \chi \leq 1, \chi \equiv 1 \) in \( B_{r_0}(0) \), we introduce the function \( PU_{\delta, \xi} \) as the unique solution of

\[
\begin{cases}
-\Delta_g PU_{\delta, \xi}(x) = \chi \nu_{\delta, \xi}(x) e^{-\nu_{\delta, \xi}(x)} e^{\nu_{\delta, \xi}(x)} - \frac{1}{|S|} \int_S \chi \nu_{\delta, \xi} e^{\nu_{\delta, \xi}} d\nu_g & \text{in } S \\
\int_S PU_{\delta, \xi} d\nu_g = 0,
\end{cases}
\]

(18)

where \( \chi \nu_{\delta, \xi}(x) = \chi(|y_\xi(x)|) \) and \( \varphi_\xi(x) = \hat{\varphi}_\xi(y_\xi(x)) \).

Notice that the R.H.S. in (18) has zero average and smoothly depends on \( x \), and then (18) is uniquely solvable by a smooth solution \( PU_{\delta, \xi} \).

Let us recall the transformation law for \( \Delta_g \) under conformal changes: if \( \tilde{g} = e^\varphi g \), then

\[
\Delta_{\tilde{g}} = e^{-\varphi}\Delta_g.
\]

(19)

Decompose now the Green function \( G(x, \xi, \xi) \), \( \xi \in S \), as

\[
G(x, \xi) = -\frac{1}{2\pi} \chi_\xi(x) \log |y_\xi(x)| + H(x, \xi),
\]

and by (3) then deduce that

\[
\begin{cases}
-\Delta_g H = \frac{1}{2\pi} \Delta_g \chi_\xi \log |y_\xi(x)| - \frac{1}{\pi} (\nabla \chi_\xi \cdot \nabla \log |y_\xi(x)|)_{\tilde{g}} = \frac{1}{|S|} \text{ in } S \\
\int_S H(\cdot, \xi) d\nu_{\tilde{g}} = \frac{1}{2\pi} \int_S \chi_\xi \log |y_\xi(\cdot)|d\nu_g.
\end{cases}
\]

We have used that

\[
\Delta_g \log |y_\xi(x)| = e^{-\hat{\varphi}_\xi(y_\xi(\cdot))} \Delta \log |y_\xi(\cdot)| \bigg|_{x = y_\xi(x)} = 2\pi \delta \chi_\xi
\]

in view of (19).
For \( r \leq 2r_0 \) define \( B_r(\xi) = y_\xi^{-1}(B_r(0)) \), \( A_r(\xi) = B_r(\xi) \setminus B_{r/2}(\xi) \), and set
\[
f_\xi = \frac{\Delta_y \chi_\xi}{|y_\xi(x)|^2} + 2 \left( \nabla \chi_\xi \cdot \nabla |y_\xi(x)|^{-2} \right)_g + \frac{2}{|S|} \int_{\mathbb{R}^2} \chi(|y|) \, dy.
\]

By (22) it follows that
\[
\int_S f_\xi \, dv_g = \frac{1}{2\delta^2} \int_S \Delta_g \Psi_{3,\delta} \, dv_g + O(\delta^2) = O(\delta^2)
\]
as \( \delta \to 0 \), where \( \Psi_{3,\delta} \in H^1(S) \) is defined in (21). Thus, \( \int_S f_\xi \, dv_g = 0 \), and then \( F_\xi \) is well defined as the unique solution of
\[
\begin{cases}
-\Delta_g F_\xi = f_\xi & \text{in } S \\
\int_S F_\xi \, dv_g = 0.
\end{cases}
\]
We have the following asymptotic expansion of \( PU_{3,\delta} \) as \( \delta \to 0 \):

**Lemma 2.1.** The function \( PU_{3,\delta} \) satisfies
\[
PU_{3,\delta} = \chi_\delta \left[ U_{3,\delta} - \log(8\delta^2) \right] + 8\pi H(x, \xi) + \alpha_{3,\delta} - 2\delta^2 F_\delta + O(\delta^4 |\log \delta|)
\]
uniquely in \( S \), where \( F_\delta \) is given in (20) and
\[
\alpha_{3,\delta} = -\frac{4\pi}{|S|} \delta^2 \log \delta + \frac{\delta^2}{|S|} \left( \int_{\mathbb{R}^2} \chi(|y|) \frac{e^{\psi}(y) - 1}{|y|^2} \, dy - \int_{\mathbb{R}^2} \chi(|y|) \log |y| \, dy \right).
\]
In particular, there holds
\[
PU_{3,\delta} = 8\pi G(x, \xi) - 2 \frac{\delta^2 \chi_\delta}{|y_\xi(x)|^2} + \alpha_{3,\delta} - 2\delta^2 F_\delta + O(\delta^4 |\log \delta|)
\]
locally uniformly in \( S \setminus \{\xi\} \).

**Proof.** Let us define
\[
\Psi_{3,\delta}(x) = PU_{3,\delta}(x) - \chi_\delta \hat{U}_{3,\delta} - 8\pi H(x, \xi),
\]
where \( \hat{U}_{3,\delta} = U_{3,\delta} - \log(8\delta^2) \), for which there holds
\[
\int_S \Psi_{3,\delta} \, dv_g = -\int_S \chi_\delta \hat{U}_{3,\delta} + 8\pi H(x, \xi) \right) \, dv_g = -\int_S \chi_\delta \log \frac{|y_\xi(x)|^4}{(\delta^2 + |y_\xi(x)|^2)^2} \, dv_g.
\]
Since \( \hat{U}_{3,\delta} \) satisfies in \( B_{2r_0}(\xi) \)
\[
-\Delta_g \hat{U}_{3,\delta} = -e^{-\psi}(y) \Delta u_{3,0} \big|_{y=y_\xi(x)} = e^{-\psi} e^{U_{3,\delta}}
\]
in view of (19), by the equation of \( H(x, \xi) \) now we have that
\[
-\Delta_g \Psi_{3,\delta} = 2 \left( \nabla \chi_\delta \cdot \nabla \hat{U}_{3,\delta} + 4 \nabla \log |y_\xi(x)| \right)_g + \Delta_g \chi_\delta \hat{U}_{3,\delta} + 4 \log |y_\xi(x)| \left( 8\pi - \int_S \chi_\delta e^{-\psi} e^{U_{3,\delta}} \, dv_g \right).
\]
Also, we have that in \( A_{2r_0}(\xi) \)
\[
\hat{U}_{3,\delta} + 4 \log |y_\xi(x)| = \log \frac{|y_\xi(x)|^4}{(\delta^2 + |y_\xi(x)|^2)^2} = -2 \frac{\delta^2}{|y_\xi(x)|^2} + O(\delta^4)
\]
and
\[
\nabla \left( \hat{U}_{3,\delta} + 4 \log |y_\xi(x)| \right) = -2\delta^2 \nabla |y_\xi(x)|^{-2} + O(\delta^4),
\]
and there holds
\[
\int_S \chi_\delta e^{-\psi} e^{U_{3,\delta}} \, dv_g = \int_{B_{2r_0}(0) \setminus B_{r_0}(0)} \chi(|y|) \frac{8\delta^2}{|y|^2} \, dy + O(\delta^4) + \int_{B_{r_0}(0)} \frac{8\delta^2}{(\delta^2 + |y|^2)^2} \, dy
\]
\[
= 8\pi - 8\delta^2 \left( \frac{\pi}{r_0^2} - \int_{B_{2r_0}(0) \setminus B_{r_0}(0)} \chi(|y|) \, dy \right) + O(\delta^4)
\]
\[
= 8\pi + 4\delta^2 \int_{\mathbb{R}^2} \frac{\chi(|y|)}{|y|^3} \, dy + O(\delta^4)
\]
in view of $dv_g = \hat{e}^{\Phi} dy$ in the coordinate system $y_\xi$ and
\[
\int_{B_{2\theta}(0) \setminus B_{\delta}(0)} \frac{\chi(|y|)}{|y|^2} dy = 2\pi \int_{r_0}^{2\delta} \frac{\chi(r)}{r^3} dr = \frac{\pi}{\theta^2} + \frac{1}{2} \int_{B_{2\theta}(0) \setminus B_{\delta}(0)} \frac{\chi'(|y|)}{|y|^3} dy.
\]
By the definition of $f_\xi$ we then have that
\[
-\Delta_g \psi_{\delta, \xi} = -2\delta^2 f_\xi + O(\delta^4) \quad \text{in } S.
\]
Since $\int_S F_\xi dv_g = 0$, by elliptic regularity theory we get that
\[
\psi_{\delta, \xi} = -2\delta^2 f_\xi + \frac{1}{|S|} \int_S \psi_{\delta, \xi} dv_g + O(\delta^4),
\]
in view of (20). On the other hand, we have that
\[
\int_S \psi_{\delta, \xi} dv_g = -\int_S \chi_\xi \log \frac{|y_\xi(x)|^4}{(\delta^2 + |y_\xi(x)|^2)^2} dv_g = 2 \int_{B_{\delta}(0)} \log \frac{\delta^2 + |y|^2}{|y|^2} e^{\hat{\Phi}(y)} dy
\]
\[
+ \int_{B_{2\theta}(0) \setminus B_{\delta}(0)} \chi(|y|) \left( 2 \frac{\delta^2}{|y|^2} + O(\delta^4) \right) e^{\hat{\Phi}(y)} dy
\]
\[
= 2 \int_{B_{\delta}(0)} \log \frac{\delta^2 + |y|^2}{|y|^2} e^{\Phi(y)} dy + 2\delta^2 \int_{B_{2\theta}(0) \setminus B_{\delta}(0)} \chi(|y|) \frac{e^{\hat{\Phi}(y)} - 1}{|y|^2} dy
\]
\[
- 4\pi \delta^2 \log r_0 - 2\delta^2 \int_{\mathbb{R}^2} \chi'(|y|) \log \frac{|y|}{|y|^2} dy + O(\delta^4)
\]
in view of
\[
\int_{B_{2\theta}(0) \setminus B_{\delta}(0)} \frac{\chi(|y|)}{|y|^2} dy = 2\pi \int_{r_0}^{2\delta} \frac{\chi(t)}{t} dt = -2\pi \log r - \int_{B_{2\theta}(0) \setminus B_{\delta}(0)} \frac{\chi'(|y|) \log |y|}{|y|} dy
\]
for $r \leq r_0$. Since
\[
\int_{B_{\delta}(0)} \log \frac{\delta^2 + |y|^2}{|y|^2} dy = \delta^2 \int_{B_{\delta}(0)} \log \frac{1 + |z|^2}{|z|^2} dz = 2\pi \delta^2 \int_0^\infty \left[ \log \frac{1 + r^2}{r^2} - \frac{1}{r^2 + 1} \right] dr
\]
\[
+ \pi \delta^2 \log \left( \frac{r^2}{\delta^2} + 1 \right) + O(\delta^4)
\]
where $y = \delta z$, by $e^{\hat{\Phi}(y)} = 1 + O(|y|^2)$ we can write that
\[
2 \int_{B_{\delta}(0)} \log \frac{\delta^2 + |y|^2}{|y|^2} e^{\hat{\Phi}(y)} dy = 2 \int_{B_{\delta}(0)} \log \left( \frac{\delta^2}{|y|^2} + 1 \right) (e^{\hat{\Phi}(y)} - 1) dy - 4\pi \delta^2 \log \delta
\]
\[
+ 4\pi \delta^2 \left[ \log r_0 + \int_0^\infty \left( \log \frac{1 + r^2}{r^2} - \frac{1}{r^2 + 1} \right) r dr \right] + O(\delta^4)
\]
\[
= 2\delta^2 \int_{B_{\delta}(0)} \log \frac{\delta^2}{|y|^2} = \delta^2 \log \left( \frac{\delta^2}{|y|^2} + 1 \right) - \int_{B_{\delta}(0) \setminus B_{\delta}(0)} \frac{\chi'(|y|) \log |y|}{|y|^2} dy)
\]
by using that
\[
\int_{B_{\delta}(0)} \log \left( \frac{\delta^2}{|y|^2} + 1 \right) - \int_{B_{\delta}(0) \setminus B_{\delta}(0)} \frac{\chi'(|y|) \log |y|}{|y|^2} dy
\]
\[
= O\left( \delta^4 \int_{B_{\delta}(0) \setminus B_{\delta}(0)} \left| \log \left( \frac{1}{|y|^2} + 1 \right) - \frac{1}{|y|^2} \right| |y|^2 dy \right) + O(\delta^4) = O(\delta^4 |\log \delta|)
\]
in view of $\log \left( \frac{1}{|y|^2} + 1 \right) - \frac{1}{|y|^2} = O\left( \frac{1}{|y|^4} \right)$ as $|y| \to +\infty$. In conclusion, we get that
\[
\int_S \psi_{\delta, \xi} dv_g = -4\pi \delta^2 \log \delta + 2\delta^2 \int_{\mathbb{R}^2} \chi(|y|) \frac{e^{\hat{\Phi}(y)} - 1}{|y|^2} dy + \pi - \int_{\mathbb{R}^2} \chi'(|y|) \log |y| dy
\]
in view of $\int_0^\infty (\log \frac{1 + r^2}{r^2} - \frac{1}{r^2 + 1}) r dr = \frac{1}{2}$. This completes the proof. \(\square\)
The ansatz will be constructed as follows. Given $m \in \mathbb{N}$, let us consider distinct points $\xi_j \in \tilde{S}$ (i.e., $\xi_j \in S$ with $k(\xi_j) > 0$) and $\delta_j > 0, j = 1, \ldots, m$. In order to have a good approximation, we will assume that

$$\delta^2_j = \delta^2 \rho_j(\xi_j) \quad \forall j = 1, \ldots, m,$$

and

$$\exists C > 1 : |\lambda - 8\pi m| \leq C\delta^2 |\log \delta|,$$

where $\delta > 0$ and $\rho_j$ is as in (6). Up to take $r_0$ smaller, we assume that the points $\xi_j$’s are well separated and $k(\xi_j)$ is uniformly far from zero, namely, we choose $\xi_j = (\xi_1, \ldots, \xi_m) \in \Xi$, where

$$\Xi = \{ (\xi_1, \ldots, \xi_m) \in \mathbb{S}^m \mid d_k(\xi_i, \xi_j) \geq 4r_0 \text{ and } k(\xi_j) \geq r_0 \forall i, j = 1, \ldots, m, i \neq j \}.$$

Denote $U_j := U_{\xi_j}$ and $W_\xi = PU_j$, $j = 1, \ldots, m$, where $P$ is the projection operator defined by (18). Thus, our approximating solution is $W(x) = \sum_{j=1}^m W_j(x)$, parametrized by $(\delta, \xi) \in (0, \infty) \times \Xi$. Notice that for $r_0$ small enough we have that $\mathcal{D} \subset \Xi \subset \tilde{S}^m \ \Delta$. We will look for a solution $u$ of (1) in the form $u = W + \phi$, for some small remainder term $\phi$. In terms of $\phi$, the problem (1) is equivalent to find $\phi \in \tilde{H}$ so that

$$L(\phi) = -[R + N(\phi)] \quad \text{in} \ S,$$

where the linear operator $L$ is defined as

$$L(\phi) = \Delta_\phi \phi + \lambda \int_S \frac{ke^W}{\int_S ke^W \phi} \left( \phi - \int_S \frac{ke^W \phi}{\int_S ke^W} \right),$$

the nonlinear part $N$ is given by

$$N(\phi) = \lambda \left( \int_S \frac{ke^W + \phi}{\int_S ke^W} - \frac{ke^W \phi}{\int_S ke^W} + \frac{ke^W}{\left( \int_S ke^W \right)^2} - \frac{ke^W}{\int_S ke^W \phi} \right),$$

and the approximation rate of $W$ is encoded in

$$R = \Delta_\phi W + \lambda \left( \frac{ke^W}{\int_S ke^W \phi} - \frac{1}{|S|} \right).$$

Notice that for all $\phi \in \tilde{H}$

$$\int_S L(\phi) d\nu_g = \int_S N(\phi) d\nu_g = \int_S R d\nu_g = 0.$$

In order to get the invertibility of $L$, let us introduce the weighted norm

$$\| h \|_\sigma = \sup_{x \in S} \left[ \sum_{j=1}^m \left( \frac{\delta^\sigma}{\lambda^2 + \chi_{B_0(\xi_j)}(x)} \left| \nabla \phi_m(\xi_j) \right|^2 \right)^\frac{1}{2} \right]^{-1} |h(x)|$$

for any $h \in L^\infty(S)$, where $0 < \sigma < 1$ is a small fixed constant and $\chi_A$ denotes the characteristic function of the set $A$. Let us evaluate the approximation rate of $W$ in $\| \cdot \|_\sigma$:

**Lemma 2.2.** Assume (24)-(25). There exists a constant $C > 0$, independent of $\delta > 0$ small, such that

$$\| R \|_\sigma \leq C \left( \delta |\nabla \phi_m(\xi) \|_g + \delta^{2-\sigma} |\log \delta| \right)$$

for all $\xi \in \Xi$, where $|\nabla \phi_m(\xi) \|_g$ stands for $\sum_{j=1}^m |\nabla \phi_m(\xi_j) \|_g$.

**Proof.** First, from Lemma 2.1 we note that for any $j \in \{1, \ldots, m\}$

$$W_j(x) = U_j(x) - \log(8\delta^2) + 8\pi H(x, \xi_j) + O(\delta^2 |\log \delta|)$$

uniformly for $x \in B_{r_0}(\xi_j)$ and

$$W_j(x) = 8\pi G(x, \xi_j) + O(\delta^2 |\log \delta|)$$

uniformly for $x$ on compact subsets of $S \ \Delta \{ \xi_j \}$. Since by symmetry and $\hat{\phi}_j(0) = 0$ we have

$$\int_{B_{r_0}(\xi_j)} \rho_j(x)e^u d\nu_g = \int_{B_{r_0}(0)} \rho_j(y_j^{-1}(y)) \frac{8\delta^2}{\delta^2 + |y|^2} e^{\hat{\phi}_j(y)} dy$$

$$= \int_{B_{r_0}(0)} \rho_j(\xi_j) \frac{8}{(1 + |y|^2)^2} (1 + O(\delta^2 |\log \delta|)) dy = 8\pi \rho_j(\xi_j) + O(\delta^2 |\log \delta|).$$
we then get that
\[
\int_S k^w \, dv_g = \sum_{j=1}^{m} \frac{1}{8\delta_j^2} \int_{B_{\rho_j}(\xi_j)} k^e \, dv_g + O(1)
\]
\[
= \sum_{j=1}^{m} \frac{1}{8\delta_j^2} \int_{B_{\rho_j}(\xi_j)} \rho_j(x)e^{U_j}(1 + O(\delta^2 |\log \delta|)) \, dv_g + O(1)
\]
\[
= \sum_{j=1}^{m} \frac{1}{8\delta_j^2} [\pi \rho_j(\xi_j) + O(\delta^2 |\log \delta|)] + O(1) = \frac{\pi m}{\delta^2} + O(|\log \delta|).
\]

By Lemma 2.1 and (24), (31) we have that
- in \( S \setminus \bigcup_{j=1}^{m} B_{\rho_j}(\xi_j) \) there holds \( 8\pi m \int_S k^w \, dv_g = O(\delta^2) \) in view of \( W(x) = O(1) \);
- in \( B_{\rho_j}(\xi_j), j \in \{1, \ldots, m\} \), there holds
\[
8\pi m \int_S k^w \, dv_g = 8\pi m \frac{ke^w}{\int_S k^w \, dv_g} = \frac{-\log(8\delta_j^2) + 8\pi H(\xi_j) + 8\pi \sum_i G_i(\xi_j) + O(\delta^2 |\log \delta|)}{8\pi m \rho_j(\xi_j) + O(\delta^2 |\log \delta|)} e^{U_j}
\]
\[
= \frac{1 + \left( \nabla(\rho_j \circ y_{\xi_j}^{-1})(0), y_{\xi_j}(x) \right) + O(|y_{\xi_j}(x)|^2 + \delta^2 |\log \delta|)}{\rho_j(\xi_j)} e^{U_j},
\]
which can be summarized as follows:
\[
8\pi m \int_S k^w \, dv_g = \sum_{j=1}^{m} \chi_j e^{-\varphi_j} e^{U_j} \, dv_g = \sum_{j=1}^{m} \chi_j e^{-\varphi_j} e^{U_j} \, dv_g = \int_{B_{\rho_j}(0)} \frac{8\delta_j^2}{\delta_j^2 + |y|^2} \, dy + O(\delta^2) = 8\pi + O(\delta^2)
\]
with \( \varphi_j = \varphi_{\xi_j} \), for
\[
R_{8\pi m} = \Delta_g W + 8\pi m \left( \frac{ke^w}{\int_S k^w \, dv_g} - \frac{1}{|S|} \right)
\]
we then have that
\[
R_{8\pi m} = -\sum_{j=1}^{m} \chi_j e^{-\varphi_j} e^{U_j} + 8\pi m \frac{ke^w}{\int_S k^w \, dv_g} + \frac{1}{|S|} \sum_{j=1}^{m} \int_S \chi_j e^{-\varphi_j} e^{U_j} \, dv_g - \frac{8\pi m}{|S|}
\]
\[
= -\sum_{j=1}^{m} \chi_j e^{-\varphi_j} e^{U_j} + 8\pi m \frac{ke^w}{\int_S k^w \, dv_g} + O(\delta^2).
\]
By (32) we now deduce that \( R_{8\pi m}(x) = O(\delta^2) \) in \( S \setminus \bigcup_{j=1}^{m} B_{\rho_j}(\xi_j) \) and
\[
R_{8\pi m} = \left[ -e^{-\varphi_j} + 1 + O(|\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)||y_{\xi_j}(x)| + \delta^2 |\log \delta|) \right] e^{U_j} + O(\delta^2)
\]
\[
e^{U_j} O \left( |\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)||y_{\xi_j}(x)| + |y_{\xi_j}(x)|^2 + \delta^2 |\log \delta| \right) + O(\delta^2)
\]
in \( B_{\rho_j}(\xi_j), j \in \{1, \ldots, m\} \), in view of \( \varphi_j(\xi_j) = 0 \) and \( \nabla \varphi_j(\xi_j) = 0 \). From the definition of \( \| \cdot \|_* \) we deduce the validity of
\[
\| R_{8\pi m} \|_* \leq C \delta \| \nabla \varphi_{\xi_j}(\xi) \|_g + \delta^{2-\sigma}
\]
in view of \( |\nabla \log(\rho_j \circ y_{\xi_j}^{-1})(0)| \leq |\nabla y_{\xi_j} \varphi_{\xi_j}(\xi) \|_g \). Since by (32)
\[
R - R_{8\pi m} = \left( \lambda - 8\pi m \right) \frac{ke^w}{\int_S k^w \, dv_g} - \frac{1}{|S|} = O \left( |\lambda - 8\pi m| \sum_{j=1}^{m} \chi_j e^{U_j} + |\lambda - 8\pi m| \right)
\]
we get that \( \| R - R_{8\pi m} \|_* = O(\delta^{-\sigma} |\lambda - 8\pi m|) \). In conclusion, by (25) and (33) we deduce the validity of (30).
3. The reduced energy

The purpose of this section is to give an asymptotic expansion of the "reduced energy" $J_\xi(W)$, where $J_\xi$ is the energy functional given by (4). For technical reasons, we will be concerned with establishing it in a $C^2$-sense in $\delta$ and just in a $C^1$-sense in $\xi$. To this aim, the following result will be very useful:

**Lemma 3.1.** Letting $f \in C^{2,\gamma}(S)$ (possibly depending on $\xi$), $0 < \gamma < 1$, denote as $P_\delta(f)$ the second-order Taylor expansion of $f(x)$ at $\xi$:

$$P_\delta(f) = f(\xi) + (\nabla f \circ y^{-1}_\xi)(0), y_\xi(x)) + \frac{1}{2}(D^2(f \circ y^{-1}_\xi)(0)(y_\xi(x), y_\xi(x)).$$

The following expansions do hold as $\delta \to 0$:

$$\int_{S \setminus B_{\delta}(\xi)} \chi_\xi e^{-\psi f(x)}e^{U_{\delta}} dv_g = 8\pi f(\xi) - 2\delta^2 \Delta f(\xi) \left[ 2\pi \log \delta + \int_{\mathbb{R}^2} \frac{\chi(\langle y \rangle)}{|y|} dy + \pi \right]$$

$$+ 8\delta^2 \int_{S \setminus B_{\delta}(\xi)} \chi_\xi e^{-\psi f(x)} e^{U_{\delta}} dv_g + 4\delta^2 f(\xi) \int_{\mathbb{R}^2} \frac{\chi(\langle y \rangle)}{|y|} dy + o(\delta^2),$$

and

$$\int_{S \setminus B_{\delta}(\xi)} \chi_\xi e^{-\psi f(x)} e^{U_{\delta}} dv_g = \frac{4\pi}{\delta^2} f(\xi) + \pi \Delta f(\xi) + O(\delta^2)$$

for $a \in \mathbb{R}$.

**Proof.** Since $dv_g = e^{\psi(y)} dy$, by symmetry observe that

$$\int_{S \setminus B_{\delta}(\xi)} \chi_\xi e^{-\psi f(x)} e^{U_{\delta}} dv_g = \frac{8\pi}{\delta^2} \int_{S \setminus B_{\delta}(\xi)} \chi_\xi e^{-\psi f(x)} e^{U_{\delta}} dv_g + O(\delta^4)$$

$$= 8\delta^2 \int_{S \setminus B_{\delta}(\xi)} \chi_\xi e^{-\psi f(x)} e^{U_{\delta}} dv_g + 8\delta^2 f(\xi) \int_{B_{2\delta}(0) \setminus B_\delta(0)} \frac{\chi(\langle y \rangle)}{|y|^4} dy$$

$$+ 2\delta^2 \Delta(f \circ y^{-1}_\xi)(0) \int_{B_{2\delta}(0) \setminus B_\delta(0)} \frac{\chi(\langle y \rangle)}{|y|^2} dy + O(\delta^4)$$

as $\delta \to 0$. On $B_\delta(\xi)$ we get that

$$\int_{B_\delta(\xi)} \chi_\xi e^{-\psi f(x)} e^{U_{\delta}} dv_g = \int_{B_\delta(\xi)} f(y^{-1}_\xi(y)) \frac{8\delta^2}{(\delta^2 + |y|^2)^2} dy$$

$$= \int_{B_\delta(\xi)} P_\delta(f)(y^{-1}_\xi(y)) \frac{8\delta^2}{(\delta^2 + |y|^2)^2} dy + \int_{B_\delta(\xi)} (f \circ y^{-1}_\xi)(y) - P_\delta(f)(y^{-1}_\xi(y)) \frac{8\delta^2}{(\delta^2 + |y|^2)^2} dy.$$

Since $f(x) - P_\delta(f)(x) = O(|y^{-1}_\xi(x)|^{2+\gamma})$, by symmetry and the Lebesgue Theorem we get that

$$\int_{B_\delta(\xi)} \chi_\xi e^{-\psi f(x)} e^{U_{\delta}} dv_g = f(\xi) \int_{B_{\delta}(0) \setminus B_\delta(0)} \frac{8}{(1 + |y|^2)^2} dy$$

$$+ \delta^2 \Delta(f \circ y^{-1}_\xi)(0) \int_{B_{\delta}(0) \setminus B_\delta(0)} \frac{2|y|^2}{(1 + |y|^2)^2} dy + 8\delta^2 \int_{B_\delta(\xi)} e^{-\psi f(x)} e^{U_{\delta}} dv_g + o(\delta^2)$$

$$= 8\pi f(\xi) \left(1 - \frac{\delta^2}{\delta^2 + \frac{\delta^2}{r_0}}\right) + 2\pi \delta^2 \Delta(f \circ y^{-1}_\xi)(0) \left(\log \frac{\delta^2 + r_0^2}{\delta^2} + \frac{\delta^2}{\delta^2 + r_0^2} - 1\right)$$

$$+ 8\delta^2 \int_{B_\delta(\xi)} e^{-\psi f(x)} e^{U_{\delta}} dv_g + o(\delta^2) = 8\pi f(\xi) \left(1 - \frac{\delta^2}{r_0^2}\right)$$

$$+ 2\pi \delta^2 (-2 \log \delta + 2 \log r_0 - 1) \Delta(f \circ y^{-1}_\xi)(0) + 8\delta^2 \int_{B_\delta(\xi)} e^{-\psi f(x)} e^{U_{\delta}} dv_g + o(\delta^2).$$
In view of (23) and
\[
\int_{B_{\rho}(0)} \frac{\chi(|y|)}{|y|^4} dy = 2\pi \int_{r}^{2\pi} \frac{\chi(t)}{t^3} dt = \frac{\pi}{r^2} + \frac{1}{2} \int_{B_{\rho}(0)} \frac{\chi(|y|)}{|y|^3} dy
\] (34)
for \( r \leq \rho_0 \), summing up the two previous expansions we get that
\[
\int_{B} \chi_{\delta} e^{-\psi_k} f(x) e^{i\xi \cdot x} dv_{\delta} = 8\pi f(\xi) - 2\delta^2 \Delta(f \circ y_{\delta}^{-1})(0) \left[ 2\pi \log \delta + \int_{B^2} \frac{\chi'(|y|) \log |y|}{|y|} dy + \pi \right] + 8\delta^2 \int_{B} \chi_{\delta} e^{-\psi_k} f(x) - P_2(f)(x) \frac{|y^2|}{|y|^4} dv_{\delta} + 4\delta^2 f(\xi) \int_{B^2} \chi'(|y|) \frac{|y|}{|y|^3} dy + o(\delta^2).
\]

Since by (19) \( \Delta g f(x) = e^{-\psi_k}(x) \Delta(f \circ y_{\delta}^{-1})(y_k(x)) \), we get that \( \Delta(f \circ y_{\delta}^{-1})(0) = \Delta g f(\xi) \), and the validity of the first expansion then follows. The other two expansions are simpler because of the stronger decay. Indeed, by the Taylor expansion of \( f \) at \( \xi \) and the symmetries we get that
\[
\int_{B^2} \frac{dy}{(1 + |y|^2)^2} = \frac{\pi}{2}
\]
in view of
\[
\int_{B^2} \frac{dy}{(1 + |y|^2)^2} = \frac{\pi}{2}.
\]
Similarly, we have that
\[
\int_{B^2} \frac{a - |y|^2}{(1 + |y|^2)^4} dv_{\delta} = \frac{8}{\delta^2} \int_{B_{\rho}(0)} \frac{|y^2|}{(1 + |y|^2)^2} dv_{\delta} + \delta^2 \int_{B^2} \frac{|y^2|}{(1 + |y|^2)^2} dy + O(\delta^2)
\]
and
\[
\int_{B^2} \frac{|y^2|}{(1 + |y|^2)^4} dy = -\int_{B^2} \frac{dy}{(1 + |y|^2)^2} + \int_{B^2} \frac{dy}{(1 + |y|^2)^3} - (1 + a) \int_{B^2} \frac{dy}{(1 + |y|^2)^3} = (a - 2) \frac{\pi}{6}.
\]
The Lemma is completely established. \( \square \)

We are now ready to establish the expansion of \( f_\delta(W) \):

**Theorem 3.2.** Assume (24)--(25). The following expansion does hold
\[
J_\delta(W) = -8\pi m - \lambda \log(\pi m) - 32\pi^2 \varphi_m(\xi) + 2(\lambda - 8\pi m) \log \delta + A(\xi) \delta^2 \log \delta - B(\xi) \delta^2 + o(\delta^2)
\] (35)
in \( C^2(\mathbb{R}) \) and \( C^1(\mathbb{S}) \) as \( \delta \to 0^+ \), where \( \varphi_m(\xi) \), \( A(\xi) \) and \( B(\xi) \) are given by (5), (7) and (12), respectively.

The proof will be divided into several steps.

**Proof of (35) in \( C(\mathbb{R} \times \mathbb{S}) \).** First, let us consider the term
\[
\int_{B^2} |\nabla W|^2 dv_{\delta} = \int_{B^2} W(-\Delta g W) dv_{\delta} = \sum_{j, l = 1}^{m} \int_{B^2} \chi_{\delta} e^{-\psi_k} W_{j l} dv_{\delta}
\]
in view of \( \int_S Wdv_g = 0 \). Since by (3) and (18)

\[
\int_S \chi_j e^{-\psi_j} e^{U_j} G(x, \xi_j) dv_g = \int_S (-\Delta_j PU_j) G(x, \xi) dv_g = PU_j(\xi)
\]

(36)

for all \( j, l = 1, \ldots, m \), by Lema 2.1, 3.1 and (36) we have that for \( l = j \)

\[
\int_S \chi_j e^{-\psi_j} e^{U_j} W_j dv_g
\]

\[
= \int_S \chi_j e^{-\psi_j} e^{U_j} \left[ \chi_j \log \left( \frac{|y_j|^2}{(1 + |y|^2)^2} \right) + 8\pi G(x, \xi_j) + \alpha_{\delta_j, \xi_j} - 2\delta_j^2 F_j \right] dv_g + O(\delta^4 |\log \delta|)
\]

\[
= \int_S \chi_j e^{-\psi_j} e^{U_j} \left[ \chi_j \log \left( \frac{|y_j|^2}{(1 + |y|^2)^2} \right) + 8\pi G(x, \xi_j) + \alpha_{\delta_j, \xi_j} - 2\delta_j^2 F_j \right] dv_g + O(\delta^4 |\log \delta|)
\]

\[
= 8 \int_{\mathbb{S}^{s_j-1}} \frac{\chi_j^2 (\delta_j |y|)}{(1 + |y|^2)^2} \log \left( \frac{|y|^2}{(1 + |y|^2)^2} \right) dy + 8\pi PU_j(\xi_j) + 8\pi \alpha_{\delta_j, \xi_j} - 16\pi \delta_j^2 F_j(\xi_j)
\]

\[
+ O(\delta^4 |\log \delta|^2) = -16\pi - 32\pi \delta_j + 64\pi^2 H(\xi_j, \xi_j) + 16\pi \alpha_{\delta_j, \xi_j} - 32\pi \delta_j^2 F_j(\xi_j) + O(\delta^4 |\log \delta|^2)
\]

in view of

\[
\int_{\mathbb{S}^2} \frac{dy}{(1 + |y|^2)^2} \log \left( \frac{|y|^2}{(1 + |y|^2)^2} \right) = 2\pi \int_0^\infty \frac{ds}{1 + s^2} \log \frac{s}{1 + s} = -2\pi \int_0^\infty \frac{ds}{1 + s^2} = -2\pi
\]

by means of an integration by parts. Similarly, by Lema 2.1, 3.1 and (36) we have that for \( l \neq j \)

\[
\int_S \chi_j e^{-\psi_j} e^{U_j} W_j dv_g = \int_S \chi_j e^{-\psi_j} e^{U_j} \left[ 8\pi G(x, \xi_j) + \alpha_{\delta_j, \xi_j} - 2\delta_j^2 F_j \right] dv_g + O(\delta^4 |\log \delta|)
\]

\[
= 64\pi^2 G(\xi_j, \xi_j) + 8\pi (\alpha_{\delta_j, \xi_j} + \alpha_{\delta_j, \xi_j}) - 16\pi (\delta_j^2 F_j(\xi_j) + \delta_j^2 F_j(\xi_j)) + O(\delta^4 |\log \delta|^2).
\]

Setting

\[
\alpha_{\delta, \xi} = \sum_{j=1}^m \alpha_{\delta_j, \xi_j} F_{\delta, \xi}(\xi) = \sum_{j=1}^m \delta_j^2 F_j(\xi),
\]

summing up the two previous expansions, for the gradient term we get that

\[
\frac{1}{2} \int_S |\nabla W_j|^2 dv_g = -8\pi m - 16\pi m \log \delta - 32\pi^2 \varphi_m(\xi) + 8\pi m \alpha_{\delta, \xi} - 16\pi \sum_{j=1}^m F_{\delta, \xi}(\xi_j) + o(\delta^2)
\]

in view of (24) and

\[
8\pi \sum_{j=1}^m \log \rho_j(\xi_j) - 32\pi^2 \left[ \sum_{j=1}^m H(\xi_j, \xi_j) + \sum_{l \neq j} G(\xi_l, \xi_j) \right] = 32\pi^2 \varphi_m(\xi).
\]

Let us now expand the potential term in \( J_k(W) \). By Lemma 2.1 for any \( j = 1, \ldots, m \) we find that

\[
\int_{B_{\delta_j}(\xi_j)} ke^{U_j} dv_g = \int_{B_{\delta_j}(\xi_j)} \rho_j e^{U_j} \left[ \log(8\delta_j^2 + \alpha_{\delta, \xi} - 2 F_{\delta, \xi}) + O(\delta^4 |\log \delta|) \right] dv_g
\]

\[
= \frac{1}{8\delta_j^2} \left[ \int_S \chi_j e^{U_j} e^{\alpha_{\delta, \xi} - 2 F_{\delta, \xi}} dv_g - 8\delta_j \int_{B_{2\delta_j}(\xi_j)} \frac{\chi_j \rho_j}{|y_j|^2} dv_g + O(\delta^4 |\log \delta|) \right].
\]

By Lemma 3.1 (with \( f(x) = e^{\psi_j} \rho_j e^{\alpha_{\delta, \xi} - 2 F_{\delta, \xi}} \)) we can now deduce that

\[
8\delta_j^2 \int_{B_{\delta_j}(\xi_j)} ke^{U_j} dv_g = 8\pi \rho_j(\xi_j) e^{\alpha_{\delta, \xi} - 2 F_{\delta, \xi}(\xi_j)} - 4\pi \left( \Delta_k \rho_j(\xi_j) - 2 K(\xi_j) \rho_j(\xi_j) \right) \delta_j^2 \log \delta_j
\]

\[
- 2 \left( \Delta_k \rho_j(\xi_j) - 2 K(\xi_j) \rho_j(\xi_j) \right) \left( \int_{\mathbb{R}^2} \frac{X'(|y|) \log |y|}{|y|} dy + \pi \right) \delta_j^2 + 4\delta_j^2 \rho_j(\xi_j) \int_{\mathbb{R}^2} \frac{X'(|y|)|y|^2}{|y|^3} dy
\]

\[
+ 8\delta_j^2 \int_{B_{\delta_j}(\xi_j)} \left[ \sum_{j=1}^m \frac{G(x, \xi_j)}{\log(8\delta_j^2 + \alpha_{\delta, \xi} - 2 F_{\delta, \xi}(\xi_j))} \right] dv_g - 8\delta_j \int_{B_{2\delta_j}(\xi_j)} \chi_j e^{U_j} \left[ \frac{P_j(e^{\psi_j} \rho_j)}{|y_j|^2} \right] dv_g + o(\delta^2)
\]
in view of $\frac{\rho}{|y_j(x)|^4} = ke^{8\pi \sum_{j=1}^m G(x,\xi_j)}$ in $B_{r_0}(\xi_j)$ and by (17)

$$\Delta_{\xi_j}[e^{\psi_i}(\rho)](\xi_j) = \Delta_{\xi_j} \rho_j(\xi_j) - 2K(\xi_j)\rho_j(\xi_j).$$

(37)

On the other hand, we have that

$$\int_{S \cap \bigcup_{j=1}^m B_{r_0}(\xi_j)} \kappa e^W d\nu_g = \int_{S \cap \bigcup_{j=1}^m B_{r_0}(\xi_j)} 8\pi \sum_{j=1}^m G(x,\xi_j) \kappa e^W d\nu_g + O(\delta^2 | \log \delta|).$$

Since

$$\sum_{j=1}^m e^{-2F_{\xi_j}(\xi_j)} = m - 2 \sum_{j=1}^m F_{\xi_j}(\xi_j) + O(\delta^4)$$

and by (24) there holds

$$\delta_j^2 \log \delta_j = \rho_j(\xi_j) \delta^2 \log \delta + \frac{1}{2} \rho_j(\xi_j) \log \rho_j(\xi_j) \delta^2,$$

we then obtain that

$$\frac{1}{\pi} e^{-\alpha \delta_j^2} \delta^2 \int_S \kappa e^W d\nu_g = m - \frac{A(\xi)}{8\pi} \delta^2 \log \delta + \frac{B_2(\xi)}{8\pi} \delta^2 - 2 \sum_{j=1}^m F_{\xi_j}(\xi_j) + o(\delta^2),$$

(38)

where

$$B_2(\xi) = -2\pi \sum_{j=1}^m [\Delta_{\xi_j} \rho_j(\xi_j) - 2K(\xi_j) \rho_j(\xi_j)] \log \rho_j(\xi_j) - \frac{A(\xi)}{2} \left( \int_{\mathbb{R}^2} \frac{\chi'(|y|)}{|y|} \log |y|^2 dy + \pi \right)$$

$$+ 4 \int_{\mathbb{R}^2} \frac{\chi'(|y|)}{|y|^4} dy \sum_{j=1}^m \rho_j(\xi_j) + 8 \int_S \kappa e^W \left[ \frac{8\pi \sum_{j=1}^m G(x,\xi_j)}{4} - \sum_{j=1}^m \frac{\chi_j e^{-\psi_j}}{|y_j(x)|^4} \right] d\nu_g.$$

By (23), (34) and the splitting of $S$ as the union of $\bigcup_{j=1}^m B_r(\xi_j)$ and $S \setminus \bigcup_{j=1}^m B_r(\xi_j), r \leq r_0$, we easily deduce that

$$B_2(\xi) = -2\pi \sum_{j=1}^m [\Delta_{\xi_j} \rho_j(\xi_j) - 2K(\xi_j) \rho_j(\xi_j)] \log \rho_j(\xi_j) - \frac{A(\xi)}{2}$$

$$+ 8 \int_{S \setminus \bigcup_{j=1}^m B_r(\xi_j)} \frac{8\pi \sum_{j=1}^m G(x,\xi_j)}{4} \kappa e^W d\nu_g - \frac{8\pi}{r^2} \sum_{j=1}^m \rho_j(\xi_j) - A(\xi) \log \frac{1}{r}$$

$$+ 8 \sum_{j=1}^m \int_{B_r(\xi_j)} e^{\psi_j(x)} \rho_j(x) - P_2(e^{\psi_j}(\rho_j)(x)) \frac{e^{-\psi_j(x)}}{|y_j(x)|^4} d\nu_g$$

in view of $\frac{\rho}{|y_j(x)|^4} = ke^{8\pi \sum_{j=1}^m G(x,\xi_j)}$ in $B_{r_0}(\xi_j)$, (37) and the definitions of $A(\xi), P_2(e^{\psi_j}(\rho_j))$. As a by-product we have that $B_2(\xi)$ does not depend on $\chi$ and $r \leq r_0$. Since

$$\lim_{r \to 0} \int_{B_r(\xi_j)} e^{\psi_j(x)} \rho_j(x) - P_2(e^{\psi_j}(\rho_j)(x)) \frac{e^{-\psi_j(x)}}{|y_j(x)|^4} d\nu_g = 0$$

in view of $e^{\psi_j(x)} \rho_j(x) - P_2(e^{\psi_j}(\rho_j)(x)) = o(|y_j(x)|^2)$ as $x \to \xi_j$, we have that $B_2(\xi)$ coincides with $B(\xi)$ as defined in (12).

Finally, we get the following expansion for $J_{8\pi m}(W)$ as $\delta \to 0$:

$$J_{8\pi m}(W) = -8\pi m (1 + \log(\pi m)) - 32\pi^2 \varphi_m(\xi_j) + A(\xi) \delta^2 \log \delta - B(\xi) \delta^2 + o(\delta^2).$$

(39)

Since

$$\log \int_S \kappa e^W d\nu_g = -2 \log \delta + \log(\pi m) + O(\delta^2 | \log \delta|),$$

by (25) we then deduce that

$$J_\delta(W) = J_{8\pi m}(W) - (\lambda - 8\pi m) \log \int_S \kappa e^W d\nu_g$$

$$= J_{8\pi m}(W) - (\lambda - 8\pi m) (-2 \log \delta + \log(\pi m)) + O(\delta^4 | \log \delta|)^2$$

and the proof is complete. \(\square\)
We establish now expansion (35) in a $C^1$-sense in $\xi$, where the derivatives in $\xi$ are with respect to a given coordinate system.

**Proof of (35) in $C^1(\Sigma)$.** We just need to expand the derivatives of $I_i(W)$ in $\xi$. Let us fix $i \in \{1, 2\}$ and $j \in \{1, \ldots, m\}$. We have that

$$\partial_{(\xi)}[I_i(W)] = -\int_S \Delta_x W + \frac{\lambda \kappa e^W}{\int_S \kappa e^W d\nu_g} \partial_{(\xi)} W d\nu_g.$$ 

Arguing as in Lemma 2.1, it is easy to show that

$$\partial_{(\xi)} W_q = -2 \frac{\chi_q}{\delta^2_q + |y_q(x)|^2} \left[ \partial_{(\xi)} |y_q(x)|^2 + \delta^2_q \partial_{(\xi)} (\log \rho_q(\xi_q)) \right]$$

$$- 4 \log |y_q(x)| \partial_{(\xi)} \chi_q + 8 \pi \partial_{(\xi)} H(x, \xi_q) + O(\delta^2 |\log \delta|)$$

(40)
does hold uniformly in $S$. In particular there hold

$$\partial_{(\xi)} W_q = -16 \pi \frac{\chi_q \delta^2_q}{\delta^2_q + |y_q(x)|^2} \partial_{(\xi)} G(\xi_q, \xi_j) + O(\delta^2 |\log \delta|) \quad \forall q \neq j$$

uniformly in $S$ and

$$\partial_{(\xi)} W_j = 8 \pi \partial_{(\xi)} G(x, \xi_j) + O(\delta^2 |\log \delta|)$$

locally uniformly in $S \setminus \{\xi_j\}$. Then we have that:

- for $q \neq i, j$

  $$\int_S \chi_l e^{-v_l} e^l \partial_{(\xi)} W_q d\nu_g = O(\delta^2 |\log \delta|)$$

  in view of $\chi_l \chi_q \equiv 0$;

- for $l \neq j$

  $$\int_S \chi_l e^{-v_l} e^l \partial_{(\xi)} W_l d\nu_g = -16 \pi \partial_{(\xi)} G(\xi_l, \xi_j) \int_S \chi_l^2 e^{-v_l} \frac{\delta^2_q}{\delta^2_q + |y_q(x)|^2} d\nu_g + O(\delta^2 |\log \delta|)$$

  $$= -128 \pi \partial_{(\xi)} G(\xi_l, \xi_j) \int_S \frac{dy}{\pi (1 + |y|^2)^3} + O(\delta^2 |\log \delta|)$$

  $$= -64 \pi^2 \partial_{(\xi)} G(\xi_l, \xi_j) + O(\delta^2 |\log \delta|)$$

and

$$\int_S \chi_l e^{-v_l} e^l \partial_{(\xi)} W_l d\nu_g = 64 \pi^2 \partial_{(\xi)} G(\xi_l, \xi_j) + O(\delta^2 |\log \delta|)$$

in view of Lemma 3.1. So we have that for $l \neq j$

$$\int_S \chi_l e^{-v_l} e^l \partial_{(\xi)} W_l d\nu_g = O(\delta^2 |\log \delta|).$$

If $l = j$, by Lemma 3.1 we have that

$$\int_S \chi_l e^{-v_l} e^l \partial_{(\xi)} W_l d\nu_g = \int_S \chi_l e^{-v_l} e^l \partial_{(\xi)} W_l d\nu_g + O(\delta^2 |\log \delta|)$$

$$= \int_S \chi_l e^{-v_l} e^l \left[ \chi_l \partial_{(\xi)} (U_l - \log(8\delta^2)) + 8 \pi \partial_{(\xi)} H(x, \xi_j) \right] d\nu_g + O(\delta^2 |\log \delta|)$$

$$= \partial_{(\xi)} \left[ \int_S \chi_l^2 e^{-v_l} e^l d\nu_g \right] + \int_S \chi_l^2 e^{-v_l} e^l \partial_{(\xi)} \partial_{(\xi)} \rho_l d\nu_g - 8 \pi \partial_{(\xi)} \log \rho_l (\xi_j) + 64 \pi^2 \partial_{(\xi)} H(x, \xi_j) \bigg|_{x=\xi_j}$$

$$+ O(\delta^2 |\log \delta|) = -8 \pi \partial_{(\xi)} \log \rho_l (\xi_j) + 64 \pi^2 \partial_{(\xi)} H(x, \xi_j) \bigg|_{x=\xi_j} + O(\delta^2 |\log \delta|)$$

in view of $\partial_{(\xi)} \log(8\delta^2) = \partial_{(\xi)} \log(8\delta^2) \equiv 0$ and

$$\partial_{(\xi)} \left( \int_S \chi_l^2 e^{-v_l} e^l d\nu_g \right) = 8 \partial_{(\xi)} \log \rho_l (\xi_j) \int_{\mathbb{R}^2} \frac{|y|^2 - 1}{(1 + |y|^2)^3} dy + O(\delta^2) = O(\delta^2).$$
In view of $\int_S \delta_{\xi_j} W dv_g = 0$ we can compute
\[
- \int_S \Delta_g W \delta_{\xi_j} W dv_g = \sum_{i=1}^n \int_S x_i e^{-\rho_i} \delta_{\xi_j} W dv_g \\
= -8\pi \delta_{\xi_j} \rho_j \log \rho_j + 64\pi^2 \delta_{\xi_j} H(x, \xi_j) |_{x=\xi_j} + O(\delta^2 | \log \delta |)
\]
\[
= -32\pi^2 \delta_{\xi_j} \varphi_m(\xi_j) + O(\delta^2 | \log \delta |)
\]  
(41)
in view of $G(x_1, x_2) = G(x_2, x_1)$ for all $x_1 \neq x_2$ and $\delta_{\xi_j} H(x, \xi_j) |_{x=\xi_j} = \frac{1}{2} \delta_{\xi_j} H(\xi_j, \xi_j)$. In order to give an expansion of the second term in $\delta_{\xi_j} [J, W]$ first observe that by Lemma 2.1 there hold $ke^W = \frac{e^{\mu_1, k}}{8\delta_j} \rho e^{U_j} [1 + O(\delta^2)]$ uniformly in $B_{\rho_0}(\xi_j)$ and $ke^W = O(1)$ uniformly in $S \setminus \cup_{j \neq 1} B_{\rho_0}(\xi_j)$. So we have that
\[
8 e^{-\mu_1, k} \int_S ke^W \delta_{\xi_j} W dv_g = \sum_{i=1}^m \int_{B_{\rho_0}(\xi_j)} \rho e^{U_j} (1 + O(\delta^2)) \delta_{\xi_j} W dv_g + O(1)
\]
\[
= \delta_j^{-2} \int_{B_{\rho_0}(\xi_j)} \rho e^{U_j} (1 + O(\delta^2)) \delta_{\xi_j} W dv_g
\]
\[
- 16\pi \sum_{i \neq j} \delta_{\xi_j} G(\xi_i, \xi_j) \int_{B_{\rho_0}(\xi_j)} \rho e^{U_j} \frac{dv_g}{\delta_j^2 + |y_j(x)|^2}
\]
\[
+ 8\pi \sum_{i \neq j} \delta_j^{-2} \int_{B_{\rho_0}(\xi_j)} \rho e^{U_j} \delta_{\xi_j} W(x, \xi_j) dv_g + O(| \log \delta |)
\]
in view of $\int_B \delta_{\xi_j} W_i = 8\pi \delta_{\xi_j} H(x, \xi_j) + O(\delta^2 | \log \delta |)$ in $B_{\rho_0}(\xi_j)$ when $q \neq l$. Since
\[
\delta_{\xi_j} W_j = \chi_j \delta_{\xi_j} [U_j - \log(8\delta^2)] - 4 \log |y_j(x)| \delta_{\xi_j} \chi_j + 8\pi \delta_{\xi_j} H(x, \xi_j) + O(\delta^2 | \log \delta |),
\]
we have that
\[
\int_{B_{\rho_0}(\xi_j)} \rho e^{U_j} \delta_{\xi_j} W dv_g = \int_S \chi_j \rho \rho e^{U_j} \delta_{\xi_j} U_j dv_g + \int_S \chi_j \rho \rho e^{U_j} [8\pi \delta_{\xi_j} H(x, \xi_j) - \delta_{\xi_j} \rho_j \log \rho_j (\xi_j)] + O(\delta^2 | \log \delta |)
\]
\[
= \delta_{\xi_j} \left( \int_S \chi_j \rho \rho e^{U_j} dv_g \right) - \delta_{\xi_j} \rho_j (\xi_j) \int_S \chi_j \rho \rho e^{U_j} + O(\delta^2 | \log \delta |)
\]
in view of $\int_B \delta_{\xi_j} \rho_j (x) = 8\pi \delta_{\xi_j} H(x, \xi_j)$. Since by the Taylor expansion of $e^{\delta_j (\rho_j o y_{\xi_j}^{-1})}$ at 0 and the symmetries we have that
\[
\int_{B_{\rho_0}(\xi_j)} \rho e^{U_j} \delta_{\xi_j} W dv_g = \int_{B_{\rho_0}(\xi_j)} \chi (|y|) \delta_{\xi_j} \left[ e^{\delta_j (\rho_j o y_{\xi_j}^{-1})} (y) \right] \frac{8\delta_j^2}{(\delta_j^2 + |y|^2)^2} dy
\]
\[
+ \delta_{\xi_j} \rho_j (\xi_j) \int_{B_{\rho_0}(\xi_j)} \chi (|y|) e^{\delta_j (\rho_j o y_{\xi_j}^{-1})} (y) \frac{8\delta_j^2 (|y|^2 - \delta_j^2)}{(\delta_j^2 + |y|^2)^3} dy = 8\pi \delta_{\xi_j} \rho_j (\xi_j) + O(\delta^2 | \log \delta |)
\]
in view of $\int_B \delta_{\xi_j} \rho_j (0) = 8\pi \delta_{\xi_j} H(0, \xi_j)$ and $\int_{\mathbb{R}^2} \frac{|y|^2 - 1}{(1 + |y|^2)^2} dy = 0$, by Lemma 3.1 we then deduce that
\[
\int_{B_{\rho_0}(\xi_j)} \rho e^{U_j} \delta_{\xi_j} W dv_g = O(\delta^2 | \log \delta |).
\]
Since the Taylor expansion of $e^{\delta_j (\rho_j o y_{\xi_j}^{-1})}$ at 0 and the symmetries we have that
\[
\int_{B_{\rho_0}(\xi_j)} \rho e^{U_j} \frac{dv_g}{\delta_j^2 + |y_{\xi_j}(x)|^2} = \delta_j^{-2} \int_{B_{\rho_0}(\xi_j)} (\rho_j o y_{\xi_j}^{-1})(\delta_j y) e^{\delta_j (\rho_j y)} \frac{8}{(1 + |y|^2)^3} dy
\]
\[
= \delta_j^{-2} \int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^3} dy + O(1) = \frac{4\pi}{\delta_j^2} + O(1),
\]
by Lemma 3.1 we obtain that
\[
e^{-\mu_1, k} \int_S ke^W \delta_{\xi_j} W dv_g = \frac{-8\pi^2}{\delta_j^2} \sum_{l \neq j} \delta_{\xi_j} G(\xi_l, \xi_j) + \frac{8\pi^2}{\delta_j^2} \sum_{l \neq j} \delta_{\xi_j} H(\xi_l, \xi_j) + O(| \log \delta |).
\]
Since by (38) \( \int_S k e^{W} dv_g = \frac{\pi m}{2^\beta} e^{\psi_k} \left( 1 + O(\delta^2 |\log \delta|) \right) \), we finally get that

\[
\int \frac{k e^{W}}{S} dv_g = -\frac{8 \pi}{m} \sum_{l \neq j} \partial_{(\xi_j)} G(\xi_l, \xi_l) + \frac{8 \pi}{m} \sum_{l \neq j} \partial_{(\xi_j)} H(\xi_l, \xi_l) + O(\delta^2 |\log \delta|) = O(\delta^2 |\log \delta|)
\]

(42)

in view of \( G(\xi_l, \xi_l) = H(\xi_l, \xi_l) \) for \( l \neq j \). In conclusion, by (41)–(42) we can write

\[
\partial_{(\xi_j)} \left[ \delta_{m}(W) \right] = -32\pi^2 \partial_{(\xi_j)} \phi_{m}(\xi) + O(\delta^2 |\log \delta|).
\]

(43)

By (25) we have that \( \partial_{(\xi_j)} \left[ \delta_{m}(W) \right] = \partial_{(\xi_j)} \left[ \delta_{m}(W) \right] + O(\delta^2 |\log \delta|) \), and the proof is complete. \( \square \)

Finally, we address the expansions for the derivatives of \( f_{\alpha}(W) \) in \( \delta \).

**Proof of (35) in \( C^{2}(\mathbb{R}) \).** We just focus on the first and second derivatives of \( f_{\alpha}(W) \) in \( \delta \). Since \( \partial_{\delta} = \frac{3}{2} \delta_{\xi} \partial_{\nu} \) in view of (24), arguing as in Lemma 2.1, it is easy to show that

\[
\rho_{\delta}^{-1}(\xi) \partial_{\delta} W_i = -8\pi \left[ \int \chi(\xi) \psi_{\delta}(\xi) + \partial_{\nu} \left( \int \chi(\xi) \partial_{\nu} \psi_{\delta}(\xi) \right) dy \right] - 4 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + O(\delta^3 |\log \delta|)
\]

(44)

\[
\rho_{\delta}^{-1}(\xi) \partial_{\delta} W_i = 4 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + 2 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + O(\delta^3 |\log \delta|)
\]

(45)

do hold uniformly in \( S \), where

\[
\beta_{\delta, \xi} = -8\pi \left[ \int \chi(\xi) \psi_{\delta}(\xi) + \partial_{\nu} \left( \int \chi(\xi) \partial_{\nu} \psi_{\delta}(\xi) \right) dy \right] - 4 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + O(\delta^3 |\log \delta|)
\]

and

\[
\gamma_{\delta, \xi} = -8\pi \left[ \int \chi(\xi) \psi_{\delta}(\xi) + \partial_{\nu} \left( \int \chi(\xi) \partial_{\nu} \psi_{\delta}(\xi) \right) dy \right] - 4 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + O(\delta^3 |\log \delta|)
\]

By Lemma 3.1 we then have that

\[
\rho_{\delta}^{-1}(\xi) \int \frac{k e^{W} e^{\nu} \nu \partial_{\nu} W_i dv_g}{S} = -\int \frac{k e^{W} e^{\nu} \nu \partial_{\nu} W_i dv_g}{S} + 8\pi \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + O(\delta^3 |\log \delta|)
\]

(46)

\[
\rho_{\delta}^{-1}(\xi) \partial_{\delta} W_i = 4 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + 2 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + O(\delta^3 |\log \delta|)
\]

and

\[
\rho_{\delta}^{-1}(\xi) \partial_{\delta} W_i = 4 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + 2 \delta_{\xi} \nu \chi(\xi) \psi_{\delta}(\xi) + O(\delta^3 |\log \delta|)
\]
in view of \( \int_{\mathbb{R}^2} \frac{|y|^2 - 1}{(1 + |y|^2)^2} \, dy = 0 \), where \( \delta \) denotes the Kronecker symbol. Since \( \int_S \partial_\delta W d\nu_g = \int_S \partial_\Delta W d\nu_g = 0 \), we then deduce the following expansions:

\[
\int_S (-\Delta_g W) \partial_\delta W d\nu_g = \sum_{j, l = 1}^{m} \int_S \chi_j e^{-\psi_j} e^{\nu_j} \partial_\delta W d\nu_g
\]

\[
= - \frac{16\pi m}{\delta} + 8\pi m \sum_{l = 1}^{m} \rho_l^{\frac{1}{2}} (\xi_l) \beta_{j, l} - 32\pi \delta \sum_{j, l = 1}^{m} \rho_l (\xi_l) F_{\xi_l} (\xi_l) + O(|\delta|^2 \log |\delta|^2),
\]

(46)

\[
\int_S (-\Delta_g W) \partial_\delta W d\nu_g = \sum_{j, l = 1}^{m} \int_S \chi_j e^{-\psi_j} e^{\nu_j} \partial_\delta W d\nu_g
\]

\[
= \frac{16\pi m}{3\delta^2} + 8\pi m \sum_{l = 1}^{m} \rho_l (\xi_l) \gamma_{j, l} - 32\pi \sum_{j, l = 1}^{m} \rho_l (\xi_l) F_{\xi_l} (\xi_l) + O(|\delta|^2 \log |\delta|^2),
\]

(47)

and

\[
\int_S -\Delta_g (\partial_\delta W) \partial_\delta W d\nu_g = \sum_{j, l = 1}^{m} \int_S \chi_j e^{-\psi_j} e^{\nu_j} \partial_\delta U_{j, l} \partial_\delta W d\nu_g = \frac{32\pi m}{3\delta^2} + O(\delta^\nu)
\]

(48)

as \( \delta \to 0 \). Since by Lemma 2.1 there hold

\[
ke^W = \frac{e^{\alpha_{x, l} - 2\alpha_{x, l} (x)}}{8\delta^2} \rho_j e^{\nu_j} [1 + O(|\delta|^4 \log |\delta|)]
\]

uniformly in \( B_{\delta_0} (\xi_l) \) and \( ke^W = O(1) \), \( \partial_\delta W = O(\delta \log \delta) \) uniformly in \( S \setminus \bigcup_{j=m}^{m} B_{\delta_0} (\xi_l) \), by Lemma 3.1 we can write that

\[
\int_S ke^W \partial_\delta W d\nu_g = \sum_{j, l = 1}^{m} \int_{B_{\delta_0} (\xi_l)} ke^W \partial_\delta W d\nu_g + O(\delta \log \delta)
\]

\[
= - \frac{m e^{\alpha_{x, l}}}{2\delta} \int_{B_{\delta_0} (\xi_l)} e^{-2\alpha_{x, l} (x)} \frac{\rho_j e^{\nu_j}}{\delta^2 + |y_l (x)|^2} d\nu_g + \pi \frac{e^{\alpha_{x, l}}}{\delta^2}
\]

\[
\times \left( m \sum_{l = 1}^{m} \rho_l^{\frac{1}{2}} (\xi_l) \beta_{j, l} - 4 \sum_{j, l = 1}^{m} \rho_l^{\frac{1}{2}} (\xi_l) F_{\xi_l} (\xi_l) \right) + O(\delta \log \delta)
\]

\[
= - \frac{m e^{\alpha_{x, l}}}{2\delta} \left( \frac{4\pi}{\delta^2} e^{-2\alpha_{x, l} (\xi_l)} + \pi (\Delta_g \rho_j (\xi_l) - 2K (\xi_l) \rho_j (\xi_l)) \right)
\]

\[
+ \pi \frac{e^{\alpha_{x, l}}}{\delta^2} \left( m \sum_{l = 1}^{m} \rho_l^{\frac{1}{2}} (\xi_l) \beta_{j, l} - 4 \sum_{j, l = 1}^{m} \rho_l^{\frac{1}{2}} (\xi_l) \delta F_{\xi_l} (\xi_l) \right) + O(\delta^{-1+\nu})
\]

\[
= \pi \frac{e^{\alpha_{x, l}}}{\delta^2} \left( -\frac{2m}{\delta} + m \sum_{l = 1}^{m} \rho_l^{\frac{1}{2}} (\xi_l) \beta_{j, l} - \frac{\delta}{8\pi} A (\xi_l) + O(\delta^{1+\nu}) \right)
\]

in view of (37) and

\[
\frac{1}{\delta} \sum_{j=1}^{m} e^{-2\alpha_{x, l} (\xi_l)} = \frac{m}{\delta} - \frac{m}{\delta} \sum_{j=1}^{m} \delta^2 F_{\xi_l} (\xi_l) + O(\delta^3) = \frac{m}{\delta} \sum_{j=1}^{m} \rho_l^{\frac{1}{2}} (\xi_l) \delta F_{\xi_l} (\xi_l) + O(\delta^3).
\]

(49)

Combining with (38) we then get that

\[
\int_S ke^W \partial_\delta W d\nu_g - \frac{2}{\delta} \sum_{l = 1}^{m} \rho_l^{\frac{1}{2}} (\xi_l) \beta_{j, l} - \delta \frac{A (\xi_l)}{4\pi m} \delta \log \delta + \delta \frac{B (\xi_l)}{4\pi m} \delta - \frac{4m}{m \delta} \sum_{j=1}^{m} F_{x, \xi_l} (\xi_l) + o(\delta),
\]

(50)

which yields to

\[
\partial_\delta [U_{8\pi m} (W)] = \int_S (-\Delta_g W) \partial_\delta W d\nu_g - 8\pi m \int_S ke^W \partial_\delta W d\nu_g
\]

\[
= 2A (\xi_l) \delta \log \delta + [A (\xi_l) - 2B (\xi_l)] \delta + o(\delta).
\]

(51)
Since by (25) and (50) there holds
\[
- (\lambda - 8\pi m) \int_S k e^W \partial_3 W d\nu_g = \frac{2(\lambda - 8\pi m)}{\delta} + O(\delta^2 |\log \delta|^2),
\]
by (51) we deduce the validity of (35) for the first derivative in $\delta$.

Towards the expansion of the second derivative, we proceed in a similar way with the aid of the expansion for $\partial_{33} W$. Since
\[
ke^W = \frac{e^{\alpha_{3,3} - 2F_{3,3}(x)}}{8\delta^2} \rho \rho e^{\gamma} [1 + O(\delta^4 |\log \delta)|]
\]
and $ke^W = O(1)$, $\partial_{33} W + (\partial_3 W)^2 = O(|\log \delta|)$ do hold uniformly in $B_0(\xi_j)$ and $S \setminus \cup_{j=1}^m B_0(\xi_j)$, respectively, by Lemma 3.1 we can write that
\[
\int_S ke^W [\partial_3 W + (\partial_3 W)^2] d\nu_g = \sum_{j=1}^m \int_{B_0(\xi_j)} ke^W [\partial_3 W + (\partial_3 W)^2] d\nu_g + O(|\log \delta|)
\]
\[
= \sum_{j=1}^m \frac{e^{\alpha_{3,3} - 2F_{3,3}(x)}}{2\delta^2} \int_{B_0(\xi_j)} e^{-2F_{3,3}(x)} \rho \rho e^{\gamma} \left( \frac{5\delta^2 - (|y_3(x)|^2}{(\delta^2 + |y_3(x)|^2)^2} \right) d\nu_g
\]
\[
- \sum_{j=1}^m \frac{e^{\alpha_{3,3} - 2F_{3,3}(x)}}{\delta^2} \int_{B_0(\xi_j)} e^{-2F_{3,3}(x)} \rho \rho e^{\gamma} \left( \sum_{l=1}^m \rho_1^2 (\xi_l) \beta_{j,\xi_l} - 4\delta \sum_{l=1}^m \rho_1 (\xi_l) F_{3,l}(\xi_l) \right) d\nu_g
\]
\[
+ \pi \frac{e^{\alpha_{3,3} - 2F_{3,3}(x)}}{\delta^2} \left( m \sum_{l=1}^m \rho_1 (\xi_l) g_{j,\xi_l} - 4 \sum_{j=1}^m \rho_1 (\xi_l) F_{3,l}(\xi_l) \right) + O(|\log \delta|^2),
\]
and then
\[
\frac{\delta^2 e^{-\alpha_{3,3} - 2F_{3,3}(x)}}{\pi} \int_S ke^W [\partial_3 W + (\partial_3 W)^2] d\nu_g = \sum_{j=1}^m \left( \frac{6}{\delta^2} \frac{e^{-2F_{3,3}(x)}}{2} + \frac{1}{2} (\Delta_3 \rho_j (\xi_j) - 2K (\rho_j (\xi_j)) \right)
\]
\[- \frac{4}{\delta} \sum_{j=1}^m \left( m \sum_{l=1}^m \rho_1^2 (\xi_l) \beta_{j,\xi_l} - 4\delta \sum_{l=1}^m \rho_1 (\xi_l) F_{3,l}(\xi_l) \right) + m \sum_{l=1}^m \rho_1 (\xi_l) g_{j,\xi_l} - 4 \sum_{j=1}^m \rho_1 (\xi_l) F_{3,l}(\xi_l) + O(\delta^{\nu})
\]
\[
= \frac{6m}{\delta^2} + m \sum_{l=1}^m \rho_1 (\xi_l) g_{j,\xi_l} - 4 \sum_{j=1}^m \rho_1^2 (\xi_l) \beta_{j,\xi_l} + \frac{A(\xi)}{4\pi m} + \frac{3B(\xi)}{4\pi m} + \frac{12}{m\delta^2} \sum_{j=1}^m F_{3,3}(\xi_j) + o(1).
\]
Combining with (38) we then get that
\[
\int_S ke^W [\partial_3 W + (\partial_3 W)^2] d\nu_g = \frac{6}{\delta^2} + \frac{3A(\xi)}{4\pi m} \log \delta + \sum_{l=1}^m \rho_1 (\xi_l) g_{j,\xi_l}
\]
\[- \frac{4}{\delta} \sum_{j=1}^m \rho_1^2 (\xi_l) \beta_{j,\xi_l} + \frac{A(\xi)}{4\pi m} + \frac{3B(\xi)}{4\pi m} + \frac{12}{m\delta^2} \sum_{j=1}^m F_{3,3}(\xi_j) + o(1).
\]
Since
\[
\partial_{33} [J_3(W)] = \int_S (-\Delta_3 W) \partial_{33} W d\nu_g - \lambda \frac{\int_S ke^W [\partial_3 W + (\partial_3 W)^2] d\nu_g}{\int_S ke^W d\nu_g}
\]
\[- \frac{4}{\delta^2} \sum_{j=1}^m \rho_1^2 (\xi_l) \beta_{j,\xi_l} + \frac{A(\xi)}{4\pi m} + \frac{3B(\xi)}{4\pi m} + \frac{12}{m\delta^2} \sum_{j=1}^m F_{3,3}(\xi_j) + o(1).
\]
by (47), (48), (50) and (52) we deduce that
\[
\partial_{33} [J_{3,m}(W)] = 2A(\xi) \log \delta + 3A(\xi) - 2B(\xi) + o(1).
\]
Since by (25), (50) and (52)
\[
(\lambda - 8\pi m) \left[ \frac{\int_S ke^W [\partial_3 W + (\partial_3 W)^2] d\nu_g}{\int_S ke^W d\nu_g} - \left( \frac{\int_S ke^W [\partial_3 W d\nu_g]}{\int_S ke^W d\nu_g} \right)^2 \right] = \frac{2(\lambda - 8\pi m)}{\delta^2} + O(\delta^2 |\log \delta|^2),
\]
by (53) we deduce the validity of (35) also for the second derivative in $\delta$, and the proof is complete. □

4. Variational reduction and the proof of main results

In the so-called nonlinear Lyapunov–Schmidt reduction, the first step is the solvability theory for the operator $L$ given in (27), obtained as the linearization of (1) at the approximating solution $W$. As $\delta \to 0$ observe that formally the operator $L$,
scaled and centered at 0 by setting \( y = y_{ij}(x)/\delta \), approaches \( \hat{L} \) defined in \( \mathbb{R}^2 \) as

\[
\hat{L}(\phi) = \Delta \phi + \frac{8}{(1 + |y|^2)^2} \left( \phi - \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\phi(z)}{(1 + |z|^2)^3} \, dz \right).
\]

Due to the intrinsic invariances, the kernel of \( \hat{L} \) in \( L^\infty(\mathbb{R}^2) \) is non-empty and is spanned by 1 and \( Y_j \), \( j = 0, 1, 2 \), where

\[
Y_1(y) = \frac{4y}{1 + |y|^2}, \quad i = 1, 2, \quad \text{and} \quad Y_0(y) = 2 \frac{1 - |y|^2}{1 + |y|^2}.
\]

Since [36,37] it is by now rather standard to show the invertibility of \( \hat{L} \), the functions \( \phi \) take the form \( \hat{L}(\phi) = \Delta \phi + \frac{8}{(1 + |y|^2)^2} \phi \) and the function 1 does not belong to its kernel, making possible to disregard the “dilation parameters” \( \delta \) in the reduction. As we will see, one additional parameter \( \delta \) is needed in the reduction and in this respect our problem displays a new feature w.r.t. Dirichlet Liouville-type equations, making our situation very similar to the one arising in the study of critical problems in higher dimension.

To be more precise, for \( i = 0 \), \( i = 1, 2 \) and \( j = 1, \ldots, m \), let \( PZ, PZ_{ij} \) be the projections of \( Z, Z_{ij} \) as the solutions in \( \hat{H} \) of

\[
\begin{align*}
\Delta_k PZ &= \chi_k \Delta_g Z - \frac{1}{|S|} \int_{S} \chi_k \Delta_k Z d\nu_k \\
\Delta_k PZ_{ij} &= \chi_k \Delta_g Z_{ij} - \frac{1}{|S|} \int_{S} \chi_k \Delta_k Z_{ij} d\nu_k.
\end{align*}
\]

In Appendix A we will prove the following result:

**Proposition 4.1.** There exists \( \delta_0 > 0 \) so that for all \( 0 < \delta \leq \delta_0, h \in C(S) \) with \( \int_{\hat{S}} h d\nu_k = 0 \), \( \xi \in \Xi \) there is a unique solution \( \phi \in \hat{H} \cap W^{2,2}(S) \) and \( c_0, c_j \in \mathbb{R} \) of

\[
\begin{align*}
L(\phi) &= h + c_0 \Delta_k PZ + \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} \Delta_k PZ_{ij} \quad \text{in} \ S \\
\int_{\hat{S}} \phi \Delta_k PZ d\nu_k &= \int_{\hat{S}} \phi \Delta_k PZ_{ij} d\nu_k = 0 \quad \forall i = 1, 2, j = 1, \ldots, m.
\end{align*}
\]

Moreover, the map \( (\delta, \xi) \mapsto (\phi, c_0, c_j) \) is twice-differentiable in \( \delta \) and one-differentiable in \( \xi \) with

\[
\begin{align*}
\|\phi\|_{\infty} &\leq C |\log \delta| \|h\|_{\ast}, \quad |c_0| + \sum_{i=1}^{m} \sum_{j=1}^{m} |c_{ij}| \leq C \|h\|_{\ast} \quad (55) \\
\|\partial_\delta \phi\|_{\infty} + \sum_{i=1}^{m} \sum_{j=1}^{m} \|\partial_{(\xi_i, \xi_j)} \phi\|_{\infty} + \frac{\delta}{|\log \delta|} \|\partial_\delta \phi\|_{\infty} &\leq C \frac{|\log \delta|^2}{\delta} \|h\|_{\ast} \quad (57)
\end{align*}
\]

for some \( C > 0 \).

Let us recall that \( u = W + \phi \) solves (1) if \( \phi \in \hat{H} \) does satisfy (26). Since the operator \( L \) is not fully invertible, in view of Proposition 4.1 one can solve the nonlinear problem (26) just up to a linear combination of \( \Delta_k PZ \) and \( \Delta_k PZ_{ij} \), as explained in the following (see Appendix B for the proof):

**Proposition 4.2.** There exists \( \delta_0 > 0 \) so that for all \( 0 < \delta \leq \delta_0, \xi \in \Xi \) problem

\[
\begin{align*}
L(\phi) &= -[R + N(\phi)] + c_0 \Delta_k PZ + \sum_{i=1}^{m} \sum_{j=1}^{m} c_{ij} \Delta_k PZ_{ij} \quad \text{in} \ S \\
\int_{\hat{S}} \phi \Delta_k PZ d\nu_k &= \int_{\hat{S}} \phi \Delta_k PZ_{ij} d\nu_k = 0 \quad \forall i = 1, 2, j = 1, \ldots, m
\end{align*}
\]

is well defined.
admits a unique solution \( \phi(\delta, \xi) \in \tilde{H} \cap W^{2,2}(S) \) and \( c_0(\delta, \xi), c_{ij}(\delta, \xi) \in \mathbb{R}, i = 1, 2 \) and \( j = 1, \ldots, m, \) where \( \delta_j > 0 \) are as in (24) and \( N, R \) are given by (28), (29), respectively. Moreover, the map \( (\delta, \xi) \mapsto (\phi(\delta, \xi), c_0(\delta, \xi), c_{ij}(\delta, \xi)) \) is twice-differentiable in \( \delta \) and one-differentiable in \( \xi \) with

\[
\|\phi\|_\infty \leq C \left( \|\log \delta\| \|\nabla \varphi_m(\xi)\|_g + \delta^{2-\sigma} \|\log \delta\|^2 \right) \tag{59}
\]
\[
\|\partial_\delta \phi\|_\infty + \sum_{i=1}^{2} \sum_{j=1}^{m} \|\partial_{\delta \theta_j}\phi\|_\infty \leq C \left( \|\log \delta\|^2 \|\nabla \varphi_m(\xi)\|_g + \delta^{1-\sigma} \|\log \delta\|^3 \right) \tag{60}
\]
\[
\|\partial_{\delta \delta} \phi\|_\infty \leq C \left( \|\log \delta\|^3 \|\nabla \varphi_m(\xi)\|_g + \delta^{-\sigma} \|\log \delta\|^4 \right). \tag{61}
\]

The function \( W + \phi(\delta, \xi) \) will be a true solution of (26) if \( \delta \) and \( \xi \) are such that \( c_0(\delta, \xi) = c_{ij}(\delta, \xi) = 0 \) for all \( i = 1, 2, \) and \( j = 1, \ldots, m. \) This problem is equivalent to finding critical points of the reduced energy \( E_\lambda(\delta, \xi) = J_\lambda(W + \phi(\delta, \xi)) \), where \( J_\lambda \) is given by (4), as stated in

**Lemma 4.3.** There exists \( \delta_0 \) such that, if \( (\delta, \xi) \in (0, \delta_0) \times \Xi \) is a critical point of \( E_\lambda, \) then \( u = W + \phi(\delta, \xi) \) is a solution of (1), where \( \delta_0 \) is given by (24).

Once Eq. (1) has been reduced to the search of c.p.'s for \( E_\lambda, \) it becomes crucial to show that the main asymptotic term of \( E_\lambda \) is given by \( J_\lambda(W), \) for which an expansion has been given in Theorem 3.2. More precisely, by the estimates in Appendix B we have that

**Theorem 4.4.** Assume (24)–(25). The following expansion does hold

\[
E_\lambda(\delta, \xi) = -8\pi \pi - \lambda \log(\pi \pi) - 32\pi \varphi_m(\xi) + 2(\lambda - 8\pi \pi) \log \delta + A(\xi) \delta^2 \log \delta - B(\xi) \delta^2 + o(\delta^2) + r_\lambda(\delta, \xi) \tag{62}
\]

in \( C^2(\mathbb{R}) \) and \( C^1(\Xi) \) as \( \delta \to 0^+, \) where \( \varphi_m(\xi), A(\xi) \) and \( B(\xi) \) are given by (5), (7) and (12), respectively. The term \( r_\lambda(\delta, \xi) \) satisfies

\[
|r_\lambda(\delta, \xi)| + \frac{\delta}{|\log \delta|} |\nabla r_\lambda(\delta, \xi)| + \frac{\delta^2}{|\log \delta|^2} |\partial_{\delta \delta} r_\lambda(\delta, \xi)| \leq C \delta^2 \|\log \delta\| |\nabla \varphi_m(\xi)\|_g^2 \tag{63}
\]

for some \( C > 0 \) independent of \( (\delta, \xi) \in (0, \delta_0) \times \Xi. \)

We are now in position to establish the main result stated in the Introduction.

**Proof of Theorem 1.5.** According to Lemma 4.3, we just need to find a critical point of \( E_\lambda(\delta, \xi). \) By Theorem 4.4 for \( \lambda > 8\pi \pi \) we have that

\[
\frac{(\partial_\delta \partial_\delta E_\lambda)}{\lambda - 8\pi \pi} \left( \sqrt{\lambda - 8\pi \pi} \frac{\mu, \xi}{\lambda - 8\pi \pi} \right) = 2 + A(\xi) \log(\lambda - 8\pi \pi) \mu^2 + 2A(\xi) \mu^2 \log \mu + (A(\xi) - 2B(\xi)) \mu^2 + o(1) + O \left( \mu^2 \|\log \left( \sqrt{\lambda - 8\pi \pi} \mu \right) \|^2 |\nabla \varphi_m(\xi)|_g^2 \right)
\]

and

\[
\frac{(\partial^2 \partial_\delta E_\lambda)}{\lambda - 8\pi \pi} \left( \sqrt{\lambda - 8\pi \pi} \frac{\mu, \xi}{\lambda - 8\pi \pi} \right) = -2 + A(\xi) \mu^2 \log(\lambda - 8\pi \pi) + 3 - 2B(\xi) \mu^2 + o(1) + O \left( \mu^2 \|\log \left( \sqrt{\lambda - 8\pi \pi} \mu \right) \|^3 |\nabla \varphi_m(\xi)|_g^2 \right)
\]

as \( \lambda \to 8\pi \pi. \) By assumption we can find \( d_0 > 0 \) small so that \( B(\xi) > 0 \) for all \( \xi \in U \) with \( |A(\xi)| \leq a_0. \) Let

\[ D_\lambda = \left\{ \xi \in U : |\nabla \varphi_m(\xi)|_g \leq \sqrt{2} \|\log(\lambda - 8\pi \pi)\|^{-1} \right\}
\]

and consider the interval

\[ I_\lambda = \left[ \frac{m_0}{\sqrt{\log(\lambda - 8\pi \pi)}}, M \right]. \]

where

\[ 0 < m_0 = \inf_{\xi \in U} |A(\xi)|^{-\frac{1}{2}}, \quad M = 2 \sup_{\xi \in U : |A(\xi)| \leq a_0} B^{-\frac{1}{2}}(\xi) < +\infty. \]

For \( \lambda \) close to \( 8\pi \pi \) and for all \( \xi \in D_\lambda, \) we have that

\[
\frac{(\partial_\delta \partial_\delta E_\lambda)}{\lambda - 8\pi \pi} \left( \sqrt{\lambda - 8\pi \pi} \frac{\mu, \xi}{\lambda - 8\pi \pi} \right) \bigg|_{\mu = \frac{m_0}{\sqrt{\log(\lambda - 8\pi \pi)}}} = 2 - A(\xi) m_0^2 (1 + o(1)) - 2B(\xi) \frac{m_0^2}{\|\log(\lambda - 8\pi \pi)\|} + o(1) > 0
\]
in view of $A(\xi)m_0^2 \leq 1$, and
\[
\frac{(\partial^2 \partial_{\xi} E_\lambda) (\sqrt{\lambda - 8\pi m} \mu, \xi)}{\lambda - 8\pi m} = 2 - A(\xi)M^2|\log(\lambda - 8\pi m)|(1 + o(1)) - 2B(\xi)M^2 + o(1) < 0
\]
since either $A(\xi) \geq a_0$ or $0 \leq A(\xi) \leq a_0, B(\xi)M^2 \geq 4$. Moreover, in $I_\lambda \times \mathcal{D}_\lambda$, we have that
\[
\frac{(\delta^2 \partial_{\xi} E_\lambda) (\sqrt{\lambda - 8\pi m} \mu, \xi)}{\lambda - 8\pi m} = -2 - A(\xi)\mu^2|\log(\lambda - 8\pi m)|(1 + o(1)) - 2B(\xi)\mu^2 + o(1) \leq -1
\]
since either $A(\xi) \geq a_0$ or $0 \leq A(\xi) \leq a_0, B(\xi) > 0$. So, for all $\lambda$ close to $8\pi m$ and $\xi \in \mathcal{D}_\lambda$ there exists a unique $\mu(\lambda, \xi) \in \text{Int} I_\lambda$ so that $\delta(\lambda, \xi) := \sqrt{\lambda - 8\pi m} \mu(\lambda, \xi)$ satisfies $\delta E_\lambda(\delta(\lambda, \xi), \xi) = 0$. Moreover, by the IFT the map $\xi \in \mathcal{D}_\lambda \to \delta(\lambda, \xi)$ is a $C^1$-function of $\xi$ with
\[
\partial_{\xi} \delta(\lambda, \xi) = \frac{-\partial^2 \partial_{\xi} E_\lambda(\delta(\lambda, \xi), \xi)}{\partial_{\xi} \partial_{\xi} E_\lambda(\delta(\lambda, \xi), \xi)} = O(|\log(\lambda - 8\pi m)|^{-3}).
\]
in view of $\mu^2(\lambda, \xi) |\partial_{\xi} \partial_{\xi} E_\lambda(\sqrt{\lambda - 8\pi m} \mu(\lambda, \xi), \xi)| \geq 1$ and $\partial_{\xi} E_\lambda(\delta, \xi) = O(\delta) |\log(\lambda - 8\pi m)|^2 + |\log \delta|^3|\nabla \varphi_m(\xi)|^2_{\xi}$ (as it can be easily shown by the methods in the proof of Theorem 3.2).

The aim now is to extend the map $\delta(\lambda, \xi)$ to the whole $U$ in a $C^1$-way. Letting $\eta \in C^\infty_0[-2, 2]$ be a cut-off function so that $\eta = 1$ in $[-1, 1]$, we define the $C^1$-extension $\tilde{\delta}$ of $\delta$ to $\mathcal{D}$ as
\[
\tilde{\delta}(\lambda, \xi) = \eta(|\log(\lambda - 8\pi m)|^2|\nabla \varphi_m(\xi)|^2_{\xi}) \delta(\lambda, \xi) + \sqrt{\lambda - 8\pi m} \left[1 - \eta\left(|\log(\lambda - 8\pi m)|^2|\nabla \varphi_m(\xi)|^2_{\xi}\right)\right]
\]
and $\tilde{E}_\lambda(\xi) = E_\lambda(\tilde{\delta}(\lambda, \xi), \xi)$. Since $|\partial_{\xi} \tilde{\delta}(\lambda, \xi)| = O(|\log(\lambda - 8\pi m)|^{-3})$, by Theorem 4.4 we have that
\[
\tilde{E}_\lambda(\xi) = -8\pi m - \lambda \log(\pi m) - 32\pi^2 \varphi_m(\xi) + O(|\log(\lambda - 8\pi m)|) \quad \text{uniformly in } \xi \in \mathcal{D}.
\]
and $\nabla_{\xi} \tilde{E}_\lambda(\xi) = \nabla_{\xi} E_\lambda(\tilde{\delta}(\lambda, \xi), \xi) + \partial_{\xi} E_\lambda(\tilde{\delta}(\lambda, \xi), \xi) |\partial_{\xi} \tilde{\delta}(\lambda, \xi)| = -32\pi^2 \nabla \varphi_m(\xi) + O\left(\sqrt{\lambda - 8\pi m} |\log(\lambda - 8\pi m)|^2\right)
\]
and then $\xi_\lambda \in \mathcal{D}_\lambda$. Moreover $\tilde{\delta}(\lambda, \xi) = \delta(\lambda, \xi)$ satisfies $\partial_{\xi} E_\lambda(\delta(\lambda, \xi), \xi) = 0$, and then $\nabla_{\xi} \tilde{E}_\lambda(\xi) = 0$ is equivalent to $\nabla_{\xi} E_\lambda(\delta(\lambda, \xi), \xi) = 0$. In conclusion, up to take $U$ smaller so that $\nabla \varphi_m(\xi) \neq 0$ for all $\xi \in U \setminus \mathcal{D}$, the pair $(\delta(\lambda, \xi), \xi)$ is a c.p. of $E_\lambda(\xi)$ and, along a sub-sequence, $\xi_\lambda \rightarrow q \in \mathcal{D}$ as $\lambda \rightarrow 8\pi m$. By construction, the corresponding solution has the required asymptotic properties.

Remark 4.5. (i) The validity of condition (13) just on $\mathcal{D}$ is enough to provide Theorem 1.5 in the case of $\mathcal{D} = \{\xi_0\}$, where $\xi_0$ is a non-degenerate local minimum of $\varphi_m$. In this case, we just consider a small ball $B_{\xi_0}(\xi_0)$ as $\mathcal{D}_\lambda$, with $s_0 = |\log(\lambda - 8\pi m)|^{-1}$. Since $A(\xi_0) \geq 0$ and $\nabla \varphi_m(\xi_0) = 0$ we have that $\Lambda(\xi) \geq -C_0s_0$ and $|\nabla \varphi_m(\xi)|_{\xi} \leq C_0s_0$, for all $\xi \in B_{\xi_0}(\xi_0)$ and some $C_0 > 0$. Since $B(\xi) > 0$ for all $\xi \in B_{\xi_0}(\xi_0)$ if $A(\xi_0) = 0$, it is easy to see as before that for $\lambda$ close to $8\pi m$ and for all $\xi \in B_{\xi_0}(\xi_0)$
\[
\partial_{\xi} E_\lambda(\sqrt{\lambda - 8\pi m} \mu, \xi) \big|_{\mu = m_0} > 0, \quad \partial_{\xi} E_\lambda(\sqrt{\lambda - 8\pi m} \mu, \xi) \big|_{\mu = M} < 0
\]
with
\[
\partial_{\xi} E_\lambda(\sqrt{\lambda - 8\pi m} \mu, \xi) \leq -\frac{1}{\mu^2}
\]
in $I_\lambda \times B_{\xi_0}(\xi_0)$. So, for all $\lambda$ close to $8\pi m$ we can still find a $C^1$-map $\xi \in B_{\xi_0}(\xi_0) \to \delta(\lambda, \xi)$ so that $\partial_{\xi} E_\lambda(\delta(\lambda, \xi), \xi) = 0$. Setting $\tilde{E}_\lambda(\xi) = E_\lambda(\delta(\lambda, \xi), \xi)$ for $\xi \in B_{\xi_0}(\xi_0)$, by Theorem 4.4 we have that
\[
\tilde{E}_\lambda(\xi) = -8\pi m - \lambda \log(\pi m) - 32\pi^2 \varphi_m(\xi) + O(|\log(\lambda - 8\pi m)|).
\]
Since by the non-degeneracy of $\xi_0$ we have that $\partial B_{\xi_0}(\xi_0) \neq 0$ and $\varphi_m(\xi) \geq \varphi_m(\xi_0) + C_1s_0^2/\varphi_m(\xi_0) \leq \varphi_m(\xi_0) - C_1s_0^2$ for some $C_1 > 0$, we can find an interior minimum/maximum point $\xi_\lambda \in B_{\xi_0}(\xi_0)$ of $\tilde{E}_\lambda(\xi)$ on $B_{\xi_0}(\xi_0)$. By $\partial_{\xi} E_\lambda(\delta(\lambda, \xi), \xi) = 0$, we also deduce that $\nabla_{\xi} E_\lambda(\delta(\lambda, \xi), \xi) = 0$, and the pair $(\delta(\lambda, \xi), \xi)$ is the c.p. of $E_\lambda(\delta, \xi)$ we were searching for.

(ii) If (13) holds just in $\mathcal{D}$, Theorem 1.5 is also valid in the special case $A(\xi) = O(|\nabla \varphi_m(\xi)|_{\xi})$. Indeed, condition (13) reduces to $B(\xi) > 0 (< 0)$ on $\mathcal{D}$ and in $\mathcal{D}_\lambda$ we have that $A(\xi) \geq -C_0|\log(\lambda - 8\pi m)|^{-3}$ for some $C_0 > 0$. Similarly as in
point (i), it is still possible to define the map \( \xi \in \mathcal{D}_\lambda \rightarrow \delta(\lambda, \xi) \), and the remaining argument in the proof of Theorem 1.5 works also in this case by extending \( \delta(\lambda, \xi) \) on a small neighborhood \( U \) of \( \mathcal{D} \) in \( S^m \setminus \Delta \).

5. Proof of Theorem 1.6

In this section, we shall study the existence of non-topological solutions of (14). To this purpose we look for a solution to the equivalent problem (15) of the form \( w = u + c_\epsilon(u) \) with \( \int_T u = 0 \) and we are lead to study (16). Assume that \( N \) is even, so that Eq. (16) is a perturbation of (1) \( \lambda = 8\pi m \) with \( m = \frac{N}{2} \). Notice that the energy functional of (15) is given by

\[
\tilde{I}_e(u) = \frac{1}{2} \int_T |\nabla w|^2 + \frac{1}{2\epsilon^2} \int_T (ke^w - 1)^2 + 4\pi N \frac{|T|}{|T|} \int_T w, \quad w \in H^1(T).
\]

Introduce the notation \( C(u) := 16\pi N \int_T \frac{k^2e^{2u}}{(|T|ke^w)^2} \), so that \( e^{c_\epsilon(u)} = \frac{8\pi N e^2}{f_\epsilon k^2e^{2w}} \)

\[
l_e(u) := \tilde{I}_e(u + c_\epsilon(u)) = J_{4\pi N}(u) - 4\pi N \log \left( 1 + \sqrt{1 - e^{2C(u)}} \right) - \frac{4\pi N}{1 + \sqrt{1 - e^{2C(u)}}} + 4\pi N \log(8\pi Ne^2) + \frac{|T|}{2\epsilon^2} - 2\pi N.
\]

(64)

Hence, if \( u \in \mathcal{A}_c = \{ u \in \tilde{H} \mid |C(u) - 1 | \leq 1 \} \) is a critical point of \( I_e \) with \( e^{2C(u)} < 1 \), then \( u + c_\epsilon(u) \) is a solution to (15) and \( u \) is a solution to (16). Observe that \( I_e \) is a perturbation of \( J_{8\pi m} \) as \( \epsilon \to 0^+ \), in view of \( 4\pi N = 8\pi m \).

Given \( m \) distinct points \( \xi_j \in T \setminus \{ p_1, \ldots, p_i \}, j = 1, \ldots, m \), we will define \( \delta_j \) according to (24) and assume

\[
\exists C > 1 : \epsilon \leq C \delta^2.
\]

(65)

Letting \( W(x) = \sum_{j=1}^m W_j(x) \), we look for a solution of (16) in the form \( u = W + \phi \), for some small remainder term \( \phi \). In terms of \( \phi \), problem (16) is equivalent to find \( \phi \in \tilde{H} \) so that \( W + \phi \in \mathcal{A}_c \) and

\[
L^*(\phi) = -[R^* + N^*(\phi)] \quad \text{in} \quad T.
\]

(66)

The linear operator \( L^* \) is defined as

\[
L^*(\phi) = \Delta \phi + 4\pi N \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} \left( \phi - \frac{f_\epsilon k^2e^{2w}}{f_\epsilon k^2e^{2w}} \right) + \Lambda^e(\phi),
\]

where

\[
\Lambda^e(\phi) = \frac{4\pi Ne^2(C(W))}{\left(1 + \sqrt{1 - e^{2C(W)}} \right)^2} \left( \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} \right)^2 \left[ \phi - \frac{f_\epsilon k^2e^{2w}}{f_\epsilon k^2e^{2w}} \right]
\]

\[
+ \frac{e^{2C(W)}}{\left(1 + \sqrt{1 - e^{2C(W)}} \right)^2} \frac{f_\epsilon k^2e^{2w}}{f_\epsilon k^2e^{2w}} \left[ \phi - \frac{f_\epsilon k^2e^{2w}}{f_\epsilon k^2e^{2w}} \right]
\]

\[
+ \frac{4\pi N e^{2}C(W)}{f_\epsilon k^2e^{2w}} \frac{e^{2C(W)}}{\left(1 + \sqrt{1 - e^{2C(W)}} \right)^2} \left( \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} - \frac{f_\epsilon k^2e^{2w}}{f_\epsilon k^2e^{2w}} \right).
\]

Observe that \( L^* \) is defined for all \( \phi \in \tilde{H} \). The nonlinear part \( N^* \) is well-defined for \( \phi \in \tilde{H} \) such that \( W + \phi \in \mathcal{A}_c \) and is given by

\[
N^e(\phi) = 4\pi N \left( \frac{k^2e^{2W + \phi}}{f_\epsilon k^2e^{2w + \phi}} - \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} + \frac{ke^W}{f_\epsilon k^2e^{2w}} \right) - \phi \Lambda^e(\phi)
\]

\[
+ \frac{4\pi N e^{2C(W + \phi)}}{\left(1 + \sqrt{1 - e^{2C(W + \phi)}} \right)^2} \left( \frac{k^2e^{2(W + \phi)}}{f_\epsilon k^2e^{2w + \phi}} - \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} \right)
\]

\[
- \frac{4\pi N e^{2C(W)}}{\left(1 + \sqrt{1 - e^{2C(W)}} \right)^2} \left( \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} - \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} \right).
\]

(67)

The approximation rate of \( W \) becomes

\[
R^e = \Delta W + 4\pi N \left( \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} - \frac{1}{|T|} \right) + \frac{4\pi Ne^{2C(W)}}{\left(1 + \sqrt{1 - e^{2C(W)}} \right)^2} \left( \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} - \frac{k^2e^{2W}}{f_\epsilon k^2e^{2w}} \right).
\]

(68)

We have that
Lemma 5.1. Let \( N \) be an even number and \( m = \frac{N}{2} \). Assume (24) and (65). There exists a constant \( C > 0 \), independent of \( \delta > 0 \) small, such that for all \( \xi \in \mathbb{R} \)
\[
\| R^\prime \|_* \leq C \left( \delta |\nabla \varphi_m(\xi)|_g + \delta^{2-\sigma} \right). \tag{69}
\]

Proof. First, note that \( R^\prime = R_{8 \pi m} + \frac{8 \pi m e^2 C(W)}{1 + \sqrt{1 - e^2 C(W)}} \left( \frac{k^2 e^{2W}}{f_T k^2 e^{2W}} - \frac{k^2 e^{2W}}{f_T k^2 e^{2W}} \right) \) in view of \( N = 2m \). As in (31) we have that
\[
\int_T k^2 e^{2W} \, dx = \sum_{j=1}^{m} \frac{1}{64 \delta^2} \int_{B_\delta(\xi_j)} \rho_j^2(x)e^{2U_j}(1 + O(\delta^2 |\log \delta|)) \, dx + O(1)
\]
which implies by (65) that for \( \epsilon \) and \( \delta \) sufficiently small \( W \in \mathcal{A}_e \) and
\[
\frac{8 \pi m e^2 C(W)}{1 + \sqrt{1 - e^2 C(W)}} = 2 \pi m e^2 C(W) + O(\epsilon^2 C(W)) = O\left( \frac{\epsilon^2}{\delta^2} \right).
\]
Therefore, by using (32) and the estimate on \( \int_T \frac{k^2 e^{2W}}{f_T k^2 e^{2W}} \) we find the following estimate
\[
R^\prime - R_{8 \pi m} = O\left( \frac{\epsilon^2}{\delta^2} \left[ \sum_{j=1}^{m} \sqrt{x_j U_j} + \delta^2 \right] \right),
\]
and then \( \| R^\prime - R_{8 \pi m} \|_* = O(\epsilon^2 \delta^{-2}) \). Thus, in view of (33) and (65) the conclusion follows. \( \square \)

Now, we are going to establish the expansion of \( I_\epsilon(W) \).

Theorem 5.2. Assume (24) and (65). The following expansion does hold
\[
I_\epsilon(W) = -16 \pi m + 8 \pi m \log(8e^2) + \frac{|T|}{2e^2} - 32 \pi^2 \varphi_m(\xi) + A(\xi) \delta^2 \log \delta - B(\xi) \delta^2 + \tilde{B}(\xi) \frac{\epsilon^2}{\delta^2} + o(\delta^2) \tag{71}
\]
in \( C^2(\mathbb{R}) \) and \( C^1(\mathbb{R}) \) as \( \delta \to 0^+ \), where \( \varphi_m(\xi) \), \( A(\xi) \) and \( B(\xi) \) are given by (5), (7) and (12), respectively, and
\[
\tilde{B}(\xi) = \frac{32 \pi^2}{3} \sum_{j=1}^{m} \frac{1}{\rho_j(\xi_j)} \tag{72}
\]

Proof. By (70) we have that
\[
1 + \sqrt{1 - \epsilon^2 C(W)}^{-1} = \frac{1}{2} + \frac{\epsilon^2}{2} C(W) + O\left( \frac{\epsilon^4}{\delta^4} \right), \quad \log\left( 1 + \sqrt{1 - \epsilon^2 C(W)} \right) = \log 2 - \frac{\epsilon^2}{4} C(W) + O\left( \frac{\epsilon^4}{\delta^4} \right).
\]
Hence, by using (64) we find that
\[
I_\epsilon(W) = I_{b,m}(W) + \pi m e^2 C(W) + 8 \pi m \log(8 \pi m e^2) + \frac{|T|}{2e^2} - 8 \pi m + O(\epsilon^4 \delta^{-4}).
\]
Thus, the expansion (71) follows by (39), (65) and $C(W) = \frac{1}{2} m s^2 |B(\xi)|\left(1 + O(\delta^2 \log \delta)\right)$ in view of (70). Finally, the expansions for the derivatives follow similarly as in the proof of Theorem 3.2, in view of

$$\partial_\beta [I_0(W) - I_{\delta \tau m}(W)] + \frac{4\pi me^2 \partial_\beta [C(W)]}{\left(1 + \sqrt{1 - e^2 C(W)}\right)^2}$$

for either $\beta = (\xi), \nu = \delta$, and

$$\partial_\beta [I_{\delta \tau m}(W)] + \frac{me^2 \partial_\beta [C(W)]}{\left(1 + \sqrt{1 - e^2 C(W)}\right)^2} + O(e^4 C(W) \partial_\beta [C(W)])$$

by using (43), (51), (53) and the expansions for the derivatives of $C(W)$ in the line of (70). □

Remark 5.4. Notice that if $\|\phi\|_* \leq 1/|\log \delta| \log \delta$ then $W + \phi \in \mathcal{A}_e$ for $\delta$ and $e$ small enough.

The function $\phi(\delta, \xi)$ will be a solution to (66), namely, $W + \phi(\delta, \xi)$ will be a true solution of (16) if $\delta$ and $\xi$ are such that $c_0(\delta, \xi) = c_0(\delta, \xi) = 0$ for all $i = 1, 2, j = 1, \ldots, m$. Similarly to Lemma 4.3, this problem is equivalent to finding critical points of the reduced energy $E^e(\delta, \xi) = I_e(W + \phi(\delta, \xi))$.

Theorem 5.5. Assume (24) and (65). The following expansion does hold

$$E^e(\delta, \xi) = -16\pi m + 8\pi m \log(8\epsilon^2) + \frac{|T|}{2\epsilon^2} - 32\pi^2 \varphi_m(\xi) + A(\xi)\delta^2 \log \delta - B(\xi)\delta^2 + \hat{B}(\xi) \frac{\epsilon^2}{\delta^2} + o(\delta^2) + r^e(\delta, \xi)$$

in $C^2(\mathbb{R})$ and $C^1(\mathbb{R})$ as $\delta \to 0^+$, where $\varphi_m(\xi)$, $A(\xi)$, $B(\xi)$ and $\hat{B}(\xi)$ are given by (5), (7), (12) and (72), respectively. The term $r^e(\delta, \xi)$ satisfies for some $C > 0$ independent of $\delta, \xi$

$$|r^e(\delta, \xi)| + \frac{\delta}{|\log \delta|} |\nabla r^e(\delta, \xi)| + \frac{\delta^2}{|\log \delta|^2} |\partial_\delta r^e(\delta, \xi)| \leq C|\log \delta| \nabla \varphi_m(\xi)^2.$$  

Proof of Theorem 1.6. Similarly to Theorem 1.5, to find a critical point of $E^e(\delta, \xi)$ the key step is to get the existence of a function $\delta = \delta(\epsilon, \xi) = \sqrt{\epsilon}\mu(\epsilon, \xi)$ such that $\partial_\delta E^e(\delta(\epsilon, \xi), \xi) = 0$ in a small neighborhood of the critical set $\mathcal{D}$. Even if $A(\xi) \geq 0$ for all $\xi \in (T \setminus \{p_1, \ldots, p_l\})^m \setminus \Delta$, this is possible in view of $B(\xi) > 0$ and “the correct sign” $B(\xi) < 0$ in $\mathcal{D}$. The argument is based on the same one explained in Remark 4.5(ii) and uses the crucial smallness property of $A(\xi)$ near $\mathcal{D}$: $A(\xi) = O(\nabla \varphi_m(\xi)^2)$. □

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Appendix A

We consider the operator

$$L_{g,\omega}(\phi) = \Delta_g \phi + \frac{8\pi m k e^W}{j \omega} v_g \left( \phi - \frac{\int_S k e^W \phi v_g}{\int_S k e^W dv_g} \right),$$

for which we first address a-priori estimates when all the $c_j$’s vanish:

**Proposition A.1.** There exists $\delta_0 > 0$ and $C > 0$ so that, for all $0 < \delta \leq \delta_0$, $h \in C(S)$ with $\int_S h dv_g = 0$, $\xi \in C$ and $\phi \in H^4_0(S) \cap W^{2,2}(S)$ a solution of (55) with $L = L_{g,\omega}$ and $c_0 = c_j = 0$, $i = 1, 2$ and $j = 1, \ldots, m$, one has

$$\|\phi\|_\infty \leq C|\log \delta|\|h\|_*.$$  \hfill (73)

**Proof.** By contradiction, assume the existence of sequences $\delta \to 0$, points $\xi \in C$ with $\xi \to \xi^*$, functions $h$ with $|\log \delta|\|h\|_* = o(1)$ and solutions $\phi$ with $\|\phi\|_\infty = 1$. Recall that $\delta_0 = \delta^2 \rho_0(\xi)$, and setting $K = \frac{8\pi m k e^W}{j \omega} v_g$ and $\psi = \phi - \frac{\int_S k e^W \phi v_g}{\int_S k e^W dv_g}$, we have that $\Delta_g \psi + K \psi = h$ in $S$ and $\psi$ does satisfy the same orthogonality conditions as $\phi$.

Since $\|\phi\|_\infty \leq 2\|\phi_0\|_\infty \leq 2$ and $\Delta_g \psi = o(1)$ in $C_0(S \setminus \{\xi_1^* \ldots, \xi_n^*\})$, we can assume that $\psi \to \psi_\infty$ in $C^1_0(S \setminus \{\xi_1^* \ldots, \xi_n^*\})$.

Since $\psi_\infty$ is bounded, it extends to a harmonic function in $S$, and then $\psi_\infty = c_0 := -\lim_{\Delta_g} \frac{\int_S k e^W \phi v_g}{\int_S k e^W dv_g}$ in view of

$$\frac{1}{j \omega} \int_S \psi dv_g = -\frac{\int_S k e^W dv_g}{\int_S k e^W dv_g}.$$

The function $\psi_1 = \psi(\nu_1^* \delta y)$ satisfies $\Delta \psi_1 + K \psi_1 = h_1$ in $B_{2\delta}^{\nu_1}(0)$, where $\nu_1 = \delta^2 K(y_1^*(-\delta))$ and $h_1 = \delta^2 h(y_1^*(\delta))$.

Since $h_1 \leq C \|h\|_*$ and $K = \frac{8}{(1 + |r|^2)^2} (1 + O(\delta^2 |\log \delta|))$ uniformly in $B_{2\delta}^{\nu_1}(0)$ in view of Lemma 2.1 and (38), up to a subsequence, by elliptic estimates $\psi_1 \to \psi_{\infty_1}$ in $C^1_0(R^2)$, where $\psi_{\infty_1}$ is a bounded solution of $\Delta \psi_{\infty_1} + \frac{8}{(1 + |r|^2)^2} \psi_{\infty_1} = 0$ of the form $\psi_{\infty_1} = \sum_{i=0}^\infty a_i Y_i$ (see for example [35]). Since $\Delta_g \psi = \psi_1 \to \psi_{\infty_1}$ in $C^1_0(R^2)$, we have that

$$0 = -\int_S \psi_1 PZ dy = 32 \int_{|y|} |y| dy - \frac{32}{|S|} \int_{|y|} dy + O(\delta^2).$$

Since $\int_{|y|} \frac{|y|}{(1 + |y|^2)^2} dy = 0$, we deduce that $a_{ij} = a_{ij} = 0$. By the other orthogonality condition $\int_S \psi \Delta_g PZ = 0$ similarly we deduce that

$$0 = -\sum_{i=1}^m \int_S \psi_1 PZ dy = 16 \int_{|y|} \frac{1 - |y|^2}{(1 + |y|^2)^3} dy - 16 \int_{|y|} \frac{1 - |y|^2}{(1 + |y|^2)^3} dy + O(\delta^2),$$

which implies $\sum_{i=1}^m a_{ij} = 0$ in view of $\int_{|y|} \frac{1 - |y|^2}{(1 + |y|^2)^3} dy = 0$. By dominated convergence we have that

$$\int_S G(y, \xi) \mathcal{K} \psi dv_g = -\frac{1}{2\pi} \log \delta \int_{B_{2\delta}(\xi)} \mathcal{K} \psi dv_g + \int_{R^2} \left[ -\frac{1}{2\pi} \log |y| + H(\xi, \xi) \right] \frac{8}{(1 + |y|^2)^2} \psi_{\infty_1} dy$$

$$+ \sum_{ij} \int_{|y|} \frac{8}{(1 + |y|^2)^2} \psi_{\infty_1} dy + o(1)) = -\frac{1}{2\pi} \log \delta \int_{B_{2\delta}(\xi)} \mathcal{K} \psi dv_g + 4a_{ij} + o(1)$$

in view of $\int_{|y|} \log |y| \frac{1 - |y|^2}{(1 + |y|^2)^3} dy = -\frac{1}{2}$. In view of $\int_S \mathcal{K} \psi = 0$ and

$$\int_S G(y, \xi) h dv_g \leq C |\log \delta| \int_S |h| dv_g + \frac{\|h\|_*}{\delta^2} \int_{B_{2\delta}(\xi)} G(y, \xi) dv_g \leq C |\log \delta| \|h\|_* = o(1),$$

$$\int_S G(y, \xi) h dv_g \leq C |\log \delta| \int_S |h| dv_g + \frac{\|h\|_*}{\delta^2} \int_{B_{2\delta}(\xi)} G(y, \xi) dv_g \leq C |\log \delta| \|h\|_* = o(1),$$

in view of $\int_S G(y, \xi) h dv_g \leq C |\log \delta| \|h\|_* = o(1).$
by Green’s representation formula
\[ \sum_{j=1}^{m} \psi_j(0) = \sum_{j=1}^{m} \psi(\xi_j) = \frac{m}{|S|} \int_S \psi \, dv_g + \sum_{j=1}^{m} \int_S G(y, \xi_j) [\mathcal{K} \psi - h] \, dv_g = mc_0 + 4 \sum_{j=1}^{m} a_{0j} + o(1) \]
which gives \( \sum_{j=1}^{m} a_{0j} = mc_0 + 4 \sum_{j=1}^{m} a_{0j} \) as \( n \to +\infty \). Since \( \sum_{j=1}^{m} a_{0j} = 0 \), we get that \( c_0 = 0 \).

Following [37], let \( PZ_j \in H^1_0(S) \) be s.t. \( \Delta_g PZ_j = \chi_j \Delta_g Z_j - \frac{4}{|S|} \int_S \chi_j \Delta_g Z_j \, dv_g \) in \( S \), where
\[ Z_j(x) = \beta_j \left( \frac{\langle y_j, x \rangle}{\delta_j} \right), \quad \beta_j(y) = 4 \frac{\delta_j}{2 \log \delta_j + \log(1 + |y|^2)} \left[ \frac{1}{1 + |y|^2} + \frac{8}{3} \frac{1}{1 + |y|^2} \right] \]
satisfies \( e^{\delta_j} \Delta_g Z_j + e^{\delta_j} Z_j = e^{\delta_j} Z_{0j} \) in \( B_{2\delta_j}(\xi_j) \). Since it is easily seen that \( PZ_j = \chi_j Z_j + \frac{H(\cdot, \xi_j) + O(\delta^2 | \log \delta |)}{d - 1} \) uniformly in \( S \), we test the equation of \( \psi \) against \( PZ_j \) to get:
\[
\int_S \left[ \chi_j \Delta_g Z_j - \frac{1}{|S|} \int_S \chi_j \Delta_g Z_j \, dv_g \right] \, dv_g + \int_S \Delta_g \psi \, PZ_g \, dv_g = \int_S \chi_j \psi \, e^{\delta_j} \Delta_g Z_j + \mathcal{K} \psi \, dv_g + o(1)
\]
\[
= \int_S \chi_j \psi e^{\delta_j} Z_{0j} \, dv_g + o(1) = 16 \int_S \psi \frac{1 - |y|^2}{(1 + |y|^2)^3} \, dv_g + o(1) = \int_S hPZ_j = o(1)
\]
view of \( \int_S \psi \, dv_g = o(1) \). Since \( \int_S \chi_j \psi \, [\mathcal{K} - e^{\delta_j}] \, Z_j \, dv_g = O(\delta^2 | \log \delta |) \) and \( \int_S hPZ_j = O(| \log \delta |) |h|_\infty \), we have that \( a_{0j} = 0 \). So far, we have shown that \( \psi \to 0 \) in \( C_0(S \setminus \{ \xi_1^*, \ldots, \xi_m^* \}) \) and uniformly in \( U_{|\underline{PZ}_j|} B_{\delta_j}(\xi_j) \), for all \( R > 0 \).

Setting \( \psi_j(y) = \psi(y_j^{-1}(y)) \), \( \xi_j(y) = \chi_j(y) \psi(y_j^{-1}(y)) \) and \( \hat{h}_j(y) = h(\psi_j^{-1}(y)) \) for \( y \in B_{2\delta_j}(0) \), we have that \( e^{\delta_j} \Delta \hat{h}_j + \mathcal{K} \hat{h}_j = \hat{h}_j \).

By now it is rather standard to show that the operator \( \hat{L}_j = e^{\delta_j} \Delta + \mathcal{K} \hat{h}_j \) satisfies the maximum principle in \( B(0) \setminus B_{\delta_j}(0) \) for \( R \) large and \( r > 0 \) small enough, see for example [36]. As a consequence, we get that \( \psi \to 0 \) in \( L^\infty_0(S) \). Since \( \int_S \phi \, dv_g \to c \) along a sub-sequence, \( \| \psi \|_\infty \to 0 \) implies \( \phi \to c \) in \( L^\infty_0(S) \) with \( c = 0 \) in view of \( \int_S \phi = 0 \), in contradiction with \( \| \phi \|_\infty = 1 \). This completes the proof. \( \square \)

We are now ready for

**Proof of Proposition 4.1.** Since \( \| \Delta_g PZ_j \| \leq C \) for all \( i = 0, 1, 2, j = 1, \ldots, m \), and
\[
\left\| \left( \lambda - 8 \pi m \right) \frac{k e^{\omega} \left( \phi - \int_S k e^{\omega} \phi \, dv_g \right) \right\|_\ast = O(\lambda - 8 \pi m) \| \phi \|_\infty.
\]
by Proposition A.1 for \( \lambda \) close to \( 8 \pi m \) any solution of (55) satisfies
\[
\| \phi \|_\infty \leq C | \log \delta | \left[ \| h \|_\ast + |c_0| + \sum_{k=1}^{m} c_k \| \right].
\]
To estimate the values of the \( c_k \)'s, test equation (55) against \( PZ_j \), \( i = 1, 2 \) and \( j = 1, \ldots, m \):
\[
\int_S \phi(L(PZ_j)) \, dv_g = \int_S hPZ_j \, dv_g + \sum_{i=0}^{m} \int_S \Delta_g PZ_0 PZ_j \, dv_g + \sum_{k=1}^{m} c_k \int_S \Delta_g PZ_k PZ_j \, dv_g.
\]
Since for \( j = 1, \ldots, m \) we have the following estimates in \( C(S) \)
\[ PZ_j = \chi_j Z_j + O(\delta), \quad i = 1, 2, \quad PZ_0 = \chi_j (Z_0 + 2) + O(\delta^2 | \log \delta |), \]
by Lemma 2.1, (25), (38) and (74) we have that for \( i = 1, 2 \)
\[ L(PZ_j) = \chi_j \Delta_g Z_j + e^{\delta^2} PZ_j + O(\delta^2 + \sum_{k=1}^{m} e^{\delta_k}) = e^{\delta_j} [PZ_j - e^{-\delta_j} X_j Z_j] + O(\delta^2 + \sum_{k=1}^{m} e^{\delta_k}) \]
in view of \( \int_S \frac{k e^{\omega} \phi \, dv_g}{k e^{\omega} \phi \, dv_g} = O(\delta) \), leading to \( \| L(PZ_j) \|_\ast = O(\delta) \).

Similarly, we have that
\[ L(PZ) = \frac{m}{\sum_{j=0}^{m} \left[ \chi_j \Delta_g Z_j + e^{\delta_j} PZ_j - \frac{2}{m} \sum_{k=1}^{m} \chi_k e^{\delta_k} \right] + O(\delta^2) + O\left( \sum_{k=1}^{m} e^{\delta_k} \right) \]
The argument below is heuristic but can be made completely rigorous. The operator \( \xi \) has only the trivial solution: for (55) with 

\[
\frac{f}{\sigma} e^{W} = \frac{2}{m} + O(\delta^{2} |\log \delta|), \text{ leading to } \|L(PZ)\| = O(\delta). \text{ Hence, we get that}
\]

\[
|c_{0}| + \sum_{i=1}^{m} |c_{i}| \leq C \|h\| + \delta O \left( \|\phi\|_{\infty} + |c_{0}| + \sum_{i=1}^{m} |c_{i}| \right) \leq C' \|h\| + \delta \log \delta O \left( |c_{0}| + \sum_{i=1}^{m} |c_{i}| \right),
\]

yielding to the desired estimates \( \|\phi\|_{\infty} = O(|\log \delta| \|h\|) \) and \( |c_{0}| + \sum_{i=1}^{m} \sum_{j=1}^{m} |c_{i}c_{j}| = O(\|h\|). \) To prove the solvability assertion, problem (55) is equivalent to finding \( \phi \in H \) such that

\[
\int_{S} (\nabla \phi, \nabla \psi)_{g} \, dv_{g} = \int_{S} \left[ \frac{\lambda ke^{W}}{\int_{S} ke^{W} dv_{g}} \phi \left( \frac{f}{\sigma} e^{W} + \frac{\lambda ke^{W}}{\int_{S} ke^{W} dv_{g}} \right) \right] \psi \, dv_{g} \quad \forall \psi \in H,
\]

where \( H = \{ \phi \in H^{1}(S) : \int_{S} \phi \Delta_{g} PZ_{i} dv_{g} = \int_{S} \phi \Delta_{g} PZ_{j} dv_{g} = 0, i = 1, 2, j = 1, \ldots, m \}. \) With the aid of the Riesz representation theorem, Fredholm’s alternative guarantees unique solvability for any \( h \) provided that the homogeneous equation has only the trivial solution: for (55) with \( h = 0 \), the a-priori estimate (56) gives that \( \phi = 0 \).

So far, we have seen that, if \( (T, h) \) denotes the unique solution \( \phi \) of (55), the operator \( T \) is a continuous linear map from \( \{ h \in L^{\infty}(S) : \int_{S} h dv_{g} = 0 \} \), endowed with the \( \| \cdot \|_{\infty} \)-norm, into \( \{ \phi \in L^{\infty}(S) : \int_{S} \phi dv_{g} = 0 \} \), endowed with \( \| \cdot \|_{\infty} \)-norm. The argument below is heuristic but can be made completely rigorous. The operator \( T \) and the coefficients \( c_{0}, c_{i} \) are differentiable w.r.t. \( \xi_{l}, l = 1, \ldots, m, \) or \( \delta \). Differentiating equation (55), we formally get that \( X = \Delta_{g} PZ_{j} \), where \( \beta = \xi_{l} \) with \( l = 1, \ldots, m \) or \( \beta = \delta \), satisfies \( L(X) = h(\phi) + d_{0} \Delta_{g} PZ + \sum_{i,j} d_{ij} \Delta_{g} PZ_{ij} \), where

\[
\begin{align*}
\tilde{h}(\phi) &= -\partial_{\beta} \left( \frac{\lambda ke^{W}}{\int_{S} ke^{W} dv_{g}} \right) + \partial_{\beta} \left( \left( \int_{S} \frac{\lambda ke^{W}}{\int_{S} ke^{W} dv_{g}} \right)^{2} \int_{S} \frac{\lambda ke^{W}}{\int_{S} ke^{W} dv_{g}} \right) \int_{S} \frac{\lambda ke^{W}}{\int_{S} ke^{W} dv_{g}} \int_{S} \frac{\lambda ke^{W}}{\int_{S} ke^{W} dv_{g}} \int_{S} \frac{\lambda ke^{W}}{\int_{S} ke^{W} dv_{g}} \\
&= c_{0} \partial_{\beta} (\Delta_{g} PZ) + \sum_{i,j} c_{ij} \partial_{\beta} (\Delta_{g} PZ_{ij})
\end{align*}
\]

and \( d_{0} = \partial_{\beta} c_{0}, d_{ij} = \partial_{\beta} c_{ij} \), and the orthogonality conditions become

\[
\int_{S} X \Delta_{g} PZ_{i} dv_{g} = -\int_{S} \phi \partial_{\beta} (\Delta_{g} PZ_{i}) dv_{g}, \quad \int_{S} X \Delta_{g} PZ_{j} dv_{g} = -\int_{S} \phi \partial_{\beta} (\Delta_{g} PZ_{j}) dv_{g},
\]

Find now coefficients \( b_{0}, b_{ij} \) so that \( Y = X + b_{0} PZ + \sum_{i,j} b_{ij} PZ_{ij} \) satisfies the orthogonality conditions \( \int_{S} Y \Delta_{g} PZ_{ij} dv_{g} = \int_{S} Y \Delta_{g} PZ_{j} dv_{g} = 0 \). The coefficients \( b_{0}, b_{ij} \) have to satisfy an almost diagonal system, and are then well-defined with \( \|b_{0}\| + \sum_{ij} |b_{ij}| \leq C \|h\|_{\infty} \), in view of \( \|\partial_{\beta} (\Delta_{g} PZ_{i})\| \leq \frac{C}{g} \), \( \|F\| \leq \frac{C}{g} \), and \( \|\partial_{\beta} \| < C \). Hence, the function \( X \) can be uniquely expressed as \( X = T(f) - b_{0} PZ - \sum_{i,j} b_{ij} PZ_{ij} \), where \( f = \tilde{h}(\phi) + b_{0} L(PZ) + \sum_{i,j} b_{ij} L(PZ_{ij}) \). Moreover, since \( \|\partial_{\beta} W\|_{\infty} + \|\partial_{\beta} \kappa\| \leq \frac{C}{g} \), \( \|\kappa\| \leq C \) and \( \|\partial_{\beta} \left( \int_{S} ke^{W} dv_{g} \right) \| \leq \frac{C}{g} \), we find that

\[
\|f\|_{\infty} \leq \|\tilde{h}(\phi)\|_{\infty} + |b_{0}| \|L(PZ)\|_{\infty} + \sum_{ij} |b_{ij}| \|L(PZ_{ij})\|_{\infty} \leq C \frac{\|h\|}{\delta} \|h\|_{\infty},
\]

and by (56) we deduce that for any first derivative

\[
\|\partial_{\beta}\|_{\infty} \leq C \left( \|h\|_{\infty} + \|\phi\|_{\infty} \right) \leq C' \frac{\|h\|_{\infty}}{\delta^{2}}.
\]

Differentiating once more in \( \delta \) the equation satisfied by \( \partial_{\beta}\) and arguing as above, we finally obtain that \( \|\partial_{\beta}\|_{\infty} \leq C \frac{\|h\|_{\infty}}{\delta^{2}} \|h\|_{\infty} \), and the proof is complete. \( \Box \)

**Appendix B**

By Proposition 4.1 we now deduce the following.

**Proof of Proposition 4.2.** In terms of the operator \( T \), problem (58) takes the form \( A(\phi) = \phi \), where \( A(\phi) := -T(R + N(\phi)) \). Given \( \nu > 0 \), let us consider the space

\[
\mathcal{F}_{\nu} = \left\{ \phi \in C(S) : \|\phi\|_{\infty} \leq \nu \left[ \delta \|f\|_{\infty} + \delta^{2} \sum_{j=1}^{m} \|\nabla \log(r_{j} \circ y_{\xi}) (0)\| + \delta^{2-\nu} \|f\|_{\infty} \right] \right\}.
\]
Notice that if $\phi \in \mathcal{F}_v$, then $W + \phi \in \mathcal{A}_v$ for $\delta$ and $\epsilon$ small enough. Since in view of Lemma 2.1 and (38) we have
\[
\left\| \frac{\lambda e^{W+\phi} \psi_1 \psi_2}{\int_S e^{W+\phi} dV_g} - \frac{\lambda e^{W+\phi} \psi_1 \int_S e^{W+\phi} \psi_2 dV_g}{\left(\int_S e^{W+\phi} dV_g\right)^2} - \frac{\lambda e^{W+\phi} \psi_2}{\int_S e^{W+\phi} dV_g} \right\| \leq C \|\phi\|_\infty \|\psi_1\|_\infty \|\psi_2\|_\infty.
\]
for any $\phi_1, \phi_2 \in \mathcal{F}_v$, we obtain that $\|N(\phi_1) - N(\phi_2)\|_\infty \leq C \|\phi_1\|_\infty \|\phi_2\|_\infty \|\phi_1 - \phi_2\|_\infty$ and then
\[
\|A(\phi_1) - A(\phi_2)\|_\infty \leq C \log \delta (\|\phi_1\|_\infty + \|\phi_2\|_\infty \|\phi_1 - \phi_2\|_\infty \leq \frac{1}{2} \|\phi_1 - \phi_2\|_\infty
\]
for $\delta$ small in view of Proposition 4.1. Moreover, we have that for any $\phi \in \mathcal{F}_v$,
\[
\|A(\phi)\|_\infty \leq C \log \delta (\|\phi\|_\infty + \|\phi\|_\infty \|\phi\|_\infty + C_0 \left[ \|\phi\|_\infty \sum_{j=1}^m |\nabla \log (\rho_j \circ \psi_j^{-1}) (0)| + \delta^{2-\sigma} |\log \delta| \right]^2
\]
in view of Lemma 2.2. Then, for $\nu = 2C_0$ and $\delta$ small $A$ is a contraction mapping of $\mathcal{F}_v$ into itself, and there has therefore a unique fixed point $\phi \in \mathcal{F}_v$.

By the Implicit Function Theorem it follows that the map $(\delta, \xi) \rightarrow (\phi(\delta, \xi), c_0(\delta, \xi), c_1(\delta, \xi))$ is (at least) twice-differentiable in $\delta$ and one differentiable in $\xi$. Differentiating $\phi = -T(R + N(\phi))$ w.r.t. $\beta = \xi$, $l = 1, \ldots, m$, or $\beta = \delta$, we get that $\partial_\beta \phi = -\partial_\beta T(R + N(\phi)) - T(\partial_\beta R + \partial_\beta N(\phi))$. By Lemma 2.2 and (57) we have that
\[
\|\partial_\beta T(R + N(\phi))\|_\infty \leq C \left[ \|\phi\|_\infty \sum_{j=1}^m |\nabla \log (\rho_j \circ \psi_j^{-1}) (0)| + \delta^{1-\sigma} |\log \delta| \right]^2.
\]
and, in view of $\|\partial_\beta W\|_\infty \leq \frac{\xi}{8}$, we can estimate
\[
\partial_\beta N(\phi) = N(\phi) \partial_\beta W + \lambda \left( \frac{e^{W+\phi} \psi_1}{\int_S e^{W+\phi} dV_g} \right) \partial_\beta \phi
\]
\[
- \lambda \left( \frac{e^{W+\phi} \psi_1}{\int_S e^{W+\phi} dV_g} \right) \partial_\beta \phi
\]
\[
- \lambda \left( \frac{e^{W+\phi} \psi_1}{\int_S e^{W+\phi} dV_g} \right) \partial_\beta \phi
\]
\[
+ 2 \lambda \left( \frac{e^{W+\phi} \psi_1}{\int_S e^{W+\phi} dV_g} \right) \partial_\beta \phi
\]

as follows
\[
\|\partial_\beta N(\phi)\|_\infty \leq C \left[ \|\partial_\beta W\|_\infty \|\phi\|_\infty \sum_{j=1}^m |\nabla \log (\rho_j \circ \psi_j^{-1}) (0)| + \delta^{1-\sigma} |\log \delta| \right]^2 + O \left( \delta^{2-\sigma} |\log \delta| \right).
\]
Since $\int_S \chi e^{-\psi_j} e^{\psi_j} dV_g = \int_{\mathbb{R}^2} \chi (|y|) \frac{8\delta^2 \rho_j(\xi)}{(\xi^2 + \rho_j(\xi))^2} dy$, we have that
\[
\partial_\xi \left( \int_S \chi e^{-\psi_j} e^{\psi_j} dV_g \right) = 8 \partial_\xi \log \rho_j(\xi) \int_{\mathbb{R}^2} \frac{1 - |y|^2}{(1 + |y|^2)^3} dy + O(\delta^2) = O(\delta^2)
\]
and
\[
\partial_y \left( \int_S \chi e^{-\psi_j} e^{\psi_j} dV_g \right) = \int_{\mathbb{R}^2} \chi (|y|) \frac{16 \delta \rho_j(\xi) (|y|^2 - \delta^2 \rho_j(\xi))}{(\delta^2 \rho_j(\xi) + |y|^2)^3} dy = \frac{16 \delta}{\delta} \int_{\mathbb{R}^2} \frac{|y|^2 - 1}{(1 + |y|^2)^3} dy + O(\delta) = O(\delta).
\]
Since $\varphi_j(\xi) = 0$ and $\nabla \varphi_j(\xi) = 0$, we have that $e^{-\psi_j} = 1 + O(|y_j(x)|^2)$ and $\partial_\beta (\chi e^{-\psi_j})(x) = O(|y_j(x)|)$, and then
\[
\Delta_\delta \partial_\beta W = - \sum_{j=1}^m \chi_j e^{\psi_j} \partial_\beta U_j + O(\delta^{1-\sigma}).
\]
in view of $|\partial_\rho U_j| = O(\frac{1}{\delta})$, where the big O is estimated in $\| \cdot \|_\infty$-norm. Since in $B_\infty(\xi_j)$

$$\partial_\xi W = \partial_\xi U_j + O(\delta^2 |\log \delta| + |\gamma_{\xi_j}(x)| + |\nabla \log (\rho_j \circ y_{\xi_j}^{-1})(0)|), \quad \partial_\beta W = \partial_\beta U_j - \frac{2}{\delta} + O(\delta |\log \delta|),$$

in the same line as Lemma 2.1 and

$$\frac{\lambda ke^W}{\int_S ke^W dv_g} = \sum_{j=1}^{m} \lambda e^{U_j}[1 + O(\|\nabla \log (\rho_j \circ y_{\xi_j}^{-1})(0)\| \|y_{\xi_j}(x)\| + \delta^2 |\log \delta|)] + O(\delta^2)$$

in view of (32), by (42) and (50) we deduce for

$$\partial_\rho R = \Delta_g \partial_\rho W + \frac{\lambda ke^W}{\int_S ke^W dv_g} \left( \partial_\rho W - \frac{\int_S ke^W \partial_\rho W}{\int_S ke^W dv_g} \right)$$

the estimate

$$\|\partial_\rho R\|_\infty = O\left( \sum_{j=1}^{m} |\nabla \log (\rho_j \circ y_{\xi_j}^{-1})(0)| + \delta^{1-\sigma} |\log \delta| \right).$$

Combining all the estimates, we then get that

$$\|\partial_\rho \phi\|_\infty = O\left( |\log \delta|^2 \sum_{j=1}^{m} |\nabla \log (\rho_j \circ y_{\xi_j}^{-1})(0)| + \delta^{1-\sigma} |\log \delta|^3 \right) + o(\|\partial_\rho \phi\|_\infty),$$

which in turn provides the validity of (60). We proceed in the same way to obtain the estimate (61) on $\partial_\beta \phi$, and the proof is complete. □

**Lemma 4.3** is rather standard and we will omit its proof. Since the problem has been reduced to find c.p.'s of the reduced energy $E_\delta(\delta, \xi) = J_\delta(W + \phi(\delta, \xi))$, where $J_\delta$ is given by (4), the last key step is to show that the main asymptotic term of $E_\delta$ is given by $J_\delta(W)$.

**Proof of Theorem 4.4.** Write

$$J_\delta(W + \phi) - J_\delta(W) = D^2 J_\delta(W)[\phi, \phi] + \frac{1}{2} \int_0^1 \left[ D^2 J_\delta(W + t\phi) - D^2 J_\delta(W) \right][\phi, \phi] \, t \, ds \, dt$$

since $D J_\delta(W)(\phi) = -\int_S R \partial_\phi dv_g$, $D^2 J_\delta(W)[\phi, \phi] = -\int_S L(\phi) \partial_\phi dv_g$ and

$$D J_\delta(W)[\phi] + D^2 J_\delta(W)[\phi, \phi] = \int S (N(\phi) \partial_\phi dv_g$$

in view of $\int_S \partial_\phi dv_g = 0$ and (58). Since $\frac{1}{2} \int_S ke^W dv_g \leq \int_S ke^{W + ts\phi} dv_g \leq 2 \int_S ke^W dv_g$ and $|e^{W + ts\phi} - e^W| \leq Ce^W \|\phi\|_\infty$, it is straightforward to see that

$$\left| D J_\delta(W)[\phi] + D^2 J_\delta(W)[\phi, \phi] + \int_0^t ds [D^2 J_\delta(W + t\phi) - D^2 J_\delta(W)][\phi, \phi] \right| = O(\|N(\phi)\|_\infty\|\phi\|_\infty + \|\phi\|_\infty^2) = O(\|\phi\|_\infty^2),$$

and then we deduce that

$$\|J_\delta(W + \phi) - J_\delta(W)\| = O(\|R\|_\infty\|\phi\|_\infty + \|\phi\|_\infty^2) = O\left( \delta^2 |\log \delta| \|\nabla \varphi_m(\xi)\|^2 + \delta^{3-\sigma} |\log \delta|^2 \right)$$

in view of (59) and $4\pi \nabla_{\xi_j} \varphi_m(\xi) = \nabla \log (\rho_j \circ y_{\xi_j}^{-1})(0)$. Differentiating w.r.t. $\beta = \xi_j, l = 1, \ldots, m, \text{ or } \beta = \delta$ we get that

$$\partial_\beta [J_\delta(W + \phi) - J_\delta(W)] = -\frac{\lambda}{2} \int_S \partial_\beta R \phi + R \partial_\beta \phi \, dv_g + \frac{1}{2} \int_S \partial_\beta [N(\phi)] \phi + N(\phi) \partial_\beta \phi \, dv_g$$

$$+ \int_0^t ds \int_0^t ds \partial_\beta [D^2 J_\delta(W + t\phi) - D^2 J_\delta(W)][\phi, \phi].$$

Since it is straightforward to see that

$$\left| \int_0^t ds \int_0^t ds \partial_\beta [D^2 J_\delta(W + t\phi) - D^2 J_\delta(W)][\phi, \phi] \right| = O(\|\phi\|_\infty^2 \|\partial_\beta \phi\|_\infty + \|\phi\|_\infty^3 \|\partial_\beta W\|_\infty),$$
by (75) we deduce that
\[
|\partial_\beta [j_\delta(W + \phi) - j_\delta(W)]| = O(\|\partial_\beta R\|_\infty \|\phi\|_\infty + \|R\|_\infty \|\partial_\beta \phi\|_\infty + \|\phi\|_\infty^2 \|\partial_\beta \phi\|_\infty + \|\phi\|_\infty \|\partial_\beta W\|_\infty)
\]
\[= O \left( \left( \delta^2 |\log \delta| |\nabla \psi_m(\xi)|^2 + \delta^{3-\alpha} |\log |\delta|^2 \right) \right).
\]
in view of (59)–(60) and \(\|\partial_\beta W\|_\infty = O(1/\delta)\). Arguing similarly for the second derivative in \(\delta\), we get that
\[
|\partial_\delta [j_\delta(W + \phi) - j_\delta(W)]| = O \left( \left( \delta^2 |\log \delta| |\nabla \psi_m(\xi)|^2 + \delta^{3-\alpha} |\log |\delta|^2 \right) \right).
\]
Combining the previous estimates on the difference \(j_\delta(W + \phi) - j_\delta(W)\) with the expansion of \(j_\delta(W)\) contained in Theorem 3.2, we deduce the validity of the expansion (62) with an error term which can be estimated (in \(C^2(\mathbb{R})\) and \(C^1(\mathbb{S})\)) like \(O(\delta^2) + r_\delta(\delta, \xi)\) as \(\delta \to 0\), where \(r_\delta(\delta, \xi)\) does satisfy (63).

References