AM110 - Analisi matematica 1

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Argomenti: limiti con formula di Taylor

Esercizio 1.
Calcolare, se esiste, il limite

$$\lim_{x\to 0}\frac{e^x-e^{\sin x}}{x\log(\cos x)}.$$

Soluzione:

$$\begin{aligned}
&\frac{e^x-e^{\sin x}}{x\log(\cos x)}\\
=&\ \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+o\left(x^3\right)\right)-\left(1+\sin x+\frac{(\sin x)^2}{2}+\frac{(\sin x)^3}{6}+o\left((\sin x)^3\right)\right)}{x\left(\cos x-1+O\left((\cos x-1)^2\right)\right)}\\
=&\ \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+o\left(x^3\right)\right)-\left(1+\left(x-\frac{x^3}{6}+o\left(x^3\right)\right)+\frac{\left(x+o\left(x^2\right)\right)^2}{2}+\frac{(x+o(x))^3}{6}+o\left(o(x)^3\right)\right)}{x\left(-\frac{x^2}{2}+o\left(x^2\right)+o\left(o(x)\right)^2\right)}\\
=&\ \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+o\left(x^3\right)\right)-\left(1+x+\frac{x^2}{2}+o\left(x^3\right)\right)}{-\frac{x^3}{2}+o\left(x^3\right)}\\
=&\ \frac{\frac{x^3}{6}+o\left(x^3\right)}{-\frac{x^3}{2}+o\left(x^3\right)}\\
\underset{x\to 0}{\to}&\ -\frac{1}{3}.
\end{aligned}$$

Esercizio 2.
Calcolare, se esiste, il limite

$$\lim_{x\to 0}\frac{e^x-1-\log\left(1+x+x^2\right)}{\log(1+x)+e^{-x}-1}.$$

Soluzione:

$$\begin{aligned}
&\frac{e^x-1-\log\left(1+x+x^2\right)}{\log(1+x)+e^{-x}-1}\\
=&\ \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+o\left(x^3\right)\right)-1-\left(x+x^2-\frac{\left(x+x^2\right)^2}{2}+\frac{\left(x+x^2\right)^3}{3}+o\left(\left(x+x^2\right)^3\right)\right)}{\left(x-\frac{x^2}{2}+\frac{x^3}{3}+o\left(x^3\right)\right)+\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+o\left(x^3\right)\right)-1}\\
=&\ \frac{\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+o\left(x^3\right)\right)-1-\left(x+\frac{x^2}{2}-\frac{2}{3}x^3+o\left(x^3\right)\right)}{\frac{x^3}{6}+o\left(x^3\right)}\\
=&\ \frac{\frac{5}{6}x^3+o\left(x^3\right)}{\frac{x^3}{6}+o\left(x^3\right)}\\
\underset{x\to 0}{\to}&\ 5.
\end{aligned}$$

Esercizio 3.
Calcolare, se esiste, il limite

$$\lim_{x\to 0}\left(\frac{1}{(\sin x)^2}-\frac{1}{\sin\left(x^2\right)}\right).$$

Soluzione:

$$\begin{aligned}
\frac{1}{(\sin x)^2}-\frac{1}{\sin\left(x^2\right)} &= \frac{\sin\left(x^2\right)-\sin^2 x}{\sin^2 x\sin\left(x^2\right)}\\
&= \frac{\left(x^2+o\left(x^6\right)\right)-\left(x-\frac{x^3}{6}+o\left(x^3\right)\right)^2}{\left(x+o(x)\right)^2\left(x^2+o\left(x^3\right)\right)}\\
&= \frac{\left(x^2+o\left(x^6\right)\right)-\left(x^2-\frac{x^4}{3}+o\left(x^4\right)\right)}{x^4+o\left(x^4\right)}\\
&= \frac{\frac{x^4}{3}+o\left(x^4\right)}{x^4+o\left(x^4\right)}\\
&\underset{x\to 0}{\to} \frac{1}{3}.
\end{aligned}$$

Esercizio 4.
Calcolare, se esiste, il limite

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{\frac{1}{x}}.$$

Soluzione:

$$\begin{aligned}
\left(\frac{e^x-1}{x}\right)^{\frac{1}{x}} &= e^{\frac{\log\left(\frac{e^x-1}{x}\right)}{x}}\\
&= e^{\frac{\log\left(\frac{\left(1+x+\frac{x^2}{2}+o\left(x^2\right)\right)-1}{x}\right)}{x}}\\
&= e^{\frac{\log\left(1+\frac{x}{2}+o(x)\right)}{x}}\\
&= e^{\frac{\frac{x}{2}+o(x)}{x}}\\
&\underset{x\to 0}{\to} e^{\frac{1}{2}}.
\end{aligned}$$

Esercizio 5 (Assegnato per casa)**.**
Calcolare, se esiste, il limite

$$\lim_{x\to 0}\frac{x\sqrt{1+x}-\log(1+x)\sqrt[3]{1+x}}{x\sqrt[3]{1+x}-\log(1+x)\sqrt{1+x}}.$$

Soluzione:

$$\begin{aligned}
\frac{x\sqrt{1+x}-\log(1+x)\sqrt[3]{1+x}}{x\sqrt[3]{1+x}-\log(1+x)\sqrt{1+x}} &= \frac{x\left(1+\frac{x}{2}+o(x)\right)-\left(x-\frac{x^2}{2}+o\left(x^2\right)\right)\left(1+\frac{x}{3}+o(x)\right)}{x\left(1+\frac{x}{3}+o(x)\right)-\left(x-\frac{x^2}{2}+o\left(x^2\right)\right)\left(1+\frac{x}{2}+o(x)\right)}\\
&= \frac{\left(x+\frac{x^2}{2}+o\left(x^2\right)\right)-\left(x-\frac{x^2}{6}+o\left(x^2\right)\right)}{\left(x+\frac{x^2}{3}+o\left(x^2\right)\right)-\left(x+o\left(x^2\right)\right)}\\
&= \frac{-\frac{2}{3}x^2+o\left(x^2\right)}{-\frac{x^2}{3}+o\left(x^2\right)}\\
&\underset{x\to 0}{\to} 2.
\end{aligned}$$

Esercizio 6 (Assegnato per casa)**.**
Calcolare, se esiste, il limite

$$\lim_{x\to 0} \frac{\arcsin x - \arctan x}{x - \arctan(\arcsin x)}.$$

Soluzione:

$$\begin{aligned}
\frac{\arcsin x - \arctan x}{x - \arctan(\arcsin x)} &= \frac{\left(x + \frac{x^3}{6} + o\left(x^3\right)\right) - \left(x - \frac{x^3}{3} + o\left(x^3\right)\right)}{x - \left(\arcsin x - \frac{\arcsin^3 x}{3} + o\left(\arcsin^3 x\right)\right)} \\
&= \frac{\frac{x^3}{2} + o\left(x^3\right)}{x - \left(x + \frac{x^3}{6} + o\left(x^3\right) - \frac{(x+o(x))^3}{3} + o\left(x^3\right)\right)} \\
&= \frac{\frac{x^3}{2} + o\left(x^3\right)}{x - \left(x - \frac{x^3}{6} + o\left(x^3\right)\right)} \\
&= \frac{\frac{x^3}{2} + o\left(x^3\right)}{\frac{x^3}{6} + o\left(x^3\right)} \\
&\underset{x\to 0}{\to} 3.
\end{aligned}$$