AM110 - Analisi matematica 1

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Argomenti: integrali

Esercizio 1.
Calcolare l'integrale

$$\int_1^{\sqrt{3}} \frac{x^3+2}{x^3+x}\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_1^{\sqrt{3}} \frac{x^3+2}{x^3+x}\mathrm{d}x &= \int_1^{\sqrt{3}} \left(1+\frac{2}{x}-\frac{1}{1+x^2}-\frac{2x}{1+x^2}\right)\mathrm{d}x \\
&= \left[x+2\log|x|-\arctan x-\log\left(1+x^2\right)\right]_1^{\sqrt{3}} \\
&= \sqrt{3}-1-\frac{\pi}{12}+\log 3-\log 2.
\end{aligned}$$

Esercizio 2.
Calcolare l'integrale

$$\int_0^4 \sqrt{3-\sqrt{x}}\mathrm{d}x.$$

Soluzione:

$$\int_0^4 \sqrt{3-\sqrt{x}}\mathrm{d}x \overset{\left(y=\sqrt{3-\sqrt{x}}\right)}{=} \int_1^{\sqrt{3}} y\left(12y-4y^3\right)\mathrm{d}y = \left[4y^3-\frac{4}{5}y^5\right]_1^{\sqrt{3}} = \frac{24}{5}\sqrt{3}-\frac{16}{5}.$$

Esercizio 3.
Calcolare l'integrale

$$\int_0^e x^2(\log x)^2\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_0^e x^2(\log x)^2\mathrm{d}x &= \left[\frac{x^3}{3}\log^2 x\right]_0^e - \int_0^e \frac{2}{3}x^2\log x\mathrm{d}x \\
&= \frac{e^3}{3}-\left(\left[\frac{2}{9}x^3\log x\right]_0^e-\frac{2}{9}\int_0^e x^2\mathrm{d}x\right) \\
&= \frac{e^3}{3}-\left(\frac{2}{9}e^3-\left[\frac{2}{27}e^3\right]_0^e\right) \\
&= \frac{5}{27}e^3.
\end{aligned}$$

Esercizio 4.
Calcolare l'integrale

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x-(\cos x)^2}\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x-(\cos x)^2}\mathrm{d}x &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\sin^2 x}\mathrm{d}x \\
&\overset{(y=\sin x)}{=} \int_{\frac{1}{2}}^{1} \frac{1}{y+y^2}\mathrm{d}y \\
&= \int_{\frac{1}{2}}^{1} \left(\frac{1}{y}-\frac{1}{1+y}\right)\mathrm{d}y \\
&= [\log|y|-\log|1+y|]_{\frac{1}{2}}^{1} \\
&= \log 3-\log 2.
\end{aligned}$$

Esercizio 5.
Calcolare l'integrale

$$\int_0^{\frac{\pi}{4}} \frac{(\sin x)^2}{1+\sin x\cos x}\mathrm{d}x.$$

Soluzione:

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \frac{(\sin x)^2}{1+\sin x\cos x}\mathrm{d}x &\overset{(y=\tan x)}{=} \int_0^1 \frac{\frac{y^2}{1+y^2}}{1+\frac{y}{1+y^2}}\frac{1}{y^2+1}\mathrm{d}y \\
&= \int_0^1 \left(\frac{1}{\sqrt{3}}\frac{\frac{1}{\sqrt{3}}}{\left(\frac{2y+1}{\sqrt{3}}\right)^2+1}+\frac{1}{2}\frac{2y}{1+y^2}-\frac{1}{2}\frac{1+2y}{1+y+y^2}\right)\mathrm{d}y \\
&= \left[\arctan\frac{2y+1}{\sqrt{3}}+\frac{\log\left(1+y^2\right)}{2}-\frac{\log\left(1+y+y^2\right)}{2}\right]_0^1 \\
&= \frac{\pi}{6\sqrt{3}}+\frac{1}{2}\log 2-\frac{1}{2}\log 3.
\end{aligned}$$