# Università degli Studi di Roma Tre, A.A. 2023/2024
# Corso di Laurea Triennale in Fisica e Matematica
# AM110 - Analisi Matematica I

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Soluzioni Tutorato 7

**Esercizio 1.** (ix) $\int \arctan x \, dx = \int 1 \cdot \arctan x \, dx = x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$

(xi) $\int x^5 e^{-x^2} dx = \int x^4 \cdot x e^{-x^2} dx = x^4 \frac{e^{-x^2}}{-2} + \frac{1}{2} \int 4x^3 e^{-x^2} \, dx = -\frac{x^4 e^{-x^2}}{2} + 2\left[x^2 \frac{e^{-x^2}}{-2} + \frac{1}{2} \int 2x e^{-x^2} \, dx\right] = -\frac{x^4 e^{-x^2}}{2} - x^2 e^{-x^2} - e^{-x^2} + C$

(xv) Osservo che:

$$(x^x)' = (e^{x \ln x})' = x^x(\ln x + 1)$$

Da cui segue immediatamente dagli integrali generalizzati (o derivata di funzione composta) che: $\int (1 + \ln x) \, x^x \, dx = x^x + C$

(xvii) $\int \frac{8x^3}{1+x^8} dx = 2 \int \frac{4x^3}{1+(x^4)^2} dx = 2 \arctan(x^4) + C$

(xxiii) Utilizzo del formule parametriche:

$$\sin x = \frac{2t}{1+t^2}, \ \cos x = \frac{1-t^2}{1+t^2}$$

e applico la sostituzione: $t = \tan\frac{x}{2}$, $dx = \frac{2}{1+t^2}dt$

Quindi: $\displaystyle\int \frac{\cot x + (\sin x)^{-1}}{3\cos x + 3 - \sin x}dx = \int \frac{\frac{1-t^2}{2t} + \frac{1+t^2}{2t}}{3\frac{1-t^2}{1+t^2} + 3 - \frac{2t}{1+t^2}} \frac{2}{1+t^2}dt = 2\int \frac{\frac{1-t^2}{2t} + \frac{1+t^2}{2t}}{3\frac{1-t^2}{1+t^2} + 3 - \frac{2t}{1+t^2}} \frac{1}{1+t^2}dt =$
$\displaystyle\int \frac{1}{t(3-t)}dt = \frac{1}{3}\int \frac{1}{t}dt + \frac{1}{3}\int \frac{1}{3-t}dt = \frac{1}{3}[\ln|t| - \ln|3-t|] + C =$
$\displaystyle\frac{1}{3}[\ln|\tan(\frac{x}{2})| - \ln|3 - \tan(\frac{x}{2})|] + C$

(xxv) $\displaystyle\int \frac{dx}{x^2\sqrt{E - \frac{c}{x} - \frac{1}{x^2}}} \overset{t=\frac{1}{x}}{=} \int \frac{t^2(-\frac{1}{t^2}dt)}{\sqrt{E - ct - t^2}}$

Ora completo il quadrato:

$E - ct - t^2 = -(t + \frac{c}{2})^2 + \frac{c^2}{4} + E = -(t+\frac{c}{2})^2 + \tilde{E}$ con $\tilde{E} := E + \frac{c^2}{4} > 0$

Applico la sostituzione: $t + \frac{c}{2} = \sqrt{\tilde{E}}\sin(z)$, $dt = \sqrt{\tilde{E}}\cos z dz$

$\displaystyle\int \frac{t^2(-\frac{1}{t^2}dt)}{\sqrt{-(t+\frac{c}{2})^2 + \tilde{E}}} = -\int \frac{\sqrt{\tilde{E}}\cos z dz}{\sqrt{\tilde{E}(1 - \sin^2 z)}} = -\int \frac{\cos z dz}{\sqrt{\cos^2 z}} = -\int dz =$
$\displaystyle -z + C = -\arcsin\left(\frac{1}{\sqrt{\tilde{E}}}\left(t + \frac{c}{2}\right)\right) + C = \arccos\left(\frac{1}{\sqrt{\tilde{E}}}\left(\frac{1}{x} + \frac{c}{2}\right)\right) - \frac{\pi}{2} + C =$
$\displaystyle\arccos\left(\frac{1}{\sqrt{\tilde{E}}}\left(\frac{1}{x} + \frac{c}{2}\right)\right) + C$

(i)

$$\int \frac{1}{x \ln x}\,dx \underset{\substack{T = \ln x \\ dT = \frac{dx}{x}}}{=} \int \frac{1}{T}\,dT = \ln|T| + c =$$

$$= \ln|\ln x| + c$$

(ii)

$$\int (\tan x + \cot x)\,dx = \int \frac{\sin x}{\cos x}\,dx + \int \frac{\cos x}{\sin x}\,dx =$$

$$= -\int \frac{-\sin x}{\cos x}\,dx + \int \frac{\cos x}{\sin x}\,dx =$$

$$= -\int \frac{d}{dx}(\cos x) \cdot \frac{1}{\cos x}\,dx + \int \frac{d}{dx}(\sin x) \cdot \frac{1}{\sin x}\,dx =$$

$$= -\ln|\cos x| + \ln|\sin x| + c = \ln|\tan x| + c$$

(iii)

$$\int \frac{x^2 - 2x}{x^3 - 3x^2 + 1}\,dx = \frac{1}{3}\int \frac{3x^2 - 6x}{x^3 - 3x^2 + 1}\,dx =$$

$$= \frac{1}{3}\int \frac{d}{dx}(x^3 - 3x^2 + 1) \cdot \frac{1}{x^3 - 3x^2 + 1}\,dx =$$

$$= \frac{1}{3}\ln|x^3 - 3x^2 + 1| + c$$

(iv)

$$\int \frac{2}{x^2+9}\,dx = 2\int \frac{1}{9\left(\frac{x^2}{9}+1\right)}\,dx = \frac{2}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx =$$

$$= \frac{2}{9}\int \frac{1}{T^2+1}\,3\,dT = \frac{2}{3}\int \frac{1}{T^2+1}\,dT =$$

$T = x/3$
$3T = x$
$3\,dT = dx$

$$= \frac{2}{3}\operatorname{ARCTAN}(T) + c = \frac{2}{3}\operatorname{ARCTAN}\left(\frac{x}{3}\right) + c$$

(v)

$$\int \frac{\ln^2 x}{x}\,dx = \int \frac{d}{dx}(\ln x)\cdot(\ln x)^2\,dx = \frac{\ln^3 x}{3} + c$$

(vi)

$$\int \frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx = -\int -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx =$$

$$= -\int \frac{d}{dx}\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right)dx = -\sin\left(\frac{1}{x}\right) + c$$

(vii)

$$\int \frac{1}{x\sqrt[3]{\ln 3x}}\,dx = \int \frac{1}{x}\cdot\frac{1}{\sqrt[3]{\ln x + \ln 3}}\,dx =$$

$$= \int \frac{d}{dx}(\ln x + \ln 3)\cdot(\ln x + \ln 3)^{-1/3}\,dx =$$

$$= \frac{(\ln x + \ln 3)^{2/3}}{2/3} + c = \frac{3}{2}\sqrt[3]{\ln^2 3x} + c$$

(viii)

$$\int 2\ln^2 x\,dx = 2\int 1\cdot \ln^2 x\,dx = *$$

UTILIZZO IL METODO DI INTEGRAZIONE PER PARTI CON

$$f(x) = \ln^2 x \longrightarrow f'(x) = \frac{2\ln x}{x}$$

$$g'(x) = 1 \longrightarrow g(x) = x$$

$$* = 2\left(x\ln^2 x - \int x\cdot\frac{2\ln x}{x}\,dx\right) =$$

$$= 2x\ln^2 x - 4\int \ln x\,dx = *$$

PER CALCOLARE $\int \ln x \, dx$ UTILIZZO IL METODO DI INTEGRAZIONE PER PARTI CON

$$f(x) = \ln x \longrightarrow f'(x) = \frac{1}{x}$$

$$g'(x) = 1 \longrightarrow g(x) = x$$

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - x$$

QUINDI

$$* = 2x \ln^2 x - 4(x \ln x - x) + c =$$

$$= 2x \ln^2 x - 4x \ln x + 4x + c$$

(x)

$$\int e^{-x} \cos x \, dx = *$$

UTILIZZO IL METODO DI INTEGRAZIONE PER PARTI CON

$$f(x) = e^{-x} \longrightarrow f'(x) = -e^{-x}$$

$$g'(x) = \cos x \longrightarrow g(x) = \sin x$$

$$* = e^{-x} \sin x - \int -e^{-x} \sin x \, dx =$$

$$= e^{-x} \sin x + \int e^{-x} \sin x \, dx = *$$

PER CALCOLARE $\int e^{-x} \sin x \, dx$ UTILIZZO IL METODO DI INTEGRAZIONE PER PARTI CON

$$f(x) = e^{-x} \longrightarrow f'(x) = -e^{-x}$$

$$g'(x) = \sin x \longrightarrow g(x) = -\cos x$$

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - \int (-e^{-x})(-\cos x) dx =$$

$$= -e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

QUINDI

$$* = e^{-x}(\sin x - \cos x) - \int e^{-x} \cos x \, dx$$

CIOÈ ABBIAMO

$$\int e^{-x} \cos x \, dx = e^{-x}(\sin x - \cos x) - \int e^{-x} \cos x \, dx$$

$$2 \int e^{-x} \cos x \, dx = e^{-x}(\sin x - \cos x)$$

$$\int e^{-x} \cos x \, dx = \frac{e^{-x}}{2}(\sin x - \cos x) + c$$

(xii)

$$\int \frac{5x-3}{x^2-5x+6}\,dx = \int \frac{5x-3}{(x-3)(x-2)}\,dx = *$$

$$\frac{5x-3}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} = \frac{Ax-2A+Bx-3B}{(x-3)(x-2)}$$

$$\to \begin{cases} A+B=5 \\ -2A-3B=-3 \end{cases} \to \begin{cases} A+B=5 \\ -B=7 \end{cases} \to \begin{cases} A=12 \\ B=-7 \end{cases}$$

$$* = 12\int \frac{1}{x-3}\,dx - 7\int \frac{1}{x-2}\,dx =$$

$$= 12\ln|x-3| - 7\ln|x-2| + c$$

(xiii)

$$\int \frac{x^3-4x+6}{x^2+6x+4}\,dx = *$$

$$\begin{array}{rrrr|l} x^3 & & -4x & +6 & x^2+6x+4 \\ \hline -(x^3 & +6x^2 & +4x & ) & x-6 \\ \hline & -6x^2 & -8x & +6 & \\ - & (-6x^2 & -36x & -24) & \\ \hline & & 28x & +30 & \end{array}$$

$$\frac{x^3-4x+6}{x^2+6x+4} = (x-6) + \frac{28x+30}{x^2+6x+4}$$

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PONIAMO $x^2+6x+4=0$

$$x_{1,2} = \frac{-6 \pm \sqrt{20}}{2} = -3 \pm \sqrt{5}$$

$$\frac{28x+30}{x^2+6x+4} = \frac{A}{x-x_1} + \frac{B}{x-x_2} =$$

$$= \frac{(A+B)x - (Ax_2 + Bx_1)}{(x-x_1)(x-x_2)}$$

$$\rightarrow \begin{cases} A+B = 28 \\ Ax_2 + Bx_1 = -30 \end{cases} \rightarrow \begin{cases} A+B = 28 \\ A(-3-\sqrt{5}) + B(-3+\sqrt{5}) = -30 \end{cases}$$

$$\rightarrow \begin{cases} A+B = 28 \\ -3(A+B) - \sqrt{5}(A-B) = -30 \end{cases} \rightarrow \begin{cases} A+B = 28 \\ A-B = -\frac{54}{\sqrt{5}} \end{cases}$$

$$\rightarrow \begin{cases} A = 14 - \frac{27}{\sqrt{5}} \\ B = 14 + \frac{27}{\sqrt{5}} \end{cases}$$

$$\frac{x^3-4x+6}{x^2+6x+4} = (x-6) + \frac{28x+30}{x^2+6x+4} =$$

$$= (x-6) + \left(14 - \frac{27}{\sqrt{5}}\right) \cdot \frac{1}{x+3-\sqrt{5}} + \left(14 + \frac{27}{\sqrt{5}}\right) \cdot \frac{1}{x+3+\sqrt{5}}$$

$$* = \int \left[ (x-6) + \left(14 - \frac{27}{\sqrt{5}}\right) \cdot \frac{1}{x+3-\sqrt{5}} + \left(14 + \frac{27}{\sqrt{5}}\right) \cdot \frac{1}{x+3+\sqrt{5}} \right] dx =$$

$$= \frac{(x-6)^2}{2} + \left(14 - \frac{27}{\sqrt{5}}\right) \ln|x+3-\sqrt{5}| + \left(14 + \frac{27}{\sqrt{5}}\right) \ln|x+3+\sqrt{5}| + C$$

## (xiv)

$$\int \frac{7x^2+14x+14}{x^3+4x^2+4x+3}\, dx = \int \frac{x^2+14x+14}{(x+3)(x^2+x+1)}\, dx = *$$

$$\frac{x^2+14x+14}{(x+3)(x^2+x+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+x+1} = \frac{Ax^2+Ax+A+Bx^2+Cx+3Bx+3C}{(x+3)(x^2+x+1)}$$

$$\longrightarrow \begin{cases} A+B=1 \\ A+3B+C=14 \\ 3C+A=14 \end{cases} \longrightarrow \begin{cases} A=-19/7 \\ B=26/7 \\ C=39/7 \end{cases}$$

$$\frac{x^2+14x+14}{(x+3)(x^2+x+1)} = -\frac{19}{7} \cdot \frac{1}{x+3} + \frac{13}{7} \cdot \frac{2x+3}{x^2+x+1} =$$

$$= -\frac{19}{7} \cdot \frac{1}{x+3} + \frac{13}{7} \cdot \frac{2x+1}{x^2+x+1} + \frac{26}{7} \frac{1}{x^2+x+1}$$

$$* = \int \left[ -\frac{19}{7} \cdot \frac{1}{x+3} + \frac{13}{7} \cdot \frac{2x+1}{x^2+x+1} + \frac{26}{7} \frac{1}{x^2+x+1} \right] dx =$$

$$= -\frac{19}{7} \ln|x+3| + \frac{13}{7} \ln|x^2+x+1| + \frac{26}{7} \int \frac{1}{x^2+x+1} dx = *$$

$$\int \frac{1}{x^2+x+1} dx = \int \frac{1}{x^2+2\cdot\frac{1}{2}x+\frac{1}{4}+\frac{3}{4}} dx =$$

$$= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} dx = \int \frac{1}{\frac{3}{4}\left\{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1\right\}} dx =$$

$$= \frac{4}{3} \cdot \frac{\sqrt{3}}{2} \int \frac{1}{T^2+1} dT = \frac{2}{\sqrt{3}} \text{ARCTAN}(T) = \frac{2}{\sqrt{3}} \text{ARCTAN}\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)$$

$$\uparrow$$

$$T = \frac{2}{\sqrt{3}} x + \frac{1}{\sqrt{3}}$$

$$dT = \frac{2}{\sqrt{3}} dx$$

$$\frac{\sqrt{3}}{2} dT = dx$$

$$* = -\frac{19}{7} \ln|x+3| + \frac{13}{7} \ln|x^2+x+1| + \frac{52}{7\sqrt{3}} \text{ARCTAN}\left(\frac{2x}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)$$

## (xix)

$$\int \ln(1+\sqrt{x})\,dx \underset{\substack{T=\sqrt{x}\\ T^2=x\\ 2T\,dT=dx}}{=} \int \ln(1+T)\,2T\,dT = 2\int T\cdot\ln(1+T)\,dT \underset{\text{PER PARTI}}{=}$$

$$= 2\left(\frac{T^2}{2}\ln(1+T) - \int \frac{T^2}{2}\,\frac{1}{1+T}\,dT\right) =$$

$$= T^2\ln(1+T) - \int \frac{T^2}{1+T}\,dT = T^2\ln(1+T) - \int \frac{T^2-1+1}{T+1}\,dT =$$

$$= T^2\ln(1+T) - \int\left[(T-1) + \frac{1}{T+1}\right]dT =$$

$$= T^2\ln(1+T) - \frac{(T-1)^2}{2} - \ln(1+T) + c =$$

$$= x\ln(1+\sqrt{x}) - \frac{1}{2}(\sqrt{x}-1)^2 - \ln(1+\sqrt{x}) + c$$

## (xx)

$$\int \frac{1}{\cosh^2 x}\,dx = \int \frac{\sinh x}{\cosh^2 x\,\sinh x}\,dx =$$

$$\underset{\substack{T=\cosh x\\ dT=\sinh x\,dx}}{=} \int \frac{1}{T^2\sqrt{1-T^2}}\,dT \underset{\substack{T=\frac{1}{\cos u}\\ dT=\frac{\sin u}{\cos^2 u}du}}{=} \int \frac{1}{\frac{1}{\cos^2 u}\sqrt{1-\frac{1}{\cos^2 u}}}\cdot\frac{\sin u}{\cos^2 u}\,du =$$

$$= \int \frac{\sin u}{\sqrt{\frac{\cos^2 u - 1}{\cos^2 u}}}\,du = \int \cos u\,du = \sin u + c = *$$

$$* = \sin\left(\arctan\left(\frac{1}{\cosh x}\right)\right) + C = *$$

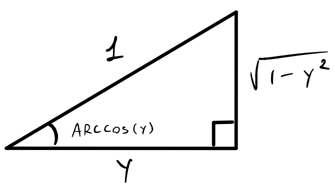
$$\cosh x = \frac{1}{\cos u}$$

$$\cos u = \frac{1}{\cosh x}$$

$$u = \arccos\left(\frac{1}{\cosh x}\right)$$

$$* = \sqrt{1 - \frac{1}{\cosh^2 x}} + C = \sqrt{\frac{\cosh^2 x - 1}{\cosh^2 x}} + C = *$$

$$\sin(\arctan(y)) = \sqrt{1 - y^2}$$




$$* = \frac{\sinh x}{\cosh x} + C = \tanh x + C$$

(xxi) 

$$\int \frac{1}{1+\sin x-\cos x}\,dx \underset{\uparrow}{=} \int \frac{1}{1+\frac{2T}{1+T^2}-\frac{1-T^2}{1+T^2}}\,\frac{2}{1+T^2}\,dT =$$

$T = \tan(x/2)$

$\cos x = \frac{1-T^2}{1+T^2}$

$\sin x = \frac{2T}{1+T^2}$

$dx = \frac{2}{1+T^2}\,dT$

$$= 2\int \frac{1}{1+T^2+2T-1+T^2}\,dT = 2\int \frac{1}{2T^2+2T}\,dT =$$

$$= \int \frac{1}{T(T+1)}\,dT = *$$

$$\frac{1}{T(T+1)} = \frac{A}{T} + \frac{B}{T+1} = \frac{(A+B)T + A}{T(T+1)}$$

$$\longrightarrow \begin{cases} A+B=0 \\ A=1 \end{cases} \longrightarrow \begin{cases} A=1 \\ B=-1 \end{cases}$$

$$\frac{1}{T(T+1)} = \frac{1}{T} - \frac{1}{T+1}$$

$$* = \int \frac{1}{T}\,dT - \int \frac{1}{T+1}\,dT =$$

$$= \ln|T| - \ln|T+1| + c = \ln\left|\frac{T}{T+1}\right| + c =$$

$$= \ln\left|\frac{\tan(x/2)}{\tan(x/2)+1}\right| + c$$

$$\int \frac{\cos^2 x}{\sin^4 x}\,dx = \int \frac{\cot^2 x}{\sin^2 x}\,dx =$$

$$= - \int T^2\,dT = -\frac{T^3}{3} + c = -\frac{\cot^3 x}{3} + C$$

$T = \cot x$

$dT = -\frac{1}{\sin^2 x}\,dx$