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Corso di Laurea Triennale in Fisica e Matematica
AM110 - Analisi Matematica I

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Soluzioni Tutorato 6

Esercizio 1.

(i) $\int \frac{1}{x \log x} dx = \int \frac{(\log x)'}{\log x} dx = \log \log x + c$

(ii) $\int (\tan x + \cot x)\, dx = \int \frac{\sin x}{\cos x} dx + \int \frac{\cos x}{\sin x} dx = -\int \frac{-(\cos x)'}{\cos x} dx + \int \frac{(\sin x)'}{\sin x} dx = -\log(|\cos x|) + \log(|\sin x|) + c = \log |\tan x| + c$

(iii) $\int \frac{x^2 - 2x}{x^3 - 3x^2 + 1} dx = \frac{1}{3} \int \frac{3x^2 - 6x}{x^3 - 3x^2 + 1} dx = \frac{1}{3} \int \frac{(x^3 - 3x^2 + 1)'}{x^3 - 3x^2 + 1} dx = \frac{1}{3} \log(|x^3 - 3x^2 + 1|) + c$

(iv) $\int \frac{2}{x^2 + 9} dx = \frac{2}{3} \int \frac{\frac{1}{3}}{1 + (\frac{x}{3})^2} dx = \frac{2}{3} \arctan\left(\frac{x}{3}\right) + c$

(v) $\int \frac{\log^2 x}{x} dx = \frac{1}{3} \int 3 \log^2 x (\log x)' dx = \frac{1}{3} \log^3 x + c$

(vi) $\int \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx = -\int \left(\frac{1}{x}\right)' \cos\left(\frac{1}{x}\right) = -\sin \frac{1}{x} + c$

(vii) Facciamo la sostituzione $y = \log 3x$, abbiamo $dy = \frac{1}{x} dx$ e quindi: $\int \frac{1}{x \sqrt[3]{\log 3x}} dx = \int y^{-\frac{1}{3}} dy = \frac{3}{2} y^{\frac{2}{3}} + c = \frac{3}{2} (\log 3x)^{\frac{2}{3}} + c$

(viii) Usiamo l'integrazione per parti $\int 2\log^2 x dx = 2x\log^2 x - 4\int \log x dx = 2x\log^2 x - 4x\log x + 4x + c$

(ix) $\int \arctan x \ dx = \int 1 \cdot \arctan x \ dx = x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2} \ dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + c$

(x) $\int e^{-x}\cos x\, dx = e^{-x}\sin x + \int e^{-x}\sin x = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x$
uguagliando il primo e l'ultimo membro e isolando l'integrale otteniamo $\int e^{-x}\cos x \ dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + c$

(xi) $\int x^5 e^{-x^2} dx = \int x^4 \cdot x e^{-x^2} dx = x^4 \frac{e^{-x^2}}{-2} + \frac{1}{2}\int 4x^3 e^{-x^2} \ dx = -\frac{x^4 e^{-x^2}}{2} + 2\left[x^2\frac{e^{-x^2}}{-2} + \frac{1}{2}\int 2xe^{-x^2} \ dx\right] = -\frac{x^4 e^{-x^2}}{2} - x^2 e^{-x^2} - e^{-x^2} + c$

(xii) $\int \frac{5x-3}{x^2-5x+6} dx = \frac{5}{2}\int \frac{2x-5+19/5}{x^2-5x+6} dx = \frac{5}{2}\int \frac{(x^2-5x+6)'}{x^2-5x+6} dx + \frac{19}{2}\int \frac{1}{(x-2)(x-3)} dx = \frac{5}{2}\log|x^2-5x+6| + \frac{19}{2}\int\left(-\frac{1}{x-2} + \frac{1}{x-3}\right) dx = \frac{5}{2}\log|(x-3)(x-2)| + \frac{19}{2}\log\left|\frac{x-3}{x-2}\right| + c = 12\log|x-3| - 7\log|x-2| + c$

(xiii) $\int \frac{x^3-4x^2+6}{x^2+6x+4} dx = \int\left(x - 10 + \frac{56x+46}{x^2+6x+4}\right) dx = \frac{x^2}{2} - 10x + 28\int \frac{2x+6-61/14}{x^2+6x+4} dx = \frac{x^2}{2} - 10x + 28\log|x^2+6x+4| - 122\int \frac{1}{(x+3-\sqrt{5})(x+3+\sqrt{5})} dx = \frac{x^2}{2} - 10x + 28\log|x^2+6x+4| - \frac{61}{\sqrt{5}}\int\left(\frac{1}{x+3-\sqrt{5}} - \frac{1}{x+3+\sqrt{5}}\right) dx = \frac{x^2}{2} - 10x + 28\log|x+3+\sqrt{5}| + 28\log|x+3-\sqrt{5}| - \frac{61}{\sqrt{5}}\log|x+3-\sqrt{5}| + \frac{61}{\sqrt{5}}\log|x+3+\sqrt{5}| + c = \frac{x^2}{2} - 10x + (28 + \frac{61}{\sqrt{5}})\log|x+3+\sqrt{5}| + (28 - \frac{61}{\sqrt{5}})\log|x+3-\sqrt{5}| + c$

(xiv) $\int \frac{1}{x^2+x+1} dx \underset{\substack{\uparrow \\ \text{il denominatore ha delta negativo}}}{=} \int \frac{1}{x^2 + 2\cdot 1/2 \cdot x + 1/4 + 3/4} dx = \int \frac{1}{(x+1/2)^2 + 3/4} dx = \int \frac{1}{\frac{3}{4}\left(\left(\frac{2}{\sqrt{3}}(x+1/2)\right)^2 + 1\right)} \underset{\substack{\uparrow \\ t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}}}{=}$

$$\frac{4\sqrt{3}}{6}\int \frac{1}{t^2+1}dt = \frac{2}{\sqrt{3}}\arctan t = \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)$$

(xv) Osservo che:

$$(x^x)' = (e^{x\ln x})' = x^x(\ln x+1)$$

Da cui segue immediatamente che: $\displaystyle\int (1+\ln x)\, x^x\, dx = x^x + c$

(xvi) $$\int \log(1+\sqrt{x})dx \underset{\underset{x=t^2}{\uparrow}}{=} \int \log(1+t)2tdt \underset{\underset{\text{per parti}}{\uparrow}}{=} t^2\log(1+t)-\int\frac{t^2}{1+t}dt =$$
$$t^2\log(1+t)-\int\frac{t^2-1+1}{1+t}dt = t^2\log(1+t)-\int\frac{(t-1)(t+1)}{1+t}dt-\int\frac{1}{1+t}dt =$$
$$t^2\log(1+t)-\int(t-1)dt-\log(1+t) = t^2\log(1+t)-\frac{(t-1)^2}{2}-\log(1+t) =$$
$$x\log(1+\sqrt{x})-\frac{(\sqrt{x}-1)^2}{2}-\log(1+\sqrt{x})+c = (x-1)\log(1+\sqrt{x})-\frac{(\sqrt{x}-1)^2}{2}+c$$