An amalgamated duplication of a ring along an ideal: the basic properties

Marco D'Anna * Marco Fontana †

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Dedicated to Luigi Salce, on his 60th birthday

Abstract

We introduce a new general construction, denoted by $R\bowtie E$, called the amalgamated duplication of a ring R along an R–module E, that we assume to be an ideal in some overring of R. (Note that, when $E^2 = 0$, $R\bowtie E$ coincides with the Nagata's idealization $R\ltimes E$.)

After discussing the main properties of the amalgamated duplication $R\bowtie E$ in relation with pullback–type constructions, we restrict our investigation to the study of $R\bowtie E$ when E is an ideal of R. Special attention is devoted to the ideal-theoretic properties of $R\bowtie E$ and to the topological structure of its prime spectrum.

1 Introduction

If R is a commutative ring with unity and E is an R-module, *the idealization* $R\ltimes E$, introduced by Nagata in 1956 (cf. Nagata's book [16], page 2), is a new ring, containing R as a subring, where the module E can be viewed as an ideal such that its square is (0).

This construction has been extensively studied and has many applications in different contexts (cf. e.g. [17], [6], [9], [11]). Particularly important is the generalization given by Fossum, in [5], where he defined *a commutative extension of a ring R by an R–module E* to be an exact sequence of abelian groups:

$$0 \to E \xrightarrow{\iota} S \xrightarrow{\pi} R \to 0$$

where S is a commutative ring, the map π is a ring homomorphism and the R–module structure on E is related to S and to the maps ι and π by the

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equation $s \cdot \iota(e) = \iota(\pi(s) \cdot e)$ (for all $s \in S$ and $e \in E$). It is easy to see that the idealization $R \ltimes E$ is a very particular commutative extension of R by the R–module E (called *trivial extension of R by E* in [5]).

In this paper, we will introduce a new general construction, called the amalgamated duplication of a ring R along an R–module E (that we assume to be an ideal in some overring of R and so E is an R-submodule of the total ring of fractions $T(R)$ of R) and denoted by $R \bowtie E$ (see Lemma 2.4).

When $E^2 = 0$, the new construction $R \bowtie E$ coincides with the idealization $R \ltimes E$. In general, however, $R \bowtie E$ it is not a commutative extension in the sense of Fossum. One main difference of this construction, with respect to the idealization (or with respect to any Fossum's commutative extension) is that the ring $R \bowtie E$ can be a reduced ring (and, in fact, it is always reduced if R is a domain).

Motivations and some applications of the amalgamated duplication $R \bowtie E$ are discussed more in detail in two recent papers [1], [2]. More precisely, M. D'Anna [1] has studied some properties of this construction in case $E = I$ is a proper ideal of R, in order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings and he has applied this contruction to curve singularities. M. D'Anna and M. Fontana in [2] have considered the case of the amalgamated duplication of a ring, in a not necessarily Noetherian setting, along a multiplicative-canonical ideal in the sense of Heinzer-Huckaba-Papick [10].

The present paper is devoted to a more systematic investigation of the general construction $R \bowtie E$, with a particular consideration to the ideal-theoretic properties and to the topological structure of its prime spectrum. More precisely, the paper is divided in two parts: in Section 2 we study the main properties of the amalgamated duplication $R \bowtie E$. In particular we give a presentation of this ring as a pullback (cf. Proposition 2.6) and from this fact (cf. also [4], [7]) we obtain several connections between the properties of R and the properties of $R \bowtie E$ and some useful information about $\mathrm{Spec}(R \bowtie E)$ (cf. Proposition 2.13).

In Section 3 we consider the case when $E = I$ is an ideal of R; this situation allows us to deepen the results obtained in Section 2; in particular we give a complete description of $\mathrm{Spec}(R \bowtie I)$ (cf. Theorems 3.6 and 3.9).

2 The general construction

In this section we will study the construction of the ring $R \bowtie E$ in a general setting. More precisely, R will always be a commutative ring with unity, $T(R)$ ($:= \{\text{regular elements}\}^{-1}R$) its total ring of fractions and E an R-submodule of $T(R)$. Moreover, in order to construct the ring $R \bowtie E$, we are interested in those R-submodules of $T(R)$ such that $E \cdot E \subseteq E$.

Lemma 2.1 *Let E be an R-submodule of $T(R)$ and let J be an ideal of R.*

(a) *$E \cdot E \subseteq E$ if and only if there exists a subring S of $T(R)$ containing R and E, such that E is an ideal of S.*

(b) *If $E \cdot E \subseteq E$ then:*

$$R{+}E := \{z = r + e \in T(R) \mid r \in R,\ e \in E\}$$

is a subring of $(E : E) := \{z \in T(R) \mid zE \subseteq E\}$ $(\subseteq T(R))$, containing R as a subring and E as an ideal.

(c) *Assume that $E \cdot E \subseteq E$; the canonical ring homomorphism $\varphi : R \hookrightarrow R{+}E \to (R{+}E)/E$, $r \mapsto r + E$, is surjective and $\mathrm{Ker}(\varphi) = E \cap R$.*

(d) *Assume that $E \cdot E \subseteq E$; the set $J{+}E := \{j{+}e \mid j \in J,\ e \in E\}$ is an ideal of $R{+}E$ containing E and $(J{+}E) \cap R = \mathrm{Ker}(R \hookrightarrow R{+}E \to (R{+}E)/(J{+}E)) = J{+}(E \cap R)$.*

Proof. (a) It is clear that the implication "if" holds. Conversely, set $S := (E : E)$. The hypothesis that $E \cdot E \subseteq E$ implies that E is an ideal of S and that S is a subring of $T(R)$ containing R as a subring.

(b) It is obvious that $R{+}E$ is an R-submodule of $(E : E)$ containing R and E. Moreover, let $r, s \in R$ and $e, f \in E$, if $z := r + e$ and $w := s + f$ $(\in R{+}E)$ then $zw = rs + (rf + se + ef) \in R{+}E$ and $zf = rf + ef \in E$.

(c) and (d) are straightforward. $\square$

From now on we will always assume that $E \cdot E \subseteq E$.

In the R-module direct sum $R \oplus E$ we can introduce a multiplicative structure by setting:

$$(r, e)(s, f) := (rs, rf + se + ef)\,, \text{ where } r, s \in R \text{ and } e, f \in E\ .$$

We denote by $R \dot{\bigoplus} E$ the direct sum $R \oplus E$ endowed also with the multiplication defined above.

The following properties are easy to check:

Lemma 2.2 *With the notation introduced above, we have:*

(a) *$R \dot{\bigoplus} E$ is a ring.*

(b) *The map $j : R \dot{\bigoplus} E \to R \times (R{+}E)$, defined by $(r, e) \mapsto (r, r + e)$, is an injective ring homomorphism.*

(c) *The map $i : R \to R \dot{\bigoplus} E$, defined by $r \mapsto (r, 0)$, is an injective ring homomorphism.* $\square$

Remark 2.3 **(a)** With the notation of Lemma 2.1, note that if $E = S$ is a subring of $T(R)$ containing as a subring R, then $R{+}S = S$. Also, if I is an ideal of R, then $R{+}I = R$.

(b) In the statement of Lemma 2.1 (d), note that, in general, $J{+}E$ does not coincide with the extension of J in $R{+}E$: we have $J(R{+}E) = \{j + \alpha \mid j \in J,\ \alpha \in JE\} \subseteq J{+}E$, but the inclusion can be strict (cf. Proposition 3.5 (a) and (b)).

(c) For an *arbitrary* R-module E, M. Nagata introduced in 1955 *the idealization of* E *in* R*,* denoted here by $R\ltimes E$, which is the R–module $R\oplus E$ endowed with a multiplicative structure defined by:

$$(r,e)(s,f) := (rs, rf+se)\,, \quad \text{where } r,s \in R \text{ and } e,f \in E$$

(cf. [15] and also Nagata's book [16, page 2] and Huckaba's book [11, Chapter VI, Section 25]). The idealization $R\ltimes E$, called also *the trivial extension of* R *by* E [5], is a ring such that the canonical embedding $R \hookrightarrow R\ltimes E$, $r \mapsto (r,0)$, defines a subring of $R\ltimes E$ isomorphic to R and the embedding $E \hookrightarrow R\ltimes E$, $e \mapsto (0,e)$, defines an ideal $E^{\ltimes}$ in $R\ltimes E$ (isomorphic as an R-module to E), which is nilpotent of index 2 (i.e. $E^{\ltimes}\cdot E^{\ltimes} = 0$). Therefore, even if R is reduced, the idealization $R\ltimes E$ is not a reduced ring, except in the trivial case for $E=(0)$, since $R\ltimes(0)=R$. Moreover, if $p_R : R\ltimes E \to R$ is the canonical projection (defined by $(r,e)\mapsto r$), then

$$0 \to E \to R\ltimes E \xrightarrow{p_R} R \to 0$$

is an exact sequence.

Note that the idealization $R\ltimes E$ coincides with the ring $R\dot{\bigoplus}E$ (Lemma 2.2) if and only if E is an R-submodule of $T(R)$ that is nilpotent of index 2 (i.e. $E\cdot E=(0)$).

Lemma 2.4 *With the notation of Lemma 2.2, note that* $\delta := j\circ i : R \hookrightarrow R\times(R{+}E)$ *is the diagonal embedding and set:*

$$R^{\vartriangle} := (j\circ i)(R) = \{(r,r) \mid r\in R\} \quad \textit{and}$$
$$R\bowtie E := j(R\dot{\bigoplus}E) = \{(r,r+e) \mid r\in R,\ e\in E\}\,.$$

We have:

(a) *The canonical maps* $R\cong R^{\vartriangle} \subseteq R\bowtie E \subseteq R\times T(R)$ *are ring homomorphisms.*

(b) $R\bowtie E$ *is a subdirect product of the ring* $R\times(R{+}E)$*, i.e. if* π_i $(i=1,2)$ *are the projections of* $R\times(R{+}E)$ *onto* R *and* $R{+}E$*, respectively, and if* $\boldsymbol{\mathfrak{O}_i} := \mathrm{Ker}(\pi_i|_{R\bowtie E})$*, then* $(R\bowtie E)/\boldsymbol{\mathfrak{O}_1}\cong R$*,* $(R\bowtie E)/\boldsymbol{\mathfrak{O}_2}\cong R{+}E$ *and* $\boldsymbol{\mathfrak{O}_1}\cap\boldsymbol{\mathfrak{O}_2}=(0)$*.*

Proof. (a) is obvious. For (b) recall that S is a subdirect product of a family of rings $\{R_i \mid i\in I\}$ if there exists a ring monomorphism $\varphi : S\hookrightarrow \prod_i R_i$ such that, for each $i\in I$, $\pi_i\circ\varphi : S\to R_i$ is a surjection (where $\pi_i : \prod_i R_i \to R_i$ is the canonical projection) [13, page 30]. Note also that $\boldsymbol{\mathfrak{O}_1} = \{(0,e)\mid e\in E\}$ and $\boldsymbol{\mathfrak{O}_2} = \{(\varepsilon,0)\mid \varepsilon\in E\cap R\}$. The conclusion is straightforward (cf. also [13, Proposition 10]). $\square$

We will call the ring $R\bowtie E$, defined in Lemma 2.4, *the amalgamated duplication of a ring along an* R *module* E; the reason for this name will be clear after studying the prime spectrum of $R\bowtie E$ and comparing it with the prime spectrum of R (see Proposition 2.13). The following is an easy consequence of the previous lemma.

Corollary 2.5 *With the notation of Lemma 2.4, the following properties are equivalent:*

(i) *R is a domain;*

(ii) *$R{+}E$ is a domain;*

(iii) *$\mathfrak{O}_\mathbf{1}$ is a prime ideal of $R\bowtie E$;*

(iv) *$\mathfrak{O}_\mathbf{2}$ is a prime ideal of $R\bowtie E$;*

(v) *$R\bowtie E$ is a reduced ring and $\mathfrak{O}_\mathbf{1}$ and $\mathfrak{O}_\mathbf{2}$ are prime ideals of $R\bowtie E$.* □

We will see in a moment that R is a domain if and only if $\mathfrak{O}_\mathbf{1}$ and $\mathfrak{O}_\mathbf{2}$ are the only minimal prime ideals $R\bowtie E$ (cf. Remark 2.8).

Proposition 2.6 *Let $v : R\times(R{+}E) \twoheadrightarrow R\times((R{+}E)/E)$ and $u : R \hookrightarrow R\times((R{+}E)/E)$ be the natural ring homomorphisms defined, respectively, by $v((x, r + e)) := (x, r + E)$ and $u(r) := (r, r + E)$, for each $x, r \in R$ and $e \in E$. Then $v^{-1}(u(R)) = R\bowtie E$. Therefore, if v' $(:= \pi_1|_{R\bowtie E}) : R\bowtie E \twoheadrightarrow R$ is the canonical map defined by $(r, r + e) \mapsto r$ (cf. Lemma 2.4) and $u' : R\bowtie E \hookrightarrow R\times(R{+}E)$ is the natural embedding, then the following diagram:*

$$\begin{array}{ccc}
R\bowtie E & \xrightarrow{v'} & R \\
{\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle u} \\
R\times(R{+}E) & \xrightarrow{v} & R\times((R{+}E)/E)
\end{array}$$

is a pullback.

Proof. Since E is an ideal of $R{+}E$ (Lemma 2.1 (b)), $\mathfrak{O}_\mathbf{1} = (0)\times E$ is a common ideal of $v^{-1}(u(R))$ and $R\times(R{+}E)$. Moreover, by definition, if $x, r \in R$ and $e \in E$, then $(x, r + e) \in v^{-1}(u(R))$ if and only if $(x, r + E) \in u(R)$, that is $x - r \in E$. Therefore we conclude that $v^{-1}(u(R)) = R\bowtie E$. The second part of the statement follows easily from the fact that $v^{-1}(u(R)) = R\bowtie E$ and $(R\bowtie E)/\mathfrak{O}_\mathbf{1} \cong R$, with $\mathfrak{O}_\mathbf{1} = \mathrm{Ker}(v')$ (Proposition 2.4 (b)). □

Corollary 2.7 *The ring $R\times(R{+}E)$ is a finitely generated $(R\bowtie E)$–module. In particular, $R\bowtie E \subseteq R\times(R{+}E)$ is an integral extension and* $\dim(R\bowtie E) = \dim(R\times(R{+}E)) = \sup\{\dim(R), \dim(R{+}E)\}$.

Proof. Clearly $u : R \hookrightarrow R\times((R{+}E)/E)$ is a finite ring homomorphism, since $R\times((R{+}E)/E)$ is generated by $(1, 0)$ and $(0, 1)$ as R–module. Since u is finite, also $u' : R\bowtie E\ (= v^{-1}(u(R))) \hookrightarrow R\times((R{+}E)/E)$ is a finite ring homomorphism [4, Corollary 1.5 (4)]. Last statement follows from [12, Theorems 44 and 48] and from the fact that $\mathrm{Spec}(R\times(R{+}E))$ is homeomorphic to the disjoint union of $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R{+}E)$ (cf. also Remark 2.8). □

Remark 2.8 Recall that every ideal of the ring $R \times (R+E)$ is a direct product of ideals $I \times J$, with I ideal of R and J ideal of $R+E$. In particular, every prime ideal Q of $R \times (R+E)$ is either of the type $I \times (R+E)$ or $R \times J$, with I prime ideal of R and J prime ideal of $(R+E)$. Therefore, in the situation of Lemma 2.4, if R is an integral domain (and so $R+E$ also is an integral domain by Corollary 2.5), then $(0) \times (R+E)$ and $R \times (0)$ are necessarily the only minimal primes of $R \times (R+E)$. By the integrality property (Corollary 2.7 and [12, Theorem 46]), then $\mathfrak{O}_1 = ((0) \times (R+E)) \cap (R \bowtie E) = (0) \times E$ and $\mathfrak{O}_2 = (R \times (0)) \cap (R \bowtie E) = (R \cap E) \times (0)$ are the only minimal primes of $R \bowtie E$.

Conversely, if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal primes of $R \bowtie E$, then clearly $R \bowtie E$ is a reduced ring (Lemma 2.4 (b)) and, by Corollary 2.5, R is an integral domain.

Corollary 2.9 *The following statements are equivalent:*

(i) *R and $R+E$ are Noetherian;*

(ii) *$R \times (R+E)$ is Noetherian;*

(iii) *$R \bowtie E$ is Noetherian.*

Proof. Clearly (i) and (ii) are equivalent. The statements (ii) and (iii) are equivalent by the Eakin-Nagata Theorem [14, Theorem 3.7], since $R \times (R+E)$ is a finitely generated $(R \bowtie E)$–module (Corollary 2.7). □

Remark 2.10 **(a)** In the situation of Proposition 2.6, the pullback degenerates in two cases:

(1) $v' : R \bowtie E \to R$ is an isomorphism if and only if $E = 0$;

(2) $u' : R \bowtie E \to R \times (R+E)$ is an isomorphism if and only if E is an overring of R (i.e., if and only if $E = R+E$).

(b) By the previous remark, we deduce easily that R Noetherian does not imply in general that $R+E$ is Noetherian and, conversely, $R+E$ Noetherian does not imply that R is Noetherian: take, for instance, E to be an arbitrary overring of R. However, if we assume that R+E is a finitely generated R-module (cf. also the following Corollary 2.11), then by the Eakin-Nagata Theorem [14, Theorem 3.7] R is Noetherian if and only if $R+E$ is Noetherian.

This same situation described above (i.e. when E is an arbitrary overring of R) shows that, in Corollary 2.7, we may have that $\dim(R \bowtie E) = \dim(R)$ or that $\dim(R \bowtie E) = \dim(R+E)$ (with $\dim(R) \neq \dim(R+E)$).

Corollary 2.11 *Assume that E is a fractional ideal of R (i.e. there exists a regular element $d \in R$ such that $dE \subseteq R$); then the following statements are equivalent:*

(i) *R is a Noetherian ring;*

(ii) *$R+E$ is a Noetherian R-module;*

(iii) *$R \times (R+E)$ is a Noetherian ring;*

(iv) *$R\bowtie E$ is a Noetherian ring.*

Proof. By Corollary 2.9 and by previous Remark 2.10 (b), it is sufficient to show that, in this case, R is a Noetherian ring if and only if $R+E$ is a Noetherian R-module. Clearly, if R is Noetherian, then E is a finitely generated R-module and so $R+E$ is also a finitely generated R-module and thus it is a Noetherian R-module. Conversely, assume that $R+E$ is a Noetherian R-module; since it is faithful, by [14, Theorem 3.5] it follows that R is a Noetherian ring. □

Corollary 2.12 *In the situation described above:*

(a) *Let R' and $(R+E)'$ be the integral closures of R and $R+E$ in $T(R)$. Then $R\bowtie E$ and $R\times(R+E)$ have the same integral closure in $T(R)\times T(R)$, which is precisely $R'\times(R+E)'$. Moreover, if $R+E$ is a finitely generated R-module, then the integral closure of R^{Δ} in $T(R)\times T(R)$ (Lemma 2.4) also coincides with $R'\times(R+E)'$.*

(b) *If $E\cap R$ contains a regular element, then $T(R\bowtie E)=T(R\times(R+E))=T(R)\times T(R)$ and, moreover, $R\bowtie E$ and $R\times(R+E)$ have the same complete integral closure in $T(R)\times T(R)$.*

Proof. (a) It is clear that $(x,y)\in T(R)\times T(R)$ is integral over $R\times(R+E)$ if and only if $(x,y)\in R'\times(R+E)'$. Since the extension $R\bowtie E\hookrightarrow R\times(R+E)$ ($\subseteq T(R)\times T(R)$) is integral (Corollary 2.7), we have the first statement. If, in addition, we assume that $R+E$ is a finitely generated R-module, then the ring extension $R^{\Delta}\hookrightarrow R\times(R+E)$ (Lemma 2.4) is finite (so, in particular, integral) and thus we have the second statement.

(b) Since E is an R-submodule of $T(R)$, then clearly $T(R)=T(R+E)$, hence it is obvious that $T(R\times(R+E))=T(R)\times T(R)$. If e is a nonzero regular element of $E\cap R$, then (e,e) is a nonzero regular element belonging to $(E\cap R)\times E$, which is a common ideal of $R\bowtie E$ and $R\times(R+E)$. From this fact it follows that $R\bowtie E$ and $R\times(R+E)$ have the same total quotient ring [8, page 326] and so $T(R\bowtie E)=T(R)\times T(R)$. The last statement follows from [8, Lemma 26.5]. □

Note that, in Corollary 2.12 (b), the assumption that $E\cap R$ contains a regular element is essential, since if E is the ideal (0) of an integral domain R with quotient field K, then $R\bowtie(0)\cong R$ and so $T(R\bowtie(0))\cong K$, but $T(R\times R)=K\times K$.

Using Proposition 2.6 and Corollary 2.7 we are now able to describe the relation between $\mathrm{Spec}(R\bowtie E)$, $\mathrm{Spec}(R\times(R+E))$ and $\mathrm{Spec}(R)$. Recall that if $f:A\to B$ is a ring homomorphism, $f^a:\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ denotes, as usual, the continuous map canonically associated to f, i.e. $f^a(Q):=f^{-1}(Q)$, for each $Q\in\mathrm{Spec}(B)$; if I is an ideal of A and if $\mathcal{X}:=\mathrm{Spec}(A)$, $V_{\mathcal{X}}(I)$ denotes the Zariski-closed set $\{P\in\mathcal{X}\mid P\supseteq I\}$ of $\mathcal{X}$.

Proposition 2.13 *In the situation of Lemma 2.4 and with the notation of Proposition 2.6, set* $X := \mathrm{Spec}(R)$*,* $Y := \mathrm{Spec}(R \bowtie E)$ *and* $Z := \mathrm{Spec}(R \times (R+E))$ *and set* $\alpha := (u')^a : Z \to Y$ *and* $\beta := (v')^a : X \to Y$*. Then the following properties hold:*

(a) *The canonical continuous map* $\alpha : Z \to Y$ *is surjective.*

(b) *The restriction of the map* $\alpha : Z \to Y$ *to* $Z \setminus V_Z(\mathfrak{O}_1)$ *gives rise to a topological homeomorphism:*

$$\alpha|_{Z\setminus V_Z(\mathfrak{O}_1)} : Z \setminus V_Z(\mathfrak{O}_1) \overset{\cong}{\longrightarrow} Y \setminus V_Y(\mathfrak{O}_1)\,.$$

Moreover, for each $Q \in \mathrm{Spec}(R \times (R+E))$*, with* $Q \not\supseteq \mathfrak{O}_1$*, if* $\mathcal{Q} := \alpha(Q) = Q \cap (R \bowtie E)$*, then the canonical map* $(R \bowtie E)_{\mathcal{Q}} \to (R \times (R+E))_Q$ *is a ring isomorphism.*

(c) $\beta : X \to Y$ *defines a canonical homeomorphism of* X *with* $V_Y(\mathfrak{O}_1)$*; moreover, for each* $\mathcal{Q} \in \mathrm{Spec}(R \bowtie E)$ *with* $\mathcal{Q} \supseteq \mathfrak{O}_1$*, the canonical ring homomorphism* $(R \bowtie E)/\mathcal{Q} \to R/v'(\mathcal{Q})$ *is an isomorphism.*

Proof. (a) Follows immediately by Corollary 2.7. (b) and (c) are consequences of Proposition 2.6 and, respectively, Theorem 1.4 (c) and Corollary 1.5 (1) of [4]. □

We conclude this section by defining some distinguished ideals of $R \bowtie E$ that are naturally associated to a given ideal J of R and by giving an example of the general construction.

Proposition 2.14 *In the situation of Proposition 2.6 and with the notation of Lemma 2.1, for each ideal* J *of* R *we can consider the following ideals of* $R \bowtie E$*:*

$$\mathcal{J}_1 := v'^{-1}(J)\,, \;\; \mathcal{J}_2 := u'^{-1}(R \times J(R+E)) \;\text{ and }\; \mathcal{J}_0 := J^e := J(R \bowtie E)\,.$$

Then we have:

(a) $\mathcal{J}_1 = u'^{-1}(J \times (R+E)) = u'^{-1}(J \times (J+E)) = \{(j, j+e) \mid j \in J,\ e \in E\}\,.$

(b) $\mathcal{J}_0 = \{(j, j+\alpha) \mid j \in J,\ \alpha \in JE\}\,.$

(c) $\mathcal{J} := \mathcal{J}_1 \cap \mathcal{J}_2 = u'^{-1}(J \times J(R+E))\,.$

(d) $\mathcal{J}_0 \subseteq \mathcal{J}_1 \cap \mathcal{J}_2\,.$

Proof. (a) and (b) are straightforward. Statement (c) is obvious, since $J \times J(R+E) = (J \times (R+E)) \cap (R \times J(R+E))$. (d) follows from (c) and from the fact that $J(R \bowtie E) \subseteq u'^{-1}(J(R \times (R+E))) = u'^{-1}(J \times J(R+E))$. □

Example 2.15 Let $R := k[t^4, t^6, t^7, t^9]$ (where k is a field and t an indeterminate), $S := k[t^2, t^3]$ and $E := (t^2, t^3)S = t^2k[t]$. We have that $R+E = S$ and hence

$$\begin{aligned} R \bowtie E =& \{(f(t), g(t)) \mid f \in R,\ g \in S \text{ and } g - f \in E\} = \\ =& \{(f(t), g(t)) \mid f \in R,\ g \in S \text{ and } f(0) = g(0)\}\,. \end{aligned}$$

Since E is a maximal ideal of S, the prime ideals in $R \times S$ containing $\mathfrak{O}_1$ are either of the form $P \times S$, for some prime ideal P of R, or $R \times E$; hence the primes not containing $\mathfrak{O}_1$ are of the form $R \times Q$, with $Q \in \mathrm{Spec}(S)$ and $Q \neq E$.

By Propositions 2.13 and 2.14, we have that if P is a prime in R, the ideal $\boldsymbol{\mathcal{P}}_1 = (v')^{-1}(P) = (u')^{-1}(P \times S) = \{(p, p+e) \mid p \in P,\ e \in E\}$ is a prime in $R \bowtie E$, containing $\mathfrak{O}_1$, and $R \bowtie E/\boldsymbol{\mathcal{P}}_1 \cong R/P$. Moreover, with the notation of Proposition 2.13, in this way we describe completely $V_Y(\mathfrak{O}_1)$. Notice also that, if we set $M := (t^4, t^6, t^7, t^9)R$, then the maximal ideals $M \times S$ and $R \times E$ of $R \times S$ have the same trace in $R \bowtie E$, i.e. $(R \times E) \cap (R \bowtie E) = \{(r, r+e) \mid r \in R \cap E,\ e \in E\} = (M \times S) \cap (R \bowtie E)$.

On the other hand, again by Proposition 2.13, we have that $Y \setminus V_Y(\mathfrak{O}_1)$ is homeomorphic to $Z \setminus V_Z(\mathfrak{O}_1)$. Hence the prime ideals of $R \bowtie E$ not containing $\mathfrak{O}_1$ are of the form $(R \times Q) \cap (R \bowtie E)$, for some prime ideal Q of S, with $Q \neq E$.

3 The prime spectrum of $R \bowtie I$

In this section we study the case when the R-module $E = I$ is an ideal of R (that we will assume to be proper and different from (0), to avoid the trivial cases); in this situation $R + I = R$. We start with applying to this case some of the results we obtained in the general situation.

Proposition 3.1 *Using the notation of Proposition 2.6, the following commutative diagram of canonical ring homomorphisms*

$$\begin{array}{ccc} R \bowtie I & \xrightarrow{v'} & R \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ R \times R & \xrightarrow{v} & R \times (R/I) \end{array}$$

is a pullback. The ideal $\mathfrak{O}_1 = (0) \times I = \mathrm{Ker}(v') = \mathrm{Ker}(v)$ is a common ideal of $R \bowtie I$ and $R \times R$, the ideal $\mathfrak{O}_2 = \mathrm{Ker}(R \bowtie I \xrightarrow{u'} R \times R \xrightarrow{\pi_2} R)$ coincides with $I \times (0) = (I \times (0)) \cap (R \bowtie I)$ and $(R \bowtie I)/\mathfrak{O}_i \cong R$, for $i = 1, 2$.

In particular, if R is a domain then $R \bowtie I$ is reduced and $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal primes of $R \bowtie I$.

Proof. The first part is an easy consequence of Lemma 2.4 (b) and Proposition 2.6; the last statement follows from Corollary 2.5. □

Remark 3.2 Note that, when $I \subseteq R$, then $R \bowtie I := \{(r, r+i) \mid r \in R,\ i \in I\} = \{(r+i, r) \mid r \in R,\ i \in I\}$. It follows that we can exchange the roles of $\mathfrak{O}_1$ and $\mathfrak{O}_2$ (and that $\mathfrak{O}_2$ is also a common ideal of $R \bowtie I$ and $R \times R$).

If we specialize to the present situation Corollary 2.7, Corollary 2.11 and Corollary 2.12, then we obtain:

Corollary 3.3 *Let R' (respectively, R^*) be the integral closure (respectively, the complete integral closure) of R in $T(R)$, we have:*

(a) $\dim(R\bowtie I) = \dim(R)$.

(b) *R is Noetherian if and only if $R\bowtie I$ is Noetherian.*

(c) *The integral closure of R^{Δ} and of $R \bowtie I$ in $T(R) \times T(R)$ coincide with $R' \times R'$.*

(d) *If I contains a regular element, then $T(R \bowtie I) = T(R) \times T(R)$ and the complete integral closure of $R\bowtie I$ in $T(R) \times T(R)$ coincide with $R^* \times R^*$, which is the complete integral closure of $R \times R$ in $T(R) \times T(R)$.*

Remark 3.4 We can now use Proposition 2.13 to describe $\mathrm{Spec}(R\bowtie I)$. Note that, in the general case (possibly with zero-divisors), if $\boldsymbol{\mathfrak{Q}} \in \mathrm{Spec}(R\bowtie I)$, then either $\boldsymbol{\mathfrak{Q}} \not\supseteq \boldsymbol{\mathfrak{O}_1}$ or $\boldsymbol{\mathfrak{Q}} \supseteq \boldsymbol{\mathfrak{O}_1}$.

Case 1. $\boldsymbol{\mathfrak{Q}} \not\supseteq \boldsymbol{\mathfrak{O}_1}$.

In this case, by Proposition 2.13, there exists a unique prime ideal Q of $R \times R$ such that $\boldsymbol{\mathfrak{Q}} = Q \cap (R \bowtie I)$ and $Q \not\supseteq (0) \times I$. Therefore $Q \not\supseteq (0) \times R$ and so $Q \supseteq R \times (0)$. Hence $Q = R \times P$ for some prime P of R such that $P \not\supseteq I$.

With a slight abuse of notation, we identify R with its isomorphic image R^{Δ} in $R \bowtie I$ $(\subseteq R \times R)$ under the diagonal embedding (Lemma 2.4) and we denote the contraction to R of an ideal $\boldsymbol{\mathcal{H}}$ of $R\bowtie I$ (or, H of $R \times R$) by $\boldsymbol{\mathcal{H}} \cap R$ (or, by $H \cap R$). Using this notation, then $P = Q \cap R = \boldsymbol{\mathfrak{Q}} \cap R$. Moreover, note that:

$$\boldsymbol{\mathfrak{Q}} = \{(p+i, p) \mid p \in P,\ i \in I\} = (R \times P) \cap (R\bowtie I)\ .$$

Furthermore, the canonical ring homomorphisms $R\bowtie I \hookrightarrow R \times R \xrightarrow{\pi_2} R$ induce for the localizations the following isomorphisms:

$$(R\bowtie I)_{\boldsymbol{\mathfrak{Q}}} \cong (R \times R)_Q = (R \times R)_{R\times P} \cong R_P\ .$$

Case 2. $\boldsymbol{\mathfrak{Q}} \supseteq \boldsymbol{\mathfrak{O}_1}$.

In this case, by Proposition 2.13, there exists a unique prime ideal P of R such that $\boldsymbol{\mathfrak{Q}} = v'^{-1}(P)$ (or, equivalently, $P = v'(\boldsymbol{\mathfrak{Q}})$); hence $\boldsymbol{\mathfrak{Q}} = \{(p, p+i) \mid p \in P,\ i \in I\} = (P \times R) \cap (R\bowtie I)$.

Now, if $P \supseteq I$, it is easy to see that $\boldsymbol{\mathfrak{Q}}$ $(= (P \times R) \cap (R \bowtie I)) = \{(p', p' + i') \mid p' \in P,\ i' \in I\} = (R \times P) \cap (R \bowtie I)$. On the other hand, if $P \not\supseteq I$, then $\boldsymbol{\mathfrak{Q}} = (P \times R) \cap (R\bowtie I) \neq (R \times P) \cap (R\bowtie I)$.

Note also that, in this case, by Proposition 2.13, $(R\bowtie I)/\boldsymbol{\mathfrak{Q}} \cong R/P$.

After studying the relation between $\mathrm{Spec}(R \times R)$ and $\mathrm{Spec}(R \bowtie I)$, under the continuous map $(u')^a$, associated the canonical embedding $u' : R \bowtie I \hookrightarrow R \times R$, the next goal is to investigate directly the relation between $\mathrm{Spec}(R\bowtie I)$ and $\mathrm{Spec}(R)$, under the canonical map associated to the diagonal embedding $\delta : R \hookrightarrow R\bowtie I$, $(r \mapsto (r, r))$. As above, we will identify R with its isomorphic image R^{Δ} in $R \bowtie I$ and we will denote the contraction to R of an ideal $\boldsymbol{\mathcal{H}}$ of $R\bowtie I$ by $\boldsymbol{\mathcal{H}} \cap R$ (instead of $\delta^{-1}(\boldsymbol{\mathcal{H}})$).

Proposition 3.5 *With the notation of Proposition 2.14, let J be an ideal of R. Then:*

(a) $\boldsymbol{\mathcal{J}_1}$ $(:= v'^{-1}(J)) = u'^{-1}(J \times R) = u'^{-1}(J \times (J{+}I)) = \{(j, j+i) \mid j \in J,\ i \in I\} =: J \bowtie I$. *If* $J = I$, *then* $I \bowtie I$ $(= I \times I)$ *is a common ideal of* $R \bowtie I$ *and* $R \times R$.

(b) $\boldsymbol{\mathcal{J}_2}$ $(:= u'^{-1}(R \times J)) = \{(j+i, j) \mid j \in J,\ i \in I\}$.

(c) $\boldsymbol{\mathcal{J}} := \boldsymbol{\mathcal{J}_1} \cap \boldsymbol{\mathcal{J}_2} = u'^{-1}(J \times J) = \{(j, j+i') \mid j \in J,\ i' \in I \cap J\} = \{(j_1, j_2) \mid j_1, j_2 \in J,\ j_1 - j_2 \in I\}$.

(d) $\boldsymbol{\mathcal{J}_0}$ $(:= J(R \bowtie I)) = \{(j, j+i'') \mid j \in J,\ i'' \in JI\}$ (cf. [1, Lemma 8]).

(e) $\boldsymbol{\mathcal{J}_0} \subseteq \boldsymbol{\mathcal{J}_1} \cap \boldsymbol{\mathcal{J}_2}$.

(f) $\boldsymbol{\mathcal{J}_1} = \boldsymbol{\mathcal{J}_2} \ \Leftrightarrow\ I \subseteq J$.

(g) $I + J = R \ \Rightarrow\ \boldsymbol{\mathcal{J}_0} = \boldsymbol{\mathcal{J}_1} \cap \boldsymbol{\mathcal{J}_2}$.

(h) $\boldsymbol{\mathcal{J}_1} \cap R = \boldsymbol{\mathcal{J}_2} \cap R = \boldsymbol{\mathcal{J}_0} \cap R = \boldsymbol{\mathcal{J}} \cap R = J$.

Proof. (a) is a particular case of Proposition 2.14 (a). The second part is straightforward.

(b) Let $r \in R$ and $j \in J$; we have that $(r, j) \in R \bowtie I$ if and only if $(r, j) = (s, s{+}i)$, for some $s \in R$ and $i \in I$. Therefore $r = s = j - i$ and $(r, j) = (j{+}i', j)$ for some $i' \in I$.

(c) Let $j_1, j_2 \in J$; we have that $(j_1, j_2) \in R \bowtie I$ if and only if $(j_1, j_2) = (s, s+i)$, for $s \in R$ and $i \in I$. Therefore $j_1 = s$, $j_2 = j_1 + i$ and $j_2 - j_1 = i \in I$.

Statements (d) and (e) are particular cases of Proposition 2.14 ((b) and (d)).

(f) follows easily from (a) and (b), since:

$$\boldsymbol{\mathcal{J}_1} = \boldsymbol{\mathcal{J}_2} \ \Rightarrow\ J{+}I = J \ \Rightarrow\ I \subseteq J \ \Rightarrow\ \boldsymbol{\mathcal{J}_1} = \boldsymbol{\mathcal{J}_2}\ .$$

(g) is a consequence of (c) and (d), since $J{+}I = R$ implies that $J \cap I = JI$.

(h) It is obvious that $\boldsymbol{\mathcal{J}_1} \cap R = J = \boldsymbol{\mathcal{J}_2} \cap R$ and hence, by (c) and (e), we also have $\boldsymbol{\mathcal{J}} \cap R = \boldsymbol{\mathcal{J}_0} \cap R = J$. $\square$

With the help of the previous proposition we can make Remark 3.4 more precise. In the following, the residue field at the prime ideal Q of a ring A (i.e. the field A_Q/QA_Q) will be denoted by $\boldsymbol{k}_A(Q)$. Part of the next theorem is contained in [1, Proposition 5].

Theorem 3.6 *Let* P *be a prime ideal of* R *and consider the ideals* $\boldsymbol{\mathcal{P}_1}$, $\boldsymbol{\mathcal{P}_2}$, $\boldsymbol{\mathcal{P}_0}$ *and* $\boldsymbol{\mathcal{P}}$ *of* $R \bowtie I$ *as in Proposition 3.5 (with* $P = J$*). Then:*

(a) $\boldsymbol{\mathcal{P}_1}$ *and* $\boldsymbol{\mathcal{P}_2}$ *are the only prime ideals of* $R \bowtie I$ *lying over* P .

(b) *If* $P \supseteq I$, *then* $\boldsymbol{\mathcal{P}_1} = \boldsymbol{\mathcal{P}_2} = \boldsymbol{\mathcal{P}} = \sqrt{\boldsymbol{\mathcal{P}_0}} = P \bowtie I$. *Moreover,* $\boldsymbol{k}_R(P) \cong \boldsymbol{k}_{R \bowtie I}(\boldsymbol{\mathcal{P}})$.

(c) *If* $P \not\supseteq I$ *then* $\boldsymbol{\mathcal{P}_1} \neq \boldsymbol{\mathcal{P}_2}$. *Moreover* $\boldsymbol{\mathcal{P}} = \sqrt{\boldsymbol{\mathcal{P}_0}}$ *and* $\boldsymbol{k}_R(P) \cong \boldsymbol{k}_{R \bowtie I}(\boldsymbol{\mathcal{P}_1}) \cong \boldsymbol{k}_{R \bowtie I}(\boldsymbol{\mathcal{P}_2})$.

(d) *If P is a maximal ideal of R then $\boldsymbol{\mathcal{P}_1}$ and $\boldsymbol{\mathcal{P}_2}$ are maximal ideals of $R \bowtie I$.*

(e) *If R is a local ring with maximal ideal M then $R \bowtie I$ is a local ring with maximal ideal $\boldsymbol{\mathcal{M}} = \sqrt{\boldsymbol{\mathcal{M}_0}} = M \bowtie I$ (using again the notation of Proposition 3.5 for $M = J$) .*

(f) *R is reduced if and only if $R \bowtie I$ is reduced.*

Proof. Note that the composition of the diagonal embedding $\delta : R \hookrightarrow R \bowtie I$, $(r \mapsto (r, r))$, with the inclusion $R \bowtie I \subseteq R \times R$, $((r, r+i) \mapsto (r, r+i))$, coincides with the diagonal embedding $R \hookrightarrow R \times R$, $(r \mapsto (r, r))$, which is a finite ring homomorphism. Thus, in particular, both $R \hookrightarrow R \bowtie I$ and $R \bowtie I \subseteq R \times R$ are integral homomorphisms. Note also that if Q is a prime ideal of $R \times R$ lying over P, then necessarily $Q \in \{P \times R, R \times P\}$ (Remark 2.8).

(a) Note that $\boldsymbol{\mathcal{P}_1} = u'^{-1}(P \times R)$ and $\boldsymbol{\mathcal{P}_2} = u'^{-1}(R \times P)$ (Proposition 3.5); hence $\boldsymbol{\mathcal{P}_1}$ and $\boldsymbol{\mathcal{P}_2}$ are prime ideals lying over P. By integrality, if $\boldsymbol{\mathcal{Q}} \in \mathrm{Spec}(R \bowtie I)$ and $\boldsymbol{\mathcal{Q}} \cap R = P$, then there exists $\overline{Q} \in \mathrm{Spec}(R \times R)$ such that $\overline{Q} \cap (R \bowtie I) = \boldsymbol{\mathcal{Q}}$ and thus $\overline{Q} \cap R = P$. Therefore $\overline{Q} \in \{P \times R, R \times P\}$ and so $\boldsymbol{\mathcal{Q}} \in \{\boldsymbol{\mathcal{P}_1}, \boldsymbol{\mathcal{P}_2}\}$.

(b) We know already by Proposition 3.5 (f) and (c) that, if $P \supseteq I$, then $\boldsymbol{\mathcal{P}_1} = \boldsymbol{\mathcal{P}_2} = \boldsymbol{\mathcal{P}}$, hence by part (a) we conclude easily that $\boldsymbol{\mathcal{P}} = \sqrt{\boldsymbol{\mathcal{P}_0}}$. Moreover we have the following sequence of canonical homomorphisms:

$$\frac{R}{P} \subseteq \frac{R \bowtie I}{\sqrt{\boldsymbol{\mathcal{P}_0}}} = \frac{R \bowtie I}{\boldsymbol{\mathcal{P}}} \subseteq \frac{R \times R}{P \times R} \cong \frac{R}{P} \cong \frac{R \times R}{R \times P} \,,$$

from which we deduce the last part of the statement.

(c) By Proposition 3.5 (e) and (f) we know that, if $P \not\supseteq I$, then $\boldsymbol{\mathcal{P}_1} \neq \boldsymbol{\mathcal{P}_2}$ and $\boldsymbol{\mathcal{P}_0} \subseteq \boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}_1} \cap \boldsymbol{\mathcal{P}_2}$. By part (a) and by the integrality of $R \hookrightarrow R \bowtie I$, we conclude easily that $\boldsymbol{\mathcal{P}} = \sqrt{\boldsymbol{\mathcal{P}_0}}$. Finally, as in part (b), it is easy to see that $\boldsymbol{k}_R(P) \cong \boldsymbol{k}_{R \bowtie I}(\boldsymbol{\mathcal{P}_1}) \cong \boldsymbol{k}_{R \bowtie I}(\boldsymbol{\mathcal{P}_2})$.

(d) follows by the integrality of $R \subseteq R \bowtie I$.

(e) follows immediately by part (d) and part (b).

(f) follows by integrality of $R \hookrightarrow R \bowtie I$ and $R \bowtie I \subseteq R \times R$ and from the fact that R is reduced if and only if $R \times R$ is reduced. $\square$

Remark 3.7 In the situation of Theorem 3.6, note that, if P is a prime ideal of R, then by integrality of $R \hookrightarrow R \bowtie I \subseteq R \times R$, inside the ring $R \times R$, the prime ideals $P \times R$ and $R \times P$ are the only minimal prime ideals of $P \times P = \boldsymbol{\mathcal{P}_0}(R \times R) = P(R \times R)$, and so

$$\boldsymbol{\mathcal{P}_0}(R \times R) = P \times P = (P \times R) \cap (R \times P) = \sqrt{\boldsymbol{\mathcal{P}_0}(R \times R)}$$

is a radical ideal of $R \times R$, with

$$(P \times P) \cap (R \bowtie I) = ((P \times R) \cap (R \times P)) \cap (R \bowtie I) = \boldsymbol{\mathcal{P}_1} \cap \boldsymbol{\mathcal{P}_2} = \boldsymbol{\mathcal{P}} \,.$$

Next example shows that in $R \bowtie I$, in general, $\boldsymbol{\mathcal{P}_0}$ is not a radical ideal (i.e. it may happen that $\boldsymbol{\mathcal{P}_0} \subsetneq \sqrt{\boldsymbol{\mathcal{P}_0}} = \boldsymbol{\mathcal{P}}$).

Example 3.8 Let V be a valuation domain with a nonzero non maximal non idempotent prime ideal P. (An explicit example can be constructed as follows: let k be a field and let X, Y be two indeterminates over k, then take $V := k[X]_{(X)} + Yk(X)[Y]_{(Y)}$ and $P := Yk(X)[Y]_{(Y)}$. It is well known that V is discrete valuation domain of dimension 2, and P is the height 1 prime ideal of V [16, (11.4), page 35], [8, page 192].)

In this situation, it is easy to see that the ideal $P \times P$ is a common (radical) ideal of $V \bowtie P$ and of its overring $V \times V$. Moreover, note that $\boldsymbol{\mathcal{P}_0} = P(V \bowtie P) = \{(p, p+x) \mid p \in P,\ x \in P^2\}$ (Proposition 3.5 (d)) and that $P(V \times V) = P \times P \subset V \bowtie P$. More precisely, by Proposition 3.5 (c), we have:

$$\begin{aligned} P \times P = &\ (P \times P) \cap (V \bowtie P) = (P \times V) \cap (V \times P) \cap (V \bowtie P) \\ = &\ \boldsymbol{\mathcal{P}_1} \cap \boldsymbol{\mathcal{P}_2} = \boldsymbol{\mathcal{P}} = \{(p, p+y) \mid p \in p,\ y \in P \cap P = P\}. \end{aligned}$$

Clearly, since $P^2 \neq P$, then $\boldsymbol{\mathcal{P}_0} \subsetneq \boldsymbol{\mathcal{P}}$; for instance if $z \in P \setminus P^2$, then $(p, p+z) \in \boldsymbol{\mathcal{P}} \setminus P(V \bowtie P)$.

The next goal is to pursue the work initiated in Remark 3.4 and to give a complete description of the affine scheme $\mathrm{Spec}(R \bowtie I)$, determining the localizations of $R \bowtie I$ in each of its prime ideals. Part of the next theorem is contained in [1, Proposition 7].

Theorem 3.9 *Let $X := \mathrm{Spec}(R)$, $Y := \mathrm{Spec}(R \bowtie I)$ and $Z := \mathrm{Spec}(R \times R) \cong \mathrm{Spec}(R) \amalg \mathrm{Spec}(R)$ and let $\alpha : Z \twoheadrightarrow Y$ and $\gamma : Y \twoheadrightarrow X$ be the canonical surjective maps associated to the integral embeddings $R \bowtie I \hookrightarrow R \times R$ and $R \cong R^{\Delta} \hookrightarrow R \bowtie I$ (proof of Theorem 3.6).*

(a) *The restrictions of α*

$$\alpha\left|_{Z \setminus V_Z(\boldsymbol{\mathfrak{O}_i})}\right. : Z \setminus V_Z(\boldsymbol{\mathfrak{O}_i}) \longrightarrow Y \setminus V_Y(\boldsymbol{\mathfrak{O}_i})$$

(for $i = 1, 2$) are scheme isomorphisms, and clearly

$$Z \setminus V_Z(\boldsymbol{\mathfrak{O}_i}) \cong X \setminus V_X(I)\ .$$

In particular, for each prime ideal P of R, such that $P \not\supseteq I$, if we set $\overline{P_1} := P \times R$ and $\overline{P_2} := R \times P$ we have $\boldsymbol{\mathcal{P}}_i := \overline{P}_i \cap (R \bowtie I)$, for $1 \leq i \leq 2$ and the following canonical ring homomorphisms are isomorphisms:

$$R_P \longrightarrow (R \bowtie I)_{\boldsymbol{\mathcal{P}}_i} \longrightarrow (R \times R)_{\overline{P}_i}\,, \quad \textit{for } 1 \leq i \leq 2.$$

(b) *The restriction of γ*

$$\gamma\left|_{V_Y(\boldsymbol{\mathfrak{O}_1}) \cap V_Y(\boldsymbol{\mathfrak{O}_2})}\right. : V_Y(\boldsymbol{\mathfrak{O}_1}) \cap V_Y(\boldsymbol{\mathfrak{O}_2}) \longrightarrow V_X(I)$$

is a scheme isomorphism.

(c) *If $P \in \operatorname{Spec}(R)$ is such that $P \supseteq I$ and $\boldsymbol{\mathcal{P}} \in \operatorname{Spec}(R \bowtie I)$ is the unique prime ideal such that $\boldsymbol{\mathcal{P}} \cap R = P$, the following diagram of canonical homomorphisms:*

$$\begin{array}{ccc} (R\bowtie I)_{\boldsymbol{\mathcal{P}}} & \longrightarrow & R_P \\ \Big\downarrow & & \Big\downarrow{\scriptstyle u_P} \\ R_P \times R_P & \xrightarrow{v_P} & R_P \times (R_P/I_P) \end{array}$$

is a pullback (where $I_P := IR_P$, $u_P(x) := (x, x+I_P)$ and $v_P((x,y)) := (x, y+I_P)$, for $x, y \in R_P$), i.e. $(R\bowtie I)_{\boldsymbol{\mathcal{P}}} \cong R_P \bowtie I_P$ (Proposition 3.1).

Proof. (a) Since $\boldsymbol{\mathfrak{O}_1} = \{0\} \times I$ (respectively, $\boldsymbol{\mathfrak{O}_2} = I \times \{0\}$) is a common ideal of $R \times R$ and $R \bowtie I$, this statement follows from the general results on pullbacks [4, Theorem 1.4], from Remark 3.4, from Theorem 3.6 (part (a) and (c) and its proof) and from the fact that the canonical ring homomorphisms $R_P \hookrightarrow (R \times R)_{\overline{P}_i}$ are isomorphisms, for $1 \leq i \leq 2$. Note that $Z \setminus V_Z(\boldsymbol{\mathfrak{O}_1}) \cong ((X \amalg X) \setminus (X \amalg V_X(I))) = X \setminus V_X(I) = ((X \amalg X) \setminus (V_X(I) \amalg X)) \cong Z \setminus V_Z(\boldsymbol{\mathfrak{O}_2})$.
(b) Note that $V_Y(\boldsymbol{\mathfrak{O}_1}) \cap V_Y(\boldsymbol{\mathfrak{O}_2}) = V_Y(\boldsymbol{\mathfrak{O}_1} + \boldsymbol{\mathfrak{O}_2})$ and $\boldsymbol{\mathfrak{O}_1} + \boldsymbol{\mathfrak{O}_2} = I \times I$. Therefore the present statement follows from the fact that the canonical surjective homomorphism $R\bowtie I \to R/I$, defined by $(r, r+i) \mapsto r + I$ (for each $r \in R$ and $i \in I$) has kernel equal to $I \times I$.
(c) If we start from the pullback diagram considered in Proposition 3.1 and we apply the tensor product $R_P \otimes_R —$, then by [4, Proposition 1.9] we get the following pullback diagram:

$$\begin{array}{ccc} R_P \otimes_R (R\bowtie I) & \xrightarrow{id\otimes v'} & R_P \otimes_R R \\ {\scriptstyle id\otimes u'}\Big\downarrow & & \Big\downarrow{\scriptstyle id\otimes u} \\ R_P \otimes_R (R \times R) & \xrightarrow{id\otimes v} & R_P \otimes_R (R \times (R/I)) \,. \end{array}$$

Note that, by the properties of the tensor product, we deduce immediately the following canonical ring isomorphisms: $R_P \otimes_R (R \times R) \cong R_P \times R_P$, $R_P \otimes_R R \cong R_P$ and that $R_P \otimes_R (R \times (R/I)) \cong R_P \times (R_P \otimes_R (R/I)) \cong R_P \times (R_P/IR_P)$. Therefore, the previous pullback diagram gives rise to the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R_P \otimes_R (R\bowtie I) & \longrightarrow & R_P \\ \Big\downarrow & & \Big\downarrow{\scriptstyle u_P} \\ R_P \times R_P & \xrightarrow{v_P} & R_P \times (R_P/I_P) \,. \end{array}$$

On the other hand, recall that $\operatorname{Spec}(R_P \otimes_R (R\bowtie I))$ can be canonically identified (under the canonical homeomorphism associated to the natural ring homomorphism $R\bowtie I \to R_P \otimes_R (R\bowtie I)$) with the set of all prime ideals $\boldsymbol{\mathcal{H}} \in \operatorname{Spec}(R\bowtie I)$

such that $\boldsymbol{\mathcal{H}} \cap R \subseteq P$. Since we know already that, in the present situation, there exists a unique prime ideal $\boldsymbol{\mathcal{P}} \in \mathrm{Spec}(R \bowtie I)$ such that $\boldsymbol{\mathcal{P}} \cap R = P$ (Theorem 3.6 (b)) and that the canonical embedding $R \hookrightarrow R \bowtie I$ has the going-up property, we deduce that $\mathrm{Spec}(R_P \otimes_R (R \bowtie I))$ can be canonically identified with the set of all the prime ideals of $R \bowtie I$ contained in $\boldsymbol{\mathcal{P}}$. Therefore $R_P \otimes_R (R \bowtie I)$ is a local ring with a unique maximal ideal corresponding to the prime ideal $\boldsymbol{\mathcal{P}}$ of $R \bowtie I$ and thus we deduce that the canonical ring homomorphism $(R \bowtie I)_{\boldsymbol{\mathcal{P}}} \to R_P \otimes_R (R \bowtie I)$ is an isomorphism. $\square$

Proposition 3.10 *The ring $R \bowtie I$ can be obtained as a pullback of the following diagram of canonical homomorphisms:*

$$\begin{tikzcd} R \bowtie I \arrow[r, "\widetilde{v}'"] \arrow[d, "\widetilde{u}'"'] & R/I \arrow[d, "\widetilde{u}"'] \\ R \times R \arrow[r, "\widetilde{v}"] & R/I \times R/I \end{tikzcd}$$

where $\widetilde{u}$ is the diagonal embedding, $\widetilde{v}$ is the canonical surjection $(x, y) \mapsto (x + I, y + I)$, $\widetilde{u}'$ is the natural inclusion and $\widetilde{v}'$ is defined by $(x, x + i) \mapsto x + I$, for all $x, y \in R$ and $i \in I$.

Proof. By Proposition 3.1 we know that

$$\begin{tikzcd} R \bowtie I \arrow[r] \arrow[d] & R \arrow[d, "u"'] \\ R \times R \arrow[r, "v"] & R \times R/I \end{tikzcd}$$

is a pullback. On the other hand, it is easy to verify that the following diagram:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "u"'] & R/I \arrow[d, "\widetilde{u}"'] \\ R \times R/I \arrow[r, "w"] & R/I \times R/I \end{tikzcd}$$

is a pullback, where w is the canonical surjection $(x, y) \mapsto (x + I, y)$ and φ is the natural proiection $x \mapsto x + I$, for each $x \in R$ and for each $y \in R/I$. The conclusion follows by juxtaposing two pullbacks. $\square$

Corollary 3.11 *If R is a local ring, integrally closed in $T(R)$ with maximal ideal M and residue field k, then $R \bowtie M$ is seminormal in its integral closure inside $T(R) \times T(R)$ (which, in this situation, coincides with $R \times R$).*

Proof. By the previous proposition $R \bowtie M$ (which is a local ring) can be obtained as a pullback of the following diagram of canonical homomorphisms:

$$\begin{tikzcd} R \bowtie M \arrow[r, "\widetilde{v}'"] \arrow[d, "\widetilde{u}'"'] & k \arrow[d, "\widetilde{u}"'] \\ R \times R \arrow[r, "\widetilde{v}"] & k \times k \end{tikzcd}$$

The statement follows from the fact that, in this case, the integral closure of $R\bowtie M$ in $T(R)\times T(R)$ coincides with $R\times R$ (Corollary 3.3 (c)). Therefore, since $\widetilde{u}$ is a minimal extension, then $\widetilde{u}'$ is also minimal [3, Lemme 1.4 (ii)], and thus the conclusion follows from [3, Théorème 2.2 (ii))] and from [18, (1.1)] (keeping in mind Theorem 3.6 (c)). $\square$

Example 3.12 **(a)** Let $R := k[[t]]$ (where k is a field and t an indeterminate) and let $I := t^n R$. Using Proposition 3.10, if we denote by $h^{(i)}(t)$ the i–th derivative of a power series $h(t) \in k[[t]]$, it is easy to see that

$$R\bowtie I = \{(f(t), g(t)) \mid f(t), g(t) \in R\,,\ f^{(i)}(0) = g^{(i)}(0)\ \forall\ i = 0, \ldots n-1\}\ .$$

(b) Let $R := k[x, y]$ and $I := xR$. In this case

$$R\bowtie I = \{(f(x, y), g(x, y)) \mid f(x, y), g(x, y) \in R\,,\ f(0, y) = g(0, y)\}\ .$$

Setting $Y = \mathrm{Spec}(R\bowtie I)$ and $X = \mathrm{Spec}(R)$, by Proposition 2.13, $V_Y(\boldsymbol{\mathfrak{O}_i}) \cong \mathrm{Spec}(k[x, y])$. On the other hand, by Theorem 3.9, $V_Y(\boldsymbol{\mathfrak{O}_1}) \cap V_Y(\boldsymbol{\mathfrak{O}_2}) = V_Y((xR \times xR)) \cong V_X(xR) \cong \mathrm{Spec}(k[y])$. Hence the ring $R\bowtie I$ is the coordinate ring of two affine planes with a common line. Note that we can present $R\bowtie I$ as quotient of a polynomial ring in the following way: consider the homomorphism $\lambda : k[x, y, z] \longrightarrow R \times R$, defined by $\lambda(x) := (x, x)$, $\lambda(y) := (y, y)$ and $\lambda(z) := (0, x)$. It is not difficult to see that $\mathrm{Im}(\lambda) = R\bowtie I$ and $\mathrm{Ker}(\lambda) = (zx - z^2)k[x, y, z]$.

References

[1] M. D'Anna, *A construction of Gorenstein rings*, J. Algebra (to appear).

[2] M. D'Anna and M. Fontana, *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Preprint 2006.

[3] D. Ferrand and J.-P. Olivier, *Homomorphismes mimimaux d'anneaux,* J. Algebra **16** (1970), 461–471.

[4] M. Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl. **123** (1980), 331–355.

[5] R. Fossum, *Commutative extensions by canonical modules are Gorenstein rings,* Proc. Am. Math. Soc. **40** (1973), 395–400.

[6] R. Fossum, P. Griffith and I. Reiten, *Trivial extensions of Abelian categories. Homological algebra of trivial extensions of Abelian categories with applications to ring theory,* Lecture Notes in Mathematics **456**, Springer-Verlag, Berlin, 1975.

[7] S. Gabelli and E.G. Houston, *Ideal theory in pullbacks,* in "Non-Noetherian Commutative Ring Theory", S.T. Chapman and S. Glaz Eds., Kluwer Academic Publishers, 2000, 199–227.

[8] R. Gilmer, Multiplicative ideal theory, M. Dekker, New York, 1972.

[9] S. Glaz, Commutative coherent rings, Lecture Notes in Mathematics **1321**, Springer-Verlag, Berlin, 1989.

[10] W. Heinzer, J. Huckaba and I. Papick, *m–canonical ideals in integral domains,* Comm. Algebra **26** (1998), 3021–3043.

[11] J. Huckaba, Commutative rings with zero divisors, M. Dekker, New York, 1988.

[12] I. Kaplansky, Commutative Rings, Allyn and Bacon, Boston, 1970.

[13] J. Lambek, Lectures on Rings and Modules, Blaisdell Publishing Company, Waltham, 1966.

[14] H. Matsumura, Commutative ring theory, Cambridge University Press, Cambridge, 1986.

[15] M. Nagata, *The theory of multiplicity in general local rings,* Proc. Intern. Symp. Tokyo-Nikko 1955, Sci. Council of Japan, Tokyo 1956, 191–226.

[16] M. Nagata, Local Rings, Interscience, New York, 1962.

[17] I. Reiten, *The converse of a theorem of Sharp on Gorenstein modules,* Proc. Amer. Math. Soc. **32** (1972), 417–420.

[18] C. Traverso, *Seminormality and Picard group,* Ann. Sc. Norm. Sup. Pisa **24** (1970), 585–595.

Marco D'Anna
Dipartimento di Matematica e Informatica
Università di Catania
Viale Andrea Doria 6
95125 Catania, Italy
mdanna@dipmat.unict.it

Marco Fontana
Dipartimento di Matematica
Università di Roma degli Studi "Roma Tre"
Largo San Leonardo Murialdo 1
00146 Roma, Italy
fontana@mat.uniroma3.it