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Going-up, Direct Limits and Universality

by

David E. DOBBS* and Marco FONTANA**

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1. Introduction and summary

This note contributes to the study of the behavior of going-up and related properties of ring-homomorphisms. The main result of Section 2, Theorem 2.1(b), establishes that going-up is preserved by direct limit. This is possibly to be expected since integrality, the most familiar sufficient condition for going-up, is known to be preserved by direct limit. Moreover, the corresponding fact for the "dual" property of going-down was obtained in [4, Theorem 2.1]. More persuasive evidence appeared in our recent study of universally going-down homomorphisms, where it was shown that UGD, a certain condition closely related to both going-up and going-down, is preserved by direct limit [3, Corollary 3.13 and Corollary 3.12(c)]. (Suffice it to recall here that in the case of an integral extension $f: R \to T$, f is UGD if and only if T is the weak normalization of R with respect to f, in the sense of Andreotti-Bombieri [1].) The proof of Theorem 2.1(b) proceeds by first establishing similar behavior for lying-over. Section 2 also contains a simple proof that lying-over is a universal property (Corollary 2.2).

The typical use in [3] of a result stating that a property P is preserved by direct limit was in characterizing the universally P property. (Cf. also Corollary 3.6 below.) However, it is folklore (cf. Lemma 3.1(b)) that universally going-up is equivalent to integrality. This fact is used, in conjunction with Theorem 2.1(b), to derive Theorem 3.2, a most tractable characterization in terms of integrality of a property, universally quasi-going-up, which figured intimately in [3]. An upshot for universally going-down is recorded in Corollary 3.4, and Section 3 then goes on to consider analogous questions of universality and stability under direct limit for the quasi-going-up property.

Throughout, rings are assumed commutative, with 1; and ring-homomorphisms are assumed unital.

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2. Direct limits and going-up

To approach assertions concerning the preservation of a property P by direct limit, we fix the following notation. Let $(I, \leq)$ be a directed set, and let (R_i, f_{ij}) and (T_i, g_{ij}) each be directed systems of rings indexed by I. For each $i \in I$, let $h_i: R_i \to T_i$ be a ring-homomorphism satisfying P, such that whenever $i \leq j$ in I, then $g_{ij}h_i = h_j f_{ij}: R_i \to T_j$. Set $R = \varinjlim R_i$, $T = \varinjlim T_i$, and $h = \varinjlim h_i$. The issue is whether $h: R \to T$ also satisfies P.

THEOREM 2.1. (a) *Lying-over is preserved by direct limit.*
(b) *Going-up is preserved by direct limit.*

Proof. (a) Consider a prime ideal p of R. To show that some prime of T lies over p, it is enough to show that for the induced map $h_p: R_p \to T_p$, some prime of T_p lies over pR_p. (As usual, T_p denotes $T_{h(R \setminus p)} \cong T \otimes_R R_p$.)

For each i, let $p_i \in \operatorname{Spec}(R_i)$ be the preimage of p under the structure map $R_i \to R$. Evidently, $(h_i)_{p_i}: (R_i)_{p_i} \to (T_i)_{p_i}$ inherits the lying-over property from h_i. In particular, $1 \notin (h_i)_{p_i} p_i (T_i)_{p_i}$.

Now, if no prime of T_p lies over (the maximal ideal) pR_p, it follows that $1 \in h_p(pR_p)T_p$. Using the canonical isomorphisms, $pR_p \cong \varinjlim p_i(R_i)_{p_i}$ and $T_p \cong \varinjlim (T_i)_{p_i}$ (cf. [6, Propositions 6.1.2(ii), 6.1.5 and 6.1.6(ii), pages 128–130]), one then finds a canonical isomorphism

$$\varinjlim (h_i)_{p_i}(p_i(R_i)_{p_i})(T_i)_{p_i} \cong h_p(pR_p)T_p$$

and thus readily produces an index j such that $1 \in (h_j)_{p_j}(p_j(R_j)_{p_j})(T_j)_{p_j}$, the desired contradiction.

(b) Consider prime ideals $p_1 \subset p_2$ of R and q_1 of T such that $h^{-1}(q_1) = p_1$. Our task is to find $q_2 \in \operatorname{Spec}(T)$ such that $q_1 \subset q_2$ and $h^{-1}(q_2) = p_2$. It is enough to show that $\bar{h}: R/p_1 \to T/q_1$ is lying-over. To this end, let p_{1i} (resp., q_{1i}) be the preimage of p_1 (resp., q_1) under the structure map $R_i \to R$ (resp., $T_i \to T$). As q_{1i} lies over p_{1i} and h_i is going-up, it follows that $\bar{h}_i: R_i/p_{1i} \to T_i/q_{1i}$ is lying-over. In view of [6, Proposition 6.1.2, page 128], one can show that $\bar{h}$ is identified with $\varinjlim \bar{h}_i$, and so an application of (a) completes the proof.

It is interesting to note that the above results may be stated topologically. For instance, by use of canonical homeomorphisms of the form $\operatorname{Spec}(\varinjlim A_i) \cong \varprojlim \operatorname{Spec}(A_i)$, we see that Theorem 2.1(a) asserts that the inverse limit of the continuous surjections $\operatorname{Spec}(T_i) \to \operatorname{Spec}(R_i)$ is the continuous surjection $\operatorname{Spec}(T) \to \operatorname{Spec}(R)$. Despite such a formulation, Theorem 2.1(a) remains an essentially algebraic, rather than topological, fact since [5, Example 4, page 431] illustrates that an arbitrary inverse limit of continuous surjections of topological spaces need not be surjective.

The next result establishes that universally lying-over is equivalent to lying-over. This is in fact already known in full generality: one need only appeal to [6, Proposition 3.6.1(ii), page 244]. We shall give a proof featuring some ideas of

McAdam [8, Proposition 1], who has proved the special case of Corollary 2.2 in which R and T are domains, f and g are inclusion maps, and $S=R[X]$, the ring of polynomials in one variable over R. We defer discussion of "universally going-up" to Section 3.

COROLLARY 2.2. *If a ring-homomorphism* $f: R\to T$ *is lying-over and if* $g: R\to S$ *is a ring-homomorphism, then the induced homomorphism* $f_S: S\to S\otimes_R T$ *is also lying-over.*

Proof. Utilize the criterion in [3, Proposition 2.2], together with the following three facts: Theorem 2.1(a); a straightforward extension of [8, Proposition 1] from the context of domains to that of rings; and the easy special case in which g is a canonical surjection $R\to R/J$ for J an ideal of R.

3. Quasi-integrality

The purpose of this section is to shed light on a concept introduced in [3]. As in [3], we shall say that a (unital) ring-homomorphism $f: R\to T$ is *quasi-going-up* (in short, QGU) if, whenever prime ideals $p_1\subset p_2$ of R and q_1 of T are such that $f^{-1}(q_1)=p_1$ and $f(p_2)T\neq T$, then there exists a prime q_2 of T such that $q_1\subset q_2$ and $f^{-1}(q_2)=p_2$. Examples of QGU homomorphisms include all integral maps, localizations, and the inclusion map of any domain into a flat overring.

A ring-homomorphism $f: R\to T$ is, of course, said to be *universally quasi-going-up* (universally QGU) if $S\to S\otimes_R T$ is QGU for each change of base, $R\to S$. Theorem 2.2 will present a more tractable view of this concept. First, we shall state a preliminary result. Its first assertion is immediate from the definitions. Its second is a piece of folklore, useful and nontrivial, for which an elegant proof appears in [2, Lemma, page 160].

LEMMA 3.1. *Let* $f: R\to T$ *be a ring-homomorphism. Then*:

(a) f *is QGU if and only if, for each prime* p *of* R *such that* $f(p)T\neq T$, *the induced homomorphism* $R_p\to T_p$ *is going-up.*

(b) f *is integral if (and only if) the induced homomorphism* $R[X]\to T[X]$ *is going-up.*

The next definition will be convenient. A ring-homomorphism $f: R\to T$ will be called *quasi-integral* if the induced homomorphism $f_p: R_p\to T_p$ is integral for each prime p of R such that $f(p)T\neq T$.

THEOREM 3.2. *A ring-homomorphism* $f: R\to T$ *is universally QGU if and only if* f *is quasi-integral.*

Proof. Suppose that f is universally QGU. By Lemma 3.1(b), to prove that f is quasi-integral, we need only show that $R_p[X]\to T_p[X]$ is going-up for each prime p of R such that $f(p)T\neq T$. We shall show, equivalently, that the inclusion map $i: f_p(R_p)[X]\to T_p[X]$ is going-up. Note first that the inclusion map $j: f_p(R_p)\to T_p$ is universally QGU. (This holds since f_p: $R_p \to T_p$ is universally QGU and, for each change of base $f_p(R_p) \to S$, one has the canonical isomorphism $S \otimes_{f_p(R_p)} T_p \cong S \otimes_{R_p} T_p$.) Hence, i is QGU. It therefore suffices to prove that i is lying-over. It is well known (cf. Corollary 2.2 or proof of [8, Proposition 1]) that this will follow once we know that j is lying-over. Since j is injective, [7, Theorem 42] shows that we need only prove j is going-up. However, this is equivalent to proving that f_p is going-up, and so an application of Lemma 3.1(a) completes the argument.

Conversely, suppose that f is quasi-integral. Since integrality implies going-up, we may use the criterion in Lemma 3.1(a) to see that f is QGU. It therefore suffices to prove that quasi-integrality is a universal property, that is, that f_S: $S \to S \otimes_R T$ is quasi-integral for each change of base, g: $R \to S$. We shall show that $S_P \to (S \otimes_R T)_P$ is integral for each prime P of S such that $f_S(P)(S \otimes_R T) \neq S \otimes_R T$. Consider $p = g^{-1}(P)$. If $f(p)T = T$, then $1 = \sum f(x_i)t_i$ for some $x_i \in p$, $t_i \in T$; then, working inside $S \otimes_R T$, we find

$$\begin{aligned} 1 &= 1 \otimes \sum f(x_i)t_i = \sum 1 \otimes x_i \cdot t_i = \sum x_i \cdot 1 \otimes t_i \\ &= \sum g(x_i) \otimes t_i = \sum (g(x_i) \otimes 1)(1 \otimes t_i) \in f_S(P)(S \otimes_R T), \end{aligned}$$

a contradiction. Therefore, $f(p)T \neq T$, and so quasi-integrality assures that $R_p \to T_p$ is integral. Since integrality is a universal property, $S_P \otimes_R R_p \to S_P \otimes_R T_p$ is also integral. The desired assertion now follows from the canonical isomorphisms, $S_P \otimes_R R_p \cong S_P$ and $S_P \otimes_R T_p \cong (S \otimes_R T)_P$. This completes the proof.

Since integrality is a local property and any injective integral map has the lying-over property, Theorem 3.2 immediately leads to

COROLLARY 3.3. *An injective ring-homomorphism is integral if and only if it is both lying-over and universally QGU.*

COROLLARY 3.4. *Let R be an integral domain, T an overring of R, and f : $R \to T$ the inclusion map. Then the following five conditions are equivalent*:

(i) *f is a universally going-down homomorphism*;

(ii) *f is both UGD and quasi-integral*;

(iii) *f is both going-down and quasi-integral. Moreover, for each prime p of R such that $pT \neq T$, if J denotes the Jacobson radical of T_p and if l is either* 1 *or* char (R/p) *according as to whether that characteristic is* 0 *or positive, then for each $x \in T$, there exists $n \geq 1$ such that $x^{l^n} \in R_p + J$.*

(iv) *f is both going-down and quasi-integral, and T is the weak normalization of R (inside T) with respect to f.*

(v) *f is quasi-integral. Moreover, for each prime p of R such that $pT \neq T$, T_p is the weak normalization of R_p inside T_p.*

Proof. (i) $\Leftrightarrow$ (ii): In view of Theorem 3.2, this assertion is a translation of [3, Theorem 3.17], the main result in [3].

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv): These assertions follow from the definitions of quasi-integral, UGD, and weak normalization.

(i) $\Rightarrow$ (v): Assume (i). Since (i) $\Rightarrow$ (ii), f is quasi-integral. Next, consider $p \in \mathrm{Spec}(R)$ such that $pT \neq T$. By (i), the inclusion map $f_p : R_p \to T_p$ is a universally going-down (overring) extension. Applying the earlier-established (i) $\Rightarrow$ (iv) to f_p, we infer (v), as desired.

(v) $\Rightarrow$ (iii): Assume (v). To see that f is going-down, it is enough to verify that $f_p: R_p \to T_p$ is going-down for each $p \in \mathrm{Spec}(R)$ such that $pT \neq T$. This condition follows from the study in [1] of weak normalization relative to integral homomorphisms, as [1] assures in fact that $\mathrm{Spec}(T_p) \to \mathrm{Spec}(R_p)$ is a homeomorphism. The final assertion in (iii) now follows routinely from the general definition of weak normalization in [3]. This completes the proof.

It was shown in [3, Corollary 3.13] that direct limits preserve UGD. Since UGD $\Rightarrow$ QGU [3, Corollary 3.12(c)], this suggests

PROPOSITION 3.5. *Direct limits preserve the property of QGU.*

Proof. We use the notation introduced at the beginning of Section 2. By Lemma 3.1(a), our task is to show that $h_p: R_p \to T_p$ is going-up for each prime p of R such that $h(p)T \neq T$. Let $p_i \in \mathrm{Spec}(R_i)$ be the preimage of p under the structure map $R_i \to R$. Since $h_i(p_i)T_i \neq T_i$, Lemma 3.1(a) assures that $(h_i)_{p_i}: (R_i)_{p_i} \to (T_i)_{p_i}$ is going-up for each i. As h_p may be identified with $\varinjlim (h_i)_{p_i}$, the assertion follows from Theorem 2.1(b).

COROLLARY 3.6. *A ring-homomorphism $f: R \to T$ is universally QGU if and only if the induced homomorphism $R[X_1, \cdots, X_n] \to T[X_1, \cdots, X_n]$ is QGU for each $n \geq 0$.*

Proof. In view of Proposition 3.5, the criterion in [3, Proposition 2.2] reduces our task to showing that if f is QGU and J is an ideal of R, then the induced map $R/J \to T/JT$ is also QGU. As this is easily verified, the proof is complete.

Remark 3.7. (a) Since tensor product commutes with direct limit, Proposition 3.5 implies that direct limits preserve the property of universally QGU. Another proof of this fact follows readily from Theorem 3.2, since direct limits preserve integrality.

(b) In view of Corollary 3.3, it seems pertinent to observe that the properties of lying-over and universally QGU are independent. Indeed, any nonalgebraic field extension exhibits lying-over but (being nonintegral) is not universally QGU. Moreover, if a domain R is properly contained in its quotient field K, then the inclusion map $R \to K$ is universally QGU but not lying-over.

It is easy to see that QGU is preserved by composition of homomorphisms. The final result of this section is a partial converse in terms of the most familiar QGU maps.

PROPOSITION 3.8. *Let R be a domain of (Krull) dimension 1 and T an overring of R, such that the inclusion map $f: R \to T$ is quasi-integral. Let S be the integral closure of R in T; let $g: R \to S$ and $h: S \to T$ be the inclusion maps. Then $f = hg$, g is integral, and h is flat.*

Proof. Only the final assertion needs a proof. It is enough to show that T_m is S_m-flat for each maximal ideal m of S. Set $M = m \cap R$. There are two cases to consider.

Suppose first that $MT \neq T$. By hypothesis, $f_M: R_M \to T_M$ is integral and, *a fortiori*, so is $S_M \to T_M$. Localizing at $S \backslash m$, we then have that $S_m \to T_m$ is integral. However S_m is integrally closed in T_m (since S is integrally closed in T), and so $S_m = T_m$, completing the first case.

In the remaining case $MT = T$, and so $mT = T$. By integrality, $\dim(S) = 1$, and so no prime of T_m can lie over a nonzero prime of S_m. Since T_m is an overring of S_m, this means that T_m has no nonzero primes; that is, $T_m = L$, the quotient field of S_m. But L is certainly S_m-flat, completing the proof.

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Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37996
U.S.A.

Dipartimento di Matematica
Università di Roma I
00185 Roma
Italy