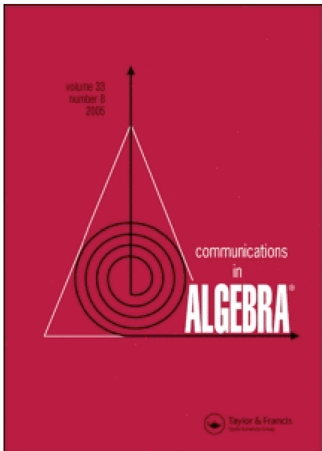

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Umt-domains and domains with prüfer integral closure

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UMT-DOMAINS AND DOMAINS WITH PRÜFER INTEGRAL CLOSURE

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An integral domain R is said to be a UMT-domain if uppers to zero in $R[X]$ are maximal t-ideals. We show that R is a UMT-domain if and only if its localizations at maximal t-ideals have Prüfer integral closure. We also prove that the UMT-property is preserved upon passage to polynomial rings. Finally, we characterize the UMT-property

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in certain pullback constructions; as an application, we show that a domain has Prüfer integral closure if and only if all its overrings are UMT-domains.

Introduction

UMT-domains were introduced in [13] and received some attention in [7] and [5]. (They were also briefly mentioned in [16].) The purpose of this paper is to study this class of rings in greater detail.

We begin by reviewing the v- and t-operations. Recall that if I is an nonzero fractional ideal of a domain R with quotient field K, then the inverse, the v- (or divisorial) closure, and the t-closure of I are given, respectively, by $I^{-1} = \{x \in K \mid xI \subseteq R\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup\{J_v \mid J$ is a nonzero finitely generated subideal of $I\}$. The v- and t-operations are examples of star-operations, and the reader is referred to [10] and to [14] for a discussion of their properties, which we shall use freely (usually without reference). Of particular importance are the standard facts that every t-ideal is contained in a maximal t-ideal, that maximal t-ideals are prime, and that any prime minimal over a t-ideal is a prime t-ideal (t-prime). In particular, height one prime ideals are t-primes. We also recall that if T is a flat overring of a domain R, then t-ideals of T contract to t-ideals of R (cf., e.g., [9, Proposition 0.7]).

Now recall that a nonzero prime ideal U of the polynomial ring $R[X]$ (in one indeterminate X) with $U \cap R = 0$ is called an <u>upper to zero</u>. The domain R is said to be a <u>UMT-domain</u> if every upper to zero in $R[X]$ is a maximal t-ideal. UMT-domains are closely related to a class of rings that has received a good deal of attention in the literature: the class of Prüfer v-multiplication domains. (See, e.g., [11], [17], [15], and [6].) A domain R is said to be a Prüfer v-multiplication domain (PVMD) if every nonzero finitely generated ideal I of R is t-invertible, i.e., satisfies $(II^{-1})_t = R$. It was observed in [13, Proposition 3.2] that a domain R is a PVMD if and only if R is an integrally closed UMT-domain. Thus UMT-domains are "PVMD's with the integrally closed hypothesis removed."

In the first section, we collect for easy reference the known facts about UMT-domains and prove a number of useful characterizations, of which perhaps the most important is that a domain R is a UMT-domain if and only if its localizations at maximal t-ideals have Prüfer integral closure. The second section is devoted to a study of polynomial rings over UMT-domains; among other things, we show that, for an arbitrary set $\{X_\alpha\}$ of indeterminates, R is a UMT-domain if and only if $R[\{X_\alpha\}]$ is. In the third section, we characterize UMT-domains in certain types of pullback constructions, and we give several applications, proving, for example, that a domain R has Prüfer integral closure $\Leftrightarrow$ each overring of R is a UMT-domain.

Notation is generally standard as in [10]. We shall use R' to denote the integral closure of a domain R.

1. Elementary Properties

Let R be a domain with quotient field K. For $f \in R[X]$, we denote the content of f, i.e., the ideal of R generated by the coefficients of f, by $c(f)$. For an ideal I of $R[X]$, $c(I)$ is the ideal generated by the contents of all the polynomials in I. It will be convenient to begin with a result which follows easily from facts about UMT-domains proved in [13].

Theorem 1.1. Let R be a domain. Then the following statements are equivalent.

(a) R is a UMT-domain.

(b) $c(U)_t = R$ for every upper to zero U in $R[X]$.

(c) For every upper to zero U in $R[X]$, $\exists\, f \in U$ with $c(f)_v = R$.

(d) U is t-invertible for every upper to zero U in $R[X]$.

(e) Every prime of $R[X]_S$ is extended from R, where $S = \{f \in R[X] \mid c(f)_v = R\}$.

(f) $U \nsubseteq P[X]$ for each upper to zero U in $R[X]$ and each t-prime P of R.

(g) $U \nsubseteq M[X]$ for each upper to zero U in $R[X]$ and each maximal t-ideal M of R.

Proof: The equivalence of statements (a)-(d) follows from [13, Theorem 1.4], and statements (a) and (e) are equivalent by [13, Theorem 3.1]. It is clear that (c) implies (f) and that (f) implies (g). Finally, (g) implies (a) by [13, Proposition 1.1]. □

An extension $R \subseteq T$ of domains is said to be <u>t-linked</u> if $I^{-1} = R$ for a finitely generated ideal I of R implies $(T{:}IT) = T$. The notion of t-linked extension (restricted to overrings) was introduced in [6] and has been studied in several papers since (cf. [5, 3]). It was shown in [13, page 1962] that a t-linked overring of a UMT-domain is again a UMT-domain (and therefore, since localizations are t-linked by [6, Proposition 2.2], the UMT-property is preserved by localization); in fact, the argument given actually works for any algebraic t-linked extension. Hence we state the following result without proof.

Proposition 1.2. Let $R \subseteq T$ be an extension of domains, and assume that (the quotient field of) T is algebraic over (the quotient field of) R. If $R \subseteq T$ is t-linked and R is a UMT-domain, then T is a UMT-domain. In particular, if R is a UMT-domain, then so is every localization of R.

Remark. The hypothesis that the extension be algebraic cannot be dispensed with in Proposition 1.2. To see this, let T be any domain with characteristic zero which is not a UMT-domain. Then, considering the ring $\mathbb{Z}$ to be a subring of T, the extension $\mathbb{Z} \subseteq T$ is t-linked since $I^{-1} \neq \mathbb{Z}$ for each nonzero proper ideal I of $\mathbb{Z}$.

Even though the UMT-property is preserved by localization, the property of being a t-ideal is not in general preserved by localization. (See [19] for a discusssion and examples of this phenomenon.) Fortunately, t-ness does localize for UMT-domains. Indeed much more is true, as Proposition 1.4 below shows. First, we need a lemma.

Lemma 1.3. Let P be a prime ideal of the domain R, and assume that P is not a t-prime. Then there is an upper to zero U in $R[X]$ with $U \subseteq P[X]$.

Proof: Since P is not a t-ideal, we may pick $a_0, a_1, \ldots, a_n \in P$ with $(a_0, \ldots, a_n)_v \nsubseteq P$. Let $f = a_0 + \cdots + a_nX^n \in R[X]$, and shrink $P[X]$ to a prime U minimal over f. Then U is a t-prime of $R[X]$. If $U = (U \cap R)[X]$, then $U \cap R$ is a (nonzero) t-prime of R (this is well known and follows easily from [12, Proposition 4.3]), and we have $c(f) \subseteq U \cap R$, whence

$c(f)_v \subseteq U \cap R \subseteq P$, a contradiction. Hence $U \neq (U \cap R)[X]$, and it now follows from [18, Theorem A] that U is an upper to zero. $\square$

Proposition 1.4. Let P be a t-prime of the UMT-domain R, let T be a domain which is an algebraic extension of R, and let Q be a prime of T with $Q \cap R = P$. Then Q is a t-prime of T.

Proof: If Q is not a t-prime of T, then by Lemma 1.3, there is an upper to zero U in $T[X]$ with $U \subseteq Q[X]$. Then since T is algebraic over R, $U \cap R[X]$ is nonzero and is therefore an upper to zero in $R[X]$ with $U \cap R[X] \subseteq P[X]$. However, since R is a UMT-domain, this contradicts (a) $\Rightarrow$ (f) of Theorem 1.1. $\square$

Recall from [6, Theorem 2.6] and [5, page 1464] that a domain R is <u>t-linkative</u> if either of the following equivalent conditions holds: (1) $R \subseteq T$ is t-linked for every overring T of R, (2) every nonzero maximal ideal of R is a t-ideal. We use the notion of t-linkativity to prove a characterization of UMT-domains which we shall find extremely useful.

Theorem 1.5. The following statements are equivalent for a domain R.

(1) R is a UMT-domain.

(2) R_P has Prüfer integral closure for each prime t-ideal P of R.

(3) R_M has Prüfer integral closure for each maximal t-ideal M of R.

(4) R_M is a t-linkative UMT-domain for each maximal t-ideal M of R.

Proof: (1) $\Rightarrow$ (2): If R is a UMT-domain, then by Proposition 1.4 PR_P is a t-prime of the UMT-domain R_P, whence R_P is also t-linkative. By [5, Theorem 2.4], R_P has Prüfer integral closure.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (4): By [5, Theorem 2.4].

(4) $\Rightarrow$ (1): For each maximal t-ideal M of R, R_M is a UMT-domain and MR_M is a t-prime of R_M. Hence by Theorem 1.1, $MR_M[X]$ contains no uppers to zero in $R_M[X]$. It follows that $M[X]$ contains no uppers to zero in $R[X]$. Therefore, again by Theorem 1.1, R is a UMT-domain. □

Remark. Griffin showed [11, Theorem 5] that a domain R is a PVMD $\Leftrightarrow$ R_M is a valuation domain for each maximal t-ideal M of R. Thus ((1) $\Leftrightarrow$ (3) of) Theorem 1.5 may be thought of as a UMT-analogue of Griffin's result.

Corollary 1.6. Let R be a UMT-domain, and let $P \subseteq Q$ be prime ideals of R. If Q is a t-prime, then P is a t-prime.

Proof: By Theorem 1.5, R_Q has Prüfer integral closure. Hence, since $R_P{}' \supseteq R_Q{}'$, $R_P{}'$ is also a Prüfer domain. By [5, Theorem 2.4], R_P is a t-linkative UMT-domain. In particular, PR_P is a maximal t-ideal of R_P. It is well known that this implies that P is a t-prime of R. □

2. Polynomial rings over a UMT-domain

Throughout this section, R denotes a domain with quotient field K, and $\{X_\alpha\}$ denotes a set of indeterminates over R. We begin with a lemma.

Lemma 2.1.

(1) If I is a fractional ideal of R, then $I_v[\{X_\alpha\}] = (I[\{X_\alpha\}])_v$, and $I_t[\{X_\alpha\}] = (I[\{X_\alpha\}])_t$.

(2) If I is a t-ideal (v-ideal) of R, then $I[\{X_\alpha\}]$ is a t-ideal (v-ideal) of $R[\{X_\alpha\}]$.

(3) If J is a t-ideal of $R[\{X_\alpha\}]$ for which $J \cap R \neq 0$, then $J \cap R$ is a t-ideal of R.

(4) If M is maximal t-ideal of R, then $M[\{X_\alpha\}]$ is a maximal t-ideal of $R[\{X_\alpha\}]$.

(5) The extension $R \subseteq R[\{X_\alpha\}]$ is t-linked.

Proof: Extending the argument from [12, Proposition 4.3] to the case of an arbitrary set of indeterminates, we have that $I^{-1}[\{X_\alpha\}] = (I[\{X_\alpha\}])^{-1}$, $I_v[\{X_\alpha\}] = (I[\{X_\alpha\}])_v$, and $I_t[\{X_\alpha\}] = (I[\{X_\alpha\}])_t$, for each nonzero fractional ideal I of R. (Here, the inverse, the v-, and the t-operations on the right sides of the equalities are taken with respect to the ring $R[\{X_\alpha\}]$.) This proves (1). Statements (2) and (5) follow easily from these equalities. Let J be a t-ideal of $R[\{X_\alpha\}]$ for which $J \cap R \neq 0$. Using (1), we have $(J \cap R)_t[\{X_\alpha\}] = ((J \cap R)[\{X_\alpha\}])_t \subseteq J$, from which it follows that $(J \cap R)_t \subseteq J \cap R$. This proves (3). To prove (4), let M be a maximal t-ideal of R. First suppose that $\{X_\alpha\}$ consists of the single indeterminate X. By (2) $M[X]$ is a t-ideal of $R[X]$. Let Q be a maximal t-ideal of $R[X]$ which contains $M[X]$. By [13, Proposition 1.1], $Q = (Q \cap R)[X]$, and by (3) $Q \cap R$ is a t-prime of R. Hence, since M is a maximal t-ideal, $M = Q \cap R$ and $M[X] = Q$ is a maximal t-ideal of $R[X]$. The case of a finite number of indeterminates now follows by induction. Now suppose that N is a maximal t-ideal of $R[\{X_\alpha\}]$ containing $M[\{X_\alpha\}]$. If the containment is proper, choose $f \in N \setminus M[\{X_\alpha\}]$. Then there is a finite subset $\{X_1,\ldots,X_n\}$ of $\{X_\alpha\}$ with $f \in N \cap R[X_1,\ldots,X_n]$ but $f \notin M[X_1,\ldots,X_n]$. By (3), $N \cap R[X_1,\ldots,X_n]$ is a t-prime of $R[X_1,\ldots,X_n]$. However, this contradicts the fact that $M[X_1,\ldots,X_n]$ is a maximal t-ideal of $R[X_1,\ldots,X_n]$. □

Proposition 2.2. Let Q be a maximal t-ideal of $R[\{X_\alpha\}]$, and let $P = Q \cap R$. Then either $P = 0$ or P is a maximal t-ideal of R and $Q = P[\{X_\alpha\}]$.

Proof: Suppose that $P \neq 0$. That $Q = P[\{X_\alpha\}]$ follows from a straightforward extension of the argument in [13, Proposition 1.1]. (This argument depends on the content formula, which is valid for any set of indeterminates [10, Corollary 28.3].) If $P \subseteq M$ for some maximal t-ideal M of R, then $Q \subseteq M[\{X_\alpha\}]$; since Q is a maximal t-ideal of $R[\{X_\alpha\}]$, we have that $P = M$, and P is a maximal t-ideal. □

Lemma 2.3. Let Q be a maximal t-ideal of $R[\{X_\alpha\}]$ with $Q \cap R = 0$. Then $R[\{X_\alpha\}]_Q$ is a valuation domain.

Proof: We first suppose that $\{X_\alpha\}$ is the finite set $\{X_1,\ldots,X_n\}$ and that Q is a maximal t-ideal of $R[X_1,\ldots,X_n]$ with $Q \cap R = 0$. If $n = 1$, the result follows from the well-known fact that a localization of $R[X]$ at an upper to zero is a valuation domain. Suppose $n > 1$, and let $P = Q \cap R[X_1,\ldots,X_{n-1}]$. If $P = 0$, the result follows from the case $n = 1$. If $P \neq 0$, then by Proposition 2.2, P is a maximal t-ideal of $R[X_1,\ldots,X_{n-1}]$ (with $P \cap R = 0$). By induction, $V = R[X_1,\ldots,X_{n-1}]_P$ is a valuation domain. Again by Proposition 2.2, $Q = P[X_n]$. Hence $R[X_1,\ldots,X_n]_Q = R[X_1,\ldots,X_n]_{P[X_n]} = V[X_n]_{PV[X_n]}$, which is easily seen to be a valuation domain. For the general case, let $u \in K(\{X_\alpha\})$, and choose finitely many indeterminates $X_1,\ldots,X_n \in \{X_\alpha\}$ such that $u \in K(X_1,\ldots,X_n)$ and $Q \cap R[X_1,\ldots,X_n] \neq 0$. By Proposition 2.2, $Q \cap R[X_1,\ldots,X_n]$ is a maximal t-ideal of $R[X_1,\ldots,X_n]$. Hence, by the case of finitely many indeterminates, u or $u^{-1} \in R[X_1,\ldots,X_n]_{Q \cap R[X_1,\ldots,X_n]} \subseteq R[\{X_\alpha\}]_Q$. ◻

Theorem 2.4. R is a UMT-domain $\Leftrightarrow$ $R[\{X_\alpha\}]$ is a UMT-domain.

Proof: We first prove the result in the case where R is a quasi-local domain whose maximal ideal M is a t-prime. Assume that R is a UMT-domain. Then (since M is a t-prime) by Theorem 1.5, R' is a Prüfer domain. Again by Theorem 1.5, to show that $R[\{X_\alpha\}]$ is a UMT-domain, it suffices to show that $(R[\{X_\alpha\}]_N)'$ is a Prüfer domain for each maximal t-ideal N of $R[\{X_\alpha\}]$. If $N \cap R = 0$, then $R[\{X_\alpha\}]_N$ is a valuation domain by Lemma 2.3. Hence by Proposition 2.2, we may assume that $N = M[\{X_\alpha\}]$. Note that $(R[\{X_\alpha\}]_N)' = R'[\{X_\alpha\}]_S$, where $S = R[\{X_\alpha\}] \setminus M[\{X_\alpha\}]$. A maximal ideal of $R'[\{X_\alpha\}]_S$ has the form $QR'[\{X_\alpha\}]_S$, where Q is a prime ideal of $R'[\{X_\alpha\}]$ which is maximal with respect to missing S. It follows from going up in the integral extension $R[\{X_\alpha\}] \subseteq R'[\{X_\alpha\}]$ that $Q \cap R[\{X_\alpha\}] = M[\{X_\alpha\}]$, and by incomparability, we have that $Q = P[\{X_\alpha\}]$ for some prime P of R' with $P \cap R = M$. Thus $(R'[\{X_\alpha\}]_S)_{QR'[\{X_\alpha\}]_S} = R'[\{X_\alpha\}]_{P[\{X_\alpha\}]}$, which is a valuation domain, since R' is a Prüfer domain. It follows that $(R[\{X_\alpha\}]_N)'$ is a Prüfer domain. Hence $R[\{X_\alpha\}]$ is a UMT-domain. Conversely, assume that

$R[\{X_\alpha\}]$ is a UMT-domain. Let P be a maximal ideal of R'. Since by Lemma 2.1, $M[\{X_\alpha\}]$ is a t-prime in the UMT-domain $R[\{X_\alpha\}]$, $(R[\{X_\alpha\}]_{M[\{X_\alpha\}]})'$ is a Prüfer domain; hence its localization $R'_P[\{X_\alpha\}]_{PR'_P[\{X_\alpha\}]}$ is a valuation domain. It follows easily that R'_P is a valuation domain (cf. [10, Theorem 33.4]). Hence R' is a Prüfer domain, i.e., R is a t-linkative UMT-domain.

We now attack the general case. Suppose that R is a UMT-domain. Let Q be a maximal t-ideal of $R[\{X_\alpha\}]$. We shall show that $R[\{X_\alpha\}]_Q$ is a t-linkative UMT-domain; and for this we may assume by Proposition 2.2 and Lemma 2.3 that $Q = M[\{X_\alpha\}]$ for some maximal t-ideal M of R. By the local case, $R_M[\{X_\alpha\}]$ is a UMT-domain, and as a localization of $R_M[\{X_\alpha\}]$, $R[\{X_\alpha\}]_{M[\{X_\alpha\}]}$ is a UMT-domain as well. By Proposition 1.4 and Lemma 2.1, $MR_M[\{X_\alpha\}]$ is a t-prime, whence by Proposition 1.4, $MR[\{X_\alpha\}]_{M[\{X_\alpha\}]} = MR_M[\{X_\alpha\}]_{MR_M[\{X_\alpha\}]}$ is a t-prime of $R[\{X_\alpha\}]_{M[\{X_\alpha\}]}$. Hence $R[\{X_\alpha\}]_{M[\{X_\alpha\}]}$ is a t-linkative UMT-domain. By Theorem 1.5, $R[\{X_\alpha\}]$ is a UMT-domain. Conversely, assume that $R[\{X_\alpha\}]$ is a UMT-domain, and let M be a maximal t-ideal in R. By Lemma 2.1, $M[\{X_\alpha\}]$ is a maximal t-ideal of $R[\{X_\alpha\}]$, whence by Theorem 1.5, $R[\{X_\alpha\}]_{M[\{X_\alpha\}]} = R_M[\{X_\alpha\}]_{MR_M[\{X_\alpha\}]}$ is a t-linkative UMT-domain. It follows that MR_M is a maximal t-ideal of R_M, and since the localization $R_M[\{X_\alpha\}]$ of $R[\{X_\alpha\}]$ is a UMT-domain, we have that R_M is a t-linkative UMT-domain by the local case. Hence, again by Theorem 1.5, R is a UMT-domain. □

For a domain R, we denote by $R\langle\{X_\alpha\}\rangle$ the ring $R[\{X_\alpha\}]_S$, where $S = \{f \in R[\{X_\alpha\}] \mid c(f)_v = R\}$. This ring has been studied by several authors. In particular, it is well known that a domain R is a PVMD $\Leftrightarrow$ $R\langle\{X_\alpha\}\rangle$ is a Prüfer domain. (See [15] for a particularly nice proof.) We generalize this result to UMT-domains.

Theorem 2.5. The following statements are equivalent for a domain R.

(1) R is a UMT-domain.

(2) $R\langle\{X_\alpha\}\rangle$ is a UMT-domain.

(3) $R\langle\{X_\alpha\}\rangle$ has Prüfer integral closure.

Proof: (1) $\Rightarrow$ (2): If R is a UMT-domain, then by Theorem 2.4, so is $R[\{X_\alpha\}]$. Since $R\langle\{X_\alpha\}\rangle$ is a localization of $R[\{X_\alpha\}]$, $R\langle\{X_\alpha\}\rangle$ is also a UMT-domain.

(2) $\Rightarrow$ (3): By [5, Theorem 2.4], it suffices to show that every maximal ideal of $R\langle\{X_\alpha\}\rangle$ is a t-ideal. By [15, Prop 2.1] a maximal ideal of $R\langle\{X_\alpha\}\rangle$ has the form $MR\langle\{X_\alpha\}\rangle$ for some maximal t-ideal M of R, and by [15, Corollary 2.3], $MR\langle\{X_\alpha\}\rangle$ is a t-ideal.

(3) $\Rightarrow$ (1): By Theorem 2.4, it suffices to show that $R[\{X_\alpha\}]$ is a UMT-domain, and for this it is enough to show that for Q a maximal t-ideal of $R[\{X_\alpha\}]$, $R[\{X_\alpha\}]_Q$ has Prüfer integral closure (Theorem 1.5). By Proposition 2.2 and Lemma 2.3, we may as well assume that $Q = M[\{X_\alpha\}]$ for some maximal t-ideal M of R, in which case $R[\{X_\alpha\}]_Q$ is an overring of $R\langle\{X_\alpha\}\rangle$ and therefore has Prüfer integral closure. $\square$

3. Pullbacks

Let T be a domain, let M be a maximal ideal of T, let D be a proper subring of $k = T/M$, let $\phi: T \longrightarrow k$ denote the canonical epimorphism, and let R be the pullback of the following diagram

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & k. \end{array}$$

(The downward arrows represent inclusion.) Thus $R = \phi^{-1}(D)$. Following [9], we shall refer to this diagram as a pullback diagram of type $\square$. The case where k is the quotient field of D is particularly important, and we shall refer to this case as a pullback diagram of type $\square^*$. It is well known (see [8] for details) that the prime spectrum of R in the diagram $\square$ is intimately related to the spectra of D and T. In particular, for each prime P of R with $P \not\supseteq M$, there is a unique prime Q of T with $Q \cap R = P$; and for this prime Q, we have $R_P = T_Q$. We shall also make use of the fact that M is the conductor of T to R and is therefore divisorial in R.

It will often be convenient to adjoin an indeterminate to the rings in the diagram $\square$, resulting in the following pullback diagram

$$\begin{array}{ccc} R[X] & \longrightarrow & D[X] \\ \downarrow & & \downarrow \\ T[X] & \xrightarrow{\phi} & k[X]. \end{array}$$

We continue to use ϕ to denote the horizontal maps.

Proposition 3.1. In a pullback diagram of type $\square$, T is a t-linked overring of R.

Proof: Let I be a (finitely generated) ideal of R with $I^{-1} = R$. Let $u \in (T:IT)$. Then $uI \subseteq T$, whence $uIM \subseteq M \subseteq R$, and $uM \subseteq I^{-1} = R$. Thus $u(M + I) \subseteq T$. However, M is maximal ideal of T and $I \not\subseteq M$ (since

M is divisorial in R and $I_v = R$). Thus $(M+I)T = T$, and we have $u \in T$. Hence $(T:IT) = T$. $\square$

Corollary 3.2. Consider a pullback diagram of type $\square$. If R is a UMT-domain, then T is a UMT-domain, and M is a maximal t-ideal of T.

Proof: That T is a UMT-domain follows immediately from Propositons 3.1 and 1.2, and Proposition 1.4 shows (since M is a divisorial ideal, and therefore a t-ideal, of R) that M is a t-ideal of T. $\square$

Lemma 3.3. Consider a pullback diagram of type $\square$, let Q be a prime ideal of T which is incomparable to M, and set $P = Q \cap R$. Then Q is a maximal t-ideal of $T \Leftrightarrow P$ is a maximal t-ideal of R.

Proof: We begin by observing that if I is a t-ideal of R which is incomparable to M, then $(IT)_t \neq T$. Otherwise, there is a finitely generated ideal J of R with $J \subseteq I$ and $(T:(T:JT)) = T$. Then $J^{-1} \subseteq T:JT = T$, whence $J_v \supseteq R:T = M$ and $I \supseteq J_v \supseteq M$, a contradiction.

Now assume that Q is a maximal t-ideal of T. By Proposition 3.1, T is t-linked over R, whence by [6, Proposition 2.1], $P_t \neq R$. Hence $P_t \subseteq N$ for some maximal t-ideal N of R. We claim that N is incomparable to M. To see this, pick $x \in Q \setminus M$. Since M is maximal in T, we can write $1 = tx + m$ for some $t \in T, m \in M$. It is easy to check that $m \in M \setminus N$ (and that $tx \in N \setminus M$). Hence, by the observation above, $(NT)_t \neq T$, and we have $NT \subseteq N'$ for some maximal t-ideal N' of T. Since $N' \supseteq NT \supseteq PT$, $N' \supseteq Q$. It follows that $N' = Q$ and $N = P$.

Conversely, assume that P is a maximal t-ideal of R. Again by the observation above, $(PT)_t \neq T$, whence $(PT)_t \subseteq Q'$ for some maximal t-ideal Q' of T (which is necessarily incomparable to M). By what we have already proved, $Q' \cap R$ is a maximal t-ideal of R, and it is easy to see that we must have $Q' \cap R = P$. It follows that $Q' = Q$. □

By Theorem 1.1, if R is a UMT-domain and U is an upper to zero in $R[X]$, then $\exists g \in U$ with $c(g)_v = R$. For convenience we insert a lemma which gives the same conclusion when U is any nonzero ideal contracted from $K[X]$ (where K is the quotient field of R).

Lemma 3.4. Let R be a domain with quotient field K. Then R is a UMT-domain $\Leftrightarrow$ for each nonconstant $f \in R[X]$, $\exists$ $g \in fK[X] \cap R[X]$ with $c(g)_v = R$.

Proof: Suppose that R is a UMT-domain. Write $f = f_1 \cdots f_r$ with each f_i irreducible in $K[X]$. Since R is a UMT-domain, for each i, $\exists$ $g_i \in f_iK[X] \cap R[X]$ with $c(g_i)_v = R$. Then $g = g_1 \cdots g_r \in fK[X] \cap R[X]$, and, via a standard argument involving the content formula [10, Theorem 28.1], $c(g)_v = R$. The converse is trivial. □

Proposition 3.5. Consider a pullback diagram of type □*. Then R is a UMT-domain $\Leftrightarrow$ D and T are UMT-domains, and M is a maximal t-ideal of T.

Proof: First assume that R is a UMT-domain, and let U be an upper to zero in $D[X]$. Then $\phi^{-1}(U)$ is an upper to M in $R[X]$. If $U \subseteq Q[X]$ for some t-prime Q of D, then $\phi^{-1}(U) \subseteq \phi^{-1}(Q)[X]$. However, $\phi^{-1}(Q)$ is a t-prime of R [9, Corollary 1.9], and by [18, Theorem A] we can find an upper to zero U' in $R[X]$ with $U' \subseteq \phi^{-1}(U) \subseteq \phi^{-1}(Q)[X]$, contradicting the fact that R is a UMT-domain (Theorem 1.1). Hence $U \nsubseteq Q[X]$ for all t-primes Q of D, and D is a UMT-domain. Corollary 3.2 shows that T is a UMT-domain and that M is a maximal t-ideal of T.

Conversely, assume that D and T are UMT-domains and that M is a t-ideal of T. Let P be a maximal t-ideal of R, and let U be an upper to zero in $R[X]$. We shall show that $U \nsubseteq P[X]$. We first suppose $P \supsetneq M$. In this case, we may as well assume that $U + M[X] \neq R[X]$. Let U' denote the upper to zero in $T[X]$ for which $U' \cap R[X] = U$. Since $M[X]$ is a common ideal of $R[X]$ and $T[X]$, it is easy to see that $U' + M[X] \neq T[X]$. Hence $\phi(U') = \phi(U' + M[X])$ is a proper ideal of $k[X]$. It follows that $\phi(U) = \phi(U') \cap D[X]$ is of the form $fk[X] \cap D[X]$ with $f \in D[X]$. By [9, Corollary 1.9], $\phi(P)$ is a t-prime of D. Hence, since D is a UMT-domain, Lemma 3.4 assures that $\phi(U) \nsubseteq \phi(P)[X]$, from which it follows that $U \nsubseteq P[X]$. This takes care of the case $P \supsetneq M$. If P does not properly contain M, then $P = M$ or P is incomparable to M. In the former case, since k is the quotient field of D, we have $R_M = T_M$, and the fact that T_M has Prüfer integral closure shows that $U \nsubseteq M[X] = P[X]$. In the latter case, let Q denote the unique prime of T with $Q \cap T = P$. By Lemma 3.3, Q is a maximal t-ideal of T. Since T is a UMT-domain, $R_P = T_Q$ has Prüfer integral closure, from which it follows that $U \nsubseteq P[X]$. By Theorem 1.5, R is a UMT-domain. $\square$

Proposition 3.6. Consider a pullback diagram of type □, and assume that $D = F$ is a field. Then R is a UMT-domain $\Leftrightarrow$ T is a UMT-domain, M is a maximal t-ideal of T, and k is algebraic over F.

Proof: Assume that R is a UMT-domain. We first show that k is algebraic over F. Let $u = \phi(t) \in k$ $(t \in T)$, and let U denote the kernel of the map from $R[X]$ to K (the quotient field of R) which sends X to t. Since R is a UMT-domain (and M is a t-prime of R), $\exists\ f = a_nX^n + \cdots + a_0 \in U \setminus M[X]$. Let m be the largest integer j for which $a_j \notin M$. Since $a_rt^r \in M \subseteq R$ for $n \geq r > m$, the equation $f(t) = 0$ shows that $m > 0$. By absorbing the terms a_rt^r, $n \geq r > m$, into the constant term, we obtain an equation $a_mt^m + \cdots = 0$. The image of this equation under ϕ is an equation showing that u is algebraic over F. This half of the proof can now be completed by appealing to Corollary 3.2.

For the converse, first observe that, since F is a field, we have the following pullback diagram.

$$\begin{array}{ccc} R_M & \longrightarrow & F \\ \downarrow & & \downarrow \\ T_M & \xrightarrow{\phi} & k. \end{array}$$

The fact that k is algebraic over F then shows that T_M is integral over R_M and therefore that $R_M{}' = T_M{}'$. Since T_M is a UMT-domain and M is a maximal t-ideal of T, this shows that $R_M{}'$ is a Prüfer domain. If P is a maximal t-ideal of R which is incomparable to M and Q is the unique prime of T with $Q \cap R = P$, then Q is a maximal t-ideal of T by Lemma 3.3. Hence, since T is a UMT-domain, $R_P = T_Q$ has Prüfer integral closure. By Theorem 1.5, R is a UMT-domain. □

Theorem 3.7. Consider a pullback diagram of type $\square$. Then R is a UMT-domain $\Leftrightarrow$ D and T are UMT-domains, M is a maximal t-ideal of T, and k is algebraic over the quotient field F of D.

Proof: The pullback diagram $\square$ can be split into two pullback diagrams as follows.

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ R_1 & \xrightarrow{\phi} & F \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & k. \end{array}$$

Here, R_1 is the pullback of the lower diagram. Assume that R is a UMT-domain. The upper diagram is a pullback diagram of type $\square^*$. By Proposition 3.5, R_1 and D are UMT-domains, and M is a maximal t-ideal of R_1. The other conclusions now follow from Proposition 3.6 applied to the lower diagram. For the converse, apply Proposition 3.6 to the lower diagram to conclude that R_1 is a UMT-domain (and note that M is divisorial and is therefore a maximal t-ideal of R_1). Then R is a UMT-domain by Proposition 3.5. $\square$

As a corollary, we recover the characterization of PVMD's in pullback constructions given in [9, Theorem 4.1] (which generalized a "$D+M$" characterization in [1]).

Corollary 3.8. Consider a pullback diagram of type $\square$. Then R is a PVMD $\Leftrightarrow$ D and T are PVMD's, M is a maximal t-ideal of T, and k is the quotient field of D.

Proof: Recall from [13, Proposition 3.2] that a domain is a PVMD $\Leftrightarrow$ it is an integrally closed UMT-domain. Assume that R is a PVMD. Then by Theorem 3.7, D and T are UMT-domains, M is a maximal t-ideal of T, and k is algebraic over the quotient field F of D. Since R is integrally closed (in T), it follows from standard pullback theory that $k = F$ and that D is integrally closed (cf. [8, Corollary 1.5]). Thus D is a PVMD. Now, since $T_Q = R_{Q \cap R}$ for each maximal ideal Q of T, T is integrally closed, whence T is a PVMD also.

For the converse, we immediately obtain from Theorem 3.7 that R is a UMT-domain. By [8, Corollary 1.5], R is also integrally closed and therefore a PVMD. $\square$

Now recall from [4] the notion of CPI(complete pre-image)-extension of a domain R with respect to a prime ideal P of R; this is denoted $R(P)$ and is defined by the following pullback diagram:

$$\begin{array}{ccc} R(P) & \longrightarrow & R/P \\ \downarrow & & \downarrow \\ R_P & \xrightarrow{\phi} & R_P/PR_P \end{array}$$

Here ϕ is the canonical homomorphism. By [5, Theorem 2.4], R_P has Prüfer integral closure $\Leftrightarrow$ it is a t-linkative UMT-domain, i.e., it is a UMT-domain and PR_P is maximal t-ideal. The following result now follows from Proposition 3.5.

Corollary 3.9. Let R be an integral domain, and let P be a t-prime of R. Then the CPI-extension $R(P)$ is a UMT-domain $\Leftrightarrow$ R/P is a UMT-domain and R_P has Prüfer integral closure.

Corollary 3.10. Consider a pullback diagram of type □. Then R has Prüfer integral closure $\Leftrightarrow$ D and T have Prüfer integral closure and k is algebraic over F.

Proof: (Again from [5, Theorem 2.4],) a domain has Prüfer integral closure $\Leftrightarrow$ it is a t-linkative UMT-domain. The theorem now follows from Theorem 3.7 and [5, Theorem 3.5]. □

Corollary 3.11. A domain R has Prüfer integral closure $\Leftrightarrow$ each overring of R is a UMT-domain.

Proof: If R has Prüfer integral closure, then R is a UMT-domain ([5, Theorem 2.4]). Since an overring of R also has Prüfer integral closure, it is also a UMT-domain. To prove the converse, we may assume that (R, M) is a quasi-local domain which is integrally closed but is not a valuation domain; we shall show that R possesses a non-UMT-overring. Choose an element u in the quotient field K of R for which neither u nor u^{-1} lies in R, and let U denote the kernel of the natural map from $R[X]$ to K sending X to u. Set $T = R[u] \simeq R[X]/U$. By the u, u^{-1} lemma [10, Lemma 19.14], we have $U \subseteq M[X]$. In the natural isomorphism $T \simeq R[X]/U$, let the prime ideal N of T correspond to $M[X]/U$. Note that $T/N \simeq (R/M)[X]$. Consider the following pullback diagram.

$$\begin{array}{ccc} S & \longrightarrow & R/M \\ \downarrow & & \downarrow \\ T_N & \xrightarrow{\phi} & T_N/NT_N \end{array}$$

The pullback S is an overring of R. Since T_N/NT_N (which is the quotient field of T/N) is not algebraic over R/M, Theorem 3.7 (or Proposition 3.6) shows that S is not a UMT-domain. □

As a final corollary, we recover the following known result (see [2, Lemme 1.2]).

Corollary 3.12. If R is a domain with Prüfer integral closure, then R/P has Prüfer integral closure for each prime P of R.

Proof: Let P be a prime of R. As an overring of R, $R(P)$, the CPI-extension of R with respect to P, is a UMT-domain by Corollary 3.11. Hence, by Corollary 3.9, R/P is also a UMT-domain. It is well known that an overring of R/P is of the form T/Q, where T is an overring of R and Q is a prime ideal of T. (This can be proved by inserting the overring of R/P between R/P and R_P/PR_P in the CPI-extension diagram above and pulling back to obtain T.) As an overring of R, T has Prüfer integral closure, whence, by what was just proved, T/Q is a UMT-domain. Thus each overring of R/P is a UMT-domain, and R/P has Prüfer integral closure by Corollary 3.11. □

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