New algebraic properties of an amalgamated algebra along an ideal

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Dedicated to Alberto Facchini on the occasion of his 60th birthday

Abstract

Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. In this paper, we study the amalgamation of A with B along J with respect to f (denoted by $A \bowtie^f J$), a construction that provides a general frame for studying the amalgamated duplication of a ring along an ideal, introduced by D'Anna and Fontana in 2007, and other classical constructions (such as the $A+XB[X]$, the $A+XB[\![X]\!]$ and the $D+M$ constructions). In particular, we completely describe the prime spectrum of the amalgamation $A \bowtie^f J$ and, when it is a local Noetherian ring, we study its embedding dimension and when it turns to be a Cohen-Macaulay ring or a Gorenstein ring.

The present version of the manuscript differs from the previous one, posted on arXiv and published in Comm. Algebra **44** (2016), 1836–1851, for an appendix –added at the end of the paper– where we observe that Proposition 4.1(2) and Theorem 4.4 hold under the assumption, not explicitly declared, that $B = f(A) + J$. Furthermore, in the same appendix, we provide the exact value for the embedding dimension of $A \bowtie^f J$, also when $B \neq f(A)+J$, under the hypothesis that J is finitely generated as an ideal of the ring $f(A) + J$. Finally, we also deleted Example 4.6.

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1 Introduction

Let A and B be commutative rings with unity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A,\ j \in J\}$$

called *the amalgamation of A with B along J with respect to f*. This construction is a generalization of the amalgamated duplication of a ring along an ideal (cf., for instance, [4], [5], [8], [12], [13], [20] and [25]). Moreover, several classical constructions (such as the $A+XB[X]$, the $A+XB[\![X]\!]$ and the $D+M$ constructions) can be studied as particular cases of the amalgamation [10, Examples 2.5 and 2.6] and other classical constructions, such as the Nagata's idealization (cf. [19, Chapter VI, Section 25], [22, page 2]), also called Fossum's trivial extension (cf. [18]), and the CPI extensions (in the sense of Boisen and Sheldon [6]) are strictly related to it [10, Example 2.7 and Remark 2.8].

On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by D.D. Anderson in [2] and motivated by a classical construction due to Dorroh [14], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [10, Section 2].

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [10, Section 4] (for a systematic study of this type of constructions, cf. [16], [17], [23]). This point of view allows us to deepen the study initiated in [10] and continued in [11] and to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. More precisely, in [10], we studied the basic properties of this construction (e.g., we provided characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and we characterized those distinguished pullbacks that can be expressed as an amalgamation and in [11] we investigated the Krull dimension of $A \bowtie^f J$. In this paper, we study in details its prime spectrum and, when $A \bowtie^f J$ is a local Noetherian ring, some of its invariants (like the embedding dimension) and relevant properties (like Cohen-Macaulyness and Gorensteinness).

In particular, after recalling (in Section 2) some basic properties proved in [10], needed in the present paper, we provide a complete description of the prime spectrum of $A \bowtie^f J$ (Corollary 2.5) and we characterize when $A \bowtie^f J$ is a local ring (Corollary 2.7). In Section 3, we prove some results on the extensions in $A \bowtie^f J$ of ideals of A (Proposition 3.1 and Lemma 3.2), that we will need in the sequel of the paper. In Sections 4 and 5, we concentrate our attention on the case when $A \bowtie^f J$ is local; in particular, we give bounds for its embedding dimension (Proposition 4.1) and we produce classes of rings $A \bowtie^f J$ satisfying the upper or the lower bound (Proposition 4.3 and Theorem 4.4). In the last section, we study when $A \bowtie^f J$ is a Cohen-Macaulay or a Gorenstein ring (Remarks 5.1, 5.4 and Proposition 5.5). Moreover, when $A \bowtie^f J$ is Cohen-Macaulay, we determine its multiplicity (Proposition 5.8).

2 The prime spectrum

Before beginning a systematic study of the ring $A\bowtie^f J$, we recall from our introductory paper to the subject [10] the notation that we will use in the present paper and some basic properties of this construction.

2.1 PROPOSITION. [10, Proposition 5.1] *Let* $f : A \to B$ *be a ring homomorphism,* J *an ideal of* B *and set* $A\bowtie^f J := \{(a, f(a)+j) \mid a \in A,\ j \in J\}$.

(1) *Let* $\iota := \iota_{_{A,f,J}} : A \to A\bowtie^f J$ *be the natural the ring homomorphism defined by* $\iota(a) := (a, f(a))$, *for all* $a \in A$. *The map* ι *is an embedding, making* $A\bowtie^f J$ *a ring extension of* A.

(2) *Let* I *be an ideal of* A *and set* $I\bowtie^f J := \{(i, f(i)+j) \mid i \in I, j \in J\}$. *Then,* $I\bowtie^f J$ *is an ideal of* $A\bowtie^f J$, *the composition of canonical homomorphisms* $A \overset{\iota}{\hookrightarrow} A\bowtie^f J \twoheadrightarrow (A\bowtie^f J)/(I\bowtie^f J)$ *is a surjective ring homomorphism and its kernel coincides with* I.

(3) *Let* $p_{_A} : A\bowtie^f J \to A$ *and* $p_{_B} : A\bowtie^f J \to B$ *be the natural projections of* $A\bowtie^f J \subseteq A \times B$ *into* A *and* B, *respectively. Then,* $p_{_A}$ *is surjective and* $\mathrm{Ker}(p_{_A}) = \{0\} \times J$. *Moreover,* $p_{_B}(A\bowtie^f J) = f(A) + J$ *and* $\mathrm{Ker}(p_{_B}) = f^{-1}(J) \times \{0\}$.

(4) *Let* $\gamma : A\bowtie^f J \to (f(A)+J)/J$ *be the natural ring homomorphism, defined by* $(a, f(a)+j) \mapsto f(a)+J$. *Then,* γ *is surjective and* $\mathrm{Ker}(\gamma) = f^{-1}(J)\times J$.

Let $f : A \to B$ be a ring homomorphism and J an ideal of B. In the present paper, we intend to further investigate the algebraic properties of the ring $A\bowtie^f J$, in relation with those of A, B, J and f. Recall that, in [10], we have shown that the ring $A\bowtie^f J$ can be represented as a pullback of natural ring homomorphisms and, using the notion of ring retraction, we have characterized which type of pullbacks are exactly of the form $A\bowtie^f J$. In this paper, we will make an extensive use of that idea for deepening the study of the ring $A\bowtie^f J$.

2.2 REMARK. (a) Recall that, if $\alpha : A \to C$, $\beta : B \to C$ are ring homomorphisms, the subring $D := \alpha \times_{_C} \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β. We denote by $p_{_A}$ (respectively, $p_{_B}$) the restriction to $\alpha \times_{_C} \beta$ of the projection of $A \times B$ onto A (respectively, B).

The following statement is a straightforward consequence of the definitions: Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. If $\pi : B \to B/J$ is the canonical projection and $\breve{f} := \pi \circ f$, then $A\bowtie^f J = \breve{f} \times_{_{B/J}} \pi$.

(b) Recall that a ring homomorphism $r : B \to A$ is called *a ring retraction* if there exists an (injective) ring homomorphism $i : A \to B$ such that $r \circ i = id_A$. In this case, we say also that A is *a retract of* B. By [10, Remark 4.6], with the previous notation, we have that A is a retract of $A\bowtie^f J$ and the map $p_{_A} : A\bowtie^f J \to A$, defined in Proposition 2.1(3), is a ring retraction. In fact, we have $p_{_A} \circ \iota = id_{_A}$, where ι is the ring embedding of A into $A \bowtie^f J$ (Proposition 2.1(1)).

(c) The pullbacks of the form $A \bowtie^f J$ form a distinguished subclass of the class of pullbacks of ring homomorphisms, as described in [10, Proposition 4.7]. Let $A, B, C, \alpha, \beta, p_{_A}, p_{_B}$ be as in (a). Then, $p_{_A} : D\ (= \alpha \times_{_C} \beta) \to A$ is a ring retraction if and only if there exists an ideal J of B and a ring homomorphism $f : A \to B$ such that $D \cong A \bowtie^f J$.

(d) Note that, using the notation in (a), we are not making any assumption on the ring homomorphism $\alpha : A \to C$ nor on the homomorphism $\breve{f} := \pi \circ f : A \to B/J$. In [1] the authors consider a new construction, called connected sum of local rings, obtained by taking a quotient of a pullback for which both the homomorphisms α and β are surjective. A particular case of this type of pullback is the amalgamated duplication $A \bowtie I$, where A is a local ring and I an ideal of A (see [12] and [13]).

(e) Note that the amalgamation $A \bowtie^f J$, even in the local case, may not be fully re-conducted to a pullback for which both the homomorphisms α and β are surjective. However, changing the data, and considering $B' := f(A) + J$, J as an ideal of B', and $f' : A \to B'$ acting as f, it is easy to see that $A \bowtie^f J = A \bowtie^{f'} J$ and $A \bowtie^{f'} J$ is a pullback of $\pi' : B' \to B'/J$ and $\breve{f}' := \pi' \circ f' : A \to B'/J$ (i.e., $A \bowtie^{f'} J = \breve{f}' \times_{B'/J} \pi'$), which are now both surjective. But, this is only apparently a simplification of the given construction, since the problem of studying $A \bowtie^f J$ from the data A, B, J, f is transformed into the problem of studying $A \bowtie^{f'} J$ and the ring inclusion $f(A) + J \hookrightarrow B$, and the last problem presents the same level of complexity of a direct investigation of the given construction (see for instance [10, Section 5] and [11, Section 4]).

Let $f : A \to B$ be a ring homomorphism, and set $X := \mathrm{Spec}(A)$, $Y := \mathrm{Spec}(B)$. Recall that $f^* : Y \to X$ denotes the continuous map (with respect to the Zariski topologies) naturally associated to f (i.e., $f^*(Q) := f^{-1}(Q)$ for all $Q \in Y$). Let S be a subset of A. Then, as usual, $V_X(S)$, or simply $V(S)$, if no confusion can arise, denotes the closed subspace of X, consisting of all prime ideals of A containing S.

In the next lemma we recall the notation and some basic properties of pullback constructions that we will use in the present paper. We refer to the paper by Fontana [17], since the subsequent work on pullbacks by Facchini [16] and, in the Noetherian setting, by Ogoma [23] is not relevant to our study.

2.3 Lemma. [17, Theorem 1.4] *With the notation of Remark 2.2 (a), set* $X := \mathrm{Spec}(A)$, $Y := \mathrm{Spec}(B)$, $Z := \mathrm{Spec}(C)$*, and* $W := \mathrm{Spec}(D)$*. Assume that* β *is surjective. Then, the following statements hold.*

(1) *If* $H \in W \backslash V(\mathrm{Ker}(p_{_A}))$*, then there is a unique prime ideal* Q *of* B *such that* $p_{_B}^{-1}(Q) = H$*. Moreover,* $Q \in Y \backslash V(\mathrm{Ker}(\beta))$ *and* $D_H \cong B_Q$*, under the canonical homomorphism induced by* $p_{_B}$*.*

(2) *The continuous map* $p_{_A}^*$ *is a closed embedding of* X *into* W*. Thus* X *is homeomorphic to its image,* $V(\mathrm{Ker}(p_{_A}))$*, under* $p_{_A}^*$*.*

(3) *The restriction of the continuous map* p_B^* *to* $Y \backslash V(\mathrm{Ker}(\beta))$ *is an homeomorphism of* $Y \backslash V(\mathrm{Ker}(\beta))$ *with* $W \backslash V(\mathrm{Ker}(p_A))$ *(hence,* a fortiori*, it is an isomorphism of partially ordered sets).*

In particular, the prime ideals of D *are of the type* $p_A^{-1}(P)$ *or* $p_B^{-1}(Q)$*, where* P *is any prime ideal of* A *and* Q *is a prime ideal of* B*, with* $Q \not\supseteq \mathrm{Ker}(\beta)$*.*

The following corollary is a direct consequence of Lemma 2.3.

2.4 Corollary. *With the notation of Remark 2.2 (a), assume that* β *is surjective. Let* H *be a prime ideal of* D $(= \alpha \times_C \beta)$*.*

(1) *Assume that* H *contains* $\mathrm{Ker}(p_A)$*. Let* P *be the only prime ideal of* A *such that* $H = p_A^*(P)$ *(Lemma 2.3(2)). Then,* H *is a maximal ideal of* D *if and only if* P *is a maximal ideal of* A*.*

(2) *Assume that* H *does not contain* $\mathrm{Ker}(p_A)$*. Let* Q *be the only prime ideal of* B $(Q \notin V(\mathrm{Ker}(\beta)))$ *such that* $p_B^*(Q) = H$ *(Lemma 2.3(1)). Then,* H *is a maximal ideal of* D *if and only if* Q *is a maximal ideal of* B*.*

(3) D $(= \alpha \times_C \beta)$ *is a local ring if and only if* A *is a local ring and* $\mathrm{Ker}(\beta)$ *is contained in the Jacobson radical* $\mathrm{Jac}(B)$*. In particular, if* A *and* B *are local rings, then* D *is a local ring. Moreover, if* D *is a local ring and* M *is the only maximal ideal of* A*, then* $\{p_A^{-1}(M)\} = \mathrm{Max}(D)$*.*

As a consequence of the previous results we can now easily describe the structure of the prime spectrum of the ring $A \bowtie^f J$. The details of the proof are omitted.

2.5 Corollary. *With the notation of Proposition 2.1, set* $X := \mathrm{Spec}(A)$*,* $Y := \mathrm{Spec}(B)$*, and* $W := \mathrm{Spec}(A \bowtie^f J)$*,* $J_0 := \{0\} \times J$ $(\subseteq A \bowtie^f J)$*, and* $J_1 := f^{-1}(J) \times \{0\}$*. For all* $P \in X$ *and* $Q \in Y$*, set:*

$$\begin{aligned} P'^f &:= P \bowtie^f J := \{(p, f(p) + j) \mid p \in P,\ j \in J\}, \\ \overline{Q}^f &:= \{(a, f(a) + j) \mid a \in A,\ j \in J,\ f(a) + j \in Q\}. \end{aligned}$$

Then, the following statements hold.

(1) *The map* $P \mapsto P'^f$ *establishes a closed embedding of* X *into* W*, so its image, which coincides with* $V(J_0)$*, is homeomorphic to* X*.*

(2) *The map* $Q \mapsto \overline{Q}^f$ *is a homeomorphism of* $Y \backslash V(J)$ *onto* $W \backslash V(J_0)$*.*

(3) *The prime ideals of* $A \bowtie^f J$ *are of the type* P'^f *or* $\overline{Q}^f$*, for* P *varying in* X *and* Q *in* $Y \backslash V(J)$*.*

(4) $W = V(J_0) \cup V(J_1)$ *and the set* $V(J_0) \cap V(J_1)$ *is homeomorphic to* $\mathrm{Spec}((f(A) + J)/J)$*, via the continuous map associated to the natural ring homomorphism* $\gamma : A \bowtie^f J \to (f(A) + J)/J$*,* $(a, f(a) + j) \mapsto f(a) + J$*. In particular, we have that the closed subspace* $V(J_0) \cap V(J_1)$ *of* W *is homeomorphic to the closed subspace* $V(J)$ *of* $Y (= \mathrm{Spec}(B))$*, when* f *is surjective.*

The following example provides a geometrical illustration of some of the material presented above.

2.6 Example. Let K be an algebraically closed field and X, Y two indeterminates over K. Set $A := K[X,Y]$, $B := K[X]$ and $f : K[X,Y] \to K[X]$ defined by $Y \mapsto 0$ and $X \mapsto X$. Let $J := XK[X]$. We want to study the ring $K[X,Y] \bowtie^f J$ (note that, from a geometrical point of view, f^* determines the inclusion of the line defined by the equation $Y = 0$ into the affine space $\mathbb{A}^2_K$.)

According to the notation of Corollary 2.5, we have $V(J_1) \cong \mathrm{Spec}(K[Y])$. Moreover, the projection p_B of $A \bowtie^f J$ into B is surjective, since f is surjective, and its kernel is J_1 (see Proposition 2.1). Thus $\mathrm{Spec}(A \bowtie^f J/J_1) \cong V(J_1) \cong \mathrm{Spec}(B) = \mathrm{Spec}(K[X])$. We have also $V(J_1) \cap V(J_2) \cong \mathrm{Spec}(B/J) = \mathrm{Spec}(K)$, by Corollary 2.5(4). Then, $A \bowtie^f J$ is the coordinate ring of the union of a plane (i.e., $\mathrm{Spec}(K[X,Y])$) and a line (i.e., $\mathrm{Spec}(K[X])$) with one common point (i.e., $\mathrm{Spec}(K)$). Note that, in this case, the ring $A \bowtie^f J$ can be also presented by a quotient of a polynomial ring. Indeed, since f is surjective and $B/J \cong K$, by a standard argument we easily obtain that $A \bowtie^f J$ is isomorphic to $K[X,Y,Z]/(ZX, YZ)$.

If we specialize Corollary 2.4 to the case of the construction $A \bowtie^f J$, then we obtain the following:

2.7 Corollary. *We preserve the notation of Corollary 2.5.*

(1) *Let $P \in X$. Then, $P'^{_f}$ is a maximal ideal of $A \bowtie^f J$ if and only if P is a maximal ideal of A.*

(2) *Let Q be a prime ideal of B not containing J. Then, $\overline{Q}^{f}$ is a maximal ideal of $A \bowtie^f J$ if and only if Q is a maximal ideal of B.*

In particular, $\mathrm{Max}(A \bowtie^f J) = \{P'^{_f} \mid P \in \mathrm{Max}(A)\} \cup \{\overline{Q}^{f} \mid Q \in \mathrm{Max}(B) \setminus V(J)\}$.

(3) *$A \bowtie^f J$ is a local ring if and only if A is a local ring and $J \subseteq \mathrm{Jac}(B)$.*

In particular, if M is the unique maximal ideal of A, then $M'^{_f} = M \bowtie^f J$ is the unique maximal ideal of $A \bowtie^f J$.

The following result, whose proof is straightforward, provides a description of the minimal prime ideals of $A \bowtie^f J$.

2.8 Corollary. *With the notation of Corollary 2.5, set*

$$\mathcal{X} := \mathcal{X}_{(f,J)} := \bigcup_{Q \in \mathrm{Spec}(B) \setminus V(J)} V(f^{-1}(Q + J))$$

The following properties hold.

(1) *The map $Q \mapsto \overline{Q}^{f}$ establishes a homeomorphism of $\mathrm{Min}(B) \setminus V(J)$ with $\mathrm{Min}(A \bowtie^f J) \setminus V(J_0)$.*

(2) *The map* $P \mapsto P'_f$ *establishes a homeomorphism of* $\mathrm{Min}(A) \setminus \mathcal{X}$ *with* $\mathrm{Min}(A \bowtie^f J) \cap V(J_0)$.

After describing the topological and ordering properties of the prime spectrum of the ring $A \bowtie^f J$, we now describe the localizations of $A \bowtie^f J$ at each of its prime ideals.

2.9 Proposition. *With the notation of Proposition 2.1 and Corollary 2.5, the following statements hold.*

(1) *For any prime ideal* $Q \in Y \setminus V(J)$, *the ring* $(A \bowtie^f J)_{\overline{Q}^f}$ *is canonically isomorphic to* B_Q.

(2) *For any prime ideal* $P \in X \setminus V(f^{-1}(J))$, *the localization* $(A \bowtie^f J)_{P'_f}$ *is canonically isomorphic to* A_P.

(3) *Let* P *be a prime ideal of* A *containing* $f^{-1}(J)$. *Consider the multiplicative subset* $S := S_{(f,P,J)} := f(A \setminus P) + J$ *of* B *and set* $B_S := S^{-1}B$ *and* $J_S := S^{-1}J$. *If* $f_P : A_P \longrightarrow B_S$ *is the ring homomorphism induced by* f, *then the ring* $(A \bowtie^f J)_{P'_f}$ *is canonically isomorphic to* $A_P \bowtie^{f_P} J_S$.

Proof. Keeping in mind the fiber product structure of $A \bowtie^f J$, (1) follows from Lemma 2.3 and (2) is straightforward. From the last part of Remark 2.2(a) we infer that, if $\breve{f}_P : A_P \longrightarrow B_S/J_S$ is the ring homomorphism induced by f_P and if $\pi_{(P)} : B_S \longrightarrow B_S/J_S$ is the canonical projection, then $A_P \bowtie^{f_P} J_S$ is isomorphic to the fiber product $\breve{f}_P \times_{B_S/J_S} \pi_{(P)}$. Moreover, it is easily verified that $p_A(A \bowtie^f J \setminus P'_f) = A \setminus P$ and $p_B(A \bowtie^f J \setminus P'_f) = S$. Then statement (3) follows from [17, Proposition 1.9]. □

3 Extension of ideals of A to $A \bowtie^f J$

In this section we pursue the study of the ideal-theoretic structure of the amalgamation $A \bowtie^f J$.

3.1 Proposition. *We preserve the notation of Proposition 2.1 and Corollary 2.5. The following properties hold.*

(1) *If* I *(respectively,* H*) is an ideal of* A *(respectively, of* $f(A) + J$*) such that* $f(I)J \subseteq H \subseteq J$, *then* $I \bowtie^f H := \{(i, f(i) + h) \mid i \in I,\, h \in H\}$ *is an ideal of* $A \bowtie^f J$.

(2) *If* I *is an ideal of* A, *then the extension* $I(A \bowtie^f J)$ *of* I *to* $A \bowtie^f J$ *coincides with* $I \bowtie^f (f(I)B)J := \{(i, f(i) + \beta) \mid i \in I,\, \beta \in (f(I)B)J\}$.

(3) *If* I *is an ideal of* A *such that* $f(I)B = B$, *then* $I(A \bowtie^f J) = I'_f = \{(i, f(i) + j) \mid i \in I,\, j \in J\} = I \bowtie^f J$.

Proof. (1) is straightforward. (2). Set $I_0 := I \bowtie^f (f(I)B)J$. By applying (1) to $H := (f(I)B)J$, it follows that I_0 is an ideal of $A \bowtie^f J$ and, by definition, $I_0 \supseteq \iota(I)$ $(= \{(i, f(i)) \mid i \in I\})$. Now, let L be an ideal of $A \bowtie^f J$ containing $\iota(I)$, and let $(i, f(i) + \beta) \in I_0$ (where $i \in I$ and $\beta \in (f(I)B)J$). Therefore, we can find $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$, $b_1, b_2, \ldots, b_n \in J$ such that $\beta = \sum_{k=1}^{n} f(\alpha_k)b_k$. Since, $(i, f(i)), (\alpha_1, f(\alpha_1)), (\alpha_2, f(\alpha_2)), \ldots, (\alpha_n, f(\alpha_n)) \in \iota(I) \subseteq L$, then

$$(i, f(i) + \beta) = (i, f(i)) + \sum_{k=1}^{n} (\alpha_k, f(\alpha_k))(0, b_k) \in L\,.$$

and so $I_0 \subseteq L$. The proof of (2) is now complete. (3) follows immediately from (2). $\square$

3.2 Corollary. *Let A be a local ring with maximal ideal M, let $f : A \to B$ be a ring homomorphism, and J be an ideal of B such that $f^{-1}(Q) \neq M$, for each $Q \in \mathrm{Spec}(B) \setminus V(J)$. If I is an ideal of A whose radical is M, then the radical of $I(A \bowtie^f J)$ is M'^f $(= M \bowtie^f J)$.*

Proof. Suppose that P is a prime ideal of A such that $P'^f \supseteq I(A \bowtie^f J)$. It follows immediately that $I \subseteq P$ and thus $P = M$, by assumption. Suppose now that $I(A \bowtie^f J) \subseteq \overline{Q}^f$, for some $Q \in \mathrm{Spec}(B) \setminus V(J)$. From Proposition 3.1(2) and the definition of $\overline{Q}^f$, we deduce that $(f(I)B)J \subseteq Q$ and, in particular, $f(I) \subseteq Q$, i.e., $I \subseteq f^{-1}(Q)$; therefore, by assumption, $f^{-1}(Q) = M$, which is a contradiction. This means that the unique prime ideal of $A \bowtie^f J$ containing $I(A \bowtie^f J)$ is M'^f. $\square$

3.3 Remark. Notice that, in case J is finitely generated as A-module and it is contained in the Jacobson radical of B, for every prime Q of B not containing J, we have $f^{-1}(Q) \neq M$. In fact, if we had $f^{-1}(Q) = M$, we would have $f(M) \subseteq Q$, that implies J/QJ is finite dimensional as A/M-vector space; now, $J \not\subseteq Q$ and Q is a prime ideal, so if $j \in J \setminus Q$ then $j^n \in J \setminus Q$, for every integer $n \geq 1$ and, since $J \subseteq \mathrm{Jac}(B)$, it is not difficult to check that the images of the elements $j, , j^2, \ldots, j^n$ in J/QJ are linearly independent over A/M for any n, that is a contradiction.

In particular, if J is finitely generated as A-module and it is contained in the Jacobson radical of B, the extension in $A \bowtie^f J$ of any M-primary ideal of A is $M \bowtie^f J$-primary.

4 The embedding dimension of $A \bowtie^f J$

Let A be a ring and I be an ideal of A. If I is finitely generated, we denote, as usual, by $\nu(I)$ the minimum number of generators of the ideal I. Assume that A is a local ring and that M is its maximal ideal. Set $\boldsymbol{k} := A/M$. If we suppose that M is finitely generated, we call the *embedding dimension of A* the natural number

$$\mathrm{embdim}(A) := \nu(M) = \dim_{\boldsymbol{k}}(M/M^2)\,.$$

We give next some bounds for the embedding dimension of $A \bowtie^f J$, when this ring is local with finitely generated maximal ideal.

4.1 PROPOSITION. *We preserve the notation of Proposition 2.1. Assume that A is a local ring with maximal ideal M and that the ideal J is contained in the Jacobson radical* $\mathrm{Jac}(B)$. *The following statements hold.*

(1) *If $A \bowtie^f J$ has finitely generated maximal ideal, then A has also finitely generated maximal ideal and*

$$\mathrm{embdim}(A) \leq \mathrm{embdim}(A \bowtie^f J).$$

(2)[1] *If A has finitely generated maximal ideal and J is finitely generated, then $A \bowtie^f J$ has finitely generated maximal ideal and*

$$\mathrm{embdim}(A \bowtie^f J) \leq \mathrm{embdim}(A) + \nu(J).$$

PROOF. By using Corollary 2.7(3), it follows that $A \bowtie^f J$ is a local ring with maximal ideal $M'^f := M \bowtie^f J := \{(m, f(m) + j) \mid m \in M,\ j \in J\}$.
(1) It suffices to note that, if $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a finite set of generators of M'^f, then $\{p_A(\mathbf{x}_i) \mid i = 1, 2, \ldots, n\}$ is a finite set of generators of M.
(2) Let $m_1, m_2, \ldots, m_r \in M$ and $j_1, j_2, \ldots, j_s \in J$ be elements such that $M = (m_1, m_2, \ldots, m_r)$ and $J = (j_1, j_2, \ldots, j_s)$, with $\nu(M) = r$ and $\nu(J) = s$. It follows immediately that $\{(m_\lambda, f(m_\lambda)); (0, j_\mu) \mid 1 \leq \lambda \leq r, 1 \leq \mu \leq s\}$ is a set of generators of $M \bowtie^f J$. Therefore, $\mathrm{embdim}(A \bowtie^f J) \leq \mathrm{embdim}(A) + \nu(J)$. □

In the next example we will provide a ring homomorphism $f : A \to B$ and an ideal $J \neq (0)$ of B such that $\mathrm{embdim}(A) = \mathrm{embdim}(A \bowtie^f J) < \mathrm{embdim}(A) + \nu(J)$.

4.2 EXAMPLE. Let p be a prime number, T be an indeterminate over $\mathbb{Q}$, and set $A := \mathbb{Z}_{(p)}$, $B := \mathbb{Q}[\![T]\!]$, $J := TB$. By [10, Example 2.6], the ring $S := A + TB$ is naturally isomorphic to $A \bowtie^f J$, where $\iota : A \to B$ is the inclusion. It is easy to see that S is a 2-dimensional valuation domain whose maximal ideal N $(:= p\mathbb{Z}_{(p)} + TB)$ is principal (namely, $N = pS$). It follows that $\mathrm{embdim}(A) = \mathrm{embdim}(A \bowtie^f J) = 1 < \mathrm{embdim}(A) + \nu(J) = 2$.

The previous example is a particular case of the following result.

4.3 PROPOSITION. *We preserve the notation of Proposition 2.1 and Corollary 2.5. Assume that A is a local ring with finitely generated maximal ideal M satisfying the property $f(M)B = B$. Then, for every ideal J of B contained in the Jacobson radical of B, the amalgamation $A \bowtie^f J$ is a local ring with finitely generated maximal ideal, and*

$$\mathrm{embdim}(A) = \mathrm{embdim}(A \bowtie^f J)\,.$$

[1] see the Appendix

PROOF. Let $r := \mathrm{embdim}(A)$ and let $\{m_1, m_2, \ldots, m_r\}$ be a minimal set of generators for M. By Corollary 2.7(3), $A \bowtie^f J$ is a local ring with maximal ideal $M'^f := \{(m, f(m) + j) \mid M \in M, j \in J\}$ and, applying Proposition 3.1(3), we get the equality $M'^f = M(A \bowtie^f J)$. It follows immediately that $\{(m_1, f(m_1)), (m_2, f(m_2)), \ldots, (m_r, f(m_r))\}$ is a finite set of generators for M'^f and, thus, $\mathrm{embdim}(A \bowtie^f J) \leq r := \mathrm{embdim}(A)$. Now, the conclusion is an immediate consequence of Proposition 4.1(1). □

The next result will provide a relevant class of rings obtained by amalgamation satisfying the equality $\mathrm{embdim}(A \bowtie^f J) = \mathrm{embdim}(A) + \nu(J)$.

4.4 THEOREM.[2] *We preserve the notation of Proposition 2.1. Suppose that A is a local ring with finitely generated maximal ideal M, and that J is a finitely generated ideal of B. If $f(M)B \subseteq \mathrm{Jac}(B)$ and $J \subseteq \mathrm{Jac}(B)$, then $A \bowtie^f J$ is a local ring with finitely generated maximal ideal, and*

$$\mathrm{embdim}(A \bowtie^f J) = \mathrm{embdim}(A) + \nu(J)\,.$$

PROOF. Let $\{m_1, m_2, \ldots, m_r\} \subseteq M$, and $\{j_1, j_2, \ldots, j_s\} \subseteq J$ be sets of generators of M and J, respectively, such that $\nu(M) = r$ and $\nu(J) = s$. By Proposition 4.1 and its proof it follows immediately the inequality $\mathrm{embdim}(A \bowtie^f J) \leq \mathrm{embdim}(A) + \nu(J)$ and, more precisely, that $G' := \{(m_\lambda, f(m_\lambda));\ (0, j_\mu) \mid 1 \leq \lambda \leq r, 1 \leq \mu \leq s\}$ is a set of generators of the maximal ideal $M' := M'^f = M \bowtie^f J$ of $A \bowtie^f J$. Notice that $\boldsymbol{k}$, the residue field of A, coincide with the residue field of $A \bowtie^f J$ (see Proposition 2.1(2)). Then, to get the equality $\mathrm{embdim}(A \bowtie^f J) = \mathrm{embdim}(A) + \nu(J)$ it suffices to show that the image $\overline{G'}$ of G' in M'/M'^2 is a basis of M'/M'^2 as a $\boldsymbol{k}-$vector space. Obviously, it is enough to check that $\overline{G'}$ is linearly independent. Pick $a_1, a_2, \ldots, a_r, \alpha_1, \alpha_2, \ldots, \alpha_s \in A$ such that

$$\sum_{\lambda=1}^{r} [a_\lambda]_M [(m_\lambda, f(m_\lambda))]_{M'^2} + \sum_{\mu=1}^{s} [\alpha_\mu]_M [(0, j_\mu)]_{M'^2} = 0\,. \quad (\star)$$

In other words, we have

$$\left(\sum_{\lambda=1}^{r} a_\lambda m_\lambda,\ \sum_{\lambda=1}^{r} f(a_\lambda m_\lambda) + \sum_{\mu=1}^{s} f(\alpha_\mu) j_\mu \right) \in M'^2$$

and, in particular, $\sum_{\lambda=1}^{r} a_\lambda m_\lambda \in M^2$. Since $r = \nu(M)$, it is easy to see that $a_\lambda \in M$, for every $\lambda = 1, 2, \ldots, r$. Thus, by $(\star)$, we have $\sum_{\mu=1}^{s} [\alpha_\mu]_M [(0, j_\mu)]_{M'^2} = 0$ and so

$$\left(0,\ \sum_{\mu=1}^{s} f(\alpha_\mu) j_\mu \right) \in M'^2\,.$$

This means that $\left(0,\ \sum_{\mu=1}^{s} f(\alpha_\mu) j_\mu\right)$ is a finite sum of elements of the form $(m,\ f(m) + j)(n,\ f(n) + \ell)$, where $m, n \in M$ and $j, \ell \in J$. Then, an easy

[2] see the Appendix

computation shows that $\sum_{\mu=1}^{s} f(\alpha_\mu)j_\mu$ is a finite sum of elements of the form $f(m)\ell+f(n)j+j\ell$ and thus the element $b := \sum_{\mu=1}^{s} f(\alpha_\mu)j_\mu \in (f(M)B)J+J^2 \subseteq \mathrm{Jac}(B)J$. Suppose, by contradiction, that some coefficient $\alpha_\mu \in A \setminus M$, say α_1, and let β_1 denote the inverse of $f(\alpha_1)$ in B. Then $\beta_1 b \in \mathrm{Jac}(B)J$, and thus there are elements $l_1, l_2, \ldots, l_s \in \mathrm{Jac}(B)$ such that

$$\beta_1 b = j_1 + \sum_{\mu=2}^{s} \beta_1 f(\alpha_\mu)j_\mu = \sum_{\mu=1}^{s} l_\mu j_\mu.$$

This shows that $(1-l_1)j_1 \in (j_2, \ldots, j_s)B$, and hence, keeping in mind that $l_1 \in \mathrm{Jac}(B)$, we have $j_1 \in (j_2, \ldots, j_s)B$, a contradiction. Thus $\alpha_\mu \in M$ for $\mu = 1, 2, \ldots, s$. The proof is now complete. □

As an application we obtain the following.

4.5 Corollary. *Let A be a local ring with finitely generated maximal ideal, and let I be a finitely generated proper ideal of A. Then, the duplicated amalgamation $A \bowtie I$ of A along I is a local ring with finitely generated maximal ideal, and furthermore* $\mathrm{embdim}(A \bowtie I) = \mathrm{embdim}(A) + \nu(I)$.

Proof. Apply [10, Example 2.4] and Proposition 4.4. □

5 Cohen–Macaulay and Gorenstein properties for the ring $A \bowtie^f J$

In this section, assuming that $A \bowtie^f J$ is local and Noetherian, we investigate the problem of when $A \bowtie^f J$ is a Cohen-Macaulay (briefly CM) ring or a Gorenstein ring. Moreover, when $A \bowtie^f J$ is Cohen-Macaulay, we determine its multiplicity.

Notation and assumptions.

In the following (unless explicitly stated to the contrary), we assume that:

- $f : A \to B$ is a ring homomorphism;
- A is Noetherian, local, with maximal ideal M;
- J is an ideal of B contained in the Jacobson radical $\mathrm{Jac}(B)$ of B;
- J is finitely generated as an A-module.

In this situation (by [10, Proposition 5.7] and by Corollary 2.7(3)) we know that the amalgamated algebra $A \bowtie^f J$ is a Noetherian local ring, with maximal ideal M'^f. Moreover, the canonical map $\iota : A \to A \bowtie^f J$ is a finite ring embedding, since J is finitely generated as an A-module [10, Proposition 5.7], and thus $\dim(A) = \dim(A \bowtie^f J)$. Moreover $\mathrm{Ann}(A \bowtie^f J) = (0)$, hence the dimension of $A \bowtie^f J$ as A–module (or, equivalently, $\dim(A/\mathrm{Ann}(A \bowtie^f J))$, since $A \bowtie^f J$ is a finite A–module) equals the Krull dimension of $A \bowtie^f J$.

5.1 Remark. We observe that, under the previous assumptions, $A \bowtie^f J$ is a CM ring if and only if it is a CM A–module if and only if J is a maximal CM A-module.

As a matter of fact, since the embedding $\iota : A \hookrightarrow A \bowtie^f J$ is finite, by [7, Exercise 1.2.26(b)] we have $\mathrm{depth}_A(A \bowtie^f J) = \mathrm{depth}(A \bowtie^f J)$, and thus, by the discussion above, $A \bowtie^f J$ is a CM ring if and only if $A \bowtie^f J$ is a CM A–module. Since $A \bowtie^f J$ is isomorphic as an A–module to $A \oplus J$, it follows that

$$\mathrm{depth}_A(A \bowtie^f J) = \mathrm{depth}(A \oplus J) = \min\{\mathrm{depth}(J), \mathrm{depth}(A)\} = \mathrm{depth}(J)$$

and, therefore, $A \bowtie^f J$ is a CM A–module if and only if J is a CM A–module of dimension equal to $\dim(A)$ (that is, if and only if J is a maximal CM A-module).

5.2 Remark. If J is not finitely generated as A–module, it is more problematic to find conditions implying $A \bowtie^f J$ CM. One can get more information if the embedding $\iota : A \to A \bowtie^f J$ is flat (or, equivalently, if the A–module J is flat). In this case, $A \bowtie^f J$ is CM if and only if both A and $A \bowtie^f J/M(A \bowtie^f J)$ are CM [7, Theorem 2.1.7]. As an example, set $A := k[[X]]$, $B := k[[X, Y]]$ (where k is a field), and let $J := M := (X, Y)$ be the maximal ideal of B. Let $f : A \hookrightarrow B$ be the inclusion. Clearly, $J = \prod_{n \geq 1} f(A)Y^n$ is flat as an A-module. Moreover, both $A \bowtie^f J$, which is isomorphic to $k[[X, Y, Z]]/(Y, Z) \cap (X - Y)$, and $A \bowtie^f J/M(A \bowtie^f J)$, which is isomorphic to $k[[Y, Z]]/(Y^2, YZ)$, are not CM.

In order to study when $A \bowtie^f J$ is a Gorenstein ring, we need to look at A endowed with a natural structure of an $A \bowtie^f J$–module.

The next proposition holds in general, without assuming the additional hypotheses on A, stated at the beginning of the section.

5.3 Proposition. *Preserve the notation of Proposition 2.1, and consider the natural map* $\Lambda : f^{-1}(J) \to \mathrm{Hom}_{A \bowtie^f J}(A, A \bowtie^f J)$, *where* $\Lambda(x) := \lambda_x : A \to A \bowtie^f J$ *is the* $A \bowtie^f J$*–linear map defined by* $\lambda_x(a) := (ax, 0)$, *for each* $a \in A$ *and* $x \in f^{-1}(J)$. *Then,* Λ *is an* A*–linear embedding and* Λ *is surjective if and only if* $\mathrm{Ann}_{f(A)+J}(J) = (0)$.

Proof. The fact that Λ is an $A-$linear embedding is straightforward. Assume $\mathrm{Ann}_{f(A)+J}(J) = (0)$. Fix now a $A \bowtie^f J-$linear map $g : A \to A \bowtie^f J$ and the elements $a_0 \in A$ and $j_0 \in J$ such that $(a_0, f(a_0) + j_0) = g(1)$. For each $j \in J$, by definition, $(1, 1 + j) \cdot 1 = 1$, hence $g(1) = g((1, 1 + j) \cdot 1) = (1, 1 + j)g(1) = (a_0, f(a_0) + j_0 + j(f(a_0) + j_0))$, and thus $j(f(a_0) + j_0) = 0$. This proves that $f(a_0) + j_0 \in \mathrm{Ann}_{f(A)+J}(J)$ and so, by hypothesis, $f(a_0) + j_0 = 0$. In particular, $a_0 \in f^{-1}(J)$ and $\Lambda(a_0) = \lambda_{a_0} = g$. Conversely, assume that Λ is surjective, take an element $f(a_0) + j_0 \in \mathrm{Ann}_{f(A)+J}(J)$, with $a_0 \in A$ and $j_0 \in J$, and consider the map $\varphi : A \to A \bowtie^f J$ defined by $\varphi(a) := (a, f(a))(a_0, f(a_0) + j_0)$, for each $a \in A$. Of course, φ is a homomorphism of (additive) abelian groups. Take now two elements $x \in A$ and $(\alpha, f(\alpha) + \beta) \in A \bowtie^f J$. Since $(\alpha, f(\alpha) + \beta) \cdot x = \alpha x$, then $\varphi((\alpha, f(\alpha) + \beta) \cdot x) = \varphi(\alpha x) = (\alpha x, f(\alpha x))(a_0, f(a_0) + j_0)$. On the other hand, we have

$$(\alpha, f(\alpha)+\beta)\varphi(x) = (\alpha, f(\alpha)+\beta)(x, f(x))(a_0, f(a_0)+j_0) = \varphi(\alpha x)$$

where the last equality holds since $\beta(f(a_0)+j_0) = 0$. Thus φ is an $A \bowtie^f J$-linear map and, since Λ is surjective, there exists an element $z \in f^{-1}(J)$ such that $\varphi = \lambda_z$. Therefore $(a_0, f(a_0)+j_0) = \varphi(1) = \lambda_z(1) = (z, 0)$, that is $f(a_0)+j_0 = 0$. □

Now we are able to give a sufficient condition and a necessary condition for the ring $A \bowtie^f J$ to be Gorenstein.

5.4 Remark. We preserve the notation of Proposition 2.1. If A is a local Cohen-Macaulay ring, with maximal ideal M, having a canonical module isomorphic (as an A–module) to J, then $A \bowtie^f J$ is Gorenstein. As a matter of fact, $\iota : A \to A \bowtie^f J$ is a local ring embedding, since, $\iota^{-1}(M'^f) = M$. The conclusion is a consequence of an unpublished result by Eisenbud [9, Theorem 12] (see also [26]), applied to the following short exact sequence of $A-$modules

$$0 \to A \xrightarrow{\iota} A \bowtie^f J \to J \to 0\,.$$

5.5 Proposition. *We preserve the notation of Proposition 2.1. Assume that A is a local Cohen-Macaulay ring and that* $\mathrm{Ann}_{f(A)+J}(J) = (0)$. *If $A \bowtie^f J$ is Gorenstein, then A has a canonical module isomorphic to $f^{-1}(J)$.*

Proof. We begin by noting that, since $A \bowtie^f J$ is Gorenstein, it has a canonical module isomorphic to $A \bowtie^f J$ as an $A \bowtie^f J-$module. Moreover, since the ring embedding ι is finite, we have $\dim(A) = \dim(A \bowtie^f J)$. Thus, keeping in mind that A is a cyclic $A \bowtie^f J-$module (via the projection of $A \bowtie^f J$ onto A) and applying Proposition 5.3 and [15, Theorem 21.15], it follows that A has a canonical module isomorphic (as an $A-$module) to

$$\mathrm{Ext}^0_{A \bowtie^f J}(A, A \bowtie^f J) \cong \mathrm{Hom}_{A \bowtie^f J}(A, A \bowtie^f J) \cong f^{-1}(J).$$

The proof is now complete. □

As a consequence of Remark 5.4 and Proposition 5.5, we deduce immediately the following.

5.6 Corollary. *We preserve the notation of Proposition 2.1. Let A be a local Cohen-Macaulay ring having a canonical module isomorphic to J as an $A-$module and such that* $\mathrm{Ann}_{f(A)+J}(J) = (0)$. *Then, $f^{-1}(J)$ and J are isomorphic as A-modules.*

With extra assumptions on the ideal $f^{-1}(J)$ and on the ring $f(A)+J$, we can obtain the following characterization of when $A \bowtie^f J$ is Gorenstein.

5.7 Proposition. *We preserve the notation and the assumptions of the beginning of the present section and, moreover, we assume that A is a CM ring, $f(A)+J$ is (S_1) and equidimensional, $J \neq 0$ and that $f^{-1}(J)$ is a regular ideal of A. Then, the following conditions are equivalent.*

(i) $A\bowtie^f J$ *is Gorenstein.*

(ii) $f(A)+J$ *is a CM ring,* J *is a canonical module of* $f(A)+J$ *and* $f^{-1}(J)$ *is a canonical module of* A.

Proof. By Remark 2.2(e), $A\bowtie^f J$ can be obtained as a fiber product of two surjective ring homomorphisms. Then, the conclusion follows by applying [24, Theorem 4]. □

We conclude this section by comparing the multiplicity of $A\bowtie^f J$ with the multiplicity of A. We assume the standing hypotheses of the present section and that A is a local Cohen-Macaulay ring of Krull dimension $n > 0$. In particular, by Remark 3.3, if I is an $M-$primary ideal, then $I(A\bowtie^f J) = I\bowtie^f (f(I)B)J$ (Proposition 3.1(2)) is M'^f-primary. Furthermore, we also assume that $A\bowtie^f J$ is a Cohen-Macaulay ring and that the residue field $\boldsymbol{k}$ of A and $A\bowtie^f J$ is infinite.

Under these assumptions, we have that the multiplicity $e(A)$ of A equals $\lambda_A(A/I)$, where I is any minimal reduction of M [21, Proposition 11.2.2] and where $\lambda_A(E)$ denotes the length of an $A-$module E. In particular, since I is a minimal reduction of M and A has infinite residue field, it is minimally generated by n elements (where $n = \dim(A) = \dim(A\bowtie^f J)$; see [21, Lemma 8.3.7]); moreover, $I = (a_1, a_2, \ldots, a_n)$ is an $M-$primary ideal of a Cohen-Macaulay local ring, hence it is generated by a regular sequence. By [21, Lemma 8.1.3], $I(A\bowtie^f J)$ is a reduction of M'^f and, since the ideal $I(A\bowtie^f J) = ((a_1, f(a_1), (a_2, f(a_2), \ldots, (a_n, f(a_n))$ is generated by n elements, it is a minimal reduction [21, Corollary 8.3.6]. Hence, the multiplicity $e(A\bowtie^f J)$ of $A\bowtie^f J$ coincides with $\lambda_{A\bowtie^f J}(A\bowtie^f J/I(A\bowtie^f J))$.

5.8 Proposition. *We preserve the notation of Proposition 2.1. Assume that both* A *and* $A\bowtie^f J$ *are Cohen-Maculay local rings. Let* I *be a minimal reduction of* M*. Then,* $e(A\bowtie^f J) = e(A) + \lambda_{f(A)+J}(J/(f(I)B)J)$.

Proof. By the previous observations, we know that the equality $e(A\bowtie^f J) = \lambda_{A\bowtie^f J}(A\bowtie^f J/I(A\bowtie^f J))$ holds. Moreover, we have

$$\lambda_{A\bowtie^f J}(A\bowtie^f J/I(A\bowtie^f J)) = \lambda_{A\bowtie^f J}(A\bowtie^f J/I\bowtie^f J) + \lambda_{A\bowtie^f J}(I\bowtie^f J/I(A\bowtie^f J))\,.$$

Now, since by Proposition 2.1(2) $A/I \cong A\bowtie^f J/I\bowtie^f J$ (as rings), we have $\lambda_{A\bowtie^f J}(A\bowtie^f J/I\bowtie^f J) = \lambda_A(A/I) = e(A)$. Moreover, again by Proposition 2.1 (3), for every ideal L of $A\bowtie^f J$ such that $I(A\bowtie^f J) = I\bowtie^f (f(I)B)J \subseteq L \subseteq I\bowtie^f J$, the image $p_B(L)$ is an ideal of $f(A)+J$ such that $(f(I)B)J \subseteq p_B(L) \subseteq J$. Conversely, for every ideal H of $f(A)+J$ such that $f(I)J \subseteq H \subseteq J$, then (by Proposition 3.1(1)) $I\bowtie^f H$ is an ideal of $A\bowtie^f J$ such that $I\bowtie^f (f(I)B)J \subseteq H \subseteq I\bowtie^f J$. Hence, we easily conclude that $\lambda_{A\bowtie^f J}(I\bowtie^f J/I(A\bowtie^f J)) = \lambda_{f(A)+J}(J/(f(I)B)J)$ and the proof is complete. □

When $A = B$, and $f = id_A$, the amalgamation along J gives rise to the amalgamated duplication $A\bowtie J$. In this case we obtain a better result about the multiplicity.

5.9 Corollary. *Let (A, M) be a Cohen-Macaulay local ring and J be an ideal of A with $\dim_A(J) = \dim(A)$. Let I be any minimal reduction of M. Then $e(A \bowtie J) = e(A) + \lambda_A(J/IJ)$. In particular, if $\dim(A) = 1$, then $e(A \bowtie J)$ $= 2e(A)$.*

Proof. The first statement is a straightforward consequence of the previous proposition. As for the one-dimensional case, any minimal reduction I of M is principal; hence $IJ = I \cap J$ and $\lambda_A(J/IJ) = \lambda_A((I + J)/I) \leq \lambda_A(A/I) = e(A)$. On the other hand, by [21, Proposition 11.1.10 and Theorem 11.2.3], $\lambda_A(J/IJ) \geq e(I; J) = e(M; J) \geq e(A)$ (where $e(I; J)$ denotes the multiplicity of I on the $A-$module J; see [21, Definition 11.1.5]). Hence, we have the equality $\lambda_A(J/IJ) = e(A)$ and the proof is complete. □

6 Appendix

Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. By Corollary 2.7(3), when A is a local ring with maximal ideal M and J is contained in the Jacobson radical of B, then $A \bowtie^f J$ is a local ring with maximal ideal $M'^{f} := \{(m, f(m) + j) \mid m \in M, j \in J\}$. As it was proved in Proposition 4.1(1) , if $A \bowtie^f J$ is a local ring with finitely generated maximal ideal, then the maximal ideal M of A is finitely generated and the following inequality $\text{embdim}(A) \leq \text{embdim}(A \bowtie^f J)$ holds. However, part 2 of Proposition 4.1 and Theorem 4.4 hold under the additional assumption, not explicitly declared, that $B = f(A)+J$. The following example shows that it is possible that $B \supsetneq f(A)+J$ and J is finitely generated as an ideal of B, but not finitely generated as an ideal of $f(A) + J$.

6.1 Example. Let $A := K$ be a field and T, U be indeterminates over K. Set $B := K(U)[T]_{(T)}$ and $J := TK(U)[T]_{(T)}$. By [10, Example 2.6], the integral domain $K+TK(U)[T]_{(T)}$ is canonically isomorphic to $A \bowtie^f J$, where $f : A \to B$ is the natural embedding. By Lemma 2.3 and Corollary 2.7(3), $f(A) + J = K+TK(U)[T]_{(T)}$ is local and 1-dimensional and the prime spectrum of $f(A)+J$ coincides with that of the DVR B. Since the field extension $K \subseteq K(U)$ is not finite, it is easy to infer that $f(A) + J$ is non Noetherian and thus its maximal ideal J, as an ideal of $f(A) + J$, is not finitely generated.

If $B \neq f(A) + J$, the correct assumption in Proposition 4.1(2) in order to ensure that M'^{f} is finitely generated is to require that M is a finitely generated ideal of A and J is a finitely generated ideal of $f(A) + J$, as shown in the next result.

6.2 Proposition. *Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. Assume that A is local with finitely generated maximal ideal M and that J is finitely generated, as an ideal of $f(A) + J$, and that J is contained in* *the Jacobson radical of B. Then, the ring $A \bowtie^f J$ is local with finitely generated maximal ideal and moreover we have*

$$\text{embdim}(A \bowtie^f J) = \text{embdim}(A) + \nu(J)\,,$$

where now $\nu(J)$ denotes the minimum number of generators of J as an ideal of the ring $f(A) + J$.

Proof. Let $\{m_1, m_2, \ldots, m_r\}$ (respectively, $\{j_1, j_2, \ldots, j_s\}$) be minimal sets of generators of $\mathfrak{m}$ (respectively, of J as an ideal of $f(A) + J$). We now claim that

$$\mathcal{G} := \{(m_i, f(m_i)), (0, j_h) \mid i = 1, 2, \ldots, r,\ h = 1, 2, \ldots, s\}$$

is a minimal set of generators of M'^f. The fact that $\mathcal{G}$ generates M'^f is straightforward and we left its easy proof to the reader. To prove that $\mathcal{G}$ is minimal with respect to the property of generating M'^f it suffices to show that the canonical image of $\mathcal{G}$ into $M'^f/(M'^f)^2$ is linearly independent over the residue field $\boldsymbol{k}$ of A. Let $a_1, a_2, \ldots, a_r,\ \alpha_1, \alpha_2, \ldots, \alpha_s \in A$ be such that

$$\sum_{i=1}^{r}[a_i]_M[(m_i, f(m_i))]_{M'^f} + \sum_{h=1}^{s}[\alpha_h]_M[(0, j_h)]_{M'^f} = 0 \qquad \text{in} \qquad M'^f/(M'^f)^2 \qquad (\star).$$

The same argument given in Theorem 4.4 proves that $a_i \in M$, for $i = 1, 2, \ldots, r$, and thus $(\star)$ is equivalent to state that

$$\mathbf{x} := \sum_{h=1}^{s}(0, f(\alpha_h)j_h) \in (M'^f)^2.$$

By definition, $\mathbf{x}$ is sum of elements of the type $(\mu_k, f(\mu_k) + u_k)(\mu'_k, f(\mu'_k) + u'_k)$, for $k = 1, 2, \ldots, t$, with $\mu_k, \mu'_k \in M$ and $u_k, u'_k \in J$. It follows that $\sum_{k=1}^{t} \mu_k\mu'_k = 0$, and then $\sum_{h=1}^{s} f(\alpha_h)j_h \in f(M)J + J^2 \subseteq J(f(M) + J)$. By contradiction, assume that there exists some index h such that $\alpha_h \in A \setminus M$. Say $h = 1$, let λ_1 be the inverse of α_1 in A. Then $f(\lambda_1)\sum_{h=1}^{s} f(\alpha_h)j_h \in J(f(M) + J)$. Take elements $\eta_1, \eta_2, \ldots \eta_s \in M$ and $v_1, v_2, \ldots, v_s \in J$ such that

$$j_1 + f(\lambda_1)\sum_{h=2}^{s} f(\alpha_h)j_h = f(\lambda_1)\sum_{h=1}^{s} f(\alpha_h)j_h = \sum_{h=1}^{s}(f(\eta_h) + v_h)j_h\,.$$

It follows that $j_1(1 - f(\eta_1) - v_1) \in (j_2, j_3, \ldots, j_s)(f(A) + J)$. Since $f(M) + J$ is the maximal ideal of the local ring $f(A) + J$, it follows that $1 - f(\eta_1) - v_1$ is invertible in $f(A) + J$, that is, $j_1 \in (j_2, j_3, \ldots, j_s)(f(A) + J)$, contradicting the minimality of $\{j_1, j_2, \ldots, j_s\}$. The proof is now complete. □

6.3 Remark. Note that if J is finitely generated as an A-module (with the structure induced by the ring homomorphism f), then it is finitely generated as an ideal of $f(A) + J$ too, as it is easily seen. The converse is not true, by [10, Remark 5.10].

6.4 Remark. If A is local with finitely generated maximal ideal M such that $f(M)B = B$ and J is finitely generated as an ideal of (the local ring) $f(A) + J$, then Nakayama's Lemma implies that $J = 0$, according to Propositions 4.3 and 6.2.

Question. Is there a local amalgamation $A \bowtie^f J$ with finitely generated maximal ideal such that J is not finitely generated as an ideal of $f(A) + J$ and $f(M)B \neq B$ (where M is the maximal ideal of A)?

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