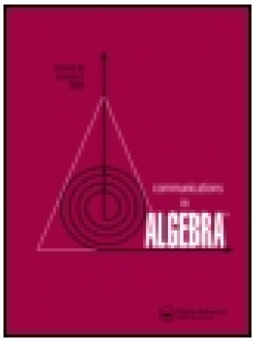

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: http://www.tandfonline.com/loi/lagb20

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COMMUNICATIONS IN ALGEBRA®
2017, VOL. 45, NO. 9, 3703–3705
http://dx.doi.org/10.1080/00927872.2016.1243699

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Corrigendum to "New algebraic properties of an amalgamated algebra along an ideal"

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ABSTRACT
At some point, after publication, we realized that Proposition 4.1(2) and Theorem 4.4 in [2] hold under the assumption (not explicitly declared) that $B = f(A) + J$. Furthermore, we provide here the exact value for the embedding dimension of $A \bowtie^f J$, also when $B \neq f(A) + J$, under the hypothesis that J is finitely generated as an ideal of the ring $f(A) + J$.

ARTICLE HISTORY
Received 22 August 2016
Communicated by S. Bazzoni

KEYWORDS
Cohen-Macaulay; $D + M$ construction; embedding dimension; Gorenstein; idealization; Krull dimension; pullback; Zariski topology

2010 MATHEMATICS SUBJECT CLASSIFICATION
13A15; 13B99; 14A05

Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. As it is well known, when A is a local ring with maximal M contained in the Jacobson radical of B, then $A \bowtie^f J$ is a local ring with maximal ideal $M'^f := \{(m, f(m) + j) \mid m \in M, j \in J\}$. As it was proved in [2, Proposition 4.1(1)], if $A \bowtie^f J$ is a local ring with finitely generated maximal ideal, then the maximal ideal M of A is finitely generated and the following inequality $\mathrm{embdim}(A) \leq \mathrm{embdim}(A \bowtie^f J)$ holds. However, part 2 of [2, Proposition 4.1] and Theorem 4.4 hold under the additional assumption, not explicitly declared, that $B = f(A) + J$.

The following example shows that it is possible that $B \supsetneq f(A) + J$ and J is finitely generated as an ideal of B, but not finitely generated as an ideal of $f(A) + J$.

Example. Let $A := K$ be a field and T, U be indeterminates over K. Set $B := K(U)[T]_{(T)}$ and $J := TK(U)[T]_{(T)}$. By [1, Example 2.6], the integral domain $K + TK(U)[T]_{(T)}$ is canonically isomorphic to $A \bowtie^f J$, where $f : A \to B$ is the natural embedding. By [2, Lemma 2.7 and Corollary 2.7(3)], $f(A) + J = K + TK(U)[T]_{(T)}$ is local and 1-dimensional and the prime spectrum of $f(A) + J$ coincides with that of the discrete valuation domain B. Since the field extension $K \subseteq K(U)$ is not finite, it is easy to infer that $f(A) + J$ is non-Noetherian and thus its maximal ideal J, as an ideal of $f(A) + J$, is not finitely generated.

If $B \neq f(A) + J$, the correct assumption in [2, Proposition 4.1(2)] in order to ensure that M'^f is finitely generated is to require that M is a finitely generated ideal of A and J is a finitely generated ideal of $f(A) + J$, as shown in the next result.

Proposition. *Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. Assume that A is local with finitely generated maximal ideal M, that J is finitely generated, as an ideal of $f(A) + J$, and that J is contained in the Jacobson radical of B. Then, the ring $A \bowtie^f J$ is local with finitely generated maximal ideal*

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and, moreover, we have

$$\text{embdim}(A \bowtie^f J) = \text{embdim}(A) + \nu(J),$$

where, now, $\nu(J)$ denotes the minimum number of generators of J as an ideal of $f(A) + J$.

Proof. Let $\{m_1, m_2, \ldots, m_r\}$ (respectively, $\{j_1, j_2, \ldots, j_s\}$) be minimal sets of generators of M (respectively, of J as an ideal of $f(A) + J$). We now claim that

$$\mathcal{G} := \{(m_i, f(m_i)), (0, j_h) \mid i = 1, 2, \ldots, r, \; h = 1, 2, \ldots, s\}$$

is a minimal set of generators of M'^f. The fact that $\mathcal{G}$ generates M'^f is straightforward and we left its easy proof to the reader. To prove that $\mathcal{G}$ is minimal with respect to the property of generating M'^f, it suffices to show that the canonical image of $\mathcal{G}$ into $M'^f/(M'^f)^2$ is linearly independent over the residue field K of A. Let $a_1, a_2, \ldots, a_r, \; \alpha_1, \alpha_2, \ldots, \alpha_s \in A$ be such that

$$\sum_{i=1}^{r} [a_i]_M [(m_i, f(m_i))]_{M'^f} + \sum_{h=1}^{s} [\alpha_h]_M [(0, j_h)]_{M'^f} = 0 \quad \text{in} \quad M'^f/(M'^f)^2. \qquad (\star)$$

The same argument given in [2, Theorem 4.4] proves that $a_i \in M$, for $i = 1, 2, \ldots, r$, and thus $(\star)$ is equivalent to state that

$$\mathbf{x} := \sum_{h=1}^{s} (0, f(\alpha_h) j_h) \in (M'^f)^2.$$

By definition, $\mathbf{x}$ is sum of elements of the type $(\mu_k, f(\mu_k) + u_k)(\mu'_k, f(\mu'_k) + u'_k)$, for $k = 1, 2, \ldots, t$, with $\mu_k, \mu'_k \in M$ and $u_k, u'_k \in J$. It follows that $\sum_{k=1}^{t} \mu_k \mu'_k = 0$, and then $\sum_{h=1}^{s} f(\alpha_h) j_h \in f(M)J + J^2 \subseteq J(f(M) + J)$. By contradiction, assume that there exists some index h such that $\alpha_h \in A \setminus M$. Say $h = 1$, let λ_1 be the inverse of α_1 in A. Then $f(\lambda_1) \sum_{h=1}^{s} f(\alpha_h) j_h \in J(f(M) + J)$. Take elements $\eta_1, \eta_2, \ldots, \eta_s \in M$ and $v_1, v_2, \ldots, v_s \in J$ such that

$$j_1 + f(\lambda_1) \sum_{h=2}^{s} f(\alpha_h) j_h = f(\lambda_1) \sum_{h=1}^{s} f(\alpha_h) j_h = \sum_{h=1}^{s} (f(\eta_h) + v_h) j_h \, .$$

It follows that $j_1(1 - f(\eta_1) - v_1) \in (j_2, j_3, \ldots, j_s)(f(A) + J)$. Since $f(M) + J$ is the maximal ideal of the local ring $f(A) + J$, it follows that $1 - f(\eta_1 - v_1)$ is invertible in $f(A) + J$, that is, $j_1 \in (j_2, j_3, \ldots, j_s)(f(A) + J)$, contradicting the minimality of $\{j_1, j_2, \ldots, j_s\}$. The proof is now complete. □

Remark. Note that if J is finitely generated as an A-module (with the structure induced by the ring homomorphism f), then it is finitely generated as an ideal of $f(A) + J$ too, as it is easily seen. The converse is not true, by [1, Remark 5.10].

Remark. If A is local with finitely generated maximal ideal M such that $f(M)B = B$ and J is finitely generated as an ideal of (the local ring) $f(A) + J$, then Nakayama's Lemma implies that $J = 0$, according to the above proposition and [2, Proposition 4.3].

Question. Is there a local amalgamation $A \bowtie^f J$ with finitely generated maximal ideal such that J is not finitely generated as an ideal of $f(A) + J$ and $f(M)B \neq B$ (where M is the maximal ideal of A)?

Acknowledgment

This work was partially supported by *GNSAGA* of *Istituto Nazionale di Alta Matematica*. The second author was also supported by a Post Doc Grant from the University of Technology of Graz - Austrian Science Fund (FWF), # P 27816.

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