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To cite this article: Marco D'Anna, Carmelo A. Finocchiaro & Marco Fontana (2017) Corrigendum to “New algebraic properties of an amalgamated algebra along an ideal”, Communications in Algebra, 45:9, 3703-3705, DOI: 10.1080/00927872.2016.1243699

To link to this article: http://dx.doi.org/10.1080/00927872.2016.1243699

Accepted author version posted online: 11 Nov 2016.
Published online: 11 Nov 2016.

Article views: 26

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Corrigendum to “New algebraic properties of an amalgamated algebra along an ideal”

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ABSTRACT
At some point, after publication, we realized that Proposition 4.1(2) and Theorem 4.4 in [2] hold under the assumption (not explicitly declared) that \( B = f(A) + J \). Furthermore, we provide here the exact value for the embedding dimension of \( A \oslash J \), also when \( B \neq f(A) + J \), under the hypothesis that \( J \) is finitely generated as an ideal of the ring \( f(A) + J \).

ARTICLE HISTORY
Received 22 August 2016
Communicated by S. Bazzoni

KEYWORDS
Cohen-Macaulay; \( D + M \) construction; embedding dimension; Gorenstein; idealization; Krull dimension; pullback; Zariski topology

2010 MATHEMATICS SUBJECT CLASSIFICATION
13A15; 13B99; 14A05

Let \( f : A \rightarrow B \) be a ring homomorphism and let \( J \) be an ideal of \( B \). As it is well known, when \( A \) is a local ring with maximal \( M \) contained in the Jacobson radical of \( B \), then \( A \oslash J \) is a local ring with maximal ideal \( M' := \{ (m, f(m) + j) \mid m \in M, j \in J \} \). As it was proved in [2, Proposition 4.1(1)], if \( A \oslash J \) is a local ring with finitely generated maximal ideal, then the maximal ideal \( M \) of \( A \) is finitely generated and the following inequality \( \text{embdim}(A) \leq \text{embdim}(A \oslash J) \) holds. However, part 2 of [2, Proposition 4.1] and Theorem 4.4 hold under the additional assumption, not explicitly declared, that \( B = f(A) + J \).

The following example shows that it is possible that \( B \supseteq f(A) + J \) and \( J \) is finitely generated as an ideal of \( B \), but not finitely generated as an ideal of \( f(A) + J \).

Example. Let \( A := K \) be a field and \( T, U \) be indeterminates over \( K \). Set \( B := K[U][T]_T \) and \( J := TK(U)[T]_T \). By [1, Example 2.6], the integral domain \( K + TK(U)[T]_T \) is canonically isomorphic to \( A \oslash J \), where \( f : A \rightarrow B \) is the natural embedding. By [2, Lemma 2.7 and Corollary 2.3], \( f(A) + J = K + TK(U)[T]_T \) is local and 1-dimensional and the prime spectrum of \( f(A) + J \) coincides with that of the discrete valuation domain \( B \). Since the field extension \( K \subseteq K(U) \) is not finite, it is easy to infer that \( f(A) + J \) is non-Noetherian and thus its maximal ideal \( J \), as an ideal of \( f(A) + J \), is not finitely generated.

If \( B \neq f(A) + J \), the correct assumption in [2, Proposition 4.1(2)] in order to ensure that \( M' \) is finitely generated is to require that \( M \) is a finitely generated ideal of \( A \) and \( J \) is a finitely generated ideal of \( f(A) + J \), as shown in the next result.

Proposition. Let \( f : A \rightarrow B \) be a ring homomorphism and let \( J \) be an ideal of \( B \). Assume that \( A \) is local with finitely generated maximal ideal \( M \), that \( J \) is finitely generated, as an ideal of \( f(A) + J \), and that \( J \) is contained in the Jacobson radical of \( B \). Then, the ring \( A \oslash J \) is local with finitely generated maximal ideal.
and, moreover, we have
\[
\text{embdim}(A \otimes f) = \text{embdim}(A) + v(f),
\]
where, now, \(v(f)\) denotes the minimum number of generators of \(f\) as an ideal of \(f(A) + J\).

**Proof.** Let \(\{m_1, m_2, \ldots, m_r\}\) (respectively, \(\{j_1, j_2, \ldots, j_s\}\)) be minimal sets of generators of \(M\) (respectively, of \(J\) as an ideal of \(f(A) + J\)). We now claim that
\[
G := \{(m_i, f(m_i)), (0, j_h) \mid i = 1, 2, \ldots, r, h = 1, 2, \ldots, s\}
\]
is a minimal set of generators of \(M^f\). The fact that \(G\) generates \(M^f\) is straightforward and we left its easy proof to the reader. To prove that \(G\) is minimal with respect to the property of generating \(M^f\), it suffices to show that the canonical image of \(G\) into \(M^f / (M^f)^2\) is linearly independent over the residue field \(K\) of \(A\). Let \(a_1, a_2, \ldots, a_r, \alpha_1, \alpha_2, \ldots, \alpha_s \in A\) be such that
\[
\sum_{i=1}^{r} [a_i, M](m_i, f(m_i))]_{M^f} + \sum_{h=1}^{s} [\alpha_h, M](0, j_h)]_{M^f} = 0 \quad \text{in} \quad M^f / (M^f)^2. \quad (\star)
\]
The same argument given in [2, Theorem 4.4] proves that \(a_i \in M\), for \(i = 1, 2, \ldots, r\), and thus \((\star)\) is equivalent to state that
\[
x := \sum_{h=1}^{s} (0, f(\alpha_h)j_h) \in (M^f)^2.
\]
By definition, \(x\) is sum of elements of the type \((\mu_k, f(\mu_k) + u_k)(\mu_k', f(\mu_k') + u_k')\), for \(k = 1, 2, \ldots, t\), with \(\mu_k, \mu_k' \in M\) and \(u_k, u_k' \in J\). It follows that \(\sum_{k=1}^{t} \mu_k \mu_k' = 0\), and then \(\sum_{h=1}^{s} f(\alpha_h)j_h \in f(M)J + f^2 \subseteq f(f(A) + J)\). By contradiction, assume that there exists some index \(h\) such that \(\alpha_h \in A \setminus M\). Say \(h = 1\), let \(\lambda_1\) be the inverse of \(\alpha_1\) in \(A\). Then \(f(\lambda_1) \sum_{h=1}^{s} f(\alpha_h)j_h \in J(f(M) + J)\). Take elements \(\eta_1, \eta_2, \ldots, \eta_s \in M\) and \(v_1, v_2, \ldots, v_s \in f(A) + J\) such that
\[
j_1 + f(\lambda_1) \sum_{h=1}^{s} f(\alpha_h)j_h = f(\lambda_1) \sum_{h=1}^{s} f(\alpha_h)j_h = \sum_{h=1}^{s} (f(\eta_h) + v_h)j_h.
\]
It follows that \(j_1 \left(1 - f(\eta_1) - v_1\right) \in (j_2, j_3, \ldots, j_s)(f(A) + J)\). Since \(f(M)J + f\) is the maximal ideal of the local ring \(f(A) + J\), it follows that \(1 - f(\eta_1) - v_1\) is invertible in \(f(A) + J\), that is, \(j_1 \in (j_2, j_3, \ldots, j_s)(f(A) + J)\), contradicting the minimality of \(\{j_1, j_2, \ldots, j_s\}\). The proof is now complete. \(\square\)

**Remark.** Note that if \(J\) is finitely generated as an \(A\)-module (with the structure induced by the ring homomorphism \(f\)), then it is finitely generated as an ideal of \(f(A) + J\) too, as it is easily seen. The converse is not true, by [1, Remark 5.10].

**Remark.** If \(A\) is local with finitely generated maximal ideal \(M\) such that \(f(M)B = B\) and \(J\) is finitely generated as an ideal of (the local ring) \(f(A) + J\), then Nakayama’s Lemma implies that \(J = 0\), according to the above proposition and [2, Proposition 4.3].

**Question.** Is there a local amalgamation \(A \otimes f J\) with finitely generated maximal ideal such that \(J\) is not finitely generated as an ideal of \(f(A) + J\) and \(f(M)B \neq B\) (where \(M\) is the maximal ideal of \(A\))?

**Acknowledgment**

This work was partially supported by GNSAGA of Istituto Nazionale di Alta Matematica. The second author was also supported by a Post Doc Grant from the University of Technology of Graz - Austrian Science Fund (FWF), # P 27816.
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