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AN AMALGAMATED DUPLICATION OF A RING ALONG AN IDEAL: THE BASIC PROPERTIES

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We introduce a new general construction, denoted by $R \bowtie E$, called the amalgamated duplication of a ring R along an R-module E, that we assume to be an ideal in some overring of R. (Note that, when $E^2 = 0$, $R \bowtie E$ coincides with the Nagata's idealization $R \ltimes E$.)

After discussing the main properties of the amalgamated duplication $R \bowtie E$ in relation with pullback-type constructions, we restrict our investigation to the study of $R \bowtie E$ when E is an ideal of R. Special attention is devoted to the ideal-theoretic properties of $R \bowtie E$ and to the topological structure of its prime spectrum.

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1. Introduction

If R is a commutative ring with unity and E is an R-module, *the idealization* $R \ltimes E$, introduced by Nagata in 1956 (cf. [16], p. 2), is a new ring, containing R as a subring, where the module E can be viewed as an ideal such that its square is (0).

This construction has been extensively studied and has many applications in different contexts (cf., e.g., [17, 6, 9, 11]). Particularly important is the generalization given by Fossum in [5], where he defined *a commutative extension of a ring R by an R-module E* to be an exact sequence of abelian groups:

$$0 \to E \xrightarrow{\iota} S \xrightarrow{\pi} R \to 0$$

where S is a commutative ring, the map π is a ring homomorphism and the R-module structure on E is related to S and to the maps ι and π by the equation $s \cdot \iota(e) = \iota(\pi(s) \cdot e)$ (for all $s \in S$ and $e \in E$). It is easy to see that the idealization $R \ltimes E$ is a very particular commutative extension of R by the R-module E (called *trivial extension* of R by E in [5]).

In this paper, we will introduce a new general construction, called the amalgamated duplication of a ring R along an R-module E (that we assume to be an ideal in some overring of R and so, E is an R-submodule of the total ring of fractions $T(R)$ of R) and denoted by $R \bowtie E$ (see Lemma 2.4).

When $E^2 = 0$, the new construction $R \bowtie E$ coincides with the idealization $R \ltimes E$. In general, however, $R \bowtie E$ is not a commutative extension in the sense of Fossum. One main difference of this construction, with respect to the idealization (or with respect to any Fossum's commutative extension), is that the ring $R \bowtie E$ can be a reduced ring (and, in fact, it is always reduced if R is a domain).

Motivations and some applications of the amalgamated duplication $R \bowtie E$ are discussed more in detail in two recent papers [1, 2]. More precisely, D'Anna [1] has studied some properties of this construction when the R-module $E = I$ is a proper ideal of R, in order to construct reduced Gorenstein rings associated to Cohen–Macaulay rings and he has applied this construction to curve singularities. D'Anna and Fontana [2] have considered the case of the amalgamated duplication of a ring, in a not-necessarily-Noetherian setting, along a multiplicative-canonical ideal in the sense of Heinzer–Huckaba–Papick [10].

The present paper is devoted to a more systematic investigation of the general construction $R \bowtie E$, with a particular consideration of the ideal-theoretic properties and the topological structure of its prime spectrum. More precisely, the paper is divided in two parts: in Sec. 2, we study the main properties of the amalgamated duplication $R \bowtie E$. In particular, we give a presentation of this ring as a pullback (cf. Proposition 2.6) and from this fact (cf. also [4, 7]), we obtain several connections between the properties of R and the properties of $R \bowtie E$, and some useful information about $\mathrm{Spec}(R \bowtie E)$ (cf. Remark 2.13).

In Sec. 3 we consider the case when $E = I$ is an ideal of R; this situation allows us to deepen the results obtained in Sec. 2; in particular we give a complete description of $\mathrm{Spec}(R \bowtie I)$ (cf. Theorems 3.5 and 3.8).

2. The General Construction

In this section, we will study the construction of the ring $R \bowtie E$ in a general setting. More precisely, R will always be a commutative ring with unity, $T(R)$ $(:= \{\text{regular elements}\}^{-1}R)$ its total ring of fractions and E an R-submodule of

$T(R)$. Moreover, in order to construct the ring $R \bowtie E$, we are interested in those R-submodules of $T(R)$ such that $E \cdot E \subseteq E$.

Lemma 2.1. *Let E be an R-submodule of $T(R)$ and let J be an ideal of R.*

(a) *$E \cdot E \subseteq E$ if and only if there exists a subring S of $T(R)$ containing R and E, such that E is an ideal of S.*
(b) *If $E \cdot E \subseteq E$, then*

$$R + E := \{z = r + e \in T(R) \mid r \in R,\ e \in E\}$$

is a subring of $(E : E) := \{z \in T(R) \mid zE \subseteq E\}$ $(\subseteq T(R))$, containing R as a subring and E as an ideal.
(c) *Assume that $E \cdot E \subseteq E$; the canonical ring homomorphism $\varphi : R \hookrightarrow R + E \to (R + E)/E$, $r \mapsto r + E$, is surjective and $\mathrm{Ker}(\varphi) = E \cap R$.*
(d) *Assume that $E \cdot E \subseteq E$; the set $J + E := \{j + e \mid j \in J,\ e \in E\}$ is an ideal of $R + E$ containing E and $(J + E) \cap R = \mathrm{Ker}(R \hookrightarrow R + E \to (R + E)/(J + E)) = J + (E \cap R)$.*

Proof. (a) It is clear that the implication "if" holds. Conversely, set $S := (E : E)$. The hypothesis that $E \cdot E \subseteq E$ implies that E is an ideal of S and that S is a subring of $T(R)$ containing R as a subring.

(b) It is obvious that $R + E$ is an R-submodule of $(E : E)$ containing R and E. Moreover, let $r, s \in R$ and $e, f \in E$, if $z := r + e$ and $w := s + f$ $(\in R + E)$, then $zw = rs + (rf + se + ef) \in R + E$ and $zf = rf + ef \in E$.

(c) and (d) are straightforward. $\square$

From now on, we will always assume that $E \cdot E \subseteq E$.

In the R-module direct sum $R \oplus E$, we can introduce a multiplicative structure by setting:

$$(r, e)(s, f) := (rs, rf + se + ef)\,, \text{ where } r, s \in R \text{ and } e, f \in E.$$

We denote by $R \dot{\oplus} E$ the direct sum $R \oplus E$ endowed with the multiplication defined above. The following properties are easy to check:

Lemma 2.2. *With the notation introduced above, we have:*

(a) *$R \dot{\oplus} E$ is a ring.*
(b) *The map $j : R \dot{\oplus} E \to R \times (R + E)$, defined by $(r, e) \mapsto (r, r + e)$, is an injective ring homomorphism.*
(c) *The map $i : R \to R \dot{\oplus} E$, defined by $r \mapsto (r, 0)$, is an injective ring homomorphism.* $\square$

Remark 2.3. (a) With the notation of Lemma 2.1, note that if $E = S$ is a subring of $T(R)$ containing a subring R, then $R + S = S$. Also, if I is an ideal of R, then $R + I = R$.

(b) In the statement of Lemma 2.1(d), note that, in general, $J + E$ does not coincide with the extension of J in $R+E$: we have $J(R+E) = \{j+\alpha \mid j \in J,\ \alpha \in JE\} \subseteq J+E$, but the inclusion can be strict (cf. Lemma 3.4(a), (d) and (e)).

(c) For an *arbitrary* R-module E, Nagata introduced in 1955 *the idealization* of E in R, denoted here by $R \ltimes E$, which is the R-module $R \oplus E$ endowed with a multiplicative structure defined by:

$$(r, e)(s, f) := (rs, rf + se)\,, \quad \text{where } r, s \in R \text{ and } e, f \in E$$

(cf. [15; 16, p. 2; 11, Chap. VI, Sec. 25]). The idealization $R \ltimes E$, called also *the trivial extension of R by E* [5], is a ring such that the canonical embedding $R \hookrightarrow R \ltimes E$, $r \mapsto (r, 0)$, defines a subring of $R \ltimes E$ isomorphic to R and the embedding $E \hookrightarrow R \ltimes E$, $e \mapsto (0, e)$, defines an ideal $E^{\ltimes}$ in $R \ltimes E$ (isomorphic as an R-module to E), which is nilpotent of index 2 (i.e. $E^{\ltimes} \cdot E^{\ltimes} = 0$). Therefore, even if R is reduced, the idealization $R \ltimes E$ is not a reduced ring, except in the trivial case for $E = (0)$, since $R \ltimes (0) = R$. Moreover, if $p_R : R \ltimes E \to R$ is the canonical projection (defined by $(r, e) \mapsto r$), then

$$0 \to E \to R \ltimes E \xrightarrow{p_R} R \to 0$$

is an exact sequence.

Note that the idealization $R \ltimes E$ coincides with the ring $R \dot{\oplus} E$ (Lemma 2.2) if and only if E is an R-submodule of $T(R)$ that is nilpotent of index 2 (i.e. $E \cdot E = (0)$).

Lemma 2.4. *With the notation of Lemma* 2.2, *note that* $\delta := j \circ i : R \hookrightarrow R \times (R+E)$ *is the diagonal embedding and set*:

$$R^{\Delta} := (j \circ i)(R) = \{(r, r) \mid r \in R\} \quad \textit{and}$$
$$R \bowtie E := j(R \dot{\oplus} E) = \{(r, r + e) \mid r \in R,\ e \in E\}.$$

We have:

(a) *The canonical maps* $R \cong R^{\Delta} \subseteq R \bowtie E \subseteq R \times T(R)$ *are ring homomorphisms.*
(b) $R \bowtie E$ *is a subdirect product of the rings* R *and* $(R + E)$, *i.e. if* π_i $(i = 1, 2)$ *are the projections of* $R \times (R + E)$ *onto* R *and* $R + E$, *respectively, and if* $\mathfrak{O}_i := \mathrm{Ker}(\pi_i|_{R\bowtie E})$, *then* $(R \bowtie E)/\mathfrak{O}_1 \cong R$, $(R \bowtie E)/\mathfrak{O}_2 \cong R + E$ *and* $\mathfrak{O}_1 \cap \mathfrak{O}_2 = (0)$.

Proof. (a) is obvious. For (b), recall that S is a subdirect product of a family of rings $\{R_i \mid i \in I\}$ if there exists a ring monomorphism $\varphi : S \hookrightarrow \prod_i R_i$ such that, for each $i \in I$, $\pi_i \circ \varphi : S \to R_i$ is a surjection (where $\pi_i : \prod_i R_i \to R_i$ is the canonical projection) [13, p. 30]. Note also that $\mathfrak{O}_1 = \{(0, e) \mid e \in E\}$ and $\mathfrak{O}_2 = \{(\varepsilon, 0) \mid \varepsilon \in E \cap R\}$. The conclusion is straightforward (cf. [13, Proposition 10]). $\square$

We will call the ring $R \bowtie E$, defined in Lemma 2.4, *the amalgamated duplication of a ring along an R-module E*; the reason for this name will be clear after studying the prime spectrum of $R \bowtie E$ and comparing it with the prime spectrum of R (see Proposition 2.13). The following is an easy consequence of the previous lemma.

Corollary 2.5. *With the notation of Lemma* 2.4, *the following properties are equivalent*:

(a) *R is a domain*;
(b) *$R+E$ is a domain*;
(c) *$\mathfrak{O}_1$ is a prime ideal of $R \bowtie E$*;
(d) *$\mathfrak{O}_2$ is a prime ideal of $R \bowtie E$*;
(e) *$R \bowtie E$ is a reduced ring and $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are prime ideals of $R \bowtie E$.* □

We will see in a moment that R is a domain if and only if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal prime ideals $R \bowtie E$ (cf. Remark 2.8).

Proposition 2.6. *Let $v : R \times (R+E) \twoheadrightarrow R \times ((R+E)/E)$ and $u : R \hookrightarrow R \times ((R+E)/E)$ be the natural ring homomorphisms defined, respectively, by $v((x, r+e)) := (x, r+E)$ and $u(r) := (r, r+E)$, for each $x, r \in R$ and $e \in E$. Then $v^{-1}(u(R)) = R \bowtie E$. Therefore, if v' $(:= \pi_1|_{R \bowtie E}) : R \bowtie E \twoheadrightarrow R$ is the canonical map defined by $(r, r+e) \mapsto r$ (cf. Lemma* 2.4) *and $u' : R \bowtie E \hookrightarrow R \times (R+E)$ is the natural embedding, then the following diagram*:

$$\begin{array}{ccc} R \bowtie E & \xrightarrow{v'} & R \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ R \times (R+E) & \xrightarrow{v} & R \times ((R+E)/E) \end{array}$$

is a pullback.

Proof. Since E is an ideal of $R+E$ (Lemma 2.1(b)), $\mathfrak{O}_1 = (0) \times E$ is a common ideal of $v^{-1}(u(R))$ and $R \times (R+E)$. Moreover, by definition, if $x, r \in R$ and $e \in E$, then $(x, r+e) \in v^{-1}(u(R))$ if and only if $(x, r+E) \in u(R)$, that is $x - r \in E$. Therefore, we conclude that $v^{-1}(u(R)) = R \bowtie E$. The second part of the statement follows easily from the fact that $v^{-1}(u(R)) = R \bowtie E$ and $(R \bowtie E)/\mathfrak{O}_1 \cong R$, with $\mathfrak{O}_1 = Ker(v')$ (Proposition 2.4(b)). □

Corollary 2.7. *The ring $R \times (R+E)$ is a finitely generated $(R \bowtie E)$-module. In particular, $R \bowtie E \subseteq R \times (R+E)$ is an integral extension and* $\dim(R \bowtie E) = \dim(R \times (R+E)) = \sup\{\dim(R), \dim(R+E)\}$.

Proof. Clearly, $u : R \hookrightarrow R \times ((R+E)/E)$ is a finite ring homomorphism, since $R \times ((R+E)/E)$ is generated by $(1,0)$ and $(0,1)$ as R-module. Since u is finite, also $u' : R \bowtie E$ $(= v^{-1}(u(R))) \hookrightarrow R \times ((R+E)/E)$ is a finite ring homomorphism [4, Corollary 1.5(4)]. The last statement follows from [12, Theorems 44 and 48] and from the fact that $Spec(R \times (R+E))$ is homeomorphic to the disjoint union of $Spec(R)$ and $Spec(R+E)$ (cf. Remark 2.8). $\square$

Remark 2.8. Recall that every ideal of the ring $R \times (R+E)$ is a direct product of ideals $I \times J$, with I ideal of R and J ideal of $R+E$. In particular, every prime ideal Q of $R \times (R+E)$ is either of the type $I \times (R+E)$ or $R \times J$, with I prime ideal of R and J prime ideal of $(R+E)$. Therefore, in the situation of Lemma 2.4, if R is an integral domain (and so $R+E$ is also an integral domain by Corollary 2.5), then $(0) \times (R+E)$ and $R \times (0)$ are necessarily the only minimal primes of $R \times (R+E)$. By the integrality property (Corollary 2.7 and [12, Theorem 46]), then $\mathfrak{O}_1 = ((0) \times (R+E)) \cap (R \bowtie E) = (0) \times E$ and $\mathfrak{O}_2 = (R \times (0)) \cap (R \bowtie E) = (R \cap E) \times (0)$ are the only minimal primes of $R \bowtie E$.

Conversely, if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal primes of $R \bowtie E$, then clearly $R \bowtie E$ is a reduced ring (Lemma 2.4(b)) and, by Corollary 2.5, R is an integral domain.

Corollary 2.9. *The following statements are equivalent*:

(a) *R and $R+E$ are Noetherian*;
(b) *$R \times (R+E)$ is Noetherian*;
(c) *$R \bowtie E$ is Noetherian.*

Proof. Clearly (a) and (b) are equivalent. The statements (b) and (c) are equivalent by the Eakin–Nagata Theorem [14, Theorem 3.7], since $R \times (R+E)$ is a finitely generated $(R \bowtie E)$-module (Corollary 2.7). $\square$

Remark 2.10. (a) In the situation of Proposition 2.6, the pullback degenerates in two cases:

(i) $v' : R \bowtie E \to R$ is an isomorphism if and only if $E = 0$;
(ii) $u' : R \bowtie E \to R \times (R+E)$ is an isomorphism if and only if E is an overring of R (i.e. if and only if $E = R+E$).

(b) By the previous remark, we deduce easily that R Noetherian does not imply, in general, that $R+E$ is Noetherian and, conversely, $R+E$ Noetherian does not imply that R is Noetherian: take, for instance, E to be an arbitrary overring of R. However, if we assume that $R+E$ is a finitely generated R-module (cf. Corollary 2.11), then by the Eakin–Nagata Theorem [14, Theorem 3.7] R is Noetherian if and only if $R+E$ is Noetherian.

This same situation described above (i.e. when E is an arbitrary overring of R) shows that, in Corollary 2.7, we may have that $\dim(R \bowtie E) = \dim(R)$ or that $\dim(R \bowtie E) = \dim(R+E)$ (with $\dim(R) \neq \dim(R+E)$).

Corollary 2.11. *Assume that E is a fractional ideal of R (i.e. there exists a regular element $d \in R$ such that $dE \subseteq R$); then the following statements are equivalent*:

(a) *R is a Noetherian ring*;
(b) *$R + E$ is a Noetherian R-module*;
(c) *$R \times (R + E)$ is a Noetherian ring*;
(d) *$R \bowtie E$ is a Noetherian ring.*

Proof. By Corollary 2.9 and Remark 2.10(b), it is sufficient to show that, in this case, R is a Noetherian ring if and only if $R + E$ is a Noetherian R-module. Clearly, if R is Noetherian, then E is a finitely generated R-module and so, $R + E$ is also a finitely generated R-module and thus, it is a Noetherian R-module. Conversely, assume that $R + E$ is a Noetherian R-module; since it is faithful, by [14, Theorem 3.5], it follows that R is a Noetherian ring. $\square$

Corollary 2.12. *In the situation described above*:

(a) *Let R' and $(R + E)'$ be the integral closures of R and $R + E$ in $T(R)$. Then $R \bowtie E$ and $R \times (R + E)$ have the same integral closure in $T(R) \times T(R)$, which is precisely $R' \times (R + E)'$. Moreover, if $R + E$ is a finitely generated R-module, then the integral closure of R^{Δ} in $T(R) \times T(R)$* (*Lemma* 2.4) *also coincides with $R' \times (R + E)'$.*
(b) *If $E \cap R$ contains a regular element, then $T(R \bowtie E) = T(R \times (R + E)) = T(R) \times T(R)$ and, moreover, $R \bowtie E$ and $R \times (R + E)$ have the same complete integral closure in $T(R) \times T(R)$.*

Proof. (a) It is clear that $(x, y) \in T(R) \times T(R)$ is integral over $R \times (R + E)$ if and only if $(x, y) \in R' \times (R + E)'$. Since the extension $R \bowtie E \hookrightarrow R \times (R + E)$ $(\subseteq T(R) \times T(R))$ is integral (Corollary 2.7), we have the first statement. If, in addition, we assume that $R + E$ is a finitely generated R-module, then the ring extension $R^{\Delta} \hookrightarrow R \times (R + E)$ (Lemma 2.4) is finite (so, in particular, it is integral) and thus, we have the second statement.

(b) Since E is an R-submodule of $T(R)$, then clearly $T(R) = T(R + E)$, hence it is obvious that $T(R \times (R + E)) = T(R) \times T(R)$. If e is a nonzero regular element of $E \cap R$, then (e, e) is a nonzero regular element belonging to $(E \cap R) \times E$, which is a common ideal of $R \bowtie E$ and $R \times (R + E)$. From this fact, it follows that $R \bowtie E$ and $R \times (R + E)$ have the same total quotient ring [8, p. 326] and so, $T(R \bowtie E) = T(R) \times T(R)$. The last statement follows from [8, Lemma 26.5]. $\square$

Note that, in Corollary 2.12(b), the assumption that $E \cap R$ contains a regular element is essential, since if E is the ideal (0) of an integral domain R with quotient field K, then $R \bowtie (0) \cong R$ and so $T(R \bowtie (0)) \cong K$, but $T(R \times R) = K \times K$.

Remark 2.13. Using [4, Theorem 1.4(c) and Corollary 1.5(1)], the previous Proposition 2.6 and Corollary 2.7 can be used to give a scheme-theoretic description of

$\mathrm{Spec}(R \bowtie E)$ and $\mathrm{Spec}(R \times (R+E))$. We do not give many details here, since in the following Sec. 3, we will prove directly and in a more elementary way the most part of the statements contained in this remark for the case $E = I$ is an ideal of R.

Recall that if $f : A \to B$ is a ring homomorphism, $f^a : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ denotes, as usual, the continuous map canonically associated to f, i.e. $f^a(Q) := f^{-1}(Q)$, for each $Q \in \mathrm{Spec}(B)$, if I is an ideal of A and if $\mathcal{X} := \mathrm{Spec}(A)$, $V_{\mathcal{X}}(I)$ denotes the Zariski-closed set $\{P \in \mathcal{X} \mid P \supseteq I\}$ of $\mathcal{X}$.

In the situation of Lemma 2.4 and with the notation of Proposition 2.6, set $X := \mathrm{Spec}(R)$, $Y := \mathrm{Spec}(R \bowtie E)$ and $Z := \mathrm{Spec}(R \times (R+E))$ and set $\alpha := (u')^a : Z \to Y$ and $\beta := (v')^a : X \to Y$. Then the following properties hold:

(a) The canonical continuous map $\alpha : Z \to Y$ is surjective.
(b) The restriction of the map $\alpha : Z \to Y$ to $Z \setminus V_Z(\boldsymbol{\mathfrak{O}_1})$ gives rise to a topological homeomorphism:

$$\alpha|_{Z \setminus V_Z(\boldsymbol{\mathfrak{O}_1})} : Z \setminus V_Z(\boldsymbol{\mathfrak{O}_1}) \xrightarrow{\cong} Y \setminus V_Y(\boldsymbol{\mathfrak{O}_1}).$$

Moreover, for each $Q \in \mathrm{Spec}(R \times (R+E))$, with $Q \not\supseteq \boldsymbol{\mathfrak{O}_1}$, if $\boldsymbol{\mathcal{Q}} := \alpha(Q) = Q \cap (R \bowtie E)$, then the canonical map $(R \bowtie E)_{\boldsymbol{\mathcal{Q}}} \to (R \times (R+E))_Q$ is a ring isomorphism.
(c) $\beta : X \to Y$ defines a canonical homeomorphism of X with $V_Y(\boldsymbol{\mathfrak{O}_1})$; moreover, for each $\boldsymbol{\mathcal{Q}} \in \mathrm{Spec}(R \bowtie E)$ with $\boldsymbol{\mathcal{Q}} \supseteq \boldsymbol{\mathfrak{O}_1}$, the canonical ring homomorphism $(R \bowtie E)/\boldsymbol{\mathcal{Q}} \to R/v'(\boldsymbol{\mathcal{Q}})$ is an isomorphism.

We conclude this section by defining some distinguished ideals of $R \bowtie E$ that are naturally associated to a given ideal J of R and by giving an example of the general construction.

Proposition 2.14. *In the situation of Proposition* 2.6 *and with the notation of Lemma* 2.1, *for each ideal J of R we can consider the following ideals of $R \bowtie E$:*

$$\boldsymbol{\mathcal{J}_1} := v'^{-1}(J)\,,\ \ \boldsymbol{\mathcal{J}_2} := u'^{-1}(R \times J(R+E)) \ \ \textit{and}\ \ \boldsymbol{\mathcal{J}_0} := J^e := J(R \bowtie E).$$

Then we have:

(a) $\boldsymbol{\mathcal{J}_1} = u'^{-1}(J \times (R+E)) = u'^{-1}(J \times (J+E)) = \{(j, j+e) \mid j \in J,\ e \in E\}$.
(b) $\boldsymbol{\mathcal{J}_0} = \{(j, j+\alpha) \mid j \in J,\ \alpha \in JE\}$.
(c) $\boldsymbol{\mathcal{J}} := \boldsymbol{\mathcal{J}_1} \cap \boldsymbol{\mathcal{J}_2} = u'^{-1}(J \times J(R+E))$.
(d) $\boldsymbol{\mathcal{J}_0} \subseteq \boldsymbol{\mathcal{J}_1} \cap \boldsymbol{\mathcal{J}_2}$.

Proof. (a) and (b) are straightforward. Statement (c) is obvious, since $J \times J(R+E) = (J \times (R+E)) \cap (R \times J(R+E))$. (d) follows from (c) and from the fact that $J(R \bowtie E) \subseteq u'^{-1}(J(R \times (R+E))) = u'^{-1}(J \times J(R+E))$. $\square$

Example 2.15. Let $R := k[t^4, t^6, t^7, t^9]$ (where k is a field and t an indeterminate), $S := k[t^2, t^3]$ and $E := (t^2, t^3)S = t^2k[t]$. We have that $R + E = S$ and hence,

$$\begin{aligned} R \bowtie E &= \{(f(t), g(t)) \mid f \in R,\ g \in S \text{ and } g - f \in E\} \\ &= \{(f(t), g(t)) \mid f \in R,\ g \in S \text{ and } f(0) = g(0)\}. \end{aligned}$$

Since E is a maximal ideal of S, the prime ideals in $R \times S$ containing $\boldsymbol{\mathfrak{O}_1}$ are either of the form $P \times S$, for some prime ideal P of R, or $R \times E$; hence the primes not containing $\boldsymbol{\mathfrak{O}_1}$ are of the form $R \times Q$, with $Q \in \mathrm{Spec}(S)$ and $Q \neq E$.

By Remark 2.13 and Proposition 2.14, we have that if P is a prime in R, the ideal $\boldsymbol{\mathcal{P}_1} = (v')^{-1}(P) = (u')^{-1}(P \times S) = \{(p, p + e) \mid p \in P,\ e \in E\}$ is a prime in $R \bowtie E$, containing $\boldsymbol{\mathfrak{O}_1}$, and $R \bowtie E/\boldsymbol{\mathcal{P}_1} \cong R/P$. Moreover, with the notation of Remark 2.13, in this way we describe completely $V_Y(\boldsymbol{\mathfrak{O}_1})$. Notice also that, if we set $M := (t^4, t^6, t^7, t^9)R$, then the maximal ideals $M \times S$ and $R \times E$ of $R \times S$ have the same trace in $R \bowtie E$, i.e. $(R \times E) \cap (R \bowtie E) = \{(r, r + e) \mid r \in R \cap E,\ e \in E\} = (M \times S) \cap (R \bowtie E)$.

On the other hand, again by Remark 2.13, we have that $Y \setminus V_Y(\boldsymbol{\mathfrak{O}_1})$ is homeomorphic to $Z \setminus V_Z(\boldsymbol{\mathfrak{O}_1})$. Hence, the prime ideals of $R \bowtie E$ not containing $\boldsymbol{\mathfrak{O}_1}$ are of the form $(R \times Q) \cap (R \bowtie E)$, for some prime ideal Q of S, with $Q \neq E$.

3. The Prime Spectrum of $\boldsymbol{R \bowtie I}$

In this section, we study the case when the R-module $E = I$ is an ideal of R (that we will assume to be proper and different from (0), to avoid the trivial cases); in this situation $R + I = R$. We start with applying to this case, some of the results we obtained in the general situation.

Proposition 3.1. *Using the notation of Proposition* 2.6, *the following commutative diagram of canonical ring homomorphisms*

$$\begin{array}{ccc} R \bowtie I & \xrightarrow{v'} & R \\ {\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ R \times R & \xrightarrow{v} & R \times (R/I) \end{array}$$

is a pullback. The ideal $\boldsymbol{\mathfrak{O}_1} = (0) \times I = \mathrm{Ker}(v') = \mathrm{Ker}(v)$ *is a common ideal of* $R \bowtie I$ *and* $R \times R$, *the ideal* $\boldsymbol{\mathfrak{O}_2} = \mathrm{Ker}(R \bowtie I \xrightarrow{u'} R \times R \xrightarrow{\pi_2} R)$ *coincides with* $I \times (0) = (I \times (0)) \cap (R \bowtie I)$ *and* $(R \bowtie I)/\boldsymbol{\mathfrak{O}_i} \cong R$, *for* $i = 1, 2$.

In particular, if R is a domain, then $R \bowtie I$ is reduced and $\boldsymbol{\mathfrak{O}_1}$ and $\boldsymbol{\mathfrak{O}_2}$ are the only minimal primes of $R \bowtie I$.

Proof. The first part is an easy consequence of Lemma 2.4(b) and Proposition 2.6; the last statement follows from Corollary 2.5. □

Remark 3.2. Note that, when $I \subseteq R$, then $R \bowtie I := \{(r, r+i) \mid r \in R,\ i \in I\} = \{(r+i, r) \mid r \in R,\ i \in I\}$. It follows that we can exchange the roles of $\mathfrak{O}_1$ and $\mathfrak{O}_2$ (and that $\mathfrak{O}_2$ is also a common ideal of $R \bowtie I$ and $R \times R$).

If we specialize to the present situation, Corollary 2.7, Corollary 2.11 and Corollary 2.12, then we obtain:

Corollary 3.3. *Let R' (respectively, R^*) be the integral closure (respectively, the complete integral closure) of R in $T(R)$, we have*:

(a) $\dim(R \bowtie I) = \dim(R)$.
(b) *R is Noetherian if and only if $R \bowtie I$ is Noetherian.*
(c) *The integral closure of R^{Δ} and of $R \bowtie I$ in $T(R) \times T(R)$ coincide with $R' \times R'$.*
(d) *If I contains a regular element, then $T(R \bowtie I) = T(R) \times T(R)$ and the complete integral closure of $R \bowtie I$ in $T(R) \times T(R)$ coincide with $R^* \times R^*$, which is the complete integral closure of $R \times R$ in $T(R) \times T(R)$.*

The next goal is to investigate directly the relations among $\mathrm{Spec}(R \times R)$, $\mathrm{Spec}(R \bowtie I)$ and $\mathrm{Spec}(R)$, under the canonical maps associated to natural embeddings, i.e. the diagonal embedding $\delta : R \hookrightarrow R \bowtie I$, $(r \mapsto (r, r))$ and the inclusion $R \bowtie I \hookrightarrow R \times R$. With a slight abuse of notation, we identify R with its isomorphic image R^{Δ} in $R \bowtie I$ $(\subseteq R \times R)$ under the diagonal embedding (Lemma 2.4) and we denote the contraction to R of an ideal $\boldsymbol{\mathcal{H}}$ of $R \bowtie I$ (or, H of $R \times R$) by $\boldsymbol{\mathcal{H}} \cap R$ (or, by $H \cap R$).

We start with an easy lemma.

Lemma 3.4. *With the notation of Proposition* 2.14, *let J be an ideal of R. Then*:

(a) $\boldsymbol{\mathcal{J}_1}$ $(:= v'^{-1}(J)) = u'^{-1}(J \times R) = u'^{-1}(J \times (J+I)) = \{(j, j+i) \mid j \in J,\ i \in I\} =: J \bowtie I$. *If $J = I$, then $I \bowtie I$ $(= I \times I)$ is a common ideal of $R \bowtie I$ and $R \times R$.*
(b) $\boldsymbol{\mathcal{J}_2}$ $(:= u'^{-1}(R \times J)) = \{(j+i, j) \mid j \in J,\ i \in I\}$.
(c) $\boldsymbol{\mathcal{J}} := \boldsymbol{\mathcal{J}_1} \cap \boldsymbol{\mathcal{J}_2} = u'^{-1}(J \times J) = \{(j, j+i') \mid j \in J,\ i' \in I \cap J\} = \{(j_1, j_2) \mid j_1, j_2 \in J,\ j_1 - j_2 \in I\}$.
(d) $\boldsymbol{\mathcal{J}_0}$ $(:= J(R \bowtie I)) = \{(j, j+i'') \mid j \in J,\ i'' \in JI\}$ (cf. [1, *Lemma* 8]).
(e) $\boldsymbol{\mathcal{J}_0} \subseteq \boldsymbol{\mathcal{J}_1} \cap \boldsymbol{\mathcal{J}_2}$.
(f) $\boldsymbol{\mathcal{J}_1} = \boldsymbol{\mathcal{J}_2} \ \Leftrightarrow\ I \subseteq J$.
(g) $I + J = R \ \Rightarrow\ \boldsymbol{\mathcal{J}_0} = \boldsymbol{\mathcal{J}_1} \cap \boldsymbol{\mathcal{J}_2}$.
(h) $\boldsymbol{\mathcal{J}_1} \cap R = \boldsymbol{\mathcal{J}_2} \cap R = \boldsymbol{\mathcal{J}_0} \cap R = \boldsymbol{\mathcal{J}} \cap R = J$.

Proof. (a) is a particular case of Proposition 2.14(a). The second part is straightforward.

(b) Let $r \in R$ and $j \in J$; we have that $(r, j) \in R \bowtie I$ if and only if $(r, j) = (s, s+i)$, for some $s \in R$ and $i \in I$. Therefore, $r = s = j - i$ and $(r, j) = (j + i', j)$ for some $i' \in I$.

(c) Let $j_1, j_2 \in J$; we have that $(j_1, j_2) \in R \bowtie I$ if and only if $(j_1, j_2) = (s, s+i)$, for $s \in R$ and $i \in I$. Therefore, $j_1 = s$, $j_2 = j_1 + i$ and $j_2 - j_1 = i \in I$.

Statements (d) and (e) are particular cases of Proposition 2.14(b) and (d).

(f) follows easily from (a) and (b), since

$$\boldsymbol{\mathcal{J}_1} = \boldsymbol{\mathcal{J}_2} \;\Rightarrow\; J + I = J \;\Rightarrow\; I \subseteq J \;\Rightarrow\; \boldsymbol{\mathcal{J}_1} = \boldsymbol{\mathcal{J}_2}\,.$$

(g) is a consequence of (c) and (d), since $J + I = R$ implies that $J \cap I = JI$.

(h) It is obvious that $\boldsymbol{\mathcal{J}_1} \cap R = J = \boldsymbol{\mathcal{J}_2} \cap R$ and hence, by (c) and (e), we also have $\boldsymbol{\mathcal{J}} \cap R = \boldsymbol{\mathcal{J}_0} \cap R = J$. □

With the help of the previous lemma, we pass to describe the prime spectrum of $R \bowtie I$. In the following, the residue field at the prime ideal Q of a ring A (i.e. the field A_Q/QA_Q) will be denoted by $\boldsymbol{k}_A(Q)$. Part of the next theorem is contained in [1, Proposition 5].

Theorem 3.5. (a) *Let P be a prime ideal of R and consider the ideals $\boldsymbol{\mathcal{P}_1}$, $\boldsymbol{\mathcal{P}_2}$, $\boldsymbol{\mathcal{P}_0}$ and $\boldsymbol{\mathcal{P}}$ of $R \bowtie I$ as in Lemma* 3.4 (*with $P = J$*). *Then*:

(i) *$\boldsymbol{\mathcal{P}_1}$ and $\boldsymbol{\mathcal{P}_2}$ are the only prime ideals of $R \bowtie I$ lying over P.*
(ii) *If $P \supseteq I$, then $\boldsymbol{\mathcal{P}_1} = \boldsymbol{\mathcal{P}_2} = \boldsymbol{\mathcal{P}} = \sqrt{\boldsymbol{\mathcal{P}_0}} = P \bowtie I$. Moreover, $\boldsymbol{k}_R(P) \cong \boldsymbol{k}_{R\bowtie I}(\boldsymbol{\mathcal{P}})$.*
(iii) *If $P \not\supseteq I$, then $\boldsymbol{\mathcal{P}_1} \neq \boldsymbol{\mathcal{P}_2}$. Moreover, $\boldsymbol{\mathcal{P}} = \sqrt{\boldsymbol{\mathcal{P}_0}}$ and $\boldsymbol{k}_R(P) \cong \boldsymbol{k}_{R\bowtie I}(\boldsymbol{\mathcal{P}_1}) \cong \boldsymbol{k}_{R\bowtie I}(\boldsymbol{\mathcal{P}_2})$.*
(iv) *If P is a maximal ideal of R, then $\boldsymbol{\mathcal{P}_1}$ and $\boldsymbol{\mathcal{P}_2}$ are maximal ideals of $R \bowtie I$.*
(v) *If R is a local ring with maximal ideal M, then $R \bowtie I$ is a local ring with maximal ideal $\boldsymbol{\mathcal{M}} = \sqrt{\boldsymbol{\mathcal{M}_0}} = M \bowtie I$ (using again the notation of Lemma* 3.4 *for $M = J$).*
(vi) *R is reduced if and only if $R \bowtie I$ is reduced.*

(b) *Let $\boldsymbol{\mathcal{Q}}$ be a prime ideal of $R \bowtie I$ and let $\boldsymbol{\mathfrak{O}_1}$ be as in Proposition* 3.1. *Two cases are possible either $\boldsymbol{\mathcal{Q}} \not\supseteq \boldsymbol{\mathfrak{O}_1}$ or $\boldsymbol{\mathcal{Q}} \supseteq \boldsymbol{\mathfrak{O}_1}$.*

(i) *If $\boldsymbol{\mathcal{Q}} \not\supseteq \boldsymbol{\mathfrak{O}_1}$, then there exists a unique prime ideal Q of $R \times R$ such that $\boldsymbol{\mathcal{Q}} = Q \cap (R \bowtie I)$ with $Q = R \times P$, where $P := \boldsymbol{\mathcal{Q}} \cap R$ (and $P \not\supseteq I$). In this case, with the notation of the previous part (a), $\boldsymbol{\mathcal{P}_1} \neq \boldsymbol{\mathcal{P}_2}$ and*

$$\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{P}_2} = \{(p+i, p) \mid p \in P,\ i \in I\}.$$

Furthermore, the canonical ring homomorphisms $R \bowtie I \hookrightarrow R \times R \xrightarrow{\pi_2} R$ induce for the localizations the following isomorphisms:

$$(R \bowtie I)_{\boldsymbol{\mathcal{Q}}} \cong (R \times R)_Q = (R \times R)_{R\times P} \cong R_P \quad (\textit{thus } \boldsymbol{k}_{R\bowtie I}(\boldsymbol{\mathcal{Q}}) \cong \boldsymbol{k}_R(P))\,.$$

(ii) *If $\boldsymbol{\mathcal{Q}} \supseteq \boldsymbol{\mathfrak{O}_1}$, then there exists a unique prime ideal P of R such that $\boldsymbol{\mathcal{Q}} = v'^{-1}(P)$ (or, equivalently, $P = v'(\boldsymbol{\mathcal{Q}})$). With the notation of the previous part* *(a), if $P \supseteq I$ then $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{P}_1} = \boldsymbol{\mathcal{P}_2}$. On the other hand, if $P \not\supseteq I$ then $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{P}_1}$ $(\neq \boldsymbol{\mathcal{P}_2})$. In both cases,*

$$\boldsymbol{\mathcal{Q}} = \{(p, p+i) \mid p \in P,\ i \in I\}.$$

Furthermore, the canonical ring homomorphism $v' : R \bowtie I \to R$ induces the following isomorphism:

$$(R \bowtie I)/\boldsymbol{\mathcal{Q}} \cong R/P \quad (\textit{thus } \boldsymbol{k}_{R\bowtie I}(\boldsymbol{\mathcal{Q}}) \cong \boldsymbol{k}_R(P)).$$

Proof. Note that the composition of the diagonal embedding $\delta : R \hookrightarrow R \bowtie I$, $(r \mapsto (r,r))$, with the inclusion $R \bowtie I \subseteq R \times R$, $((r, r+i) \mapsto (r, r+i))$, coincides with the diagonal embedding $R \hookrightarrow R \times R$, $(r \mapsto (r,r))$, which is a finite ring homomorphism. Thus, in particular, both $R \hookrightarrow R \bowtie I$ and $R \bowtie I \subseteq R \times R$ are integral homomorphisms. Note also that if Q is a prime ideal of $R \times R$ lying over P, then necessarily $Q \in \{P \times R, R \times P\}$ (Remark 2.8).

(a)(i) Note that $\boldsymbol{\mathcal{P}_1} = u'^{-1}(P \times R)$ and $\boldsymbol{\mathcal{P}_2} = u'^{-1}(R \times P)$ (Lemma 3.4); hence $\boldsymbol{\mathcal{P}_1}$ and $\boldsymbol{\mathcal{P}_2}$ are prime ideals lying over P. By integrality, if $\boldsymbol{\mathcal{Q}} \in \mathrm{Spec}(R \bowtie I)$ and $\boldsymbol{\mathcal{Q}} \cap R = P$, then there exists $\bar{Q} \in \mathrm{Spec}(R \times R)$ such that $\bar{Q} \cap (R \bowtie I) = \boldsymbol{\mathcal{Q}}$ and thus, $\bar{Q} \cap R = P$. Therefore, $\bar{Q} \in \{P \times R, R \times P\}$ and so, $\boldsymbol{\mathcal{Q}} \in \{\boldsymbol{\mathcal{P}_1}, \boldsymbol{\mathcal{P}_2}\}$.
(a)(ii) We know already by Lemma 3.4(f) and (c) that, if $P \supseteq I$, then $\boldsymbol{\mathcal{P}_1} = \boldsymbol{\mathcal{P}_2} = \boldsymbol{\mathcal{P}}$, hence by part (a)(i), we conclude easily that $\boldsymbol{\mathcal{P}} = \sqrt{\boldsymbol{\mathcal{P}_0}}$. Moreover, we have the following sequence of canonical homomorphisms:

$$\frac{R}{P} \subseteq \frac{R \bowtie I}{\sqrt{\boldsymbol{\mathcal{P}_0}}} = \frac{R \bowtie I}{\boldsymbol{\mathcal{P}}} \subseteq \frac{R \times R}{P \times R} \cong \frac{R}{P} \cong \frac{R \times R}{R \times P},$$

from which we deduce the last part of the statement.
(a)(iii) By Lemma 3.4(e) and (f), we know that, if $P \not\supseteq I$, then $\boldsymbol{\mathcal{P}_1} \neq \boldsymbol{\mathcal{P}_2}$ and $\boldsymbol{\mathcal{P}_0} \subseteq \boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}_1} \cap \boldsymbol{\mathcal{P}_2}$. By part (a)(i) and by the integrality of $R \hookrightarrow R \bowtie I$, we conclude easily that $\boldsymbol{\mathcal{P}} = \sqrt{\boldsymbol{\mathcal{P}_0}}$. Finally, as in part (a)(ii), it is easy to see that $\boldsymbol{k}_R(P) \cong \boldsymbol{k}_{R\bowtie I}(\boldsymbol{\mathcal{P}_1}) \cong \boldsymbol{k}_{R\bowtie I}(\boldsymbol{\mathcal{P}_2})$.
(a)(iv) follows by the integrality of $R \subseteq R \bowtie I$.
(a)(v) follows immediately by part (a)(iv) and part (a)(ii).
(a)(vi) follows by integrality of $R \hookrightarrow R \bowtie I$ and $R \bowtie I \subseteq R \times R$ and from the fact that R is reduced if and only if $R \times R$ is reduced.
(b) If $P = \boldsymbol{\mathcal{Q}} \cap R$, then necessarily $\boldsymbol{\mathcal{Q}} \in \{\boldsymbol{\mathcal{P}_1}, \boldsymbol{\mathcal{P}_2}\}$ by (a)(i).
(b)(i) Since $\boldsymbol{\mathcal{Q}} \not\supseteq \boldsymbol{\mathfrak{O}_1}$, then $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{P}_2}$, because $\boldsymbol{\mathcal{P}_1} \supseteq \boldsymbol{\mathfrak{O}_1}$. Note that $\boldsymbol{\mathcal{P}_2} = (R \times P) \cap R \bowtie I$; it is easy to see that $Q := R \times P$ is the unique prime of $R \times R$ contracting over $\boldsymbol{\mathcal{Q}}$. The elementwise description of $\boldsymbol{\mathcal{P}_2}$ is a particular case of Lemma 3.4(b). Last statement follows from the following canonical inclusions of localizations $R_P \hookrightarrow (R \bowtie I)_{\boldsymbol{\mathcal{Q}}} \hookrightarrow (R \times R)_Q = (R \times R)_{R \times P} \cong R_P$.
(b)(ii) The first and the last statements are trivial consequences of the fact that v' induces an isomorphism between $R \bowtie I/\boldsymbol{\mathfrak{O}_1}$ and R. It is easy to see that the prime P is such that $P = \boldsymbol{\mathcal{Q}} \cap R$. Therefore, the second statement follows from (b)(i).

If $P \not\supseteq I$ (and $\boldsymbol{\mathcal{Q}} \supseteq \boldsymbol{\mathfrak{O}_1}$), then $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{P}_1}$ ($\neq \boldsymbol{\mathcal{P}_2}$), since $\boldsymbol{\mathcal{Q}}$ does not contain $\boldsymbol{\mathfrak{O}_2}$ (note that a prime ideal of $R \bowtie I$ containing both $\boldsymbol{\mathfrak{O}_1}$ and $\boldsymbol{\mathfrak{O}_2}$ has a trace in R containing I). The elementwise description of $\boldsymbol{\mathcal{P}_1}$ is a particular case of Lemma 3.4(a). □

Remark 3.6. In the situation of Theorem 3.5, note that, if P is a prime ideal of R, then by integrality of $R \hookrightarrow R \bowtie I \subseteq R \times R$, inside the ring $R \times R$, the prime ideals $P \times R$ and $R \times P$ are the only minimal prime ideals of $P \times P = \boldsymbol{\mathcal{P}_0}(R \times R) = P(R \times R)$, and so

$$\boldsymbol{\mathcal{P}_0}(R \times R) = P \times P = (P \times R) \cap (R \times P) = \sqrt{\boldsymbol{\mathcal{P}_0}(R \times R)}$$

is a radical ideal of $R \times R$, with

$$(P \times P) \cap (R \bowtie I) = ((P \times R) \cap (R \times P)) \cap (R \bowtie I) = \boldsymbol{\mathcal{P}_1} \cap \boldsymbol{\mathcal{P}_2} = \boldsymbol{\mathcal{P}}.$$

Next example shows that in $R \bowtie I$, in general, $\boldsymbol{\mathcal{P}_0}$ is not a radical ideal (i.e. it may happen that $\boldsymbol{\mathcal{P}_0} \subsetneq \sqrt{\boldsymbol{\mathcal{P}_0}} = \boldsymbol{\mathcal{P}}$).

Example 3.7. Let V be a valuation domain with a nonzero nonmaximal nonidempotent prime ideal P. (An explicit example can be constructed as follows: let k be a field and let X, Y be two indeterminates over k, then take $V := k[X]_{(X)} + Yk(X)[Y]_{(Y)}$ and $P := Yk(X)[Y]_{(Y)}$. It is well known that V is the discrete valuation domain of dimension 2, and P is the height 1 prime ideal of V [16, (11.4), p. 35]; [8, p. 192].)

In this situation, it is easy to see that the ideal $P \times P$ is a common (radical) ideal of $V \bowtie P$ and of its overring $V \times V$. Moreover, note that $\boldsymbol{\mathcal{P}_0} = P(V \bowtie P) = \{(p, p + x) \mid p \in P,\ x \in P^2\}$ (Lemma 3.4(d)) and that $P(V \times V) = P \times P \subset V \bowtie P$. More precisely, by Lemma 3.4(c), we have:

$$\begin{aligned} P \times P &= (P \times P) \cap (V \bowtie P) = (P \times V) \cap (V \times P) \cap (V \bowtie P) \\ &= \boldsymbol{\mathcal{P}_1} \cap \boldsymbol{\mathcal{P}_2} = \boldsymbol{\mathcal{P}} = \{(p, p + y) \mid p \in p,\ y \in P \cap P = P\}. \end{aligned}$$

Clearly, since $P^2 \neq P$, then $\boldsymbol{\mathcal{P}_0} \subsetneq \boldsymbol{\mathcal{P}}$; for instance, if $z \in P \setminus P^2$, then $(p, p + z) \in \boldsymbol{\mathcal{P}} \setminus P(V \bowtie P)$.

We complete now the description of the affine scheme $\mathrm{Spec}(R \bowtie I)$, initiated in Theorem 3.5, determining in particular the localizations of $R \bowtie I$ in each of its prime ideals. Part of the next theorem is contained in [1, Proposition 7].

Theorem 3.8. *Let* $X := \mathrm{Spec}(R)$, $Y := \mathrm{Spec}(R \bowtie I)$ *and* $Z := \mathrm{Spec}(R \times R) \cong \mathrm{Spec}(R) \amalg \mathrm{Spec}(R)$ *and let* $\alpha : Z \twoheadrightarrow Y$ *and* $\gamma : Y \twoheadrightarrow X$ *be the canonical surjective maps associated to the integral embeddings* $R \bowtie I \hookrightarrow R \times R$ *and* $R \cong R^{\Delta} \hookrightarrow R \bowtie I$ (*proof of Theorem* 3.5).

(a) *The restrictions of* α

$$\alpha\big|_{Z \setminus V_Z(\boldsymbol{\mathfrak{O}_i})} : Z \setminus V_Z(\boldsymbol{\mathfrak{O}_i}) \longrightarrow Y \setminus V_Y(\boldsymbol{\mathfrak{O}_i})$$

(for $i = 1, 2$) are scheme isomorphisms, and clearly

$$Z \setminus V_Z(\boldsymbol{\mathfrak{O}_i}) \cong X \setminus V_X(I).$$

In particular, for each prime ideal P of R, such that $P \not\supseteq I$, if we set $\bar{P}_1 := P \times R$ and $\bar{P}_2 := R \times P$, we have $\boldsymbol{\mathcal{P}}_i := \bar{P}_i \cap (R \bowtie I)$, for $1 \leq i \leq 2$, and the following canonical ring homomorphisms are isomorphisms:

$$R_P \to (R \bowtie I)_{\boldsymbol{\mathcal{P}}_i} \to (R \times R)_{\bar{P}_i} \quad \textit{for } 1 \leq i \leq 2.$$

(b) *The restriction of γ*

$$\gamma\,|_{V_Y(\boldsymbol{\mathfrak{O}_1}) \cap V_Y(\boldsymbol{\mathfrak{O}_2})} : V_Y(\boldsymbol{\mathfrak{O}_1}) \cap V_Y(\boldsymbol{\mathfrak{O}_2}) \to V_X(I)$$

is a scheme isomorphism.

(c) *If $P \in \operatorname{Spec}(R)$ is such that $P \supseteq I$ and $\boldsymbol{\mathcal{P}} \in \operatorname{Spec}(R \bowtie I)$ is the unique prime ideal such that $\boldsymbol{\mathcal{P}} \cap R = P$, the following diagram of canonical homomorphisms:*

$$\begin{tikzcd} (R \bowtie I)_{\boldsymbol{\mathcal{P}}} \arrow[r] \arrow[d] & R_P \arrow[d, "u_P"'] \\ R_P \times R_P \arrow[r, "v_P"] & R_P \times (R_P/I_P) \end{tikzcd}$$

is a pullback (where $I_P := IR_P$, $u_P(x) := (x, x + I_P)$ and $v_P((x, y)) := (x, y + I_P)$, for $x, y \in R_P$), i.e. $(R \bowtie I)_{\boldsymbol{\mathcal{P}}} \cong R_P \bowtie I_P$ (Proposition 3.1).

Proof. (a) Since $\boldsymbol{\mathfrak{O}_1} = \{0\} \times I$ (respectively, $\boldsymbol{\mathfrak{O}_2} = I \times \{0\}$) is a common ideal of $R \times R$ and $R \bowtie I$, this statement follows from the general results on pullbacks [4, Theorem 1.4] and from Theorem 3.5 (and its proof). Note that $Z \setminus V_Z(\boldsymbol{\mathfrak{O}_1}) \cong ((X \amalg X) \setminus (X \amalg V_X(I))) = X \setminus V_X(I) = ((X \amalg X) \setminus (V_X(I) \amalg X)) \cong Z \setminus V_Z(\boldsymbol{\mathfrak{O}_2})$.
(b) Note that $V_Y(\boldsymbol{\mathfrak{O}_1}) \cap V_Y(\boldsymbol{\mathfrak{O}_2}) = V_Y(\boldsymbol{\mathfrak{O}_1} + \boldsymbol{\mathfrak{O}_2})$ and $\boldsymbol{\mathfrak{O}_1} + \boldsymbol{\mathfrak{O}_2} = I \times I$. Therefore, the present statement follows from the fact that the canonical surjective homomorphism $R \bowtie I \to R/I$, defined by $(r, r + i) \mapsto r + I$ (for each $r \in R$ and $i \in I$) has kernel equal to $I \times I$.
(c) If we start from the pullback diagram considered in Proposition 3.1 and we apply the tensor product $R_P \otimes_R -$, then by [4, Proposition 1.9], we get the following pullback diagram:

$$\begin{tikzcd} R_P \otimes_R (R \bowtie I) \arrow[r, "id\otimes v'"] \arrow[d, "id\otimes u'"'] & R_P \otimes_R R \arrow[d, "id\otimes u"'] \\ R_P \otimes_R (R \times R) \arrow[r, "id\otimes v"] & R_P \otimes_R (R \times (R/I))\,. \end{tikzcd}$$

Note that, by the properties of the tensor product, we deduce immediately the following canonical ring isomorphisms: $R_P \otimes_R (R \times R) \cong R_P \times R_P$, $R_P \otimes_R R \cong R_P$ and that $R_P \otimes_R (R \times (R/I)) \cong R_P \times (R_P \otimes_R (R/I)) \cong R_P \times (R_P/IR_P)$. Therefore, the previous pullback diagram gives rise to the following pullback of canonical homomorphisms:

$$\begin{tikzcd} R_P \otimes_R (R \bowtie I) \arrow[r] \arrow[d] & R_P \arrow[d, "u_P"'] \\ R_P \times R_P \arrow[r, "v_P"] & R_P \times (R_P/I_P)\,. \end{tikzcd}$$

On the other hand, recall that $Spec(R_P \otimes_R (R \bowtie I))$ can be canonically identified (under the canonical homeomorphism associated to the natural ring homomorphism $R \bowtie I \to R_P \otimes_R (R \bowtie I)$) with the set of all prime ideals $\boldsymbol{\mathcal{H}} \in Spec(R \bowtie I)$ such that $\boldsymbol{\mathcal{H}} \cap R \subseteq P$. Since we know already that, in the present situation, there exists a unique prime ideal $\boldsymbol{\mathcal{P}} \in \mathrm{Spec}(R \bowtie I)$ such that $\boldsymbol{\mathcal{P}} \cap R = P$ (Theorem 3.5 (a)(ii)) and that the canonical embedding $R \hookrightarrow R \bowtie I$ has the going-up property, we deduce that $Spec(R_P \otimes_R (R \bowtie I))$ can be canonically identified with the set of all the prime ideals of $R \bowtie I$ contained in $\boldsymbol{\mathcal{P}}$. Therefore, $R_P \otimes_R (R \bowtie I)$ is a local ring with a unique maximal ideal corresponding to the prime ideal $\boldsymbol{\mathcal{P}}$ of $R \bowtie I$ and thus, we deduce that the canonical ring homomorphism $(R \bowtie I)_{\boldsymbol{\mathcal{P}}} \to R_P \otimes_R (R \bowtie I)$ is an isomorphism. $\square$

Proposition 3.9. *The ring $R \bowtie I$ can be obtained as a pullback of the following diagram of canonical homomorphisms*:

$$\begin{tikzcd} R \bowtie I \arrow[r, "\tilde{v}'"] \arrow[d, "\tilde{u}'"'] & R/I \arrow[d, "\tilde{u}"'] \\ R \times R \arrow[r, "\tilde{v}"] & R/I \times R/I \end{tikzcd}$$

where $\tilde{u}$ is the diagonal embedding, $\tilde{v}$ is the canonical surjection $(x, y) \mapsto (x+I, y+I)$, $\tilde{u}'$ is the natural inclusion and $\tilde{v}'$ is defined by $(x, x+i) \mapsto x+I$, for all $x, y \in R$ and $i \in I$.

Proof. By Proposition 3.1, we know that

$$\begin{tikzcd} R \bowtie I \arrow[r] \arrow[d] & R \arrow[d, "u"'] \\ R \times R \arrow[r, "v"] & R \times R/I \end{tikzcd}$$

is a pullback. On the other hand, it is easy to verify that the following diagram:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "u"'] & R/I \arrow[d, "\tilde{u}"'] \\ R \times R/I \arrow[r, "w"] & R/I \times R/I \end{tikzcd}$$

is a pullback, where w is the canonical surjection $(x, y) \mapsto (x + I, y)$ and φ is the natural proiection $x \mapsto x + I$, for each $x \in R$ and for each $y \in R/I$. The conclusion follows by juxtaposing two pullbacks. $\square$

Corollary 3.10. *If R is a local ring, integrally closed in $T(R)$ with maximal ideal M and residue field k, then $R \bowtie M$ is seminormal in its integral closure inside $T(R) \times T(R)$ (which, in this situation, coincides with $R \times R$).*

Proof. By the previous proposition, $R \bowtie M$ (which is a local ring) can be obtained as a pullback of the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R \bowtie M & \xrightarrow{\tilde{v}'} & k \\ {\scriptstyle \tilde{u}'}\big\downarrow & & {\scriptstyle \tilde{u}}\big\downarrow \\ R \times R & \xrightarrow{\tilde{v}} & k \times k. \end{array}$$

The statement follows from the fact that, in this case, the integral closure of $R \bowtie M$ in $T(R) \times T(R)$ coincides with $R \times R$ (Corollary 3.3(c)). Therefore, since $\tilde{u}$ is a minimal extension, then $\tilde{u}'$ is also minimal [3, Lemme 1.4(ii)], and thus the conclusion follows from [3, Thèoréme 2.2(ii))] and from [18, (1.1)] (keeping in mind Theorem 3.5(c)). $\square$

Example 3.11. (a) Let $R := k[[t]]$ (where k is a field and t an indeterminate) and let $I := t^n R$. Using Proposition 3.9, if we denote by $h^{(i)}(t)$ the ith derivative of a power series $h(t) \in k[[t]]$, it is easy to see that

$$R \bowtie I = \{(f(t), g(t)) \mid f(t), g(t) \in R,\ f^{(i)}(0) = g^{(i)}(0)\ \forall\ i = 0, \ldots n-1\}.$$

(b) Let $R := k[x, y]$ and $I := xR$. In this case,

$$R \bowtie I = \{(f(x, y), g(x, y)) \mid f(x, y), g(x, y) \in R,\ f(0, y) = g(0, y)\}.$$

Setting $Y = \mathit{Spec}(R \bowtie I)$ and $X = \mathit{Spec}(R)$, by Proposition 2.13, $V_Y(\boldsymbol{\mathfrak{O}_i}) \cong \mathit{Spec}(k[x, y])$. On the other hand, by Theorem 3.8, $V_Y(\boldsymbol{\mathfrak{O}_1}) \cap V_Y(\boldsymbol{\mathfrak{O}_2}) = V_Y((xR \times xR)) \cong V_X(xR) \cong \mathit{Spec}(k[y])$. Hence, the ring $R \bowtie I$ is the coordinate ring of two affine planes with a common line. Note that we can present $R \bowtie I$ as quotient of a polynomial ring in the following way: consider the homomorphism $\lambda : k[x, y, z] \rightarrow R \times R$, defined by $\lambda(x) := (x, x)$, $\lambda(y) := (y, y)$ and $\lambda(z) := (0, x)$. It is not difficult to see that $\mathrm{Im}(\lambda) = R \bowtie I$ and $\mathit{Ker}(\lambda) = (zx - z^2)k[x, y, z]$.

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