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The Zariski topology on the set of semistar operations on an integral domain

Candidato:

Giovan Battista Pignatti Morano di Custoza

Relatore:

Prof. Marco Fontana

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To my beloved grandmothers, Irma and Dialma.

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Contents

Introduction

The notion of star operation was introducted by Krull in [19]. If A is integral domain and $\mathbf{F}(A)$ is the set of all the nonzero fractional ideals of A, he defined a star operation on A to be a map $\star\colon \mathbf{F}(A) \to \mathbf{F}(A), I \mapsto I^\star$ such that, for any nonzero element x of the quotient field of A and any $I, J \in \mathbf{F}(A)$, the following properties hold:

(star.1) $A^\star = A$ and $(xI)^\star = xI^\star$.

(star.2) $I \subseteq J$ implies $I^\star \subseteq J^\star$.

(star.3) $I \subseteq I^\star$.

(star.4) $(I^\star)^\star = I^\star$.

Star operations and the related theory of ideal systems based on the works of E. Noether, H. Prüfer and P. Lorentzen from the 1930's, play a central role in the study of the multiplicative structure of the ideals of a ring. These notions represent a natural and abstract setting that allows a new approach for characterizing several classes of integral domains. For example, it is proved in [14, Proposition 34.12] that an integrally closed domain A is a Prüfer domain if and only if $I^t = I$, where I is any nonzero ideal of A and t is a particular star operation (we will return on this topic in Example 1.0.1 (d)).

An enlightening motivation for trying to generalize the notion of star operation can be found if we consider the map $b\colon \mathbf{F}(A) \to \mathbf{F}(A), I \mapsto \bigcap\{IV : V \text{ is a valuation overring of } A\}$. By [17, Section 6.8], we note that b sends each fractional ideal of A in its integral closure. This map is a star operation if and only if A is integrally closed. In fact it follows by the famous Krull's Theorem that A^b is the integral closure of A and thus b can satisfy property **(star.1)** if and only if it is integrally closed (we will return on the properties of the operation b in Remark 4.0.10).

Since the other conditions that characterize star operations are easily satisfied by b, it is natural to look for a class of operations that includes the "closure of ideals" even if A is not supposed to be integrally closed.

So we do not require anymore that $A^\star$ coincides with A and, as a consequence, we need to define $\star$ on the larger set $\overline{\mathbf{F}}(A)$ of all the A-submodules of the quotient field of A, since the integral closure of a domain is not necessarily a fractional ideal.

These considerations led A. Okabe and R. Matsuda to introduce in [24] the notion of *semistar operation*. This is a more flexible notion that gives a more appropriate context for approaching several questions of multiplicative ideal theory. Semistar operations are also closely related to the theory of Kronecker function rings. For a deeper insight on the recent developments on this topic see [8], [11], [22] and [23].

Following [9], in our paper we use a topological approach to study the algebraic properties of semistar operations. A similar topological perspective was adopted in [8] for studying the multiplicative structure of the ideals of an integral domain. In that paper the authors endowed the Riemann-Zariski space $\mathrm{Zar}(A)$ (see [31]) of all the valuation overrings of an integral domain A with several topological structures (the Zariski, the constructible and the inverse topology) and studied the interplay between the topological properties of a given subspace of $\mathrm{Zar}(A)$ and the algebraic properties of the semistar operation determined by such a subspace. By using this approach, the authors investigated, from a topological point of view, the representations of an integrally closed domain as an intersection of valuation overrings.

In the meanwhile, B. Olberding in [25] defined the Zariski topology on the space $\mathrm{Over}(A)$ of all overrings of an integral domain A in such a way that both the set of localizations of A (with the topology induced by the Zariski topology of $\mathrm{Spec}(A)$) and $\mathrm{Zar}(A)$ become subspaces of $\mathrm{Over}(A)$. The main idea of Chapter 1 is to consider the set $\mathrm{SStar}(A)$ of all semistar operations on A and to endow it with a "natural" Zariski topology, in such a way that $\mathrm{Over}(A)$ is identifiable canonically with a subspace of $\mathrm{SStar}(A)$ (Proposition 1.2.1). After giving the main properties of the Zariski topology on $\mathrm{SStar}(A)$, we relate the compactness of the subspaces of $\mathrm{SStar}(A)$ with the finite type property of their infimum (with respect to a natural order on $\mathrm{SStar}(A)$ induced by the set inclusion). We show (Proposition 1.1.7) that the infimum of a compact family of semistar operations of finite type is of finite type and that the converse is true when each operation of the family is induced by a localization of A or by a valuation ring (Corollary 4.0.8 and Proposition 4.0.9).

Then, we specialize the study of the Zariski topology on the subspace $\mathrm{SStar}_f(A)$ of all the semistar operations on A of finite type and we show that this space is spectral (Theorem 1.3.1). To prove that result we need a specific characterization of spectral spaces. Therefore, in Chapter 2 (following [7]), we construct a new topology defined through ultrafilters on an arbitrary set X. This topology can be seen as a generalization of the theory developed in [12]. In this paper the authors considered, for each subset Y of $\mathrm{Spec}(A)$ and any $\mathscr{U}$ ultrafilter on Y, the set $\mathfrak{p}_{Y,\mathscr{U}} := \{x \in A : V(x) \cap Y \in \mathscr{U}\}$ (where $V(x)$ is the canonical closed set of the Zariski topology on $\mathrm{Spec}(A)$). This set is proved to be a prime ideal of A in [4, Lemma 2.4] and is called *ultrafilter limit point of Y, with respect to $\mathscr{U}$*. Ultrafilter limit points are not always elements of Y: for example, if A is the ring of integers, $Y := \mathrm{Max}(A)$ and $\mathscr{U}$ is a nontrivial ultrafilter on Y (i.e., an ultrafilter whose elements are infinite sets), then it is easy to check that $\mathfrak{p}_{Y,\mathscr{U}} = (0)$. Thus, it is shown that the subsets of $\mathrm{Spec}(A)$ which contain all their ultrafilter limit points are the closed sets of a topology on $\mathrm{Spec}(A)$, called *the ultrafilter topology*. Moreover, the authors also prove that the ultrafilter topology is identical to the patch topology on $\mathrm{Spec}(A)$ introduced by M. Hochster in [16].

Our construction extends this setting to an arbitrary set X and replaces ultrafilter limit points with other subsets of X built through a collection $\mathcal{F}$ of fixed subsets of X. As a result, we get the ultrafilter topology as a particular case of our new topology (Example 2.1.2 (c)). We then use the results contained in [16] to relate our topology to spectral spaces (Theorem 2.2.3) and we finally obtain a new characterization of spectral spaces (Corollary 2.2.4).

In Chapter 3, we show that if $A \subseteq B$ is an extension of integral domains, then there is a natural continuous map $\sigma\colon \mathrm{SStar}(B) \to \mathrm{SStar}(A)$. Moreover, in case A and B have the same quotient field, we show that σ is an embedding (Proposition 3.1.3) and we give an interpretation through the Theory of Categories.

The last Chapter is dedicated to a deeper study of the semistar operations induced by localizations of A, which are usually called *spectral semistar operations*. We see that it is possible to associate to each semistar operation $\star$ another semistar operation $\tilde{\star}$ which is always a spectral operation of finite type. We then find a necessary and sufficient condition for the equality of $\widetilde{\star_1}$ and $\widetilde{\star_2}$ in case $\star_1$ and $\star_2$ are spectral semistar operations (Proposition 5.1.1) and we find an explicit form of $\tilde{\star}$ when $\star$ is a semifinite semistar operation (this definition will be recalled later) (Proposition 5.2.3). Respecting the focus of our paper, in order to get these results, we choose a topological approach, using the inverse topology of the Zariski topology introduced by M. Hochster in [16].

Chapter 0

Background material

0.1 Prerequisites from General Topology

In this section we want to recall some of the main definitions and basic results from General Topology that will occur more frequently in our paper. For a deeper look into these notions the reader can refer to any good book in General Topology (see for example [30]).

0.1.1 Topological spaces

Definition 0.1.1. Let X be a nonempty set. A *topology on* X is a nonempty family τ of subsets of X which satisfies the following conditions:
(O1) $\emptyset, X \in \tau$.
(O2) If $\{A_i\}_{i\in I}$ is a family of elements of τ then $\bigcup_{i\in I} A_i \in \tau$.
(O3) If $A, B \in \tau$ then $A \cap B \in \tau$.
The subsets of X belonging to the family τ are called *open sets.* The set X together with a topology τ is called *topological space* and is denoted by (X, τ) or simply by X, when no confusion can arise.
If τ, τ' are two topologies on X, we will say that τ' is *finer than* τ (or equivalently that τ is *coarser than* τ') if every open set in the topology τ is open also in the topology τ', i.e., if $\tau \subseteq \tau'$.

The two most basic examples of topologies are presented next.

Example 0.1.1. (a) If X is a nonempty set, the family τ of all subsets of X is a topology on X called the *discrete topology.* It is easy to see that the discrete topology is the finest topology on X.

(b) If X is a nonempty set, the family $\tau = \{X, \emptyset\}$ is a topology on X called the *indiscrete topology.* It is easy to see that the indiscrete topology is the coarsest topology on X.

As we have seen so far, a topological space X is determined by the collection of its open sets. However, there is a more convenient way to describe topologies which allows to consider just a smaller collection of open sets.

Definition 0.1.2. Let τ be a topology on the nonempty set X. A *basis of* τ is a family of open sets $\mathcal{B} \subseteq \tau$ such that every open set of τ is union of elements of $\mathcal{B}$.

Thus, if we want to study a given topology τ on a set X, it is enough to provide a basis $\mathcal{B}$ of τ; in fact all the open sets of τ can be described through elements of $\mathcal{B}$. As it is shown in [30, Theorem 5.3], things works also on the converse way: a family of subsets of a set X which satisfies two particular properties will always be the basis of a topology on X.

Proposition 0.1.3. *A family $\mathcal{B}$ of subsets of X is a basis for a topology on X if and only if the two following conditions are satisfied:*
(i) $\bigcup_{B\in\mathcal{B}} B = X$.
(ii) For any $A, B \in \mathcal{B}$, $A \cap B$ is union of elements of $\mathcal{B}$.

In our paper, when we will need to define topologies, rather than basis we will often prefer to use a more general tool: the *subbasis.*

Definition 0.1.4. Let τ be a topology on the nonempty set X. A *subbasis of* τ is a family of open sets $\mathcal{S} \subseteq \tau$ such that the family $\mathcal{B} := \{U_1 \cap \cdots \cap U_n : n \in \mathbb{N},\ U_1, \ldots, U_n \in \mathcal{S}\}$ is a basis of τ.

From this definition it is clear that every basis is also a subbasis. However the converse statement does not hold (see [30, Example 5A 1] for a classic counterexample).

Definition 0.1.5. Let X be a topological space. If $x \in X$, a *neighborhood of* x is a subset N of X such that there exists an open set A such that $x \in A \subseteq N$.

It is clear that an open set A is a neighborhood of each one of its elements. The converse is also true.

Proposition 0.1.6. *A subset A of the topological space X is open if and only if A is a neighborhood of x for each $x \in A$.*

Definition 0.1.7. Let X be a topological space; we say that a subset C of X is *closed* if $X \setminus C$ is open in X.

It is clear that a topology is completely described once either a basis for the open or for the closed sets is provided.
A subset of a topological space X which is both open and closed is called *clopen*. Obviously X and $\emptyset$ are trivially clopen in any topological space X.

Definition 0.1.8. Let S be a subset of a topological space X. The *closure of S* is the intersection of all the closed subsets of X which contain S and is denoted by $\mathrm{Ad}(S)$.

The following Proposition fixes some of the most important properties of the set closure. The proof follows easily noting that the closure of a set S is by definition the smallest closed set containing S.

Proposition 0.1.9. *Let S be a subset of a topological space X. Then:*
(i) $S \subseteq \mathrm{Ad}(S)$.
(ii) S is closed if and only if $S = \mathrm{Ad}(S)$.
(iii) $\mathrm{Ad}(S) = \mathrm{Ad}(\mathrm{Ad}(S))$.
(iv) If T is a subset of X such that $S \subseteq T$, then $\mathrm{Ad}(S) \subseteq \mathrm{Ad}(T)$.

If we regard $\mathrm{Ad}(\cdot)$ as a function on the power set of the topological space X partially ordered with the set inclusion, then we can see that properties (i), (iii) and (iv) from Proposition 0.1.9 (called extensivity, idempotency and order preservance, respectively) make sense in any partially ordered set. Therefore the set closure is used as the prototype to define the so called *closure operations*.
The proof of next Proposition follows easily having in mind the definitions we have introduced so far.

Proposition 0.1.10. *Let S be a subset of a topological space X. Then, the following conditions are equivalent:*
(i) $\mathrm{Ad}(S) = X$.
(ii) S intersects each nonempty open set.
(iii) There exist a basis $\mathcal{B}$ for the topology on X such that S intersects each nonempty open set of $\mathcal{B}$.

A subset of a topological space X which satisfies any of the equivalent conditions listed

in Proposition 0.1.10 is called *dense*. As a consequence of condition (iv) from Proposition 0.1.9 we have that if $S \subseteq T \subseteq X$ and S is dense in X, then also T is dense in X.
Let (X, τ) be a topological space and let S be a nonempty subset of X. If we consider the family τ_S of subsets of S defined as follows

$$\tau_S := \{S \cap A : A \in \tau\}$$

it is easy to see that τ_S is a topology on S. We will call τ_S *the topology induced on S by X* (or *subspace topology*) and (S, τ_S) will be called *topological subspace*.
The following Proposition is easily proved and shows that most of the topological notions, that we have introduced so far, descend to a subspace of X just by intersection.

Proposition 0.1.11. *Let X be a topological space and S be a subspace of X. Then,*
(i) A is open in S if and only if $A = S \cap G$, where G is open in X.
(ii) F is closed in S if and only if $F = S \cap K$, where K is closed in X.
(iii) If $\mathcal{B}$ is a basis of the topology on X, then the family $\mathcal{B}_S := \{B \cap S : B \in \mathcal{B}\}$ is a basis of the topology on S.
(iv) If $\mathcal{S}$ is a subbasis of the topology on X, then the family $\mathcal{S}_S := \{A \cap S : A \in \mathcal{S}\}$ is a subbasis of the topology on S.
(v) Let $x \in S$; a subset U of S is a neighborhood of x in S if and only if there exists a neighborhood V of x in X such that $U = V \cap S$.
(vi) If W is a subset of S and we denote by $\mathrm{Ad}_S(W)$ the closure of W in S and by $\mathrm{Ad}_X(W)$ the closure of W in X, then $\mathrm{Ad}_S(W) = \mathrm{Ad}_X(W) \cap S$.

0.1.2 Separation properties

A common way to distinguish different classes of topological spaces is through the so called *separation axioms*.

Definition 0.1.12. Let X be a topological space. We say that X is a T_0 space if for any $x, y \in X, x \neq y$, there exist U, V open subsets of X such that $x \in U$, $y \notin U$ or $y \in V$, $x \notin V$.
We say that X is a T_1 space if for any $x, y \in X, x \neq y$, there exist U, V open subsets of X such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.
We say that X is an Hausdorff space (or a T_2 space) if for any $x, y \in X, x \neq y$, there exist U, V open subsets of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

It is immediate to see that an Hausdorff space is also a T_1 space and that a T_1 space is also a T_0 space. Thus we have the chain of implications $\mathrm{T}_2 \Longrightarrow \mathrm{T}_1 \Longrightarrow \mathrm{T}_0$.
The following Proposition shows that satisfying any of the aforementioned separation axioms is an hereditary property. Its proof follows immediately recalling the definition of subspace topology (see also Proposition 0.1.11).

Proposition 0.1.13. *Let X be a T_i space, with $i \in \{0, 1, 2\}$. Then every subspace of X is a T_i space too.*

We provide next two alternative characterizations of T_0 and T_1 spaces. A proof of the result for T_1 spaces can be found in [30, Theorem 13.4].

Lemma 0.1.14. *Let X be a topological space. Then, X is a T_0 space if and only if for any x,$y \in X$, $\mathrm{Ad}(\{x\}) = \mathrm{Ad}(\{y\})$ implies $x = y$.*

Proof. Let us suppose that for any $x, y \in X$, $\mathrm{Ad}(\{x\}) = \mathrm{Ad}(\{y\})$ implies $x = y$. Let x, y be distinct points of X; then $x \notin \mathrm{Ad}(\{y\})$ or $y \notin \mathrm{Ad}(\{x\})$. Without loss of generality, we can suppose that $x \notin \mathrm{Ad}(\{y\})$. Then $X \setminus \mathrm{Ad}(\{y\})$ is an open subset of X which contains x and not y. This proves that X is a T_0 space.
Conversely, let X be a T_0 space and let $x, y \in X$ such that $\mathrm{Ad}(\{x\}) = \mathrm{Ad}(\{y\})$. If $x \neq y$ then there exists U open subset of X such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. In any case $X \setminus U$ is a closed subset of X that contains just one betweem x and y; this implies that $\mathrm{Ad}(\{x\}) \neq \mathrm{Ad}(\{y\})$, a contradiction of our hypothesis. □

Lemma 0.1.15. *The following are equivalent, for a topological space X*
(i) X is a T_1 space.
(ii) Every singleton of X is closed.
(iii) Each subset of X is the intersection of the open sets containing it.

Remark 0.1.16. In light of Lemma 0.1.14 we can see that if X is a topological space which satisfies the T_0 axiom, then the topology on X induces a natural partial order defined by the law:

$$x \leq y \Longleftrightarrow x \in \mathrm{Ad}(\{y\})$$

for any $x, y \in X$. Note that it follows by Lemma 0.1.15 that if X is a T_1 space then the order is trivial.

Example 0.1.2. (a) If a set X with at least two points is endowed with the indiscrete topology, then X is not T_0.
(b) Let X be an infinite set and let us endow X with a topology whose closed sets are the finite sets and X (this topology is usually called *cofinite topology*). Since singletons are closed, it follows that X is a T_1 space. However, any two nonempty open sets of X intersect, thus X cannot be Hausdorff.

0.1.3 Continuous functions

Definition 0.1.17. Let X, Y be topological spaces. A function $f\colon X \to Y$ is called continuous at the point $x \in X$ if for each neighborhood N of $f(x) \in Y$ there exists a neighborhood M of x such that $f(M) \subseteq N$. The function f is called *continuous on* X if it is continuous at every point of X.

The following Proposition presents a well known characterization of continuous functions (see [30, Theorem 7.2] for the proof).

Proposition 0.1.18. *Let $f\colon X \to Y$ be a function between topological spaces. Then, the following conditions are equivalent:*
(i) f is continuous.
(ii) $f^{-1}(A)$ is open in X for each open subset A of Y.
(iii) $f^{-1}(C)$ is closed in X for each closed subset C of Y.

Next definition introduces one of the most crucial notions from General Topology.

Definition 0.1.19. Let X, Y be topological spaces. A function $f\colon X \to Y$ is called *an homeomorphism* if f is continuous, bijective and $f^{-1}\colon Y \to X$ is continuous. Two topological spaces are called *homeomorphic* if there exists an homeomorphism between them.

Homeomorphisms are important because they preserve topological properties. This means that if a space X satisfies a topological property, then all the spaces homeomorphic to X satisfy that property too. Hence, we can consider homeomorphic spaces as topologically equivalent.
An easy condition we will often use to check whether a continuous bijection between topological spaces is an homeomorphism or not is presented in [30, Theorem 7.9].

Proposition 0.1.20. *Let $f\colon X \to Y$ be a continuous bijection between topological spaces. Then, the following conditions are equivalent* *(i) f is an homeomorphism.*

(ii) if $A \subseteq X$, then $f(A)$ is open in Y if and only if A is open in X.

(iii) if $F \subseteq X$, then $f(F)$ is closed in Y if and only if F is closed in X.

Remark 0.1.21. If $f \colon X \to Y$ is an homeomorphism between T_0 spaces and we denote by $\leq_X$ and $\leq_Y$ the orders induced by the topologies on X and Y respectively as in Remark 0.1.16, then f preserves ordered couples, in the sense that if x and y are elements of X such that $x \leq_X y$, then $f(x) \leq_Y f(y)$. As a matter of fact it is enough to note that by Proposition 0.1.20 we have $f(\mathrm{Ad}(\{y\})) = \mathrm{Ad}(\{f(y)\})$.

In our work we will often deal with functions that are not necessarily bijective but are still very close to be homeomorphisms.

Definition 0.1.22. Let X, Y be topological spaces. A function $f \colon X \to Y$ is called *a topological embedding* if the function $f' \colon X \to f(X)$ defined by $f'(x) := f(x)$ for each $x \in X$ is an homeomorphism.

Example 0.1.3. (a) Every inclusion map of a subspace into the space which contains it is clearly a topological embedding.
(b) If τ_1 and τ_2 are two different topologies on the nonempty set X such that $\tau_1 \lneq \tau_2$, then the identity map $(X, \tau_2) \to (X, \tau_1)$ is a continuous bijection which is not a topological embedding.

0.1.4 Compactness

Definition 0.1.23. Let X be a set. A collection $\{A_i\}_{i \in I}$ of subsets of X is called *cover of* X if $X = \bigcup_{i \in I} A_i$.

Definition 0.1.24. A topological space X is called *compact* if every open cover of X admits a finite subcover, i.e., if every family of open sets which is a cover of X admits a finite subfamily which is still a cover of X. A subset K of the space X is called compact if it is a compact space with respect to the subspace topology.

Remark 0.1.25. Let X be a topological space and let τ, τ' be two different topologies on X. If τ' is finer than τ and X is a compact space with respect to the topology τ', then X is clearly also compact with respect to the topology τ, since every open cover of (X, τ) is in particular also an open cover of (X, τ').

Some easy examples of compact and non compact spaces are listed below.

Example 0.1.4. (a) Every finite space is compact.

(b) Every space with the cofinite topology is compact. To see this, let X be endowed with the cofinite topology and let $\{A_i\}_{i\in I}$ be an open cover of X. If we fix $\bar{i} \in I$, then the complement in X of $A_{\bar{i}}$ is a finite subset $\{x_1, \dots, x_n\}$ of X. For each $k = 1, \dots, n$ there exists i_k such that $x_k \in A_{i_k}$ and thus $\{A_{\bar{i}}\} \cup \{A_{i_k} : k = 1, \dots, n\}$ is a finite subcover of X.
(c) Every infinite discrete space X is non compact because the open cover $\{\{x\}\}_{x\in X}$ does not admit any finite subcover.

Definition 0.1.26. We say that a family $\mathcal{F}$ of subsets of a fixed set X has *the finite intersection property* if the intersection of any finite subfamily of elements of $\mathcal{F}$ is nonempty.

Last definition allows us to find a new characterization of compact spaces.

Lemma 0.1.27. *A topological space X is compact if and only if each family $\mathcal{C}$ of closed subsets of X with the finite intersection property has nonempty intersection.*

Proof. Suppose that X is compact and let $\{C_\alpha\}_{\alpha\in A}$ be a family of closed sets of X with the finite intersection property and such that $\bigcap_{\alpha\in A} C_\alpha = \emptyset$. Then, the family $\{X \setminus C_\alpha\}_{\alpha\in A}$ is an open cover of X. By compactness, there is a finite subcover $\{X \setminus C_{\alpha_i}\}_{i=1}^n$. Thus $\bigcap_{i=1}^n C_{\alpha_i} = \emptyset$, so that $\{C_\alpha\}_{\alpha\in A}$ does not have the finite intersection property.
Conversely, suppose that each family of closed subsets of X with the finite intersection property has nonempty intersection. If $\{E_\alpha\}_{\alpha\in A}$ is an open cover of X, then $\bigcap_{\alpha\in A} X \setminus E_\alpha = \emptyset$. Thus, by hypothesis, there exists a finite subfamily $\{X \setminus E_{\alpha_i}\}_{i=1}^n$ such that $\bigcap_{i=1}^n X \setminus E_{\alpha_i} = \emptyset$. It follows that $\{E_{\alpha_i}\}_{i=1}^n$ is a finite subcover of and so X is compact. □

We have already mentioned that in our paper we will rather define topologies through subbasis. Thus, when we need to check the compactness of a topological space, we will often use the following result.

Proposition 0.1.28 (*Alexander's subbasis Theorem*)**.** *Let X be a topological space with a subbasis $\mathcal{S}$ such that every cover by elements of $\mathcal{S}$ has a finite subcover. Then X is compact.*

Proof. Let us assume by contradiction that X is not compact. Using Zorn's lemma (see [18, Theorem 5.4]) we can find an open cover $\mathcal{U}$ of X which is maximal among the set of open covers of X that have no finite subcovers. This means that if $V \notin \mathcal{U}$, then $\mathcal{U} \cup \{V\}$ has a finite subcover which is clearly of the form $\mathcal{U}_0 \cup \{V\}$ with $\mathcal{U}_0$ finite subset of $\mathcal{U}$. If

$\mathcal{U} \cap \mathcal{S}$ covers X then, by hypothesis, it has a finite subcover; but this would also be a finite subcover of $\mathcal{U}$, which is a contradiction by the choice of $\mathcal{U}$. Thus $\mathcal{U} \cap \mathcal{S}$ doesn't cover X and we can find $x \in X \setminus (\bigcup(\mathcal{U} \cap \mathcal{S}))$. Since $\mathcal{U}$ covers X and $\mathcal{S}$ is a subbasis, we can find $U \in \mathcal{U}$ such that $x \in U$ and $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in \bigcap_{i=1}^n S_i \subseteq U$. Since $x \notin \bigcup(\mathcal{U} \cap \mathcal{S})$, we have that $S_i \notin \mathcal{U}$ for every $i = 1, \ldots, n$. This means that for every $i = 1, \ldots, n$ we can find a finite subset $\mathcal{U}_i$ of $\mathcal{U}$ such that $\mathcal{U}_i \cup \{S_i\}$ covers X. It follows that $(\bigcup_{i=1}^n \mathcal{U}_i) \cup \{U\}$ is a finite subcover of $\mathcal{U}$. From the maximality of $\mathcal{U}$ we can conclude that X is compact. □

Next two Propositions are classic results about compact spaces. For complete proofs the reader can refer to [30, Theorems 17.5 and 17.7] respectively.

Proposition 0.1.29. *(i) Let X be a compact space and K be a closed subset of X. Then, K is compact.*
(ii) Let X be an Hausdorff space and K be a compact subset of X. Then, K is closed.

Proposition 0.1.30. *Let X, Y be topological spaces and let $f\colon X \to Y$ be a continuous function. If K is a compact subset of X, then $f(K)$ is compact in Y.*

Propositions 0.1.29 and 0.1.30 have the following useful consequence.

Corollary 0.1.31. *Let X_1 be a compact space, X_2 be an Hausdorff space and $f\colon X_1 \to X_2$ be a continuous bijection. Then, f is an homeomorphism.*

Proof. Let $g := f^{-1}\colon X_2 \to X_1$ and let us show that g is a continuous mapping. Let C be a closed subset of the compact space X_1. Then, C is compact by Proposition 0.1.29 (i). Since f is continuous it follows by Proposition 0.1.30 that $f(C) = g^{-1}(C)$ is a compact subspace of X_2. Since X_2 is Hausdorff, then $g^{-1}(C)$ is closed by Proposition 0.1.29 (ii). The proof is complete recalling the characterization of continuous functions given in Proposition 0.1.18. □

0.2 Prerequisites from Commutative Algebra

In this section we present an overview of those main notions from Commutative Algebra that will appear more often in our paper.

In the following with the term *ring* we always mean a commutative ring with identity. Moreover, $\mathrm{Spec}(A)$ and $\mathrm{Max}(A)$ denote the set of prime ideals and maximal ideals of the ring A respectively. If A is an integral domain and K is its quotient field, any other ring B such that $A \subseteq B \subseteq K$ is called *overring of A*.

0.2.1 Krull dimension of a ring

Our first goal is introducing the notion of *Krull dimension* which is a useful tool to distinguish among different kinds of rings. In order to get the final definition, we need some preliminary notions.

Definition 0.2.1. Let A be a ring; if $\mathfrak{p}_0, \dots, \mathfrak{p}_n$ are prime ideals of A such that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \subsetneq ... \subsetneq \mathfrak{p}_n$, we say that these prime ideals form a *chain of lenght n.*

Definition 0.2.2. If $\mathfrak{p} \in \mathrm{Spec}(A)$ we define the *height of* $\mathfrak{p}$*,* and we write $\mathrm{ht}(\mathfrak{p})$, to be the supremum of the lenghts of all chains of the form $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$, allowing $\mathrm{ht}(\mathfrak{p})$ to be infinite.

We are now ready to define the *Krull dimension of a ring.*

Definition 0.2.3. Let A be a ring; the *Krull dimension of A,* denoted by dim(A), is the supremum of the lenghts of all chains of prime ideals of A or, equivalently, the supremum of the heights of all the prime ideals of A, allowing the supremum to be infinite.

The equivalence of the two definitions is easy to check.

Lemma 0.2.4. *Let A be an integral domain. Then,* $\dim(A) = 1$ *if and only if every nonzero prime ideal of A is a maximal ideal.*

0.2.2 Localizations of a ring

We recall that a ring is called *local* if it has a unique maximal ideal. One of the most notable classes of local rings is represented by the *localizations of a ring A*. For a complete look on their construction the reader can refer to [3, Chapter 3]. In general, if A is a ring and $\mathfrak{p}$ is a prime ideal of A, we call *localization of A in the prime ideal* $\mathfrak{p}$ the set

$A_{\mathfrak{p}} := \{\frac{a}{b} : a \in A, b \in A \setminus \mathfrak{p}\}$. It is clear that, if A is an integral domain and K is its quotient field, then $A \subseteq A_{\mathfrak{p}} \subsetneq K$, for every $\mathfrak{p}$ non zero prime ideal of A, and $K = A_{(0)}$. Moreover it is well known that $A_{\mathfrak{p}}$ is local and that its maximal ideal is $\mathfrak{p}A_{\mathfrak{p}}$.

0.2.3 Modules and fractional ideals of an integral domain

Definition 0.2.5. Let A be a ring; an A-*module* is an abelian group (M,+) together with an operation, called *scalar multiplication,* $A \times M \to M$, $(a, m) \mapsto a \cdot m$ such that for every $a, b \in A$ and $m, n \in M$ we have:

- $a \cdot (m + n) = a \cdot m + a \cdot n$;
- $(a + b) \cdot m = a \cdot m + b \cdot m$;
- $(ab) \cdot m = a \cdot (b \cdot m)$;
- $1_A \cdot m = m$.

In the following we write am instead of $a \cdot m$, if no confusion can arise.
By the properties of the scalar multiplication, we see that modules are a generalization of vector spaces. In fact, if A is equal to a field K, then the A-module structure coincides with the one of K-vector space.
Some easy examples of modules are reported below.

Example 0.2.1. (a) Every abelian group G is a $\mathbb{Z}$-module with scalar multiplication $\mathbb{Z} \times G \to G$, $(\pm n, g) \mapsto \pm(ng) := \underbrace{\pm(g + \cdots + g)}_{n-times}$ for each $n > 0$ and $0g := 0$.
(b) If A is a ring, A^n is an A-module for any $n \in \mathbb{N}$ with the component-wise scalar multiplication $A \times A^n \to A^n$, $[a, (a_1, \ldots, a_n)] \mapsto (aa_1, \ldots, aa_n)$.
(c) Any ideal of a ring A is an A-module with scalar multiplication equal to the multiplication in A.
(d) If $A \subseteq B$ is an extension of rings, then B is a A-module with the scalar multiplication induced by the multiplication in B.

Definition 0.2.6. Let M be a A-module; a subgroup N of M is called A-*submodule* if for any $a \in A$, $n \in N$ the product $an \in N$.

In case A is an integral domain, among all the A-modules we distinguish some of them that are, in some way, very close to be ideals in A.

Definition 0.2.7. Let A be an integral domain and K be its quotient field. A *fractional ideal of* A is a A-submodule I of K such that $dI \subseteq A$ for some nonzero $d \in A$.

It is clear that I is a fractional ideal of A if and only if $I = d^{-1}J$ where $d \in A \setminus \{0\}$ and $J \subseteq A$ is an ideal of A.
From now on, in contexts where fractional ideals and ordinary ring ideals are both under discussion, the latter will be called *integral ideals*, in order to prevent misunderstandigs.
Easy ways to recognize and to build new fractional ideals are presented in [3], last section of Chapter 9 and are recalled by the following two Propositions.

Proposition 0.2.8. *Let A be an integral domain and K be its quotient field. Then, every finitely generated A-submodule of K is a fractional ideal of A.*

Proposition 0.2.9. *Let I,J be fractional ideals of the integral domain A, K be the quotient field of A and let $(I : J) := \{x \in K : xJ \subseteq I\}$. Then $(I : J)$ is a fractional ideal of A.*

We end this section by briefly mentioning the notion of *invertible ideal.*

Definition 0.2.10. A nonzero fractional ideal I of an integral domain A is called *invertible* if there exists a fractional ideal J of A such that $IJ = A$.

The inverse of a fractional ideal has a well known form.

Proposition 0.2.11. *If a fractional ideal I of an integral domain A is invertible, then its inverse is (A:I). Thus, I is invertible if and only if $I(A : I) = A$.*

A useful characterization of invertible fractional ideals is given in [13, Proposition 16.15].

Proposition 0.2.12. *If I is an invertible fractional ideal of an integral domain A, then I is finitely generated.*

The converse pf Proposition 0.2.12 does not hold (see for example [13, Proposition 16.17]).

0.2.4 Integral dependence

In this section we just want to recall the definitions of *integral element, integral extension* and *integral closure.* These tools are useful to describe some notable algebraic structures that we will discuss in the next section.

Definition 0.2.13. Let B be a ring and A be a subring of B; an element $b \in B$ is called *integral over* A if there exists an integer n and elements $a_i \in A$, $i = 1, \dots, n$, such that

$$b^n + a_1 b^{n-1} + a_2 b^{n-2} + \cdots + a_{n-1} b + a_n = 0.$$

If I is an ideal of a ring B we say that an element $b \in B$ is *integral over* I if there exists a polynomial expression like the one given before with the only exception that this time we require each of the coefficients a_i to be in I^i.

Note that, in case both A and B are fields, the notion of integral element coincides with the one of algebraic element since the leading coefficient of any nonzero polynomial over a field can always be scaled to 1.

Definition 0.2.14. Let $A \subseteq B$ be a ring extension; the set of elements of B that are integral over A is called the *integral closure of* A *in* B and is denoted by $\overline{A}^B$. If $\overline{A}^B = B$, then we say that B is *integral over* A and that $A \subseteq B$ is an *integral extension.* If $\overline{A}^B = A$, then we say that A is *integrally closed in* B. A similar notation will be used for the integral closure of ideals.

In the following, if we say that a domain A is integrally closed without specifying the extension we are considering, we always refer to the extension $A \subseteq K$, where K is the quotient field of A. Some notable results about integral closures will be presented when we introduce valuation domains. Here we just want to report the classic characterization of integral elements (see [3, Proposition 5.1] for the proof).

Theorem 0.2.15. *Let $A \subseteq B$ be a ring extension and x an element of B. Then, the following statements are equivalent:*
(i) x is integral over A.
(ii) $A[x]$ is a finitely generated A-module.
(iii) There exists a ring C such that $A[x] \subseteq C \subseteq B$ and C is a finitely generated A-module.
(iv) There exists an $A[x]$ faithful module M (i.e. $Ann_{A[x]}(M) := \{f \in A[x] : fM = 0\} = \{0\}$) such that M is a finitely generated A-module.

0.2.5 Main algebraic structures

We are now ready to introduce some of the main algebraic structures that will often recur in our examples.

0.2.5.1 Noetherian rings

Definition 0.2.16. A ring A is called *Noetherian* if every ideal of A is finitely generated.

The following Proposition introduces a well known characterization of Noetherian rings (see [3, Propositions 6.1 and 6.2] for the proof).

Proposition 0.2.17. *Let A be a ring. Then the following statements are equivalent:*
(i) A is a Noetherian ring.
(ii) A satisfies the ascending chain condition i.e. for any chain $I_1 \subseteq I_2 \subseteq ... \subseteq I_k \subseteq ...$ of ideals of A there exists $n \geq 1$ such that $I_j = I_n$ for each $j \geq n$.
(iii) Every nonempty set of ideals of A has at least one maximal element.

There exists a very complete theory about Noetherian rings which describes the decomposition of their ideals. For an insight on this topic the reader can refer to [3, Chapter 7]. However, this theory results redundant for our task of studying semistar operations. Therefore, we limit ourselves to state three main results on Noetherian rings: the first one suggests us that being Noetherian is a local property; the second one is the famous *Hilbert's basis theorem* which is a fundamental result, also in Algebraic Geometry (see [3, Theorem 7.5] for the proof); the last one is a consequence of *Krull's generalized principal ideal theorem*, whose complete statement can be found in [29, Theorem 15.4].

Proposition 0.2.18. *Let A be a Noetherian ring. Then for any I ideal of A and any $\mathfrak{p} \in \mathrm{Spec}(A)$, both A/I and $A_{\mathfrak{p}}$ are Noetherian rings.*

Theorem 0.2.19 (Hilbert's basis Theorem)**.** *Let A be a Noetherian ring and $x_1, \ldots, x_n$ be indeterminates over A. Then, $A[x_1, \ldots, x_n]$ is a Noetherian ring.*

Proposition 0.2.20 ([29], Corollary 15.5 (i))**.** *Let A be a local Noetherian ring. Then, A has finite dimension.*

0.2.5.2 Valuation domains

In [14, §18] the author constructs *valuation domains* as domains associated to special functions, called *valuations,* that satisfy some specific properties. However, we prefer to give a more direct, ring-theoretic definition.

Definition 0.2.21. Let A be a domain and K be its quotient field; then A is called *valuation domain* if, for every nonzero $x \in K$, we have that either $x \in A$ or $x^{-1} \in A$.

Some notable results about valuation domains are collected in [3, Chapter V]. We report the most important ones next.

Proposition 0.2.22. *Let A be a valuation domain and K be its quotient field. Then, the following statements hold:*
(i) If B is a domain and $A \subseteq B \subseteq K$, then B is a valuation domain.
(ii) A is a local domain.
(iii) A is an integrally closed domain.

Proposition 0.2.23. *Let A be an integral domain and K be its quotient field. Then, the following statements are equivalent:*
(i) A is a valuation domain.
(ii) The principal ideals of A are totally ordered by inclusion.
(iii) All ideals of A are totally ordered by inclusion.

One of the most outstanding results about valuation domains is *Krull's Theorem*; its proof requires few technical prerequisites so we suggest the reader to refer to [3, Theorem 5.21 and Corollary 5.22]

Theorem 0.2.24 (Krull's Theorem)**.** *Let A be a domain. Then $\overline{A} = \bigcap\{V : V$ is a valuation domain, $A \subseteq V \subseteq K\}$*

Krull's Theorem has the following immediate corollary.

Corollary 0.2.25. *If A is an integrally closed domain then $A = \bigcap\{V : V$ is a valuation domain, $A \subseteq V \subseteq K\}$*

The following Proposition shows that also the integral closure of the modules of a domain A in K, the quozient field of A, has an easy representation through valuation domains (see [17, Section 6.8] for the complete proof).

Proposition 0.2.26. *Let A be an integral domain and let I be an A-submodule of K, the quotient field of A. Then $\overline{I}^K = \bigcap\{IV : V$ is valuation domain, $A \subseteq V \subseteq K\}$.*

0.2.5.3 DVRs

Among valuation domains, those that are also Noetherian are of particular interest. Using valuation theory, they can be constructed as rings associated to a specific class of valuation maps, called *discrete valuations*. However, as we have done for valuation domains, we prefer to characterize them through their algebraic properties referring to [13, Proposition 3.2].

Theorem 0.2.27. *Let A be a ring. Then, the following statements are equivalent:*

(i) A is a Noetherian valuation domain.

(ii) A is a valuation domain and every ideal of A is principal.

(iii) A is a local domain and every ideal of A is principal.

(iv) A is a local domain, $dim(A) = 1$ and the maximal ideal of A is principal.

(v) There exists $t \in A$ such that, for any nonzero element $x \in A$, we have $x = ut^n$, where u is invertible in A and $n \in \mathbb{N}$ are uniquely determined.

(vi) There exists $t \in A$ such that, for any nonzero ideal I of A, $I = (t^n)$, where $n \in \mathbb{N}$.

(vii) A is a local domain and, if $\mathfrak{m}$ is its maximal ideal, the set of all nonzero ideals of A corresponds to the set $\{\mathfrak{m}^n : n \in \mathbb{N}\}$.

(viii) A is a local, Noetherian domain, it is integrally closed and $dim(A) = 1$.

Definition 0.2.28. A ring A which satisfies any of the conditions listed above is called *discrete valuation ring (DVR).* Moreover, the element t that appears in (v)-(vi) is called *uniformizing parameter.*

By Theorem 0.2.27 it follows that if A is a DVR and $t \in A$ is a uniformizing parameter, then the only maximal ideal of A is (t) and every other ideal of A is of the form (t^n).

0.2.5.4 Dedekind domains

Definition 0.2.29. A ring A is called *Dedekind domain* if it is a Noetherian integrally closed domain and $\dim(A) = 1$.

Having in mind property (viii) from Theorem 0.2.27, we obtain the following result.

Observation 0.2.30. *Let A be an integral domain. Then, A is a DVR if and only if A is a local Dedekind domain.*

Sometimes it easy to characterize Dedekind domains through the properties of their localizations.

Proposition 0.2.31 ([13], Theorem 13.7)**.** *Let A be an integral domain. Then, A is a Dedekind domain if and only if A is a Noetherian ring and $A_{\mathfrak{m}}$ is a DVR for any $\mathfrak{m} \in$* $\mathrm{Max}(A)$.

0.2.5.5 Prüfer domains

Definition 0.2.32. An integral domain A is called *Prüfer domain* if $A_{\mathfrak{m}}$ is a valuation ring for any $\mathfrak{m} \in \mathrm{Max}(A)$.

The relation between Prüfer and Dedekind domains follows easily by Propositions 0.2.31 and 0.2.18.

Proposition 0.2.33. *Let A be an integral domain. Then, A is a Dedekind domain if and only if A is a Noetherian, Prüfer domain.*

Moreover, next Proposition describes Prüfer domains using their invertible ideals (see [13, Theorem 14.1] for the complete proof). In the next Chapter we will need this result to characterize Prüfer domains through a particular semistar operation that will be introduced later (see Example 1.0.1 (d)).

Theorem 0.2.34. *Let A be a ring. Then the following conditions are equivalent:*
(i) A is a Prüfer domain.
(ii) $A_{\mathfrak{p}}$ is a valuation ring for each prime ideal $\mathfrak{p}$ of A.
(iii) Every nonzero finitely generated ideal of A is invertible.

0.2.6 The Zariski topology on the prime spectrum of a ring

In our paper we will refer with the same term, *Zariski topology,* to different topologies on different sets: the prime spectrum of a ring, the overrings of a given ring, the local overrings of a given ring and, of course, the semistar operations on an integral domain. Among these topologies, the *Zariski topology on the prime spectrum of a ring* is probably the most famous since it is a central notion in Commutative Algebra and Algebraic Geometry. Therefore, in this section, we want to recall its definition and some of its properties.
First of all, recall once again that the *prime spectrum* of a ring A is the collection of all the prime ideals of A. For each $\mathfrak{a}$ ideal of A, define $V(\mathfrak{a}) := \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\}$. Next Proposition lists some properties of these sets; its proof follows by basic set inclusions considerations.

Proposition 0.2.35. *Preserve the notation given before. Then:*
(i) $V((0)) = \mathrm{Spec}(A)$.
(ii) $V((1)) = \emptyset$.
(iii) If $\mathfrak{a}_1 \subseteq \mathfrak{a}_2$ *then* $V(\mathfrak{a}_2) \subseteq V(\mathfrak{a}_1)$.
(iv) $\bigcap_{i \in I} V(\mathfrak{a}_i) = V(\sum_{i \in I} \mathfrak{a}_i)$.
(v) $V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2) = V(\mathfrak{a}_1\mathfrak{a}_2)$.

It follows by properties (i), (ii), (iv) and (v) that the family $\{V(\mathfrak{a}) : \mathfrak{a} \text{ is an ideal of } A\}$ is the collection of the closed sets of a topology on $\mathrm{Spec}(A)$. As we have already mentioned before, we will call this topology *Zariski topology.* If we now set $D(\mathfrak{a}) := \operatorname{Spec}(A) \setminus V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathfrak{a} \not\subseteq \mathfrak{p}\}$, for any $\mathfrak{a}$ ideal of A, then it is clear that $D(\mathfrak{a})$ is an open set in the Zariski topology. We are just one step away from finding a basis for the Zariski topology and the next Proposition will show us how to do it.

Proposition 0.2.36. *Preserve the notation given before and set $D(f) := D((f))$, for every $f \in A$. Then:*
(i) $D(f) = \emptyset$ if and only if there exists $n \in \mathbb{N}$ such that $f^n = 0$.
(ii) $D(f) = \operatorname{Spec}(A)$ if and only if f is invertible in A.
(iii) $D(f) \cap D(g) = D(fg)$.
(iv) $D(\mathfrak{a}) = \bigcup_{f \in \mathfrak{a}} D(f)$.

Corollary 0.2.37. *The collection $\{D(f) : f \in A\}$ is a basis for the Zariski topology on $\operatorname{Spec}(A)$.*

We now focus on the topological properties of the Zariski topology. Let us start with compactness.

Proposition 0.2.38. *Let $\operatorname{Spec}(A)$ be endowed with the Zariski topology. Then:*
(i) $\operatorname{Spec}(A)$ is compact.
(ii) $D(f)$ is compact for any $f \in A$.
(iii) An open set U of $\operatorname{Spec}(A)$ is compact if and only if $U = D(f_1) \cup D(f_2) \cup \cdots \cup D(f_n)$, with $f_1, \ldots, f_n \in A$.

Proof. (i) Let $\mathcal{U}$ be an open cover of $\operatorname{Spec}(A)$. Without loss of generality we can suppose that all the elements of $\mathcal{U}$ are basic open sets. Thus, let $\operatorname{Spec}(A) = \bigcup_{f \in E} D(f)$, for some subset E of the ring A. Then it follows by Proposition 0.2.35 (v) that $\emptyset = \bigcap_{f \in E} V(f) = V((E))$. Thus it must be $1 \in (E)$ and so we can find $a_1, \ldots, a_n \in A$, $f_1, \ldots, f_n \in E$ such that $a_1 f_1 + \cdots + a_n f_n = 1$. Therefore $1 \in (f_1, \ldots, f_n)$ and it follows again by Proposition 0.2.35 that $\emptyset = V((f_1, \ldots, f_n)) = \bigcap_{i=1}^n V(f_i)$. Thus $\operatorname{Spec}(A) = \bigcup_{i=1}^n D(f_i)$ and so $\operatorname{Spec}(A)$ is compact.
(ii) Let $\mathcal{U}$ be an open cover of $D(f)$. Like we did in the previous point, we can suppose that all the elements of $\mathcal{U}$ are basic open sets of $D(f)$. Thus, let $D(f) = \bigcup_{g \in E}(D(g) \cap D(f))$ for some subset E of the ring A.. By Proposition 0.2.36 (iii) we have that $D(f) = \bigcup_{g \in E} D(fg) = \bigcup_{h \in E'} D(h)$, where E' is a suitable subset of (f). Using the set complement on both sides and recalling Proposition 0.2.35 (v) we get $V(f) = \bigcap_{h \in E'} V(h) = V((E'))$.

Thus we have in particular that $(E') \in V(f)$ and so we can find $a_1, \dots, a_n \in A$, $h_1, \dots, h_n \in E'$ such that $f = a_1h_1 + \cdots + a_nh_n$. Therefore $(f) \subseteq (h_1, \dots, h_n)$; if we recall that E' was a subset of (f) we can also see that the inclusion $(h_1, \dots, h_n) \subseteq (f)$ holds. Then it follows by Proposition 0.2.35 (v) that $V(f) = V((h_1, \dots, h_n)) = \bigcap_{i=1}^{n} V(h_i)$. Using again set complement to go back to the open sets, we finally obtain $D(f) = \bigcup_{i=1}^{n} D(h_i)$. The proof is now complete.

(iii) It follows easily by the previous point. $\square$

Next Proposition provides an easy way to find the closure of a set in the Zariski topology.

Proposition 0.2.39. *Let* Spec*(A) be endowed with the Zariski topology,* $Z \subseteq$ Spec*(A) and denote by* Ad*(Z) the closure of Z. Then* Ad*(Z)=* $V(\bigcap_{\mathfrak{p} \in Z} \mathfrak{p})$.

Proof. For each $\mathfrak{q}$ in Z we clearly have that $\bigcap_{\mathfrak{p} \in Z} \mathfrak{p} \subseteq \mathfrak{q}$. It follows by Proposition 0.2.35 (iii) that $\mathfrak{q} \in V(\mathfrak{q}) \subseteq V(\bigcap_{\mathfrak{p} \in Z} \mathfrak{p})$. Thus $Z \subseteq V(\bigcap_{\mathfrak{p} \in Z} \mathfrak{p})$. To complete the proof we need to show that if $V(I)$ is a closed set of $\mathrm{Spec}(A)$ which contains Z, then $V(\bigcap_{\mathfrak{p} \in Z} \mathfrak{p}) \subseteq V(I)$. Again by Proposition 0.2.35 (iii), it suffices to show that $I \subseteq \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$; for each $\mathfrak{q}$ in Z we have that $\mathfrak{q} \in V(I)$ and thus $I \subseteq \mathfrak{q}$. It follows that $I \subseteq \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$ as required. $\square$

Remark 0.2.40. The Zariski topology on $\mathrm{Spec}(A)$ in general is not T_1. In fact it follows by Proposition 0.2.39 that for each prime ideal $\mathfrak{p}$ of A we have $\mathrm{Ad}(\{\mathfrak{p}\}) = V(\mathfrak{p})$ and $\mathrm{Ad}(\{\mathfrak{p}\}) = \{\mathfrak{p}\}$ if and only if $\mathfrak{p}$ is a maximal ideal of A. Thus the only closed singletons of $\mathrm{Spec}(A)$ are the maximal ideals and it follows by Lemma 0.1.15 that $\mathrm{Spec}(A)$ is not a T_1 space.

Remark 0.2.40 suggests that the Zariski topology on $\mathrm{Spec}(A)$ is not Hausdorff. Indeed, it is possible to show that $\mathrm{Spec}(A)$ is Hausdorff if and only if it is T_1 if and only if $\dim(A) = 0$ (see [3, Exrecise 3.11] for an hint on how to prove this). However, we show in the next Lemma that $\mathrm{Spec}(A)$ is always a T_0 space.

Lemma 0.2.41. Spec*(A) is a* T_0 *space i.e. for any* $\mathfrak{p}, \mathfrak{q} \in$ Spec*(A),* $\mathfrak{p} \neq \mathfrak{q}$*, there exists* $f \in A$ *such that* $\mathfrak{p} \in D(f)$ *and* $\mathfrak{q} \notin D(f)$.

Proof. Since $\mathfrak{p} \neq \mathfrak{q}$ then there exists $f \in \mathfrak{q} \setminus \mathfrak{p}$ (or vice versa). Then $D(f)$ is the required open set of $\mathrm{Spec}(A)$. $\square$

Remark 0.2.42. Since $\mathrm{Spec}(A)$ is a T_0 space we can consider the partial order relation $\leq$ we introduced in Remark 0.1.16. By Proposition 0.2.39 it follows easily that this order is exactly the reverse set inclusion, meaning that for any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ we have $\mathfrak{p} \leq \mathfrak{q}$ if and only if $\mathfrak{p} \supseteq \mathfrak{q}$.

0.2.7 Spectral spaces

Definition 0.2.43. A topological space X is called *spectral* if there exists a ring A such that X is homeomorphic to $\mathrm{Spec}(A)$, endowed with the Zariski topology.

The theory of spectral spaces is rather rich and complex. For a deeper look into this topic the reader can refer to [16]. For our task we just want to recall the characterization of spectral spaces that the author gives in [16, Proposition 4]. Since it includes the notions of *irreducible set* and *generic point,* we first recall these two definitions.

Definition 0.2.44. Let X be a topological space; a subset F of X is called *irreducible* if $F \subseteq C_1 \cup C_2$, with C_1, C_2 closed subsets of X, implies $F \subseteq C_1$ or $F \subseteq C_2$.

Definition 0.2.45. Let X be a topological space; a point $P \in X$ is called *generic point* if $\mathrm{Ad}(P) = X$.

Theorem 0.2.46 ([16], Proposition 4)**.** *A topological space X is spectral if and only if it satisfies the following conditions:*
(i) X is compact and T_0.
(ii) X admits a basis of open and compact subspaces that is closed under intersection.
(iii) Every irreducible closed subset of X has a unique generic point.

Chapter 1

The Zariski topology on the set of all semistar operations

In this chapter we introduce a "natural" topology on the set of all semistar operations on an integral domain and we investigate the relations between the algebraic properties of semistar operations and the topological properties of this new space. The results we give represent a recent contribution to the theory of semistar operations and they have been first proved by C.A. Finocchiaro and D. Spirito in [9].
We begin with an introductive section which presents the notion of semistar operation and offers an overview of its main features.

1.0 Notation and preliminaries

In the following with the term *ring* we always mean a commutative ring with identity. If A is an integral domain, we will denote by K the quotient field of A. Any ring B such that $A \subseteq B \subseteq K$ will be called *overring of* A. We will denote by $\mathrm{Over}(A)$ the set of all the overrings of A. Moreover, we will use the following additional notation:

- $\mathbf{f}(A)$ will denote the set of all nonzero finitely generated fractional ideals of A.
- $\mathbf{F}(A)$ will denote the set of all nonzero fractional ideals of A.
- $\overline{\mathbf{F}}(A)$ will denote the set of all nonzero A-submodules of K.

Of course, we have $\mathbf{f}(A) \subseteq \mathbf{F}(A) \subseteq \overline{\mathbf{F}}(A)$.

Definition 1.0.1. Let A be a integral domain; a *semistar operation on* A is a function $\star\colon \overline{\mathbf{F}}(A) \to \overline{\mathbf{F}}(A)$, $F \mapsto F^\star$ such that, for any nonzero element $k \in K$ and every $F, G \in \overline{\mathbf{F}}(A)$:

- $(kF)^\star = kF^\star$;

- $F \subseteq G$ implies $F^\star \subseteq G^\star$;

- $F \subseteq F^\star$;

- $(F^\star)^\star = F^\star$.

Recalling what we said at the end of Proposition 0.1.9, we can note that semistar operations are closure operators.

We will denote the set of all semistar operations on A by $\mathrm{SStar}(A)$.

The following Lemma explains how the semistar operations distribute with respect to the sum, the intersection and the product of modules.

Lemma 1.0.2. *Let A be an integral domain and $\star$ be a semistar operation on A. Then for any $E, F \in \overline{\mathbf{F}}(A)$ and any subset $\{E_i\}_{i\in I} \subseteq \overline{\mathbf{F}}(A)$:*
(1) $(E+F)^\star = (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star$
(2) $\bigcap_{i\in I} E_i^\star = (\bigcap_{i\in I} E_i^\star)^\star$, if $\bigcap_{i\in I} E_i^\star \neq (0)$
(3) $(EF)^\star = (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star$.

Proof. (1) $E, F \subseteq (E+F)$ and thus $E^\star, F^\star \subseteq (E+F)^\star$. It follows that $(E^\star+F^\star)^\star \subseteq [(E+F)^\star]^\star = (E+F)^\star$. The converse inclusion is clear and so we have that $(E^\star+F^\star)^\star = (E+F)^\star$. The other equalities follow using the same argument.

(2) $\bigcap_{i\in I} E_i^\star \subseteq E_i^\star$ for each $i \in I$ and thus $(\bigcap_{i\in I} E_i^\star)^\star \subseteq (E_i^\star)^\star = E_i^\star$. It follows that $(\bigcap_{i\in I} E_i^\star)^\star \subseteq \bigcap_{i\in I} E_i^\star$. The converse inclusion is clear and so we have that $\bigcap_{i\in I} E_i^\star = (\bigcap_{i\in I} E_i^\star)^\star$.

(3) We will show that $(EF^\star)^\star \subseteq (EF)^\star$ by proving that $EF^\star \subseteq (EF)^\star$. If $x \in EF^\star$, then there exist $a_1, \dots, a_r \in E$ such that $x \in a_1F^\star + \cdots + a_nF^\star$. By the proof of point (1) we have that:

$$\sum_{i=1}^r a_iF^\star = \sum_{i=1}^r (a_iF)^\star \subseteq (\sum_{i=1}^r a_iF)^\star \subseteq (EF)^\star.$$

It follows that $(EF^\star)^\star \subseteq (EF)^\star$. The converse inclusion is clear and thus we have that $(EF^\star)^\star = (EF)^\star$. The other equalities follow using the same argument. □

As a consequence of Lemma 1.0.2 (3) we have the following result.

Proposition 1.0.3. *Let A be an integral domain and $\star$ be a semistar operation on A. Then:*

(1) Let B be an overring of A. Then $B^\star$ is an overring of A. In particular, $A^\star$ is an overring of A.

(2) If $E \in \overline{\mathbf{F}}(A)$, then $E^\star \in \overline{\mathbf{F}}(A^\star)$.

(3) If $E \in \mathbf{F}(A)$, then $E^\star \in \mathbf{F}(A^\star)$.

Proof. It suffices to prove that each of those sets is closed under multiplication. By Lemma 1.0.2 (3) we have:

(1) $B^\star B^\star \subseteq (B^\star B^\star)^\star = (BB)^\star = B^\star$.

(2) $E^\star A^\star \subseteq (E^\star A^\star)^\star = (EA)^\star = E^\star$.

(3) Since $E \in \mathbf{F}(A)$, there exists a nonzero element $d \in A$ such that $dE \subseteq A$. Then, $dE^\star = (dE)^\star \subseteq A^\star$. Since $d \in A \subseteq A^\star$, we have $E^\star \in \mathbf{F}(A^\star)$. $\square$

The set $\mathrm{SStar}(A)$ is endowed with a natural partial order which is induced by the set inclusion. It is defined in the following way:

$$\star \leq \star' \Leftrightarrow F^\star \subseteq F^{\star'}, \quad \forall F \in \overline{\mathbf{F}}(A).$$

Let $\star$ be a semistar operation on A. We can associate to $\star$ a new semistar operation $\star_f$ on A by setting

$$F^{\star_f} := \bigcup \{G^\star : G \in \mathbf{f}(A), G \subseteq F\},$$

for any $F \in \overline{\mathbf{F}}(A)$. We call $\star_f$ *the semistar operation of finite type associated to* $\star$. We say that $\star$ is *a semistar operation of finite type* if $\star = \star_f$. In the following, we shall denote by $\mathrm{SStar}_f(A)$ the set of the semistar operations of finite type on A. It is immediate to see that $(\star_f)_f = \star_f$, and thus $\star_f$ is a semistar operation of finite type. It is clear that the semistar operations of finite type are completely determined by the image of the elements of $\mathbf{f}(A)$ and thus if $\star$ and $\star'$ are two semistar operations of finite type such that $\star|_{\mathbf{f}(A)} = \star'|_{\mathbf{f}(A)}$, then it must be $\star = \star'$. Moreover, since semistar operations preserve set inclusions, we have that $\star_f \leq \star$, for any semistar operation $\star$ on A, and by the definition of the order on $\mathrm{SStar}(A)$ it follows that $\star_1 \leq \star_2$ implies $(\star_1)_f \leq (\star_2)_f$.

If $\star \in \mathrm{SStar}(A)$ and $F \in \overline{\mathbf{F}}(A)$ is such that $F = F^\star$, then F is called $\star$-*closed*. If A is $\star$-closed, then $\star$ is called a *(semi)star operation*, while $\star|_{\mathbf{F}(A)}$ is called a *star operation*. For background on star operations, see [14].

An integral ideal $\mathfrak{a}$ of A is called a *quasi-$\star$-ideal of* A if either $\mathfrak{a} = (0)$ or $\mathfrak{a}^\star \cap A = \mathfrak{a}$. It is not hard to see that $\mathfrak{a}^\star \cap A$ is a quasi-$\star$-ideal of A. In fact, $(\mathfrak{a}^\star \cap A)^\star \cap A \subseteq (\mathfrak{a}^\star)^\star \cap A^\star \cap A = \mathfrak{a}^\star \cap A$ and the converse inclusion is trivial.

By the definition of the order $\leq$, it is clear that $\star \leq \star'$ if and only if every nonzero $\star'$-closed A-submodule of K is also $\star$-closed. Moreover, if $\star \leq \star'$, then every quasi-$\star'$-ideal of A is also a quasi-$\star$-ideal. In fact for any ideal $\mathfrak{a}$ of A we have that $\mathfrak{a} \subseteq \mathfrak{a}^\star \subseteq \mathfrak{a}^{\star'}$ and then $\mathfrak{a} \subseteq \mathfrak{a}^\star \cap A \subseteq \mathfrak{a}^{\star'} \cap A = \mathfrak{a}$. However the fact that every quasi-$\star'$-ideal of A is also a quasi-$\star$-ideal is not sufficient to conclude that $\star \leq \star'$ because no information is given on the image of the A-submodules of K that are not integral ideals of A.

A prime ideal of A that is also a quasi-$\star$-ideal is called a *quasi-$\star$-prime ideal of A*. A maximal element in the set of all the proper quasi-$\star$-ideals is called *quasi-$\star$-maximal ideal of A*. We will use the following notation:

- $\mathrm{QSpec}^\star(A)$ will denote the set of all the quasi-$\star$-prime ideals of A.
- $\mathrm{QMax}^\star(A)$ will denote the set of all the quasi-$\star$-maximal ideals of A.

The following Proposition presents two important results about quasi-$\star$-maximal ideals.

Proposition 1.0.4. *Let $\star$ be a semistar operation on the integral domain A. Then,*
(1) $\mathrm{QMax}^\star(A) \subseteq \mathrm{QSpec}^\star(A)$.
(2) If $\star$ is a semistar operation of finite type, then each proper quasi-$\star$-ideal is contained in a quasi-$\star$-maximal ideal of A.

Proof. (1) Let $\mathfrak{m}$ be a quasi-$\star$-maximal ideal of A and let x, y be elements of A such that $xy \in \mathfrak{m}$ and $x \notin \mathfrak{m}$. Let us consider the ideal $(\mathfrak{m}, x)$. We have that $(\mathfrak{m}, x)^\star = A^\star$, otherwise $(\mathfrak{m}, x)^\star \cap A$ would be a quasi-$\star$-ideal of A which strictly contains $\mathfrak{m}$. Let us now consider the ideal $y(\mathfrak{m}, x) = (y\mathfrak{m}, yx) \subseteq \mathfrak{m}$. Then we have that $y \in yA^\star \cap A = y(\mathfrak{m}, x)^\star \cap A = (y\mathfrak{m}, yx)^\star \cap A \subseteq \mathfrak{m}^\star \cap A = \mathfrak{m}$. Hence $y \in \mathfrak{m}$ and so $\mathfrak{m}$ is a prime ideal.

(2) Let I be a proper quasi-$\star$-ideal of A and let $\{I_k\}_{k\in K}$ be a chain of proper quasi-$\star$-ideals of A which contain I. Then we have that $\bigcup_{k\in K} I_k^\star \subseteq (\bigcup_{k\in K} I_k)^\star$. On the other hand, since $\star$ is of finite type, if $x \in (\bigcup_{k\in K} I_k)^\star$, then there exists a finitely generated ideal J of A such that $x \in J^\star$ and $J \subseteq \bigcup_{k\in K} I_k$. Since $\{I_k\}_{k\in K}$ is a chain, there exists $\overline{k} \in K$ such that $J \subseteq I_{\overline{k}}$ and thus $x \in J^\star \subseteq I_{\overline{k}}^\star \subseteq \bigcup_{k\in K} I_k^\star$. Therefore we finally have that $(\bigcup_{k\in K} I_k)^\star \cap A = (\bigcup_{k\in K} I_k^\star) \cap A = \bigcup_{k\in K}(I_k^\star \cap A) = \bigcup_{k\in K} I_k$. This means that $\bigcup_{k\in K} I_k$ is a quasi-$\star$-ideal of A which contains I. The conclusion follows from a straightforward application of Zorn's lemma (see [18, Theorem 5.4]). □

We provide next a list of examples of some useful semistar operations and several techniques to construct specific classes of semistar operations and new semistar operations from old ones.

Example 1.0.1. (a) We denote by $d := d_A$ the identity semistar operation on A. It is easy to see that d is the minimum of the partially ordered set $(\mathrm{SStar}(A), \leq)$.

(b) If B is an overring of A, we denote by $\star_{\{B\}}$ the semistar operation on A defined by setting $F^{\star_{\{B\}}} := FB$, for any $F \in \overline{\mathbf{F}}(A)$. It is easy to see that $\star_{\{B\}}$ is of finite type (see Section 1.2 for a direct argument).

(c) We denote by v the *divisorial semistar operation on A,* defined by $F^v := (A : (A : F))$, for any $F \in \overline{\mathbf{F}}(A)$. Recalling Proposition 0.2.11, we can also write $F^v := (F^{-1})^{-1}$. It is shown in [26, Lemma 1.11] that v (resp. $v|_{\mathbf{F}(A)}$) is the biggest (semi)star operation (resp. star operation). The finite type semistar operation associated to v is usually denoted by t. It follows easily by what we have said before that t (resp. $t|_{\mathbf{F}(A)}$) is the biggest (semi)star operation (resp. star operation) of finite type. Furthermore, if A is integrally closed, then it is possible to prove, using the characterization of Prüfer domains we gave in Theorem 0.2.34, that A is a Prüfer domain if and only if $t|_{\mathbf{F}(A)}$ is the identity (see also [14, Proposition 34.12]).

(d) Let $\mathcal{S}$ be a nonempty collection of semistar operations on A. Then $\bigwedge(\mathcal{S})$ is the semistar operation on A defined by setting

$$F^{\bigwedge(\mathcal{S})} := \bigcap\{F^\star : \star \in \mathcal{S}\}, \quad \forall F \in \overline{\mathbf{F}}(A)$$

It is easy to see that $\bigwedge(\mathcal{S})$ is the infimum of $\mathcal{S}$ in the partially ordered set $(\mathrm{SStar}(A),\leq)$. Moreover, the semistar operation

$$\bigvee(\mathcal{S}) := \bigwedge(\{\sigma \in \mathrm{SStar(A)} : \sigma \geq \star, \forall \star \in \mathcal{S}\})$$

is the supremum of $\mathcal{S}$ in the partially ordered set $(\mathrm{SStar}(A),\leq)$.

(e) Let Y be a nonempty collection of overrings of A. By (e), $\wedge_Y := \bigwedge(\{\star_{\{B\}} : B \in Y\})$ is a semistar operation on A. In other words, the semistar operation $\wedge_Y$ is defined by setting

$$F^{\wedge_Y} := \bigcap\{FB : B \in Y\}, \quad \forall F \in \overline{\mathbf{F}}(A).$$

(f) If Y is a collection of valuation overrings of A, then $\wedge_Y$ is called a *valutative semistar operation.* In particular, when Y is the set of all the valuation overrings of A, $\wedge_Y$ is called *the b-semistar operation* (or *integral closure*) on A.

(g) Let X be a nonempty collection of prime ideals of A. The semistar operation $s_X := \wedge_{\{A_{\mathfrak{p}}:\mathfrak{p}\in X\}}$ is called a *spectral semistar operation.*

(h) We say that a semistar operation is *stable* if $(F \cap G)^\star = F^\star \cap G^\star$ for any $F, G \in \overline{\mathbf{F}}(A)$.

1.1 Construction and main properties

Now that we have set the main tools we will use, we are able to start constructing our new topology on the set of all semistar operations. In the following, as usual, A will denote an integral domain and K will be the quotient field of A.

Definition 1.1.1. The *Zariski topology on* $\mathrm{SStar}(A)$ is the topology which has as a subbasis of open sets the collection of all sets of the form $V_F := V_F^{(A)} := \{\star \in \mathrm{SStar}(A) : 1 \in F^\star\}$, as F ranges among the nonzero A-submodules of K. The *Zariski topology on* $\mathrm{SStar}_f(A)$ is just the subspace topology induced by the Zariski topology on $\mathrm{SStar}(A)$.

In the following Remark we focus on the most basic properties of the Zariski topology on $\mathrm{SStar}(A)$.

Remark 1.1.2. (a) Let us consider the semistar operation $\star_{\{K\}}$ which sends every nonzero submodule H to the quotient field K of A. Since clearly $1 \in K$, it follows that $\star_{\{K\}} \in V_F$ for any F nonzero A-submodule of K; then, by the definition of the Zariski topology on $\mathrm{SStar}(A)$, every open set of $\mathrm{SStar}(A)$ contains $\star_{\{K\}}$. As a result we have that $\star_{\{K\}}$ is a generic point of $\mathrm{SStar}(A)$. Moreover, since $\star_{\{K\}}$ is obviously of finite type, we can conclude that $\mathrm{SStar}_f(A)$ is a dense subspace of $\mathrm{SStar}(A)$.

(b) The identity operation d is contained in V_F if and only if $1 \in F$. Recalling that $F \subseteq F^\star$ for any $\star \in \mathrm{SStar}(A)$, the previous condition is equivalent to say that $V_F = \mathrm{SStar}(A)$. Hence every nonempty closed set contains d; therefore, if $\star \in \mathrm{SStar}(A) \setminus \{d\}$, then $\{\star\}$ is not closed. We will see that $\{d\}$ is closed in $\mathrm{SStar}(A)$ in Proposition 1.1.3 where we will describe the closure of the singletons of $\mathrm{SStar}(A)$.

(c) The topology of $\mathrm{SStar}(A)$ is naturally linked to the order $\leq$, in the following sense. If $U \subseteq \mathrm{SStar}(A)$ is an open neighborhood of $\star$ and $\star' \geq \star$, then U is also an open neighborhood of $\star'$. As a matter of fact, by definition there are A-submodules $F_1, \ldots, F_n$ of K such that $\star \in \bigcap_{i=1}^n V_{F_i} \subseteq U$. Since $\star' \geq \star$, we have $1 \in F_i^\star \subseteq F_i^{\star'}$, for each $i = 1, \ldots, n$, and thus $\star' \in \bigcap_{i=1}^n V_{F_i} \subseteq U$ and U is an open neighborhood of $\star'$.

(d) The Zariski topology of $\mathrm{SStar}_f(A)$ is determined by the finitely generated fractional ideals of A, in the sense that the collection of the sets of the form $U_F := V_F \cap \mathrm{SStar}_f(A)$, where F ranges among the finitely generated fractional ideals of A, is a subbasis. To show this, it suffices to recall the definition of semistar operation of finite type and deduce that, for any A-submodule G of K, we have

$$V_G \cap \mathrm{SStar}_f(A) = \bigcup\{U_F : F \in \mathbf{f}(A), F \subseteq G\}.$$

Proposition 1.1.3. *Let $\star$ be a semistar operation on the integral domain A. Then,*

$$\mathrm{Ad}(\{\star\}) = \{\star' \in \mathrm{SStar}(A) : \star' \leq \star\}.$$

In particular, $\{d\}$ is the unique closed point in $\mathrm{SStar}(A)$.

Proof. Let $\star' \leq \star$. By Remark 1.1.2 (c), if U is an open neighborhood of $\star'$ then $\star \in U$. It follows that $\star' \in \mathrm{Ad}(\{\star\})$. Conversely, fix $\star' \in \mathrm{Ad}(\{\star\})$; if we suppose that $\star' \nleq \star$, then there exists a $F \in \overline{\mathbf{F}}(A)$ such that $F^{\star'} \nsubseteq F^{\star}$; hence, there exists $x \in F^{\star'} \setminus F^{\star}$; thus $\star \notin V_{x^{-1}F}$ while $\star' \in V_{x^{-1}F}$. Therefore, $\mathrm{SStar}(A)\setminus V_{x^{-1}F}$ is a closed set containing $\star$ but not $\star'$; this contradicts the fact that $\star' \in \mathrm{Ad}(\{\star\})$.
The last statement follows immediately recalling that d is the minimum of ($\mathrm{SStar}(A)$, $\leq$). □

Other easy consequences of Remark 1.1.2 include the facts that $\mathrm{SStar}(A)$ is a compact space (in fact if $\mathcal{U}$ is an open cover of $\mathrm{SStar}(A)$ then there exists $U_0 \in \mathcal{U}$ such that $d \in U_0$ and thus $\mathrm{SStar}(A) = U_0$ is a finite subcover of $\mathcal{U}$) and that, in general, it does not satisfy the T_1 axiom (and thus it is even less an Hausdorff space) (in fact if $\star_1 \leq \star_2$, then we cannot find an open set which contains $\star_1$ and not $\star_2$; it follows in particular that $\mathrm{SStar}(A)$ is a T_1 space, i.e. an Hausdorff space, if and only if it contains only the identity and this happens exactly when A is a field, since otherwise $\star_{\{K\}}$ would be a semistar operation different from d). However, in the next Proposition we show that the space $\mathrm{SStar}(A)$ satisfies the weaker separation property of being a T_0 space.

Proposition 1.1.4. *The set* $\mathrm{SStar}(A)$*, endowed with the Zariski topology, satisfies the T_0 axiom.*

Proof. By Proposition 1.1.3, for any pair of semistar operations $\star, \star'$ on A, we have $\mathrm{Ad}(\{\star\})$ $= \mathrm{Ad}(\{\star'\})$ if and only if $\star = \star'$ which, by Lemma 0.1.14, is equivalent to say that $\mathrm{SStar}(A)$ is a T_0 space. □

Definition 1.1.5. Let X be a topological space and Y be a subspace of X. A *topological retraction* is a continuous map $f\colon X \to Y$ such that $f|_Y = id_Y$.

Proposition 1.1.6. *The canonical map* $\Phi\colon \mathrm{SStar}(A) \to \mathrm{SStar}_f(A)$, $\star \mapsto \star_f$ *is a topological retraction.*

Proof. It follows immediately by the definition of semistar operation of finite type that $\Phi(\star) = \star$ for any $\star \in \mathrm{SStar}_f(A)$. In light of Remark 1.1.2 (d), in order to show that Φ is continuous it suffices to prove that for any $F \in \mathbf{f}(A)$ we have that $\Phi^{-1}(U_F)$ is open in $\mathrm{SStar}(A)$. Let $\star \in \Phi^{-1}(U_F)$; then $\Phi(\star) = \star_f \in U_F$. This means that there exists $G \in \mathbf{f}(A)$ such that $G \subseteq F$ and $1 \in G^\star$. It follows that $1 \in F^\star$ and $\star \in V_F$. Furthermore, since $F \in \mathbf{f}(A)$ it is easy to see that $V_F \subseteq \Phi^{-1}(U_F)$. Therefore, $\Phi^{-1}(U_F) = V_F$ is open in $\mathrm{SStar}(A)$ and Φ is a topological retraction. $\square$

In the following Proposition we investigate the algebraic interpretation of compactness for the subspaces of $\mathrm{SStar}_f(A)$.

Proposition 1.1.7. *Let* Δ *be a compact subspace of* $\mathrm{SStar}_f(A)$*. Then, the semistar operation* $\bigwedge(\Delta)$ *is of finite type.*

Proof. Set $\Delta := \{\star_i : i \in I\}$, $\star := \bigwedge(\Delta)$, fix an A-submodule F of K and let $x \in F^\star$. Since $F^\star = \bigcap_{i\in I} F^{\star_i}$, and each $\star_i$ is of finite type, there are finitely generated ideals $G_i \subseteq F$ such that $x \in G_i^{\star_i}$; thus, for any $i \in I$, $1 \in x^{-1}G_i^{\star_i} = (x^{-1}G_i)^{\star_i}$ and $\star_i \in U_{x^{-1}G_i} =: \Omega_i$. Therefore, $\{\Omega_i : i \in I\}$ is an open cover of Δ; by compactness it admits a finite subcover $\{\Omega_{i_1}, \dots, \Omega_{i_n}\}$. Set $G := G_{i_1} + \cdots + G_{i_n} \subseteq F$; we claim that $x \in G^\star$, and this implies that $\star$ is of finite type.
For each $i \in I$, there is at least a Ω_{i_j} such that $\star_i \in \Omega_{i_j}$; hence $\star_i \in U_{x^{-1}G_{i_j}}$ and so $1 \in (x^{-1}G_{i_j})^{\star_i}$ which implies $x \in G_{i_j}^{\star_i} \subseteq G^{\star_i}$. Therefore, $x \in \bigcap_{i\in I} G^{\star_i} = G^\star$. $\square$

The next result shows that the order structure of the intersection of a nonempty family of subbasic open sets of the Zariski topology is particularly simple. We first to recall a definition.

Definition 1.1.8. A partially ordered set $(L,\leq)$ is called *a complete lattice* if every subset of L admits both a supremum and a infimum in L.

Proposition 1.1.9. *Let* $\{V_{F_i} : i \in I\}$ *be a nonempty family of subbasic open sets of the Zariski topology of* SStar*(A). Then the following statements hold.*
(1) $\bigcap_{i\in I} V_{F_i}$ *is a complete lattice (as a subset of the partially ordered set* $(\mathrm{SStar}(A),\leq)$*).*
(2) $\bigcap_{i\in I} V_{F_i}$ *is a compact subspace of* SStar*(A). In particular* V_F *is compact for every* $F \in \overline{\mathbf{F}}(A)$*.*

Proof. (1) Set $V := \bigcap_{i\in I} V_{F_i}$ and let Δ be a nonempty subset of V. By Example 1.1.2 (e), $\sharp := \bigvee(\Delta)$ and $\flat := \bigwedge(\Delta)$ are, respectively, the supremum and the infimum of Δ in $(\mathrm{SStar}(A),\leq)$. Thus it suffices to show that $\sharp, \flat \in V$. By Remark 1.1.2 (c) it follows that $\sharp \in V$, since $\sharp \geq \star$, for any $\star \in \Delta \subseteq V$. Furthermore, since $\Delta \subseteq V_{F_i}$ for any $i \in I$, we have

$$1 \in \bigcap_{\star \in V_{F_i}} F_i^\star \subseteq \bigcap_{\star\in\Delta} F_i^\star =: F_i^\flat,$$

and thus $\flat \in V$.

(2) Now let $\mathcal{U}$ be an open cover of V. By (1), $\flat \in V$, and thus there exists an open set $U_0 \in \mathcal{U}$ such that $\flat \in U_0$. It follows again by Remark 1.1.2 (c) that U_0 must contain the whole V. Therefore V is compact. $\square$

1.2 Relation with the Zariski topology on Over(A)

Our goal in this section is to find a relation between the space $\mathrm{SStar}(A)$ and the set $\mathrm{Over}(A)$ of all overrings of A. Classicaly $\mathrm{Over}(A)$ can be endowed with a topology, called again *Zariski topology*, whose basic open sets are of the form $B_F := \{C \in \mathrm{Over}(A) : F \subseteq C\}$, where F varies among the finite subsets of the quotient field of A. For an insight on this topic the reader can refer to [31]. Additionally, a similar construction can be found in [25] where the author defines this topology on the set of all the integrally closed overrings of a given integral domain A. An easy way to associate to each overring of A a semistar operation on A is provided by Example 1.0.1 (b): given D overring of A we can consider $\star_{\{D\}} \in \mathrm{SStar}(A)$ such that $F^{\star_{\{D\}}} := FD$, for any $F \in \overline{\mathbf{F}}(A)$. Additionally, we note that for any $D \in \mathrm{Over}(A)$ the semistar operation $\star_{\{D\}}$ is of finite type since

$$F^{\star_{\{D\}}} := FD = \bigcup\{(f_1,\dots,f_n)^{\star_{\{D\}}} : f_i \in F \quad \forall i = 1,\dots,n,\ n \in \mathbb{N}\} =: F^{(\star_{\{D\}})_f}$$

for any $F \in \overline{\mathbf{F}}(A)$. Therefore, we can define a natural map $\phi\colon \mathrm{Over}(A) \to \mathrm{SStar}_f(A)$, $D \mapsto \star_{\{D\}}$. We show in the following Proposition that this map allows us to identify $\mathrm{Over}(A)$ with a subspace of $\mathrm{SStar}_f(A)$.

Proposition 1.2.1. *If both* Over*(A) and* SStar_f*(A) are endowed with their Zariski topologies, then the natural map* $\phi\colon \mathrm{Over}(A) \to \mathrm{SStar}_f(A)$*,* $D \mapsto \star_{\{D\}}$ *is a topological embedding.*

Proof. Since $A^{\star_{\{D\}}} = AD = D$ for any $D \in \mathrm{Over}(A)$, we can immediately infer that ϕ is injective. Let us show that ϕ is continuous by proving that, for any finitely generated fractional ideal F of A, the set $\phi^{-1}(U_F)$ is open in $\mathrm{Over}(A)$. Since

$$D \in \phi^{-1}(U_F) \quad \Longleftrightarrow \quad \phi(D) = \star_{\{D\}} \in U_F \quad \Longleftrightarrow \quad 1 \in F^{\star_{\{D\}}} = FD$$

we have $\phi^{-1}(U_F) = \{D \in \mathrm{Over}(A) : 1 \in FD\}$. Fix a ring $D \in \phi^{-1}(U_F)$. Then there are $d_1, \ldots, d_n \in D$ and $f_1, \ldots, f_n \in F$ such that $1 = f_1d_1 + \cdots + f_nd_n$. Hence $1 \in FC$, for each $C \in B_{\{d_1,\ldots,d_n\}}$, and thus $C \in \phi^{-1}(U_F)$. Therefore, $B_{\{d_1,\ldots,d_n\}}$ is an open neighborhood of D contained in $\phi^{-1}(U_F)$. Thus, $\phi^{-1}(U_F)$ is open. Finally we show that the image via ϕ of an open set V of $\mathrm{Over}(A)$ is open in $\phi(\mathrm{Over}(A))$ (with respect to the subspace topology). Without loss of generality, we can assume that $V = B_F$, for some finite subset $F := \{f_1, \ldots, f_n\}$ of $K \setminus \{0\}$. First, consider the open set $U := \bigcap_{i=1}^n U_{(f_i^{-1})}$ of $\mathrm{SStar}_f(A)$. If $\star \in \phi(B_F)$, then $\star = \star_{\{C\}}$ for some overring C of A such that $C \supseteq F$; hence $f_i \in C$ for every i and $1 \in f_i^{-1}C = (f_i^{-1})^{\star_{\{C\}}}$. Thus we have $\phi(B_F) \subseteq U \cap \phi(\mathrm{Over}(A))$. Conversely, if $\star \in U \cap \phi(\mathrm{Over}(A))$, then $\star = \star_{\{C\}}$, for some $C \in \mathrm{Over}(A)$, and $1 \in (f_i^{-1})^{\star_{\{C\}}}$ for every i; it follows that $f_i \in C$ for every i, and thus $C \in B_F$. The equality $\phi(B_F) = U \cap \phi(\mathrm{Over}(A))$ shows that $\phi(B_F)$ is open in $\phi(\mathrm{Over}(A))$.

Since we have proved that ϕ is a continuous injection which is open on its image, it follows that ϕ is a topological embedding. □

One of the main conveniences of Proposition 1.2.1 is that we can link the topological properties of the overrings of A to the algebraic properties of the semistar operations over A, using the results we found in the previous section. The following Proposition is an example of how this strategy can be used effectively.

Proposition 1.2.2. *Let Y be a compact subspace of* Over*(A). Then, the semistar operation $\wedge_Y$ is of finite type.*

Proof. It suffices to apply Proposition 1.1.7 to $\phi(Y)$ (which is compact by Proposition 1.2.1) noting that $\bigwedge(\phi(Y)) = \wedge_Y$. □

The following result generalizes [1, Theorem 2(4)]. Before stating it let us recall a definition.

Definition 1.2.3. Let A be a ring; a subset Y of $\mathrm{Over}(A)$ is called *locally finite* if any nonzero element of A is non-invertible only in finitely many rings of Y.

Proposition 1.2.4. *Let $\{B_i : i \in I\}$ be a locally finite family of overrings of A and, for any $i \in I$, let $\star_i$ be a semistar operation of finite type on B_i. Then the map $\star\colon F \mapsto \bigcap_{i\in I}(FB_i)^{\star_i}$ is a semistar operation of finite type on A.*

Proof. Let $\star_i^\sharp$ be the map $\star_i^\sharp \colon F \mapsto (FB_i)^{\star_i}$. Borrowing Proposition 3.1.1 (2), we notice that $\star_i^\sharp$ is a semistar operation on A of finite type, since $\star_i$ is of finite type on B_i. Moreover, $\star = \bigwedge(\Delta)$, where $\Delta = \{\star_i^\sharp : i \in I\}$, and by Proposition 1.1.7 it suffices to show that Δ is compact.

Let $\mathcal{U}$ be an open cover of Δ. By Alexander's subbasis Theorem (see Proposition 0.1.28), we can assume that each set in $\mathcal{U}$ is a subbasic open set of the Zariski topology on $\mathrm{SStar}_f(A)$. Thus, choose an ideal $F \in \mathbf{f}(A)$ such that $U_F \in \mathcal{U}$ and let $x_0 \in F \setminus \{0\}$. By local finiteness, there is a finite subset $I' \subseteq I$ such that $x_0, x_0^{-1} \in B_i$ for any $i \in I \setminus I'$. Thus we have $1 = x_0 x_0^{-1} \in FB_i \subseteq (FB_i)^{\star_i} =: F^{\star_i^\sharp}$, for any $i \in I \setminus I'$. For every $i \in I'$, there is an $F_i \in \mathbf{f}(A)$ such that $U_{F_i} \in \mathcal{U}$ and $\star_i^\sharp \in U_{F_i}$; hence, $\{U_{F_i} : i \in I'\} \cup \{U_F\}$ is a finite subcover of $\mathcal{U}$, and thus Δ is compact. □

Corollary 1.2.5. *Let Y be a locally finite subset of $\mathrm{Over}(A)$. Then, $\wedge_Y$ is of finite type.*

Proof. Apply the previous proposition by taking $\star_i = id|_{B_i}$ for every $i \in I$. □

1.3 Spectral structure of $\mathrm{SStar}_f(A)$ (first part)

Our next goal is to prove the following Theorem.

Theorem 1.3.1. *The set $\mathrm{SStar}_f(A)$, endowed with the Zariski topology, is a spectral space.*

However, in order to prove this result, we need to make a digression that will keep us busy for the entire next chapter. More specifically, we will introduce a new topology based on ultrafilters and we will use its properties to obtain a new characterization for spectral spaces. For now, we limit ourselves to state a Lemma we will need for proving Theorem 1.3.1 and whose proof is based on the argument given in [2, p.1628].

Lemma 1.3.2. *Let Y be a nonempty collection of semistar operations of finite type on an integral domain A. Then $\bigvee(Y)$ is of finite type and, for any $F \in \overline{\mathbf{F}}(A)$, we have*

$$F^{\bigvee(Y)} = \bigcup\{F^{\sigma_1 \circ \cdots \circ \sigma_n} : \sigma_1, \ldots, \sigma_n \in Y, n \in \mathbb{N}\},$$

where $\sigma_1 \circ \cdots \circ \sigma_n$ denotes the usual composition of $\sigma_1, \ldots, \sigma_n$ as functions.

Proof. Having in mind that $\bigvee(Y)$ is the supremum of Y in the partially ordered set $(\mathrm{SStar}(A),\leq)$, we get $\bigvee(Y) \geq \sigma$, for any $\sigma \in Y$. Thus it follows easily that $(\bigvee(Y))_f \geq \sigma_f = \sigma$, for any $\sigma \in Y$ (where the last equation holds since Y is a collection of semistar operations of finite type); hence $(\bigvee(Y))_f \geq \bigvee(Y)$. The converse inequality is trivial and thus we have that $\bigvee(Y) = (\bigvee(Y))_f$. This proves that $\bigvee(Y)$ is of finite type.

For the second part it is enough to show that the function

$$\star\colon \overline{\mathbf{F}}(A) \to \overline{\mathbf{F}}(A), F \mapsto F^{\star} := \bigcup\{F^{\sigma_1\circ\cdots\circ\sigma_n} : \sigma_1, \ldots, \sigma_n \in Y, n \in \mathbb{N}\}$$

is a semistar operation. In fact, it is easy to see that $\star \geq \sigma$, for each $\sigma \in Y$; thus we have $\star \geq \bigvee(Y)$. Conversely, let us take arbitrary $\sigma_1, \ldots, \sigma_n \in Y$ and let $* \in \mathrm{SStar}(A)$ be any upper bound for Y, i.e., $* \geq \sigma$, for any $\sigma \in Y$; then, in particular $* \geq \sigma_n$ and so

$$(F^{\sigma_1\circ\cdots\circ\sigma_{n-1}})^{\sigma_n} = F^{\sigma_1\circ\cdots\circ\sigma_n} \subseteq F^{*}$$

for any $F \in \overline{\mathbf{F}}(A)$. It is now clear that $* \geq \star$ and, by the definition of $\bigvee(Y)$, it follows that $\star \leq \bigvee(Y)$.

The only property of semistar operations which is not trivial to show for $\star$ is $(F^{\star})^{\star} = F^{\star}$, for any $F \in \overline{\mathbf{F}}(A)$, and in particular the inclusion $(F^{\star})^{\star} \subseteq F^{\star}$. Therefore, let $0 \neq x \in (F^{\star})^{\star}$; then there exist $\sigma_1, \ldots, \sigma_n \in Y$ such that $x \in (F^{\star})^{\sigma_1\circ\cdots\circ\sigma_n}$. On the other hand, σ_n is of finite type thus there exist nonzeros $x_1, \ldots, x_m \in (F^{\star})^{\sigma_1\circ\cdots\circ\sigma_{n-1}}$ such that $x \in (x_1, \ldots, x_m)^{\sigma_n}$. Now, σ_{n-1} is also of finite type, so there exist nonzeros $x'_1, \ldots, x'_{m'} \in (F^{\star})^{\sigma_1\circ\cdots\circ\sigma_{n-2}}$ such that $x_i \in (x'_1, \ldots, x'_{m'})^{\sigma_{n-1}}$, for each $i = 1, \ldots, m$. Hence $x \in (x_1, \ldots, x_m)^{\sigma_n} \subseteq (x'_1, \ldots, x'_{m'})^{\sigma_{n-1}\circ\sigma_n}$. Reiterating the same steps starting from $x'_1, \ldots, x'_{m'}$ we finally get nonzeros $y_1, \ldots, y_r \in F^{\star}$ such that $x \in (y_1, \ldots, y_r)^{\sigma_1\circ\cdots\circ\sigma_n}$. By the definiton of $\star$, for each $i = 1, \ldots, r$, we can find $\varphi^i_1, \ldots, \varphi^i_{k(i)} \in Y$ such that $y_i \in F^{\varphi^i_1\circ\cdots\circ\varphi^i_{k(i)}}$. Hence $(y_1, \ldots, y_r) \subseteq \bigcup_{i=1}^r F^{\varphi^i_1\circ\cdots\circ\varphi^i_{k(i)}}$ and $x \in (y_1, \ldots, y_r)^{\sigma_1\circ\cdots\circ\sigma_n} \subseteq \bigcup_{i=1}^r F^{\varphi^i_1\circ\cdots\circ\varphi^i_{k(i)}\circ\sigma_1\circ\cdots\circ\sigma_n} \subseteq F^{\star}$. This argument shows that $(F^{\star})^{\star} \subseteq F^{\star}$ and completes the proof. $\square$

Chapter 2

The $\mathcal{F}$-ultrafilter topology and spectral spaces

As we have mentioned at the end of last section, the goal of this chapter is to define a new topology and to develop a theory that will allow us to get a new characterization for spectral spaces. We will then use this characterization to prove Theorem 1.3.1. The topology we will construct has been introduced by C.A. Finocchiaro in [7] as a generalization of the *ultrafilter topology* on the set Spec(A), where A is a ring, discussed in [12]. Therefore, we will reobtain through a more general setting some of the results contained in [12].

2.0 Preliminary notions

As we have already forewarned, we will deal heavily with ultrafilters. Ultrafilters play an important role in several mathematical disciplines such as algebra, logic and, of course, topology. For instance it is possible to characterize compactness using ultrafilters and there exists a representation of the Stone-Cêch compactification through ultrafilters (for a deeper look into these topics see [30, Section 12 and Example 19I]). For our purposes, we just want to recall some easy properties of ultrafilters (for a more general view the reader can refer to [18, Chapter 7]).

Definition 2.0.1. Let X be a set; a nonempty collection $\mathscr{F}$ of subsets of X is called *a filter on* X if the following properties hold:
(i) $\emptyset \notin \mathscr{F}$.
(ii) if $Y, Z \in \mathscr{F}$, then $Y \cap Z \in \mathscr{F}$.
(iii) if $Z \in \mathscr{F}$ and $Z \subseteq Y \subseteq X$, then $Y \in \mathscr{F}$.

An *ultrafilter on* X is a filter $\mathscr{F}$ on X which is maximal (with respect to set inclusion) in the set of all filters on X.

Note that every filter is a family of sets that satisfies the finite intersection property and that each filter on X contains X as an element.
If X is a set and $x \in X$, the easiest example of filter on X is *the trivial ultrafilter generated by* x. It is the collection

$$\beta_X^x := \{Y \subseteq X : x \in Y\}.$$

Note that β_X^x is in particular an ultrafilter since if $A \subseteq X$ does not belong to β_X^x then $\beta_X^x \cup \{A\}$ is not a filter since $A \cap \{x\} = \emptyset \notin \beta_X^x$.

Lemma 2.0.2. *If $\mathscr{F}$ is a collection of subsets of a set X which satisfies the finite intersection property, then there exists an ultrafilter $\mathscr{U}$ on X containing $\mathscr{F}$.*

Proof. If we set

$$\mathscr{G} := \{A \subseteq X : \text{there exists } n \in \mathbb{N} \text{ and } A_1, \dots, A_n \in \mathscr{F} \text{ such that } A_1 \cap \dots \cap A_n \subseteq A\},$$

then it is not hard to see that $\mathscr{G}$ is a filter on X which contains $\mathscr{F}$. The conclusion follows by applying Zorn's lemma after noting that if $\{\mathscr{F}_\alpha\}_{\alpha \in A}$ is a chain of filters, then $\bigcup_{\alpha \in A} F_\alpha$ is a filter too. $\square$

By a straightforward application of Zorn's lemma it is also possible to prove Tarski's Theorem which states that every filter is contained in some ultrafilter.
Next Proposition introduces a very useful and common characterization of ultrafilter.

Proposition 2.0.3. *If $\mathscr{F}$ is a filter on X, the following conditions are equivalent:*
(i) $\mathscr{F}$ is an ultrafilter on X.
(ii) If Y is any subset of X, then $Y \notin \mathscr{F}$ implies $X \setminus Y \in \mathscr{F}$.
(iii) If Y and Y_0 are subsets of X such that $Y \cup Y_0 \in \mathscr{F}$, then $Y \in \mathscr{F}$ or $Y_0 \in \mathscr{F}$.

Proof. (i) $\iff$ (ii). Let us suppose by contradiction that there exists $Y \subseteq X$ such that both Y and $X \setminus Y$ do not belong to $\mathscr{F}$. Set $\mathscr{G} := \mathscr{F} \cup \{Y\}$ and let us show that $\mathscr{G}$ satisfies the finite intersection property. If $U \in \mathscr{F}$, then $U \cap Y \neq \emptyset$, since otherwise we would have that $U \subseteq (X \setminus Y)$ and so $(X \setminus Y) \in \mathscr{F}$. Since for any finite family $\{U_i\}_{i=1}^n$ of elements of $\mathscr{F}$ we have that $\bigcap_{i=1}^n U_i \in \mathscr{F}$, then it follows that $Y \cap (\bigcap_{i=1}^n U_i) \neq \emptyset$. Hence $\mathscr{G}$ satisfies the finite intersection property and thus, by Lemma 2.0.2, it is contained in an ultrafilter, against the maximality of $\mathscr{F}$.

If conversely we suppose that condition (ii) holds and that $Y \subseteq X$ is not an element of $\mathscr{F}$, then $\mathscr{F} \cup \{Y\}$ is not a filter since otherwise it would contain $\emptyset = Y \cup (X \setminus Y)$. It follows that $\mathscr{F}$ is an ultrafiler.

(ii) $\iff$ (iii). Suppose that condition (ii) holds and assume by contradiction that there exist $Y, Y_0 \subseteq X$ such that $Y \cup Y_0 \in \mathscr{F}$ but both Y and Y_0 are not elements of $\mathscr{F}$. Then $(X \setminus Y), (X \setminus Y_0) \in \mathscr{F}$ and thus $(X \setminus Y) \cap (X \setminus Y_0) \cap (Y \cup Y_0) = \emptyset \in \mathscr{F}$, which is clearly a contradiction.

If conversely we suppose that condition (iii) holds, then, for every $Y \subseteq X$, we have that $Y \cup (X \setminus Y) = X \in \mathscr{F}$ and thus $Y \in \mathscr{F}$ or $(X \setminus Y) \in \mathscr{F}$. □

The following Proposition is easily proved and shows how to build new ultrafilters from old ones.

Proposition 2.0.4. *(1) If $\mathscr{U}$ is an ultrafilter on a set X and $Y \in \mathscr{U}$, then*

$$\mathscr{U}_Y := \{Y \cap U : U \in \mathscr{U}\}$$

is an ultrafilter on Y contained in $\mathscr{U}$.

(2) If $f \colon X \to Y$ is a function and $\mathscr{U}$ is an ultrafilter on a set X, then

$$\mathscr{U}^f := \{T \subseteq Y : f^{-1}(T) \in \mathscr{U}\}$$

is an ultrafilter on Y. In particular, if $Z \subseteq Y$ and $\mathscr{U}$ is an ultrafilter on Z, then $\mathscr{U}^Y := \{T \subseteq Y : T \cap Z \in \mathscr{U}\}$ is an ultrafilter on Y containing $\mathscr{U}$. In fact, if $\iota \colon Z \to Y$ is the inclusion, then $\mathscr{U}^Y = \mathscr{U}^\iota$.

2.1 Construction and main properties

We are now ready to start constructing our new topology. For this purpose, let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. For each $Y \subseteq X$ and each ultrafilter $\mathscr{U}$ on Y, we define

$$Y_{(X,\mathcal{F})}(\mathscr{U}) := Y_{\mathcal{F}}(\mathscr{U}) := \{x \in X : [\forall F \in \mathcal{F}, x \in F \iff F \cap Y \in \mathscr{U}]\}.$$

Since $\mathcal{F}$ will be almost always a fixed collection of subsets of a fixed set X, we will denote the set $Y_{(X,\mathcal{F})}(\mathscr{U})$ simply by $Y(\mathscr{U})$, when no confusion can arise.

Example 2.1.1. Let X be a set, $\mathcal{F}$ be a nonempty collection of subsets of X and Y be a subset of X.

(a) If $y \in Y$ and β_Y^y is the trivial ultrafilter on Y generated by y, then $y \in Y_{\mathcal{F}}(\beta_Y^y)$.

(b) Let A be a ring, Y be a subset of $\mathrm{Spec}(A)$ and $\mathscr{U}$ be an ultrafilter on Y. Let

$$\mathfrak{p}_{Y,\mathscr{U}} := \{x \in A : V(x) \cap Y \in \mathscr{U}\}$$

be the set we have already introduced in page 8. If $\mathcal{P} := \{D(a) : a \in A\}$ is the collection of the so called principal open subsets of the Zariski topology on $\mathrm{Spec}(A)$, then the equality $Y_{\mathcal{P}}(\mathscr{U}) = \{\mathfrak{p}_{Y,\mathscr{U}}\}$ holds. As a matter of fact, fix $\mathfrak{p} \in \mathrm{Spec}(A)$. Then, by definition, $\mathfrak{p} \in Y_{\mathcal{P}}(\mathscr{U})$ if and only if the following statement is true:

$$\mathfrak{p} \in D(a) \Longleftrightarrow D(a) \cap Y \in \mathscr{U},$$

for any $a \in A$. By denying both sides of this statement, we get the following equivalent form:

$$a \in \mathfrak{p} \Longleftrightarrow V(a) \cap Y \in \mathscr{U},$$

for any $a \in A$. Now it follows immediately that $\mathfrak{p} \in Y_{\mathcal{P}}(\mathscr{U})$ if and only if $\mathfrak{p} = \mathfrak{p}_{Y,\mathscr{U}}$.

Definition 2.1.1. Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then we say that a subset Y of X is *$\mathcal{F}$-stable under ultrafilters* if $Y_{\mathcal{F}}(\mathscr{U}) \subseteq Y$, for each ultrafilter $\mathscr{U}$ on Y.

Remark 2.1.2. Let A be a ring. Keeping in mind Example 2.1.1 (b), it follows immediately that a subset Y of $\mathrm{Spec}(A)$ is $\mathcal{P}$-stable under ultrafilters if and only if it is closed in the ultrafilter topology of $\mathrm{Spec}(A)$ introduced in [12] and recalled in the Introduction.

The following technical Lemma will allow us to show that it is possible to build a topology on a set X starting from the $\mathcal{F}$-stable under ultrafilters subsets of X.

Lemma 2.1.3. *Let X be a set, $\mathcal{F}$ be a given nonempty collection of subsets of X and $Y \subseteq Z \subseteq X$. Let $\mathscr{U}$ be an ultrafilter on Y, T be an element of U and, as in Proposition 2.0.4 set*

$$\mathscr{U}_T := \{U \cap T : U \in \mathscr{U}\} \qquad \mathscr{U}^Z := \{Z' \subseteq Z : Z' \cap Y \in \mathscr{U}\}.$$

Then we have

$$Y(\mathscr{U}) = T(\mathscr{U}_T) = Z(\mathscr{U}^Z).$$

Proof. We will prove only the inclusion $Y(\mathscr{U}) \subseteq T(\mathscr{U}_T)$. The others can be shown using the same argument. Let $x \in Y(\mathscr{U})$ and $F \in \mathcal{F}$. We need to show that $x \in F$ if and only if $F \cap T \in \mathscr{U}_T$. Assume $x \in F$. Then, $F \cap Y \in \mathscr{U}$ and $F \cap T = (F \cap Y) \cap T \in \mathscr{U}_T$, by the definition of $\mathscr{U}_T$. Conversely, assume that $F \cap T \in \mathscr{U}_T$. Since $\mathscr{U}_T \subseteq \mathscr{U}$ and $F \cap T \subseteq F \cap Y$, then $F \cap Y \in \mathscr{U}$ and thus $x \in F$. $\square$

Proposition 2.1.4. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then, the family of all the subsets of X that are $\mathcal{F}$-stable under ultrafilters is the collection of the closed sets for a topology on X. We will call this topology the* $\mathcal{F}$-ultrafilter topology on X, *and we will denote by $X^{\mathcal{F}-ultra}$ the set X endowed with the $\mathcal{F}$-ultrafilter topology.*

Proof. It is clear that both $\emptyset$ and X are trivially $\mathcal{F}$-stable under ultrafilters.
Let C, C_0 be $\mathcal{F}$-stable under ultrafilters subsets of X and $\mathscr{U}$ be an arbitrary ultrafilter on $Y := C \cup C_0$. By Proposition 2.0.3 (b) we can assume that $C \in \mathscr{U}$. Then, by Lemma 2.1.3, we have $Y(\mathscr{U}) = C(\mathscr{U}_C) \subseteq C \subseteq Y$, and thus Y is $\mathcal{F}$-stable under ultrafilters.
Now, let $\mathcal{G}$ be a collection of $\mathcal{F}$-stable under ultrafilters subsets of X and let $\mathscr{U}$ be an arbitrary ultrafilter on $Z := \bigcap \mathcal{G}$. For each $C \in \mathcal{G}$, again by Lemma 2.1.3, we have $Z(\mathscr{U}) = C(\mathscr{U}^C) \subseteq C$, and thus $Z(\mathscr{U}) \subseteq Z$. This completes the proof. $\square$

Example 2.1.2. Let X be a set and denote by $\mathcal{B}(X)$ the power set of X, i.e., the set containing all subsets of X.
(a) The $\mathcal{B}(X)$-ultrafilter topology on X is the discrete topology on X. In fact, for any $x \in X$ the singleton $Y := \{x\}$ is closed in the $\mathcal{B}(X)$-ultrafilter topology on X. As a matter of fact, the only possible ultrafilter on Y is $\mathscr{U} := \{Y\}$. Therefore, if $y \in X$ and $y \neq x$, then $y \in \{y\} \in \mathcal{B}(X)$ but $\{y\} \cap Y = \emptyset \notin \mathscr{U}$. It follows that Y is $\mathcal{B}(X)$-stable under ultrafilters and thus $X^{\mathcal{B}(X)-ultra}$ is the discrete space.
(b) The $\{X\}$-ultrafilter topology on X is the indiscrete topology. In fact, let Y be a $\{X\}$-stable under ultrafilters subset of X and $\mathscr{U}$ be an ultrafilter on Y. Since $X \cap Y = Y \in \mathscr{U}$, it follows that $Y = X$. Since $\emptyset \subseteq X$ is trivially stable under ultrafilters, we have proved that $X^{\{X\}-ultra}$ is the indiscrete space.
(c) Let A be a ring and $\mathcal{P}$ be the collection of all the principal open subsets of the Zariski topology on $X := \mathrm{Spec}(A)$. Then it follows by Remark 2.1.2 that the $\mathcal{P}$-ultrafilter topology of X is equal to the ultrafilter topology studied in [12].
(d) If $\mathcal{F} \subseteq \mathcal{G}$ are two collections of subsets of X, then the $\mathcal{G}$-ultrafilter topology is finer than or equal to the $\mathcal{F}$-ultrafilter topology. In fact, for each subset Y of X and each ultrafilter

$\mathscr{U}$ on Y, we have $Y_{\mathcal{G}}(\mathscr{U}) \subseteq Y_{\mathcal{F}}(\mathscr{U})$.

Proposition 2.1.5. *Let X be a set, $\mathcal{F}$ be a nonempty collection of subsets of X and denote by $\mathcal{B}_{\mathit{fin}}(\mathcal{F})$ the collection of all the finite subsets of $\mathcal{F}$. Set*

$$\mathcal{F}_{\sharp} := \left\{ \bigcap \mathcal{G} : \mathcal{G} \in \mathcal{B}_{\mathit{fin}}(\mathcal{F}) \right\}, \quad \mathcal{F}^{\sharp} := \left\{ \bigcup \mathcal{G} : \mathcal{G} \in \mathcal{B}_{\mathit{fin}}(\mathcal{F}) \right\}.$$

Then, the $\mathcal{F}$-ultrafilter topology, the $\mathcal{F}_{\sharp}$-ultrafilter topology and the $\mathcal{F}^{\sharp}$- ultrafilter topology are the same.

Proof. By Example 2.1.2 (d) and the obvious inclusion $\mathcal{F} \subseteq \mathcal{F}_{\sharp}$, it is enough to show that the $\mathcal{F}$-ultrafilter topology is finer than or equal to the $\mathcal{F}_{\sharp}$ -ultrafilter topology. Let Y be an $\mathcal{F}_{\sharp}$-stable under ultrafilters subset of X and $\mathscr{U}$ be an ultrafilter on Y: we will show that $Y_{\mathcal{F}}(\mathscr{U}) \subseteq Y_{\mathcal{F}_{\sharp}}(\mathscr{U})$. For this purpose let $x \in Y_{\mathcal{F}}(\mathscr{U})$, $\mathcal{G} := \{F_1, \dots, F_n\} \in \mathcal{B}_{\mathsf{fin}}(\mathcal{F})$ and $G := \bigcap \mathcal{G}$. We need to show that $x \in G$ if and only if $G \cap Y \in \mathscr{U}$. If $x \in G$, then $F_i \cap Y \in \mathscr{U}$, for each $i = 1, \dots, n$, and thus $\bigcap_{i=1}^{n}(F_i \cap Y) = G \cap Y \in \mathscr{U}$. Conversely, since $G \cap Y \subseteq F_i \cap Y$, for each $i = 1, \dots, n$, if $G \cap Y \in \mathscr{U}$, then it follows that $F_i \cap Y \in \mathscr{U}$, for each $i = 1, \dots, n$; this implies $x \in G$, by the choice of x. We can conclude that the $\mathcal{F}$-ultrafilter topology and the $\mathcal{F}_{\sharp}$-ultrafilter topology are the same. By a similar argument it can be shown that $Y_{\mathcal{F}}(\mathscr{U}) = Y_{\mathcal{F}^{\sharp}}(\mathscr{U})$. Therefore the proof is complete. □

The next Proposition uses the notion of *boolean algebra* to show that the elements of $\mathcal{F}$ are clopen in the $\mathcal{F}$-ultrafilter topology. For an insight on boolean algebras see [15]. Here we limit ourselves to consider a boolean algebra on a fixed set X as a collection of subsets of X which is closed under finite unions, intersections and set complement. Following this definition, it is easy to see that if X is a topological space, then the collection of all the clopen subsets of X is an example of boolean algebra. For sake of completeness we also mention that *Stone's representation Theorem* allows to invert last statement and to show that every boolean algebra S is isomorphic to the collection of clopen subsets of a particular topological space associated to S, called *Stone space of S* (see [15, Theorem 17]). Moreover, we recall that a topological space X is called *connected* if the only clopen subsets of X are $\emptyset$ and X; the space X is called *totally disconnected* if the only connected subsets of X are the singletons.

Proposition 2.1.6. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X; let* $\mathrm{Bool}(\mathcal{F})$ *be the boolean algebra of X generated by $\mathcal{F}$, i.e., the intersection of all the boolean algebras on X which contain $\mathcal{F}$. Then, the following statements hold.*

(1) $\mathrm{Bool}(\mathcal{F}) \subseteq \mathrm{Clop}(X^{\mathcal{F}-ultra})$.

(2) If, for each pair of distinct points $x, y \in X$ there exists a set $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$, then $X^{\mathcal{F}-ultra}$ is an Hausdorff and totally disconnected space.

Proof. (1) Since $\mathrm{Clop}(X^{\mathcal{F}-ultra})$ is a boolean algebra, it is enough to show that $\mathcal{F} \subseteq \mathrm{Clop}(X^{\mathcal{F}-ultra})$. Pick a set $E \in \mathcal{F}$. If $\mathscr{U}$ is an ultrafilter on E and $x \in E(\mathscr{U})$, then the statement $x \in F \iff F \cap E \in \mathscr{U}$ holds for each $F \in \mathcal{F}$, and in particular for $F := E$. Since clearly $E \in \mathscr{U}$, it follows that $x \in E$. This proves that E is closed in $X^{\mathcal{F}-ultra}$.
Now let $\mathscr{V}$ be an ultrafilter on $Z := X \setminus E$ and $x \in Z(\mathscr{V})$. The statement $x \in E \iff E \cap Z \in \mathscr{V}$ holds. Since $E \cap Z = \emptyset \notin \mathscr{V}$, then $x \in Z$. Thus, Z is closed and E is open in the $\mathcal{F}$-ultrafilter topology.
(2) The fact that $X^{\mathcal{F}-ultra}$ is an Hausdorff space follows immediately by (1) and the extra assumption on $\mathcal{F}$. To show that $X^{\mathcal{F}-ultra}$ is totally disconnected let us assume by contradiction that Y is a connected subset of X with more than one point. Then, let $x, y \in X$ be distinct points of Y and let $F \in \mathcal{F}$ be such that $x \in F$ and $y \notin F$. It follows by (1) that $Y \cap F$ is a nontrivial clopen subset of Y and this contradicts the assumption that Y is connected. Therefore $X^{\mathcal{F}-ultra}$ is totally disconnected. □

In the next Theorem we discuss the compactness of the $\mathcal{F}$-ultrafilter topology. For this task, we will use the characterization of compact spaces we recalled in Lemma 0.1.27.

Theorem 2.1.7. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X. Then, the following conditions are equivalent:*
(i) *$X^{\mathcal{F}-ultra}$ is a compact topological space.*
(ii) *$X(\mathscr{U}) \neq \emptyset$, for each ultrafilter $\mathscr{U}$ on X.*
(iii) *If $\mathcal{F}^- := \{X \setminus F : F \in \mathcal{F}\}$ and $\mathcal{H}$ is a subcollection of $\mathcal{G} := \mathcal{F} \cup \mathcal{F}^-$ with the finite intersection property, then $\bigcap \mathcal{H} \neq \emptyset$.*

Proof. (i)$\Longrightarrow$(iii). It is enough to use Proposition 2.1.6 (1) and compactness of $X^{\mathcal{F}-ultra}$.
(iii)$\Longrightarrow$(ii). Let $\mathscr{U}$ be an ultrafilter on X. Assume, by contradiction, that $X(\mathscr{U}) = \emptyset$. This means that, for each $x \in X$ there exists a set $F_x \in \mathcal{F}$ such that exactly one of the following statements is true:
(a) $x \in F_x$ and $F_x \notin \mathscr{U}$.
(b) $x \notin F_x$ and $F_x \in \mathscr{U}$.
Now, for each $x \in X$, set $C_x := X \setminus F_x$, if $x \in F_x$, and $C_x := F_x$, if $x \notin F_x$. Then, it is clear

that $\mathcal{H} := \{C_x : x \in X\}$ is a subcollection of $\mathcal{G}$ and it has the finite intersection property, since $\mathcal{H} \subseteq \mathscr{U}$. Thus, by assumption, there exists $x_0 \in \bigcap \mathcal{H}$. This is a contradiction , since $x_0 \notin C_{x_0}$ by construction.

(ii)$\Longrightarrow$(i). Let $\mathcal{C}$ be a collection of closed subsets of $X^{\mathcal{F}-ultra}$ with the finite intersection property. By Lemma 2.0.2, there exists an ultrafilter $\mathscr{U}^*$ on X such that $\mathcal{C} \subseteq \mathscr{U}^*$. By assumption, we can pick a point $x^* \in X(\mathscr{U}^*)$. Now, let $C \in \mathcal{C}$. Since $C \in \mathscr{U}^*$, we have $x^* \in X(\mathscr{U}^*) = C(\mathscr{U}^*_C) \subseteq C$, keeping in mind Lemma 2.1.3. Thus $x^* \in \bigcap \mathcal{C}$. This completes the proof. □

Remark 2.1.8. Let A be a ring. By Example 2.1.1 (b) and Theorem 2.1.7 we get immediately the fact that the ultrafilter topology on $\mathrm{Spec}(A)$ is compact.

Proposition 2.1.9. *Let X be a set and $\mathcal{F}$ be a nonempty collection of subsets of X such that, for each pair of distinct points $x, y \in X$, there exists a set $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$. If $X^{\mathcal{F}-ultra}$ is a compact topological space, then the $\mathcal{F}$-ultrafilter topology is the coarsest topology for which $\mathcal{F}$ is a collection of clopen sets.*

Proof. Denote by $X_\star$ the set X endowed with the coarsest topology for which $\mathcal{F}$ is a collection of clopen sets. Then, the identity map $id_X \colon X^{\mathcal{F}-ultra} \to X_\star$ is continuous, by Proposition 2.1.6 (1). Moreover, by the assumption we made on $\mathcal{F}$ it follows immediately that $X_\star$ is an Hausdorff space. Thus, it is enough to apply Corollary 0.1.31 with $f := id_X$. □

In [12, Theorem 8] the authors show that if A is a ring, then the ultrafilter topology on $X := \mathrm{Spec}(A)$ is equal to the *patch topology*, i.e., the topology whose subbasis of open sets is the set of all the open and compact sets of X (with respect to the usual Zariski topology) and their complements.

Having in mind Example 2.1.2 (c), we can get this same result using the $\mathcal{P}$-ultrafilter topology.

Corollary 2.1.10. *Preserve the notation of Example 2.1.2 (c). Then, the $\mathcal{P}$-ultrafilter topology and the patch topology of* $\mathrm{Spec}(A)$ *are the same.*

Proof. The family $\mathcal{P}$ of principal open subsets of $\mathrm{Spec}(A)$ satisfies the condition of Proposition 2.1.9; in fact if $\mathfrak{p}_1$, $\mathfrak{p}_2$ are distinct prime ideals of A and $x \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$, then $D(x) \in \mathcal{P}$ and $\mathfrak{p}_1 \in D(x)$, $\mathfrak{p}_2 \notin D(x)$. Moreover, $X^{\mathcal{P}-ultra}$ is a compact topological space by Remark 2.1.8. Finally, since $\mathcal{P}$ is a collection of compact and basic open sets of the Zariski topology on

Spec(A) (see Proposition 0.2.38), it follows that the patch topology is the coarsest topology for which $\mathcal{P}$ is a collection of clopen sets. Thus, it sufficies to apply Proposition 2.1.9 to prove our statement. $\square$

2.2 Relations with spectral spaces

In this section we want to show what happens if we endow a topological space $(X,\mathcal{T})$ with the $\mathcal{F}$-ultrafilters topology, choosing $\mathcal{F}$ to be a basis of $\mathcal{T}$. We will show that this case is particularly interesting since it allows to relate the $\mathcal{F}$-ultrafilter topology to spectral spaces.

Proposition 2.2.1. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B}$ be a basis of open sets of X. Then, the following statements hold:*
(1) *The $\mathcal{B}$-ultrafilter topology on X is finer than or equal to $\mathcal{T}$.*
(2) *If X satisfies the T_0 axiom, then the $\mathcal{B}$-ultrafilter topology is Hausdorff and totally disconnected. In particular, if X satisfies the T_0 axiom but it is not Hausdorff, then the $\mathcal{B}$-ultrafilter topology is strictly finer than the given topology $\mathcal{T}$.*
Assume additionally that $X^{\mathcal{B}-ultra}$ is compact. Then the following statement holds.
(3) The $\mathcal{B}$-ultrafilter topology is the coarsest topology on X for which $\mathcal{B}$ is a collection of clopen sets.

Proof. Statements (1) and (2) are immediate consequences of Proposition 2.1.6 (1) and (2) respectively.
Statement (3) follows by applying Proposition 2.1.9. $\square$

Proposition 2.2.2. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B}$ be a basis of open sets of X. Assume also that X satisfies the T_0 axiom and that $X^{\mathcal{B}-ultra}$ is compact. Then the patch topology induced by $\mathcal{T}$ is equal to the $\mathcal{B}$-ultrafilter topology.*

Proof. Let us recall one more time that the patch topology induced by $\mathcal{T}$ is the topology whose subbasis of open sets is the set $\mathcal{S}_0$ of all the open and compact subsets of X (with respect to the topology $\mathcal{T}$) and their complements. By Proposition 2.2.1 (3), it follows that $\mathcal{S} := \mathcal{B} \cup \mathcal{B}^-$ is a subbasis of open sets of $X^{\mathcal{B}-ultra}$. Moreover, by Proposition 2.1.6 (1), each member of $\mathcal{S}$ is closed in $X^{\mathcal{B}-ultra}$. By compactness of $X^{\mathcal{B}-ultra}$ and Proposition 2.2.1 (1), it follows that each member of $\mathcal{S}$ is compact with respect to the topology $\mathcal{T}$. Therefore, $\mathcal{S} \subseteq \mathcal{S}_0$. This proves that the patch topology induced by $\mathcal{T}$ is finer than or equal to the $\mathcal{B}$-ultrafilter topology. Now, let $S_0 \in \mathcal{S}_0$. Since $\mathcal{B}$ is a basis of open sets of X (with respect to $\mathcal{T}$), then there exists a finite subcollection $\mathcal{C} \subseteq \mathcal{B}$ such that $S_0 = \bigcup \mathcal{C}$ or $S_0 = \bigcap \mathcal{C}^-$. Thus S_0 is an open set of the $\mathcal{B}$-ultrafilter topology. Therefore, the two topologies must be the same. □

The following Theorem enlighten us about the relation with spectral spaces we were looking for.

Theorem 2.2.3. *Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B}$ be a basis of open sets of X. Assume also that X satisfies the T_0 axiom and that $X^{\mathcal{B}-ultra}$ is compact. Then X, equipped with the topology $\mathcal{T}$, is a spectral space.*

Proof. If we denote by X^{patch} the set X endowed with the patch topology induced by $\mathcal{T}$, then Proposition 2.2.2 tells us that $X^{patch} = X^{\mathcal{B}-ultra}$ and thus it is compact by hypothesis and Hausdorff by Proposition 2.2.1 (2). Our statement results proved by applying [16, Corollary to Proposition 7]. □

Corollary 2.2.4. *Let X be a topological space. Then, the following conditions are equivalent.*

(i) X is a spectral space.

(ii) There is a basis $\mathcal{B}$ of X such that $X^{\mathcal{B}-ultra}$ is a compact and Hausdorff topological space.

(iii) X satisfies the T_0 axiom and there is a basis $\mathcal{B}$ of X such that $X_{\mathcal{B}}(\mathscr{U}) \neq \emptyset$, for any ultrafilter $\mathscr{U}$ on X.

(iv) X satisfies the T_0 axiom and there is a subbasis $\mathcal{S}$ of X such that $X_{\mathcal{S}}(\mathscr{U}) \neq \emptyset$, for any ultrafilter $\mathscr{U}$ on X.

Proof. (i)$\Longrightarrow$(iii). We can assume, without loss of generality, that $X = \mathrm{Spec}(A)$, for some ring A. We have noted in Lemma 0.2.41 that X is a T_0 space. Let $\mathscr{U}$ be an ultrafilter on X and $\mathcal{P}$ be the basis of X consisting of the principal open subsets. Keeping in mind Example 2.1.1 (b), we have $X_{\mathcal{P}}(\mathscr{U}) = \{\mathfrak{p}_{X,\mathscr{U}}\}$. Thus, it sufficies to choose $\mathcal{B} := \mathcal{P}$.

(iii)$\Longrightarrow$(ii). Apply Theorem 2.1.7 and Proposition 2.2.1 (2) to the basis $\mathcal{B}$ given in condition (iii).

(ii)$\Longrightarrow$(i). It is the statement of Theorem 2.2.3.

(iii)$\Longleftrightarrow$(iv). It is trivial by Proposition 2.1.5 and Theorem 2.1.7. □

In Theorem 2.2.3 we used a result from [16] to complete our proof. In the following Proposition we present a straightforward argument to prove the same result using the characterization of spectral spaces given by Hochster in [16] and which we recalled in Theorem 0.2.46.

Proposition 2.2.5 ([16], Corollary to Proposition 7)**.** *A space $(X,\mathcal{T})$ with a basis of compact and open sets is spectral if and only if X, endowed with the patch topology induced by $\mathcal{T}$, is a compact and Hausdorff space.*

Proof. If X is a spectral space we can assume, without loss of generality, that $X = \mathrm{Spec}(A)$, where A is a ring, and that $\mathcal{T}$ is the usual Zariski topology on $\mathrm{Spec}(A)$ with its basis $\mathcal{P}$ of principal open and compact subsets (see Proposition 0.2.38). It follows by Remark 2.1.8, Theorem 2.1.7 and Proposition 2.2.1 that the $\mathcal{P}$-ultrafilter topology is compact and Hausdorff. Moreover, by Corollary 2.1.10, this topology is equal to the patch topology induced by $\mathcal{T}$.

Conversely, let us suppose that the space $(X,\mathcal{T})$ has a basis $\mathcal{A}$ of open and compact sets and that the patch topology induced by $\mathcal{T}$ is compact and Hausdorff. Let us denote by X^{patch} the space X endowed with the patch topology. Since the patch topology is by definition finer than $\mathcal{T}$, then the space X is compact. Moreover, let $x, y \in X$ be distinct points. Since X^{patch} is an Hausdorff space, we can find U, V disjoint open subsets of X^{patch} such that $x \in U$, $y \in V$. By the definition of patch topology, we can assume $U = (\bigcap \mathcal{U}_1) \cap (\bigcap \mathcal{U}_2^-)$, where both $\mathcal{U}_1$ and $\mathcal{U}_2$ are finite families of open and compact subsets of X. Since $U \cap V = \emptyset$, it follows that $y \in U^- = (\bigcup \mathcal{U}_1^-) \cup (\bigcup \mathcal{U}_2)$. Thus, at least one of the following statements holds

(a) there exists $U_1' \in \mathcal{U}_1$ such that $y \in X \setminus U_1'$.

(b) there exists $U_2' \in \mathcal{U}_2$ such that $y \in U_2'$.

If (a) is true, then U_1' is an open subset of X which contains x and not y; in case (b) is true, then U_2' is an open subset of X which contains y and not x. Either case we can conclude that X satisfies the T_0 axiom.

Now let us show that the basis $\mathcal{A}$ of compact and open subsets of X is closed under finite intersections. Let $A, B \in \mathcal{A}$. Then A, B are clopen subsets of X^{patch}; thus, also $A \cap B$ is a clopen subset of X^{patch}. By compactness of X^{patch}, it follows that $A \cap B$ is compact in X^{patch}; from the fact that the patch topology is finer than $\mathcal{T}$, it follows that $A \cap B$ is compact also in X. Since $A \cap B$ is clearly open in X, we get $A \cap B \in \mathcal{A}$.

To finish our proof it just remains to show that every irreducible closed subset of X has a generic point. Let $C \subseteq X$ be closed and irreducible. Let us consider the family $\mathcal{F} :=$

$\{U \in \mathcal{A} : U \cap C \neq \emptyset\}$. This family is clearly nonempty, since $X \in \mathcal{F}$. Let $U, V \in \mathcal{F}$. As we saw before, $U \cap V \in \mathcal{A}$. Furthermore, $U \cap C$ and $V \cap C$ are nonempty open subsets of C, with respect to the subspace topology induced by X. Since C is irreducible, we get $U \cap V \cap C \neq \emptyset$. This proves that $\mathcal{F}$ is closed under finite intersections. Thus the family $\mathcal{G} := \{U \cap C : U \in \mathcal{F}\}$ has the finite intersection property. Since $\mathcal{F}$ is by definition a family of clopen subsets of X^{patch} and since C, being closed in X, results closed in X^{patch}, it follows that $\mathcal{G}$ is a family of closed subsets of X^{patch} with the finite intersection property. Since X^{patch} is compact, this implies that $\bigcap \mathcal{G} \neq \emptyset$. Let $x \in \bigcap \mathcal{G}$ and let us show that $\mathrm{Ad}(\{x\})= C$. Since $x \in C$ the inclusion $\subseteq$ is trivial. Conversely, let $c \in C$ and let Ω be an open neighborhood of c. Since $\mathcal{A}$ is a basis of X, there exist $U \in \mathcal{A}$ such that $c \in U \subseteq \Omega$. Thus $U \cap C \in \mathcal{G}$. It follows that $x \in U \subseteq \Omega$. This proves the opposite inclusion. □

2.3 Spectral structure of $\mathbf{SStar}_f(A)$ (final part)

We are now finally able to prove that $\mathrm{SStar}_f(A)$ is a spectral space.

Theorem 2.17. *The set $SStar_f(A)$, endowed with the Zariski topology, is a spectral space.*

Proof. Let $\mathscr{U}$ be an ultrafilter on $X := \mathrm{SStar}_f(A)$ and let $\mathcal{S} := \{U_F : F \in \mathbf{f}(A)\}$ be the canonical subbasis of the Zariski topology on X. Since we have already proved in Proposition 1.1.4 that the space $\mathrm{SStar}(A)$ satisfies the T_0 axiom, then, having in mind Proposition 0.1.13 and in view of Corollary 2.2.4, it suffices to show that the set

$$X_{\mathcal{S}}(\mathscr{U}) := \{\star \in X : [\forall U_F \in \mathcal{S}, \star \in U_F \Longleftrightarrow U_F \in \mathscr{U}]\}$$

is nonempty. By Propositions 1.1.7 and 1.1.9, any semistar operation of the form $\bigwedge(U_F)$ (where $F \in \mathbf{f}(A)$) is of finite type. Thus, by Lemma 1.3.2, the semistar operation $\star := \bigvee(\{\bigwedge(U_F) : U_F \in \mathscr{U}\})$ is of finite type. We claim that $\star \in X_{\mathcal{S}}(\mathscr{U})$. For this purpose, fix a finitely generated fractional ideal F of A. It suffices to show that $\star \in U_F$ if and only if $U_F \in \mathscr{U}$. First, we assume $\star \in U_F$, i.e., $1 \in F^\star$. Again, by Lemma 1.3.2, there exist finitely generated fractional ideals $F_1, \ldots, F_n$ of A (not necessarily distinct) such that $1 \in F^{\bigwedge(U_{F_1}) \circ \cdots \circ \bigwedge(U_{F_n})}$ and $U_{F_i} \in \mathscr{U}$, for any $i = 1, \ldots, n$. Take a semistar operation $\sigma \in \bigcap_{i=1}^n U_{F_i}$. By definition, $\sigma \geq \bigwedge(U_{F_i})$, for any $i = 1, \ldots, n$, and thus

$$1 \in F^{\bigwedge(U_{F_1}) \circ \cdots \circ \bigwedge(U_{F_n})} \subseteq F^{\sigma \circ \cdots \circ \sigma} = F^\sigma,$$

i.e., $\sigma \in U_F$. This shows that $\bigcap_{i=1}^n U_{F_i} \subseteq U_F$ and thus, by the properties of filters, $U_F \in \mathscr{U}$, since $U_{F_1}, \ldots, U_{F_n} \in \mathscr{U}$. Conversely, assume that $U_F \in \mathscr{U}$. This implies that $\bigwedge(U_F) \leq \star$.

By definition, $1 \in F^\sigma$, for each $\sigma \in U_F$, and thus

$$1 \in \bigcap_{\sigma \in U_F} F^\sigma =: F^{\bigwedge(U_F)} \subseteq F^\star.$$

The conclusion is now clear. □

The proof we have just given is really not constructive. In fact a canonical way to find a ring D whose prime spectrum is homeomorphic to $\mathrm{SStar}_f(A)$ has not been found yet. However, in the last part of this section, we want to infer about some of the properties that the ring D must satisfy.

First of all, let us note that it follows easily by Proposition 1.1.3 that the order relation induced by the Zariski topology on $\mathrm{SStar}(A)$ (in the same canonical way we showed in Remark 0.1.16) coincides with the natural order $\leq$ we introduced in Chapter 1. Since it is clear that the semistar operations d and $\star_{\{K\}}$ are respectively the minimum and the the maximum of $(\mathrm{SStar}_f(A), \leq)$, it follows from what we observed in Remarks 0.1.21 and 0.2.42 that D has both a unique maximal and minimal ideal. Thus we can take D as a local domain since if we consider $D/\mathfrak{p}$, where $\mathfrak{p}$ is the unique minimal ideal of D, we obtain a domain whose prime spectrum is in 1 to 1 correspondence with the prime spectrum of D. Moreover it is possible to give a lower bound to the dimension of D.

Lemma 2.3.1. *Let A and D be two integral domains such that $\mathrm{SStar}_f(A)$ is homeomorphic to $\mathrm{Spec}(D)$. Then, $\dim(D) \geq |\mathrm{Spec}(A)|$.*

Proof. Let us first assume that $|\mathrm{Spec}(A)| < \infty$. We can rearrange $\mathrm{Spec}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ in a way such that $\mathfrak{p}_i$ is a minimal element of the set $\{\mathfrak{p}_i, \dots, \mathfrak{p}_n\}$, for each $i = 1, \dots, n$. If we now set $\Delta_k := \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$, we have a descending chain $\star_{\{K\}} = s_\emptyset \gneq s_{\Delta_1} \gneq s_{\Delta_2} \gneq \cdots \gneq s_{\Delta_n} = d$, where the inequalities are justified by the defintion of spectral semistar operation and by the way we rearranged $\mathrm{Spec}(A)$. If we recall once again that homeomorphisms preserve ordered couples, we are able to obtain a chain of prime ideals of D of lenght $n = |\mathrm{Spec}(A)|$. The conclusion is now clear.

If $|\mathrm{Spec}(A)| = \infty$, we can use the same argument passages to construct arbitrary long chains of prime ideals of D so that it must be $\dim(D) = \infty$. □

Lemma 2.3.1 leads to the following result.

Corollary 2.3.2. *Let A and D be two integral domains such that A has infinitely many prime ideals and $\mathrm{SStar}_f(A)$ is homeomorphic to $\mathrm{Spec}(D)$. Then D is not Noetherian.*

Proof. It follows by what we have said so far that D is a local domain of infinite dimension. Thus, by Proposition 0.2.20, it cannot be Noetherian. $\square$

Chapter 3

Functorial properties

In this chapter we show that, if $A \subseteq B$ is an extension of integral domains, it is always possible to associate to each $\star \in \mathrm{SStar}(B)$ a semistar operation $\sigma(\star)$ on A. We will focus in particular on the case when B is an overring of A and we will see how the properties of $\sigma(\star)$ are related to those of $\star$. This corrispondence between $\mathrm{SStar}(A)$ and $\mathrm{SStar}(B)$ can be interpretated through the Theory of Categories; thus, in the next section, we recall some basic definitions and provide some easy examples of categories.

3.0 Prerequisites and basic notions

3.0.1 Categories

Following [20, Chapter 1], we say that a *graph* consists of a set O of *objects*, a set A of *arrows* and two functions *dom* and *cod* which assign to each $f \in A$ respectively a domain and a codomain belonging to O. If $f \in A$, $a, b \in O$ and $dom(f) = a$, $cod(f) = b$, we will also write $f : a \to b$.
In a graph *the set of composable pairs of arrows* is the set

$$A \times_O A := \{(g, f) : g, f \in A \quad \text{and} \quad cod(f) = dom(g)\}.$$

A *category* is a graph with two additional functions: the *identity* $id \colon O \to A, c \mapsto id_c$, and the *composition* $\circ \colon A \times_O A \to A, (g, f) \mapsto g \circ f$. These functions are defined through the following axioms:

$$dom(id_a) = a = cod(id_a), \quad dom(g \circ f) = dom(f), \quad cod(g \circ f) = cod(g)$$

$$k \circ (g \circ f) = (k \circ g) \circ f \quad \textit{(associative law)}$$

$$id_b \circ h = h, \quad j \circ id_b = j \quad \textit{(unit law)}$$

for any $a, b \in O$, any $(g, f) \in A \times_O A$, $k \in A$ such that $dom(k) = cod(g)$ and any $h, j \in A$ such that $cod(h) = b = dom(j)$.

Example 3.0.1. (a) An easy example of category is the *category of sets* whose objects are all the sets and whose arrows are all the functions. In this case *id* sends each set in its identity function and $\circ$ sends any pair of composable functions in their usual composed function. We will denote this category by **Set**.
(b) A category is called *discrete* if every arrow is the identity arrow. It is easy to see that each set X is the set of objects of a discrete category (just define the identities on the elements of X to be the only arrows of the category) and that every discrete category is uniquely defined by its set of objects. Thus, discrete categories consist of sets.
(c) A category is called *a preorder* if for each pair of objects (p, p') there is at most one arrow $p \to p'$. Let P be a preorder and define on the set of objects of P the binary relation $\leq$ by

$$p \leq p' \quad :\Longleftrightarrow \quad \text{there exists in } P \text{ an arrow} \quad p \to p'.$$

It is easy to see that $\leq$ is reflexive (because for each object of P there is the identity map) and transitive (because arrows can be composed). Hence a preorder is a set equipped with a reflexive and transitive binary relation. Conversely any set P with such a relation is a preorder, in which the arrows $p \to p'$ correspond to the pairs (p, p') such that $p \leq p'$.

3.0.2 Functors

Let B and C be two categories. A *covariant functor T with domain C and codomain B* is the collection of two functions: the first one is called *object function* and maps each object c of C into an object $T(c)$ of B; the second one is called *arrow function* and maps each arrow $f : c \to c'$ of C into an arrow $T(f) : T(c) \to T(c')$ of B. Since we want T to somehow preserve the structure of the two categories, we also require that

$$T(id_c) = id_{T(c)}, \quad T(g \circ f) = T(g) \circ T(f)$$

for each object c of C and any pair of composable arrows (g, f) of C. A *controvariant functor* is defined in the same way but it inverts the arrows, which means that if $f : c \to c'$ is an arrow in C, then $T(f) : T(c') \to T(c)$.

Example 3.0.2. An easy example of functor is the *power set functor* $\mathcal{P}\colon \mathbf{Set} \to \mathbf{Set}$. If X is a set, then $\mathcal{P}(X)$ is the usual power set of X whose elements are all the possible subsets of X. Moreover if $f\colon X \to Y$ is a function between sets, then $\mathcal{P}(f)\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ is the function which sends each subset S of X in the subset $f(S)$ of Y. It is immediate to note that $\mathcal{P}$ satisfies the two axioms which define a functor and that it is indeed a covariant functor. Having in my mind this instance, it is easy to construct an example of controvariant functor. In fact, if we consider the map $\mathcal{P}^{con}\colon \mathbf{Set} \to \mathbf{Set}$ whose object function is the same one of $\mathcal{P}$ and such that, if $f\colon X \to Y$ is a function between sets, then $\mathcal{P}^{con}(f)\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ is the function which sends each subset S of Y in the subset $f^{-1}(S)$ of X, then it is clear that $\mathcal{P}^{con}$ is a controvariant functor.

3.1 The semistar operation $\sigma(\star)$

Let $A \subseteq B$ be an extension of integral domains, and let K be the quotient field of A. For any semistar operation $\star \in \mathrm{SStar}(B)$ we can define a semistar operation $\sigma(\star) \in \mathrm{SStar}(A)$ by setting

$$F^{\sigma(\star)} := (FB)^{\star} \cap K,$$

for each nonzero A-submodule F of K. In fact we have already noted in Example 0.2.1 (d) that FB is a B-submodule of the quotient field of B.

Some basic properties of the map σ are analyzed in the following Proposition.

Proposition 3.1.1. *Preserve the notation given at the beginning of the present section. Then the following statements hold.*

(1) The map σ is continuous.

(2) If $\star \in \mathrm{SStar}(B)$ is of finite type, then so is $\sigma(\star)$; equivalently, we can say that σ restricts to a map $\mathrm{SStar}_f(B) \longrightarrow \mathrm{SStar}_f(A)$.

Proof. (1) Let $F \in \overline{\mathbf{F}}(A)$. Then we have:

$$\begin{aligned}\sigma^{-1}(V_F^{(A)}) &= \{\star \in \mathrm{SStar}(B) : \sigma(\star) \in V_F\} = \{\star \in \mathrm{SStar}(B) : 1 \in F^{\sigma(\star)}\}\\ &= \{\star \in \mathrm{SStar}(B) : 1 \in (FB)^{\star}\} = V_{FB}^{(B)}.\end{aligned}$$

Since we already noticed that $FB \in \overline{\mathbf{F}}(B)$, then σ is continuous.

(2) Let $I \in \overline{\mathbf{F}}(A)$ and $x \in I^{\sigma(\star)}$; then $x \in (IB)^{\star}$, and, since $\star$ is of finite type, there are $y_1, \dots, y_n \in IB$ such that $x \in (y_1, \dots, y_n)^{\star}$. For every y_i, there is, by the definition of product of ideals, a finitely generated A-module $F_i \subseteq I$ such that $y_i \in F_iB$; let $F :=$

$F_1 + \cdots + F_n$. Then $F \subseteq I$ is finitely generated (as an A-module), and $(y_1, \ldots, y_n) \subseteq FB$; therefore, $x \in (FB)^\star \cap K = F^{\sigma(\star)}$. Thus $\sigma(\star)$ is of finite type. □

The map σ defined above exhibits better properties when A and B have the same quotient field K. In fact, following the theory we recalled in the previous section, we can regard $\sigma\colon \mathrm{SStar}(B) \to \mathrm{SStar}(A)$, $\star \mapsto \sigma(\star)$ as the map induced by the inclusion $A \subseteq B$ through a controvariant functor. Indeed, fix a ring A. The set $\mathrm{Over}(A)$ of all the overrings of A, together with the set inclusions, clearly forms a category. Moreover, we can consider the set $\{\mathrm{SStar}(B) : B \in \mathrm{Over}(A)\}$ as the object set of a category whose arrows are again the inclusions (or, more properly, the topological embeddings). Therefore if we set $T(B) := \mathrm{SStar}(B)$, for any $B \in \mathrm{Over}(A)$, and, if $f : C \subseteq D$ is an inclusion between elements of $\mathrm{Over}(A)$, we define $T(f)\colon \mathrm{SStar}(D) \hookrightarrow \mathrm{SStar}(C)$, $\star \mapsto \star'$ such that $F^{\star'} := (FD)^\star$ (where F is a C-submodule of K), then it is not hard to see that T is a controvariant functor and that $\sigma = T(\iota)$, where $\iota : A \subseteq B$.

The next two Propositions will help us to fix in a more tangible form what we have just noted.

Proposition 3.1.2. *We preserve the notation of the beginning of the present section, and suppose in addition that B is an overring of A. If $\star \in \mathrm{SStar}(B)$, then the following statements hold.*

(1) $\sigma(\star)|_{\overline{\mathbf{F}}(B)} = \star$.

(2) σ is injective.

(3) $\sigma(\star)$ is of finite type if and only if so is $\star$.

Proof. (1) It is straightforward, since if I is a B-module, then $I^{\sigma(\star)} = (IB)^\star = I^\star$.

(2) It follows immediately by (1) since $\overline{\mathbf{F}}(B) \subseteq \overline{\mathbf{F}}(A)$.

(3) The sufficient condition is Proposition 3.1.1 (2). Conversely, suppose that $\sigma(\star)$ is of finite type and let $I \in \overline{\mathbf{F}}(B)$ and $x \in I^\star$. By (1) we have $x \in I^{\sigma(\star)}$, and thus there is a finitely generated A-module $F \subseteq I$ such that $x \in F^{\sigma(\star)}$. Hence, $x \in (FB)^\star$, and, since FB is finitely generated as a B-module and $FB \subseteq IB = I$, then $\star$ is of finite type. □

Proposition 3.1.3. *We preserve the notation of the beginning of the present section and assume that B is an overring of A. Then the map $\sigma\colon \mathrm{SStar}(B) \to \mathrm{SStar}(A)$ is a topological embedding.*

Proof. Let $F \in \overline{\mathbf{F}}(B)$. By Proposition 3.1.1 (1) and 3.1.2 (2), it is enough to show that

$\sigma(V_F^{(B)})$ is open in $\sigma(\mathrm{SStar}(B))$. Since B is an overring of A, then F is an A-module and thus is defined the open set $V_F^{(A)}$. However, since by Proposition 3.1.2 (1) we have that $F^\star = F^{\sigma(\star)}$ for every $\star \in \mathrm{SStar}(B)$, it follows that $\sigma(V_F^{(B)}) = V_F^{(A)} \cap \sigma(\mathrm{SStar}(B))$, which is an open set of $\sigma(\mathrm{SStar}(B))$. The proof is now complete. $\square$

The following counterexamples show that, in Proposition 3.1.2, the hypothesis that A and B have the same quotient field is crucial.

Example 3.1.1. (a) Let $A \subseteq B$ be a ring extension. If K, the quotient field of A, is strictly contained in B, then σ is not injective. In fact, for every $\star \in \mathrm{SStar}(B)$ and every $F \in \overline{\mathbf{F}}(A)$, we have

$$F^{\sigma(\star)} = (FB)^\star \cap K = B^\star \cap K = K$$

(b) Let Z be an indeterminate over $\mathbb{C}$, set $A = \mathbb{C}[Z]$ and let B be the ring of all the entire functions (i.e., the functions that are holomorphic on the entire complex plane). It is easy to see that $A \subseteq B$ and that $\mathbb{C}(Z)$, the quotient field of A, is strictly contained in the quotient field of B.

Let $\star$ be the map defined by

$$F \mapsto F^\star := \bigcap_{\alpha\in\mathbb{C}} FB_{(Z-\alpha)} \quad \text{for any } F \in \overline{\mathbf{F}}(B).$$

Then $\star = s_{\{(Z-\alpha):\alpha\in\mathbb{C}\}}$ is a semistar operation on B. We want to show that $\star$ is not of finite type. As a consequence of the fact that B is a GCD-domain where the *Wedderburn's lemma* holds, we get that B is a Bezout domain (see [28, Chapter 6, §3*] for more complete notions). It follows that every finitely generated ideal of B is principal and thus it is in particular a quasi-$\star$-ideal. Let $\mathfrak{b} \subsetneq B$ be a *free ideal* (i.e., all the functions belonging to $\mathfrak{b}$ have no common zeros). Then $\mathfrak{b}^\star = B$; in fact for every $\alpha \in \mathbb{C}$ and every $g(Z) \in B$ we can always find $h_\alpha(Z) \in \mathfrak{b}$ such that $h_\alpha(\alpha) \neq 0$ and we can write $g(Z) = h_\alpha(Z) \cdot [g(Z)h_\alpha(Z)^{-1}] \in \mathfrak{b}B_{(Z-\alpha)}$. On the other hand, for any finite subset $\{f_1, \dots, f_n\}$ of $\mathfrak{b}$, we have $(f_1, \dots, f_n)^\star = (f_1, \dots, f_n) \subseteq \mathfrak{b} \subsetneq B$. Suppose by contradiction that $\star$ is of finite type and pick $f \in B \setminus \mathfrak{b}$. Then $f \in \mathfrak{b}^\star$ and so there exists $a \in \mathfrak{b}$ such that $f \in (a)^\star$. It follows that $f \in (a)^\star \cap B = (a) \subseteq \mathfrak{b}$. This contradicts our hypothesis. Thus $\star$ is not of finite type.
Since it is clear that $A \subseteq B$, we can consider $\sigma(\star)$. For every $F \in \overline{\mathbf{F}}(A)$ we have that

$$F^{\sigma(\star)} = (FB)^\star \cap \mathbb{C}(Z) = \bigcap_{\alpha\in\mathbb{C}} FBB_{(Z-\alpha)} \cap \mathbb{C}(Z) = \bigcap_{\alpha\in\mathbb{C}} FB_{(Z-\alpha)} \cap \mathbb{C}(Z) = F.$$

Therefore $\sigma(\star) = id_{\overline{\mathbf{F}}(A)}$ and thus is of finite type even if $\star$ is not.

Proposition 3.1.4. *We preserve the notation given at the beginning of the present section and suppose that B is an overring of A. Then:*

$$\sigma(\mathrm{SStar}(B)) = \{\star \in \mathrm{SStar}(A) : \star \geq \star_{\{B\}}\} = \{\star \in \mathrm{SStar}(A) : B \subseteq A^{\star}\}.$$

Proof. If $\star \in \sigma(\mathrm{SStar}(B))$, then $\star = \sigma(\star')$, for some $\star' \in \mathrm{SStar}(B)$, and, for every $I \in \overline{\mathbf{F}}(A)$,

$$I^{\star} = I^{\sigma(\star')} = (IB)^{\star'} \supseteq IB = I^{\star_{\{B\}}}$$

and thus $\star \geq \star_{\{B\}}$.

If $\star \geq \star_{\{B\}}$, then $A^{\star} \supseteq A^{\star_{\{B\}}} = AB = B$.

If $B \subseteq A^{\star}$ and $I \in \overline{\mathbf{F}}(A)$, then $I^{\star}$ is a B-module, since for every $b \in B$ we have $bI^{\star} = (bI)^{\star} \subseteq (A^{\star}I)^{\star} = (AI)^{\star} = I^{\star}$ (last equation follows by Lemma 1.0.2 (3)). Hence, $\star|_{\overline{\mathbf{F}}(B)} \in \mathrm{SStar}(B)$ and thus it suffices to show that $\star = \sigma(\star|_{\overline{\mathbf{F}}(B)})$. Let $F \in \overline{\mathbf{F}}(A)$; since $F \subseteq FB$ it follows that $F^{\star} \subseteq (FB)^{\star} = F^{\sigma(\star|_{\overline{\mathbf{F}}(B)})}$. Conversely, $F^{\sigma(\star|_{\overline{\mathbf{F}}(B)})} = (FB)^{\star} \subseteq (FA^{\star})^{\star} = (FA)^{\star} = F^{\star}$ (we applied Lemma 1.0.2 (3) again). The proof is now complete. $\square$

Chapter 4

Spaces of local rings

In this chapter we consider a ring extension $A \subseteq B$ and we introduce a new topology on the set of the local subrings of B containing A. We then note that if B is the quotient field of A, then it is possible to embed $\mathrm{Spec}(A)$ in the space of all local overrings of A and we will use this result to find conditions for when it is possible to invert the implication of Proposition 1.2.2.

Let $A \subseteq B$ be a ring extension and let $\mathrm{L}(B|A)$ denote the set (possibly empty) of the local subrings of B containing A. We can define on $\mathrm{L}(B|A)$ a topology just by taking as a basis of open sets the collection of the sets of the form $\mathrm{L}(B|A[F])$, where F runs in the family of all the finite subsets of B. We will call this topology *the Zariski topology on* $\mathrm{L}(B|A)$. When A is an integral domain and B is the quotient field of A, then $\mathrm{L}(A) := \mathrm{L}(B|A)$ is simply the space of all the local overrings of A. It is clear that $\mathrm{L}(A)$ is a subset of the space $\mathrm{Over}(A)$ we introduced in Section 1.2 and that the Zariski topology on $\mathrm{L}(A)$ is just the topology induced by the Zariski topology on $\mathrm{Over}(A)$. As an immediate consequence we have that the inclusion $\mathrm{L}(A) \longrightarrow \mathrm{Over}(A)$ is a topological embedding, when the two sets are endowed with their respective Zariski topologies.
The following Lemma is a result of the theory developed in [6, Chapter 2] and allows to relate the spaces $\mathrm{L}(B|A)$ and $\mathrm{Spec}(A)$ through a map with good topological properties.

Lemma 4.0.5. *Let $A \subseteq B$ be a ring extension and consider the canonical map $\lambda\colon \mathrm{L}(B|A) \longrightarrow \mathrm{Spec}(A)$ sending a local ring $C \in \mathrm{L}(B|A)$ with maximal ideal $\mathfrak{m}_C$ into the prime ideal $\mathfrak{m}_C \cap A$ of A. Then,*
(1) λ is continuous.

(2) If A is an integral domain and B is the quotient field of A, then $\mathrm{Spec}(A)$ *is homeomorphic to a subspace of* $\mathrm{L}(A)$. *Moreover, λ is a topological retraction.*

Proof. (1) It suffices to show that $\lambda^{-1}(D(f)) = L(B|A[f^{-1}])$, for any element $f \in A$. Thus, it is enough to prove that $f \notin \mathfrak{m}_C$ if and only if $f^{-1} \in C$, for any $C \in L(B|A)$ and $f \in A$. Let us suppose that $f^{-1} \in C$. If we assume that $f \in \mathfrak{m}_C$, then $f \cdot f^{-1} = 1 \in \mathfrak{m}_C$ which is a contradiction. Conversely, if $f \notin \mathfrak{m}_C$ and we assume that $f^{-1} \notin C$, then $(\mathfrak{m}_C, f)$ is a proper ideal of C which contains $\mathfrak{m}_C$ and thus we have a contradiction for the maximality of $\mathfrak{m}_C$.

(2) Let us denote by $\mathrm{Loc}(A)$ the set $\{A_\mathfrak{p} : \mathfrak{p} \in \mathrm{Spec}(A)\}$. It is easy to see that $\lambda(A_\mathfrak{p}) = \mathfrak{p}$, for any $\mathfrak{p} \in \mathrm{Spec}(A)$. Thus, $h := \lambda|_{\mathrm{Loc}(A)}$ is a bijection and it follows from the previous point that it is also continuous. To show that h is open we want to prove that $Y := \mathrm{Spec}(A) \setminus h\big(\mathrm{Loc}(A) \cap \mathrm{L}(A[x_1, \dots, x_n])\big)$ is closed in $\mathrm{Spec}(A)$ for each finite subset $\{x_1, \dots, x_n\}$ of the quotient field of A. In order to make our notation easier, we will denote by $\Gamma(x_1, \dots, x_n)$ the set $\mathrm{L}(A) \setminus \mathrm{L}(A[x_1, \dots, x_n])$. Then,

$$Y = h\big(\mathrm{Loc}(A) \cap \Gamma(x_1, \dots, x_n)\big) = \bigcup_{i=1}^{n} h\big(\mathrm{Loc}(A) \cap \Gamma(x_i)\big)$$

and thus it is enough to study the case when the set $\{x_1, \dots, x_n\}$ is a singleton $\{x\}$. Let us consider the ideal $J := \{r \in A : rx \in A\}$ of A. It is easy to see that for each $\mathfrak{p} \in \mathrm{Spec}(A)$, $J \subseteq \mathfrak{p}$ if and only if $x \notin A_\mathfrak{p}$ and this last condition is equivalent to say that $A_\mathfrak{p} \in \Gamma(x)$. Therefore, $Y = V(J)$ is closed in the Zariski topology on $\mathrm{Spec}(A)$. This proves that $\mathrm{Spec}(A)$ is homeomorphic to $\mathrm{Loc}(A)$. Since $\lambda(A_\mathfrak{p}) = \mathfrak{p}$, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, it follows immediately that λ is a topological retraction. $\square$

Next Proposition, combined with Lemma 4.0.5, represents the crucial result that allows to see when it is possible to invert the implication of Proposition 1.2.2.

Proposition 4.0.6. *Let A be an integral domain, Y be a nonempty subspace of* $\mathrm{L}(A)$ *and assume that $\wedge_Y$ is a semistar operation of finite type. If $\lambda\colon\ \mathrm{L}(A) \longrightarrow \mathrm{Spec}(A)$ is the canonical continuous map, then $\lambda(Y)$ is compact.*

Proof. Let $\{D(f_i) : i \in I\}$ ($f_i \in A$) be a collection of basic open sets of $\mathrm{Spec}(A)$ such that $\bigcup\{D(f_i) : i \in I\} \supseteq \lambda(Y)$. Then for any $B \in Y$ there is an element $i_B \in I$ such that $f_{i_B} \notin \mathfrak{m}_B \cap A$. Since $f_{i_B} \in A \subseteq B$ and $\mathfrak{m}_B$ is a maximal ideal of B, it must be $f_{i_B}^{-1} \in B$. Thus, if $\mathfrak{a}$ is the ideal of A generated by the set $\{f_i : i \in I\}$ we have that $1 \in \mathfrak{a}B$, for any

$B \in Y$, i.e., $1 \in \mathfrak{a}^{\wedge_Y}$. Since $\wedge_Y$ is of finite type, there is a finitely generated ideal $\mathfrak{b}$ of A contained in $\mathfrak{a}$ such that $1 \in \mathfrak{b}^{\wedge_Y}$. Let J be a finite subset of I such that the set $\{f_j : j \in J\}$ generates $\mathfrak{b}$. It suffices to show that $\bigcup\{D(f_j) : j \in J\} \supseteq \lambda(Y)$. If, for some $B \in Y$, we had $\{f_j : j \in J\} \subseteq \mathfrak{m}_B \cap A$, it would follow that $\mathfrak{b}B \subseteq \mathfrak{m}_B$. Since $1 \in \mathfrak{b}^{\wedge_Y}$ this would mean that $1 \in \mathfrak{m}_B$ which is clearly a contradiction. The proof is now complete. □

Corollary 4.0.7. *Preserve the notation of Proposition 4.0.6 and let Y be a subspace of* $\mathrm{L}(A)$ *such that $\lambda|_Y$ is a topological embedding. Then $\wedge_Y$ is a semistar operation of finite type if and only if Y is compact.*

Proof. It is enough to apply Proposition 1.2.2 and Proposition 4.0.6 remembering that λ is a continuous map. □

Corollary 4.0.8. *Let A be an integral domain and let $\Delta \subseteq \mathrm{Spec}(A)$. Then s_Δ is a semistar operation of finite type if and only if Δ is compact.*

Proof. It is enough to apply Corollary 4.0.7 noting that $s_\Delta = \wedge_{\lambda^{-1}(\Delta)}$ and remembering that, by Lemma 4.0.5 (2), $\lambda|_{\mathrm{Loc}(A)} \colon \mathrm{Loc}(A) \longrightarrow \mathrm{Spec}(A)$ is an homeomorphism. □

The following Proposition extends even more the class of overrings for which the converse of Proposition 1.2.2 holds.

Proposition 4.0.9. *Let A be an integral domain and Y be a nonempty collection of valuation overrings of A. Then, $\wedge_Y$ is of finite type if and only if Y is a compact subspace of* $\mathrm{L}(A)$.

Proof. The sufficient condition follows immediately by Proposition 1.2.2.
Conversely, assume that $\wedge_Y$ is of finite type and let $\mathcal{U}$ be an open cover of Y. Clearly, a subbasis of open sets of Y, as a subspace of $\mathrm{L}(A)$, consists of the sets of the form $B_f := \{V \in Y : f \in V\}$, for any element f of the quotient field K of A. Thus, by Alexander's subbasis Theorem (see Theorem 0.1.28), we can assume that $\mathcal{U} = \{B_{f_i} : i \in I\}$ and that every f_i is nonzero. If F is the A-submodule of K generated by the set $\{f_i^{-1} : i \in I\}$, then, by definition, $1 \in F^{\wedge_Y}$ and, since $\wedge_Y$ is of finite type, there is a finite subset J of I such that $1 \in G^{\wedge_Y}$, where $G := (\{f_j^{-1} : j \in J\})$. We want to show that $\{B_{f_j} : j \in J\}$ is a (finite) subcover of Y. If not, then there exists a valuation domain $V \in Y$ such that $f_j \notin V$ for any $j \in J$, and hence f_j^{-1} is an element of the maximal ideal $\mathfrak{m}$ of V, for any $j \in J$. It follows that $G \subseteq \mathfrak{m}$. Since $1 \in G^{\wedge_Y}$, we infer, in particular, that $1 \in GV \subseteq \mathfrak{m}$, which is clearly a contradiction. The proof is now complete. □

Summing up the results we have proved in this chapter, in view of Proposition 1.2.2 and 4.0.9 and Corollary 4.0.8, we can state that if Y is either a collection of localizations of an integral domain A or a collection of valuation overrings of A, then compactness of Y is equivalent to the requirement that the semistar operation $\wedge_Y$ on A is of finite type. This gives evidence to the following

Conjecture. *Let Y be any subspace of* $\mathrm{L}(A)$*. If $\wedge_Y$ is of finite type, then Y is compact.*

We end this chapter with a notable example of how it possible to use Proposition 4.0.9 to prove interesting topological results about the space $\mathrm{L}(A)$.

Remark 4.0.10. Let Y be the space of all the valuation overrings of an integral domain A and set $\wedge_Y =: b$. Having in mind Proposition 0.2.26 we can note that b is the semistar operation which sends each $I \in \overline{\mathbf{F}}(A)$ in its integral closure $\overline{I}$. Thus an element $x \in K$ is in I^b if and only if there is an integer n and there are elements $a_i \in I^i$, $i = 1, \ldots, n$, such that $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$. Now it is easy to see that b is of finite type, and thus Proposition 4.0.9 implies that Y is compact.

Chapter 5

An insight on spectral semistar operations

In Example 1.0.1 (g), (h) we introduced the notions of spectral semistar operation and stable semistar operation. In this last chapter we see that it is always possible to associate to each semistar operation another semistar operation which is stable and of finite type. We will then focus our attention on spectral semistar operations and semifinite semistar operations (this definition will be recalled later). In particular we will study the behavior of the stable semistar operations of finite type associated to them.
We start by introducing the main tools we will deal with and by recalling their most important properties.

5.0 Background results

5.0.1 The semistar operation $\tilde{\star}$

Let A be an integral domain. Following Example 1.0.1 (h), we say that $\star \in \mathrm{SStar}(A)$ is *stable* if $(F \cap G)^\star = F^\star \cap G^\star$ for any $F, G \in \overline{\mathbf{F}}(A)$. Given $\star \in \mathrm{SStar}(A)$ set

$$F^{\tilde{\star}} := \bigcup\{(F : \mathfrak{a}) : \mathfrak{a} \text{ is a finitely generated ideal of } A \text{ and } \mathfrak{a}^\star = A^\star\}$$

for any $F \in \overline{\mathbf{F}}(A)$. Then it is not hard to see that $\tilde{\star}$ is a stable semistar operation on A of finite type. We will call $\tilde{\star}$ *the stable semistar operations of finite type associated to* $\star$. The two following crucial results come from [11, Corollaries 2.7(2,a) and 3.5(2)].

Proposition 5.0.1. *Let A be an integral domain and let $\star$ be a semistar operation on A. Then $\tilde{\star} = s_{\mathrm{QMax}^{\star_f}(A)}$ and $\mathrm{QMax}^{\star_f}(A) = \mathrm{QMax}^{\tilde{\star}}(A)$.*

This new, alternative form for $\tilde{\star}$ allows us to get the following Corollary.

Corollary 5.0.2. *Let A be an integral domain and let $\star \in \mathrm{SStar}(A)$. Then,*

(i) $\tilde{\star}$ is a spectral semistar operation.

(ii) $\tilde{\star} \leq \star_f \leq \star$.

(iii) If $\star_1, \star_2 \in \mathrm{SStar}(A)$ are such that $\star_1 \leq \star_2$, then $\widetilde{\star_1} \leq \widetilde{\star_2}$.

In [10, Corollary 3.9(2)] the authors find an important characterization for those semistar operations $\star$ such that $\star = \tilde{\star}$.

Proposition 5.0.3. *Let A be an integral domain and let $\star \in \mathrm{SStar}(A)$. Then the following conditions are equivalent:*

(i) $\star = \tilde{\star}$

(ii) $\star$ is stable and of finite type.

(iii) $\star$ is spectral and of finite type.

As a consequence of last Proposition we have the easy fact that for any $\star \in \mathrm{SStar}(A)$, $(\tilde{\star})_f = \tilde{\star} = \tilde{\tilde{\star}}$.

5.0.2 The inverse topology

To prove the main results of this chapter we will use a topological approach. In particular we will deeply rely on *the inverse topology* of the Zariski topology on $\mathrm{Spec}(A)$. In general if $(X, \mathcal{T})$ is a spectral space, then the inverse topology of $\mathcal{T}$ is the topology whose basis of closed sets is the collection of all the open and compact subspaces of $(X, \mathcal{T})$. Given a space X, we will denote by X^{inv} the set X endowed with the inverse topology and by $\mathrm{Ad}^i(Y)$ the closure of a subset Y of X with respect to the inverse topology.
In [16, Proposition 8] the author proves an important result which also justifies the choice of the name given to the inverse topology.

Proposition 5.0.4. *Let X be a spectral space. Then, X^{inv} is a spectral space. Moreover, for any $x, y \in X$, $\mathrm{Ad}^i(\{x\}) \subseteq \mathrm{Ad}^i(\{y\})$ if and only if $\mathrm{Ad}(\{y\}) \subseteq \mathrm{Ad}(\{x\})$.*

Furthermore, the following result is a consequence of [8, Remark 2.2 and Proposition 2.6] combined with the fact that the patch topology is, by definition, finer than the topology it is induced by (see the end of Section 2.1) and, as we saw in Proposition 2.2.5, the patch topology of a spectral space is always compact.

Proposition 5.0.5. *Let X be a spectral space. Then, for every subset $Y \subseteq X$, $\mathrm{Ad}^i(Y)$ is compact in X.*

Since, as we mentioned, we will use the inverse topology of the Zariski topology on $\mathrm{Spec}(A)$, we focus in the following Remark on two of its basic properties.

Remark 5.0.6. If A is an integral domain and $\mathfrak{a} := (f_1, \dots, f_n)$ is a finitely generated ideal of A, then $D(\mathfrak{a}) = \bigcup_{i=1}^n D(f_i)$ is an open and compact subspace of $\mathrm{Spec}(A)$ (see Proposition 0.2.38) and thus it is, by definition, a basic closed set, with respect to the inverse topology. Conversely, if $D(\mathfrak{a})$ is compact and $\mathfrak{a} = (f_\lambda)_{\lambda\in\Lambda}$, then $D(\mathfrak{a}) = \bigcup_{\lambda\in\Lambda} D(f_\lambda) = \bigcup_{i=1}^n D(f_{\lambda_i}) = D(\mathfrak{b})$, where $\mathfrak{b} = (f_{\lambda_1}, \dots, f_{\lambda_n})$ is finitely generated.

5.1 The stable semistar operation of finite type associated to a spectral semistar operation

Recalling Example 1.0.1 (g), we say that a semistar operation $\star$ on an integral domain A is *spectral* if there is a nonempty set Y of prime ideals of A such that

$$F^\star = \bigcap_{\mathfrak{p}\in Y} FA_{\mathfrak{p}}, \quad \text{for any } F \in \overline{\mathbf{F}}(A)$$

In this case we will denote $\star$ by s_Y. Since localizations commute with finite intersections, it follows that every spectral semistar operation is stable.

Next Proposition focuses on $\tilde{\star}$ in case $\star$ is a spectral semistar operation. In the following we will say that two semistar operations $\star_1, \star_2$ are *weakly equivalent* if $\widetilde{\star_1} = \widetilde{\star_2}$.

Proposition 5.1.1. *Let A be an integral domain and let Y, Z be nonempty subsets of* $\mathrm{Spec}(A)$*. Then, the following conditions are equivalent.*

(i) s_Y and s_Z are weakly equivalent.

(ii) $\mathrm{Ad}^i(Y) = \mathrm{Ad}^i(Z)$.

(iii) If $\mathfrak{a}$ is a finitely generated ideal of A, then $Y \subseteq D(\mathfrak{a})$ if and only if $Z \subseteq D(\mathfrak{a})$.

Proof. (ii) $\Longrightarrow$ (iii). Having in mind Remark 5.0.6, we have that, for any finitely generated ideal $\mathfrak{a}$ of A, $Y \subseteq D(\mathfrak{a})$ if and only if $Z \subseteq \mathrm{Ad}^i(Z) = \mathrm{Ad}^i(Y) \subseteq D(\mathfrak{a}) = \mathrm{Ad}^i(D(\mathfrak{a}))$.

(iii) $\Longrightarrow$ (ii). Let Ω be an open and compact subspace of $\mathrm{Spec}(A)$. Still by Remark 5.0.6, we get that $\Omega = D(\mathfrak{a})$, for some finitely generated ideal $\mathfrak{a}$ of A. Then $\Omega \supseteq Y$ if and only if $\Omega \supseteq Z$, i.e, by definition, $\mathrm{Ad}^i(Y) = \mathrm{Ad}^i(Z)$.

(i) $\Longleftrightarrow$ (iii). By [27, Proposition 2.4 (iii)], s_Y and s_Z are weakly equivalent if and only if for any finitely generated ideal $\mathfrak{a}$ of A, we have $\mathfrak{a}^{s_Y} = A^{s_Y} \iff \mathfrak{a}^{s_Z} = A^{s_Z}$. Then, the conlcusion is an immediate consequence of the following Lemma.

Lemma 5.1.2. *Let A be an integral domain and $Y \subseteq \mathrm{Spec}(A)$. Then, for any prime ideal $\mathfrak{a}$ of A, $\bigcap_{\mathfrak{p}\in Y} \mathfrak{a}A_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in Y} A_{\mathfrak{p}}$ if and only if $Y \subseteq D(\mathfrak{a})$.*

Proof. Let us suppose that $\bigcap_{\mathfrak{p}\in Y} \mathfrak{a}A_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in Y} A_{\mathfrak{p}}$. If, by way of contradiction, $Y \nsubseteq D(\mathfrak{a})$, then there exists $\mathfrak{p} \in Y$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Let $\frac{a}{b} \in \bigcap_{\mathfrak{p}\in Y} A_{\mathfrak{p}}$, with $a \in A\setminus\mathfrak{p}$. If $\frac{a}{b} \in \bigcap_{\mathfrak{p}\in Y} \mathfrak{a}A_{\mathfrak{p}}$, then there exist $c \in \mathfrak{a}, d \in A$ and $e \in A\setminus\bigcup_{\mathfrak{p}\in Y} A_{\mathfrak{p}}$ such that $\frac{a}{b} = c\frac{d}{e}$. Then $\mathfrak{a} \ni c(db) = ae \notin \mathfrak{p} \supseteq \mathfrak{a}$. Thus we have a contradiction.
Conversely, assume that $Y \subseteq D(\mathfrak{a})$ and let us show that $\bigcap_{\mathfrak{p}\in Y} \mathfrak{a}A_{\mathfrak{p}} = \bigcap_{\mathfrak{p}\in Y} A_{\mathfrak{p}}$. The inclusion $\subseteq$ always holds; then let $\frac{a}{b} \in \bigcap_{\mathfrak{p}\in Y} A_{\mathfrak{p}}$. Since $Y \subseteq D(\mathfrak{a})$, then for each $\mathfrak{p} \in Y$ we can find an element $f_{\mathfrak{p}} \in \mathfrak{a} \setminus \mathfrak{p}$. Thus $\frac{a}{b} = f_{\mathfrak{p}}\frac{a}{f_{\mathfrak{p}}b} \in \mathfrak{a}A_{\mathfrak{p}}$ for any $\mathfrak{p} \in Y$. It follows that also the inclusion $\supseteq$ holds. The proof is complete. □

□

Corollary 5.1.3. *Let A be an integral domain and Y be a subset of $\mathrm{Spec}(A)$. Then $\widetilde{s_Y} = s_{\mathrm{Ad}^i(Y)}$.*

Proof. By Proposition 5.0.5, the set $\mathrm{Ad}^i(Y)$ is compact, with respect to the Zariski topology. Hence the stable semistar operation $s_{\mathrm{Ad}^i(Y)}$ is also of finite type, by Proposition 4.0.8, and thus $s_{\mathrm{Ad}^i(Y)} = \widetilde{s_{\mathrm{Ad}^i(Y)}}$, by Proposition 5.0.3. The conclusion follows now immediately by Proposition 5.1.1. □

In [21] the author introduces and studies those domains, called *DW domains*, where the semistar operation $w := \tilde{v}$ and the identity are the same. In particular, in Proposition 2.2 he shows that the DW domains are exactly those domains A with no proper *GV ideal*, i.e., no proper finitely generated ideal J such that $J^{-1} = A$. Respecting the focus of our paper, we provide a more topological characterization of DW domains.

Proposition 5.1.4. *For an integral domain A, the following conditions are equivalent.*
(i) A is a DW domain.
(ii) Every $Y \subseteq \mathrm{Spec}(A)$ such that $\bigcap_{\mathfrak{p}\in Y} A_{\mathfrak{p}} = A$ is dense in $\mathrm{Spec}(A)$, with respect to the inverse topology.
(iii) $\mathrm{QMax}^t(A)$ is dense in $\mathrm{Spec}(A)$, with respect to the inverse topology.

Proof. (i) $\Longrightarrow$ (ii). If A is a DW domain and $\bigcap_{\mathfrak{p}\in Y} A_{\mathfrak{p}} = A$, then s_Y is a star operation and therefore, by Example 1.0.1 (c) and Corollary 5.0.2 (iii), $\widetilde{s_Y} \leq \tilde{v} = w = d = s_{\mathrm{Spec}(A)}$; hence, by Corollary 5.1.3, Y is dense in Spec(A), with respect to the inverse topology.
(ii) $\Longrightarrow$ (iii). Since each principal ideal is a quasi-$\star$-ideal, for any $\star \in \mathrm{SStar}(A)$ and t is, by definition, a semistar operation of finite type, it follows from what we noted in Proposition 1.0.4 (2) that for each non-invertible element a of A there exists a quasi-t-maximal ideal $\mathfrak{m}$ such that $a \in \mathfrak{m}$. Therefore $\bigcap_{\mathfrak{p}\in \mathrm{QMax}^t(A)} A_{\mathfrak{p}} = A$. The conclusion is now clear.
(iii) $\Longrightarrow$ (i). By last part of Section 5.0.1 and Corollary 5.1.3, remembering also the alternative form of $\tilde{\star}$ that we gave in Proposition 5.0.1 and that $\mathrm{QMax}^t(A)$ is, by hypothesis, dense with respect to the inverse topology, we finally have that $w = \tilde{v} = s_{\mathrm{QMax}^t(A)} = s_{\widetilde{\mathrm{QMax}^t}(A)} = s_{\mathrm{Spec}(A)} = d$. $\square$

5.2 Semifinite semistar operations

Following [11], we say that a semistar operation $\star$ on an integral domain A is *quasi-spectral* or *semifinite* if every proper quasi-$\star$-ideal is contained in some quasi-$\star$-prime ideal. An alternative easy characterization of semifinite semistar operations is given in the following Lemma.

Lemma 5.2.1. *Let A be an integral domain and $\star$ be a semistar operation on A. Then, $\star$ is semifinite if and only if for any ideal $\mathfrak{a}$ of A, we have* $\mathrm{QSpec}^\star(A) \subseteq D(\mathfrak{a}) \Longleftrightarrow \mathfrak{a}^\star \cap A = A$.

Proof. Let us assume that $\star$ is a semifinite semistar operation on A and that, for any ideal $\mathfrak{a}$ of A, $\mathrm{QSpec}^\star(A) \subseteq D(\mathfrak{a})$. If by contradiction $\mathfrak{a}^\star \cap A \neq A$, then $\mathfrak{a}^\star \cap A$ is a proper quasi-$\star$-ideal; hence, by hypothesis, there exists $\mathfrak{p} \in \mathrm{QSpec}^\star(A)$ such that $\mathfrak{a}^\star \cap A \subseteq \mathfrak{p}$. Thus, $\mathfrak{a} \subseteq \mathfrak{p}$. It follows that $\mathrm{QSpec}^\star(A) \nsubseteq D(\mathfrak{a})$, a contradiction.
Conversely, if $\mathfrak{a}^\star \cap A = A$ and we assume by contradiction that there exists $\mathfrak{p} \in \mathrm{QSpec}^\star(A)$ such that $\mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{a}^\star \cap A \subseteq \mathfrak{p}^\star \cap A = \mathfrak{p} \subsetneq A$ and again we have a contradiction with the fact that $\mathfrak{a}^\star \cap A = A$.
Finally, assume that for any ideal $\mathfrak{a}$ of A, we have $\mathrm{QSpec}^\star(A) \subseteq D(\mathfrak{a}) \Longleftrightarrow \mathfrak{a}^\star \cap A = A$. If by contradiction $\star$ is not semifinite, i.e., there exists a proper quasi-$\star$-ideal $\mathfrak{a}$ of A such that $\mathfrak{a} \nsubseteq \mathfrak{p}$, for any $\mathfrak{p}$ quasi-$\star$-prime ideal, then $\mathrm{QSpec}^\star(A) \subseteq D(\mathfrak{a})$ and, by hypothesis, $\mathfrak{a}^\star \cap A = A$. This means that $\mathfrak{a}$ is not a quasi-$\star$-ideal, a contradiction. $\square$

We present next some notable examples of semifinite semistar operations.

Example 5.2.1. (a) If $\star$ is a semistar operation of finite type, then we have already observed in Proposition 1.0.4 (2) that every proper quasi-$\star$-ideal is contained in some quasi-$\star$-maximal ideal, which is quasi-$\star$-prime. Thus every semistar operation of finite type is semifinite.

(b) If $\mathcal{S} := \{\star_i : i \in I\}$ is a nonempty collection of semifinite semistar operations on A, then $\star := \bigwedge(\mathcal{S})$ is semifinite. Indeed, let $\mathfrak{a}$ be a proper quasi-$\star$-ideal of A. Then, by definition, we have $1 \notin \mathfrak{a}^{\star_{i_0}}$, for some $i_0 \in I$. Thus, $\mathfrak{a}^{\star_{i_0}} \cap A$ is a proper quasi-$\star_{i_0}$-ideal of A. By assumption, there is a quasi-$\star_{i_0}$-prime ideal $\mathfrak{p}$ containing $\mathfrak{a}^{i_0} \cap A$ and, since $\star \leq \star_{i_0}$, $\mathfrak{p}$ is also a quasi-$\star$-prime ideal. Finally $\mathfrak{a} = \mathfrak{a}^{\star} \cap A \subseteq \mathfrak{a}^{\star_{i_0}} \cap A \subseteq \mathfrak{p}$.

(c) By part (a) and (b), every semistar operation of the form $\wedge_Y$, where Y is a nonempty subspace of $\mathrm{Over}(A)$, is semifinite. In particular every spectral semistar operation is semifinite.

(d) Not all semistar operations are semifinite: to see this let V be a valuation domain of dimension 1 which is not discrete and let M be its maximal ideal. From the characterization of DVRs we gave in Theorem 0.2.27, we can infer that V is not a Noetherian ring. Thus M is not finitely generated. By Proposition 0.2.12, it follows that M is not invertible; hence $M^v := (M^{-1})^{-1} \neq M$. Since $\dim(A)$=1, M is the only prime ideal of A and therefore we can conclude that every nonzero principal ideal is a quasi-v-ideal and is not contained in any quasi-v-prime ideal.

Next Remark shows that the implications in Example 5.2.1 (a), (b) work one way only.

Remark 5.2.2. (a) To get an example of a semifinite semistar opereration which is not of finite type, by Corollary 4.0.8 and Example 5.2.1 (c), it suffices to consider s_Δ, where Δ is a non-compact collection of prime ideals.

For a concrete example, let us consider the ring $A := K[x_0, \dots, x_n, \dots]$ of the polynomials in infinitely many indeterminates over a field K, and let Δ be the set of all the finitely generated prime ideals of A. If we show that there is no quasi-s_Δ-maximal ideal of A, then it will follow by Proposition 1.0.4 (2) that s_Δ is a semifinite semistar operation which is not of finite type. Let us suppose that there exists a quasi-s_Δ-maximal ideal $\mathfrak{m}$ of A. Then, $\mathfrak{m}^{s_\Delta} \neq A^{s_\Delta}$, and thus $\mathfrak{m}A_\mathfrak{p} \neq A_\mathfrak{p}$ for some finitely generated prime ideal $\mathfrak{p}$ of A. Therefore, it follows by Lemma 5.1.2 and by the maximality of $\mathfrak{m}$ that $\mathfrak{m} = \mathfrak{p}$. Thus $\mathfrak{m}$ is finitely generated and so we can find $f_1, \dots, f_n$ polynomials in A such that $\mathfrak{m} = (f_1, \dots, f_n)$. Choose an indeterminate x_n which does not appear in any f_i. Recalling the easy fact that

(x_i) is a prime ideal of A for any $i \in \mathbb{N}$, we have that $\mathfrak{m}' = (\mathfrak{m}, x_n)$ is a prime and finitely generated ideal of A, i.e., $\mathfrak{m}' \in \Delta$. Therefore $(\mathfrak{m}')^{s_\Delta} \cap A \subseteq \mathfrak{m}' A_{\mathfrak{m}'} \cap A = \mathfrak{m}'$ and thus $\mathfrak{m}'$ is a quasi-s_Δ-ideal of A which strictly contains $\mathfrak{m}$, in contradiction with the maximality of $\mathfrak{m}$.

(b) Let A be a non-local Dedekind domain and let K be its quotient field; then, A admits proper overrings different from K (take for example the localizations in the maximal ideals of A). Moreover, every proper overring of A is not a fractional ideal. In fact, if $B \in \mathrm{Over}(A)$ is a fractional ideal of A, then it is finitely generated as an A-module, because A is Noetherian. By the classic characterization of integral elements we recalled in Theorem 0.2.15, it follows that B is integral over A. Since A is a Dedekind domain, it is in particular integrally closed and thus $A = B$.

Define $\star \in \mathrm{SStar}(A)$ by

$$F^\star := \begin{cases} F, & \text{if } F \in \mathbf{F}(\mathrm{A}) \\ K, & \text{if } F \in \overline{\mathbf{F}}(A) \setminus \mathbf{F}(\mathrm{A}) \end{cases}$$

Then, every maximal ideal of A is $\star$-closed and this is enough to make $\star$ a semifinite semistar operation. We want to show that $\star$ is not the infimum of a family of semistar operation of finite type.

First of all, $\star$ is not of finite type. In fact if B is a proper overring of A different from K, then $B^\star = K$, but

$$\bigcup\{F^\star : F \in \mathbf{f}(A), F \subseteq B\} = \bigcup\{F : F \in \mathbf{f}(A), F \subseteq B\} = B \subsetneq K.$$

Let now $\sharp$ be a semistar operation of finite type such that $\sharp \geq \star$. We will show that $\sharp = \wedge_{\{K\}}$. Since $\sharp \geq \star$, then every $\sharp$-closed nonzero A-submodule of K is $\star$-closed. Obviously $A^\sharp$ is $\sharp$-closed and thus $A^\sharp = (A^\sharp)^\star$. Since A has no proper overring which is a also a fractional ideal and since by Proposition 1.0.3 (1) we have that $A^\sharp$ is an overring of A, then either $A^\sharp = A$ or $A^\sharp = K$. In the first case, $\sharp|_{\mathbf{F}(A)}$ is a star operation of finite type. We recalled in Proposition 0.2.33 that a Dedekind domain is in particular a Prüfer domain and thus, by the final part of Example 1.0.1 (c), $\sharp|_{\mathbf{F}(A)}$ is the identity. It follows that for any $F \in \overline{\mathbf{F}}(A)$

$$F^\sharp = \bigcup\{G^\sharp : G \subseteq F, G \in \mathbf{f}(A)\} = \bigcup\{G : G \subseteq F, G \in \mathbf{f}(A)\} = F.$$

Therefore $\sharp$ is the identity and we have a contradiction with the fact that $\sharp \geq \star > d$. Hence it can only be $A^\sharp = K$. In this case, $\sharp = \star_{\{K\}}$ because if $f \in F \in \mathbf{F}(A)$, then $K = fA^\sharp = (fA)^\sharp \subseteq F^\sharp$; if $F \in \overline{\mathbf{F}}(A) \setminus \mathbf{F}(A)$, then $K = A^\sharp \subseteq F^\sharp$. Since $\star \neq \star_{\{K\}}$, it follows that $\star$ is not the infimum of a family of semistar operations of finite type.

Proposition 5.2.3. *Let A be an integral domain and let $\star$ be a semifinite semistar operation on A. Then $\tilde{\star} = s_{\mathrm{Ad}^i(\mathrm{QSpec}^\star(A))}$.*

Proof. Since $Y := \mathrm{Ad}^i(\mathrm{QSpec}^\star(A))$ is compact by Proposition 5.0.5, it follows by Proposition 5.0.3 that $s_Y = \widetilde{s_Y}$. Having in mind the alternative form of $\tilde{\star}$ given in Proposition 5.0.1, we get that $\tilde{\star} = s_{\mathrm{QMax}^{\star_f}(A)}$ and thus by Proposition 5.1.1 it suffices to show that $Y = \mathrm{Ad}^i(\mathrm{QMax}^{\star_f}(A))$. Let $\mathfrak{a}$ be a finitely generated ideal of A. By Lemma 5.1.2 and Lemma 5.2.1 we have

$$\mathrm{QMax}^{\star_f}(A) \subseteq D(\mathfrak{a}) \quad \Longleftrightarrow \quad \mathfrak{a}^\star \cap A = A \quad \Longleftrightarrow \quad \mathrm{QSpec}^\star(A) \subseteq D(\mathfrak{a}).$$

The conclusion follows immediately recalling that $D(\mathfrak{a})$ is, by definition, a basic closed set of the inverse topology. $\square$

The following corollary is now immediate.

Corollary 5.2.4. *Let A be an integral domain and let $\star_1, \star_2$ be two semifinite semistar operations on A. Then $\widetilde{\star_1} = \widetilde{\star_2}$ if and only if $\mathrm{Ad}^i(\mathrm{QSpec}^{\star_1}(A)) = \mathrm{Ad}^i(\mathrm{QSpec}^{\star_2}(A))$. In particular, $\tilde{\star} = d$ if and only if $\mathrm{QSpec}^\star(A)$ is dense in $\mathrm{Spec}(A)$ with respect to the inverse topology.*

Bibliography

[1] D.D. Anderson, *Star-operations induced by overrings*, Comm. Algebra 16 (1988 2535-2553).

[2] F. Anderson, D. Anderson, *Examples of star operations on integral domains*, Comm. Algebra 18 (5) (1990) 1621-1643.

[3] M.F. Atiyah, I.G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[4] P.J. Cahen, K.A. Loper, F.Tartarone, *Integer-valued polynomials and Prüfer v-multiplication domains*, Journal of Algebra 226 (2000), 765-787.

[5] O. Cámpoli, *A principal ideal domain that is not a Euclidean domain-operations on integral domains*, Amer. Math. Monthly, 95 (9) (1988), 868871.

[6] D. Dobbs, R. Fedder, M. Fontana, *Abstract Riemann surfaces of integral domains and spectral spaces*, Ann. Mat. Pura Appl. (4) 148 (1987) 101-115.

[7] C.A. Finocchiaro, *Spectral spaces and ultrafilters*, Comm. Algebra 42 (4) (2014) 1496-1508.

[8] C.A. Finocchiaro, M. Fontana, K.A. Loper, *The constructible topology on spaces of valuation domains*, Trans. Ameri. Math. Soc. 365 (12) (2013) 6199-6216.

[9] C.A. Finocchiaro, D. Spirito, *Some topological considerations on semistar operations*, Journal of Algebra 409 (2014) 199-218.

[10] M. Fontana, J. Huckaba, *Localizing systems and semistar operations*, in: S. T. Chapman, S. Glaz (Eds.), *Non-Noetherian Commutative Ring Theory*, Kluwer Academic Publishers, 2000, pp 169-198.

[11] M. Fontana, K.A. Loper, *Nagata Rings, Kronecker function rings and related semistar operations*, Comm. Algebra 31 (10) (2003) 4775-4805.

[12] M. Fontana, K.A. Loper, *The patch topology and the ultrafilter topology on the prime spectrum of a commutative ring*, Comm. Algebra 36 (2008) 2917-2922.

[13] S. Gabelli, *Introduzione alla teoria delle valutazioni*, Lezioni del corso AL430 - Anelli Commutativi e Ideali (2011).

[14] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Appl. Math, vol. 90, Queen's University, Kingston, Ontario, Canada, 1992.

[15] S. Givant, P. Halmos, *Introduction to Boolean algebras*, Undergraduate Texts in Mathematics, Springer, New York, 2009.

[16] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. 142 (1969) 43-60.

[17] C. Huneke, I. Swanson,, *Integral closure of ideals, rings and modules*, London Math. Soc. Lecture Note Ser., vol. 336, Cambridge University Press, Cambridge, 2006.

[18] T. Jech, *Set Theory*, Springer, New York, 1997 (First Edition, Academic Press, 1978).

[19] W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche*, I-II, Math. Z. 41 (1936), 545-577, 665-679.

[20] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Springer, 1978.

[21] A. Mimouni, *Integral domains in which each ideal is a w-ideal*, Comm. Algebra 33 (5) (2005) 1345-1355.

[22] A. Okabe, R. Matsuda, *Kronecker function rings of semistar-operations*, Tsukuba J. Math. 21 (2) (1997), 529-540.

[23] A. Okabe, R. Matsuda, *On generalized Kronecker function rings*, II, Math. J. Ibaraki Univ. 44 (2012), 7-16.

[24] A. Okabe, R. Matsuda, *Semistar-operations on integral domains*, Toyama Math. J. 17 (1994) 1-21.

[25] B. Olberding, *Noetherian spaces of integrally closed rings with an application to intersections of valuation rings*, Comm. Algebra 38 (9) (2010) 3318-3332.

[26] G. Picozza, *Semistar operations and multiplicative ideal theory*, PhD thesis, University Roma Tre, 2004.

[27] G. Picozza, F. Tartarone, *When the semistar operation* $\tilde{\star}$ *is the identity*, Comm. Algebra 36 (5) (2008) 1954-1975.

[28] R. Remmert, *Classical Topics in Complex Function Theory*, Graduate Texts in Mathematics, Springer, New York, 1998.

[29] R.Y. Sharp, *Steps in Commutative Algebra*, London Mathematical Society Student Texts, Cambridge University Press, 2000.

[30] S. Willard, *General Topology*, Dover Publications, New York, 2004.

[31] O. Zariski, P. Samuel, *Commutative algebra*, vol. II, Reprint of the 1960 edition, Grad. Texts in Math., vol. 29, Springer-Verlag, New York, 1975.