Amalgamated algebras along an ideal

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Abstract. Let $f : A \to B$ be a ring homomorphism and J an ideal of B. In this paper, we initiate a systematic study of a new ring construction called the "amalgamation of A with B along J with respect to f". This construction finds its roots in a paper by J.L. Dorroh appeared in 1932 and provides a general frame for studying the amalgamated duplication of a ring along an ideal, introduced and studied by D'Anna and Fontana in 2007, and other classical constructions such as the $A + XB[X]$ and $A + XB[\![X]\!]$ constructions, the CPI-extensions of Boisen and Sheldon, the $D + M$ constructions and the Nagata's idealization.

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1 Introduction

Let A and B be commutative rings with unity, let J ba an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can define the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A,\ j \in J\}$$

called *the amalgamation of A with B along J with respect to f*. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied in [8] and [9]). Moreover, other classical constructions (such as the $A+XB[X]$ construction, the $D + M$ construction and the Nagata's idealization) can be studied as particular cases of the amalgamation.

On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by D.D. Anderson in [1] and motivated by a classical construction due to Dorroh [8], concerning the embedding of a ring without identity in a ring with identity.

The level of generality that we have choosen is due to the fact that the amalgamation can be studied in the frame of pullback constructions. This point of view allows us to provide easily an ample description of the properties of $A \bowtie^f J$, in connection with the properties of A, J and f.

In this paper, we begin a study of the basic properties of $A \bowtie^f J$. In particular, in Section 2, we present all the constructions cited above as particular cases of the amalgamation. Moreover, we show that the CPI extensions (in the sense of Boisen and Sheldon [3]) are related to amalgamations of a special type and we compare Nagata's

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idealization with the amalgamation. In Section 3, we consider the iteration of the amalgamation process, giving some geometrical applications of it.

In the last two sections, we show that the amalgamation can be realized as a pullback and we characterize those pullbacks that arise from an amalgamation (Proposition 4.7. Finally we apply these results to study the basic algebraic properties of the amalgamation, with particular attention to the finiteness conditions.

2 The genesis

Let A be a commutative ring with identity and let $\mathcal{R}$ be a ring without identity which is an A-module. Following the construction described by D.D. Anderson in [1], we can define a multiplicative structure in the A–module $A \oplus \mathcal{R}$, by setting $(a,x)(a',x') := (aa', ax' + a'x + xx')$, for all $a, a' \in A$ and $x, x' \in \mathcal{R}$. We denote by $A\dot{\oplus}\mathcal{R}$ the direct sum $A \oplus \mathcal{R}$ endowed also with the multiplication defined above.

The following properties are easy to check.

Lemma 2.1. [1, Theorem 2.1] *With the notation introduced above, we have:*

(1) *$A\dot{\oplus}\mathcal{R}$ is a ring with identity $(1,0)$, which has an A–algebra structure induced by the canonical ring embedding $\iota_A : A \hookrightarrow A\dot{\oplus}\mathcal{R}$, defined by $a \mapsto (a,0)$ for all $a \in A$.*

(2) *If we identify $\mathcal{R}$ with its canonical image $(0) \times \mathcal{R}$ under the canonical embedding $\iota_{\mathcal{R}} : \mathcal{R} \hookrightarrow A\dot{\oplus}\mathcal{R}$, defined by $x \mapsto (0,x)$, for all $x \in \mathcal{R}$, then $\mathcal{R}$ becomes an ideal in $A\dot{\oplus}\mathcal{R}$.*

(3) *If we identify A with $A \times (0)$ (respectively, $\mathcal{R}$ with $(0) \times \mathcal{R}$) inside $A\dot{\oplus}\mathcal{R}$, then the ring $A\dot{\oplus}\mathcal{R}$ is an A–module generated by $(1,0)$ and $\mathcal{R}$, i.e., $A(1,0) + \mathcal{R} = A\dot{\oplus}\mathcal{R}$. Moreover, if $p_A : A\dot{\oplus}\mathcal{R} \twoheadrightarrow A$ is the canonical projection (defined by $(a,x) \mapsto a$ for all $a \in A$ and $x \in \mathcal{R}$), then*

$$0 \to \mathcal{R} \xrightarrow{\iota_{\mathcal{R}}} A\dot{\oplus}\mathcal{R} \xrightarrow{p_A} A \to 0$$

is a splitting exact sequence of A–modules. □

Remark 2.2. (1) The previous construction takes its roots in the classical construction, introduced by Dorroh [8] in 1932, for embedding a ring (with or without identity, possibly without regular elements) in a ring with identity (see also Jacobson [14], page 155). For completeness, we recall Dorroh's construction starting with a case which is not the motivating one, but that leads naturally to the relevant one (Case 2).

Case 1. Let R be a commutative ring (with or without identity) and let $\mathrm{Tot}(R)$ be its total ring of fractions, i.e., $\mathrm{Tot}(R) := N^{-1}R$, where N is the set of regular elements of R. If we assume that R has a regular element r, then it is easy to see that $R \subseteq \mathrm{Tot}(R)$, and $\mathrm{Tot}(R)$ has identity $1 := \frac{r}{r}$, even if R does not. In this situation we can consider $R[1] := \{x + m \cdot 1 \mid x \in R, m \in \mathbb{Z}\}$. Obviously, if R has an identity, then $R = R[1]$; otherwise, we have that $R[1]$ is a commutative ring with identity, which

contains properly R and it is the smallest subring of $\mathrm{Tot}(R)$ containing R and 1. It is easy to see that:

(a) R and $R[1]$ have the same characteristic,

(b) R is an ideal of $R[1]$ and

(c) if $R \subsetneq R[1]$, then the quotient-ring $R[1]/R$ is canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$, where $n\ (\geq 0)$ is the characteristic of $R[1]$ (or, equivalently, of R).

Case 2. Let R be a commutative ring (with or without identity) and, possibly, without regular elements. In this situation, we possibly have $R = \mathrm{Tot}(R)$, so we cannot perform the previous construction. Following Dorroh's ideas, we can consider in any case R as a $\mathbb{Z}$-module and, with the notation introduced at the beginning of this section, we can construct the ring $\mathbb{Z}\dot{\oplus}R$, that we denote by $\mathtt{Dh}(R)$ in Dorroh's honour. Note that $\mathtt{Dh}(R)$ is a commutative ring with identity $1_{\mathtt{Dh}(R)} := (1,0)$. If we identify, as usual, R with its canonical image in $\mathtt{Dh}(R)$, then R is an ideal of $\mathtt{Dh}(R)$ and $\mathtt{Dh}(R)$ has a kind of minimal property over R, since $\mathtt{Dh}(R) = \mathbb{Z}(1,0) + R$. Moreover, the quotient-ring $\mathtt{Dh}(R)/R$ is naturally isomorphic to $\mathbb{Z}$.

On the bad side, note that if R has an identity 1_R, then the canonical embedding of R into $\mathtt{Dh}(R)$ (defined by $x \mapsto (0,x)$ for all $x \in R$) does not preserve the identity, since $(0,1_R) \neq 1_{\mathtt{Dh}(R)}$. Moreover, in any case (whenever R is a ring with or without identity), the canonical embedding $R \hookrightarrow \mathtt{Dh}(R)$ may not preserve the characteristic.

In order to overcome this difficult, in 1935, Dorroh [9] gave a variation of the previous construction. More precisely, if R has positive characterisitic n, then R can be considered as a $\mathbb{Z}/n\mathbb{Z}$-module, so $\mathtt{Dh}_n(R) := (\mathbb{Z}/n\mathbb{Z})\,\dot{\oplus}R$ is a ring with identity, having characteristic n. Moreover, as above, $\mathtt{Dh}_n(R) = (\mathbb{Z}/n\mathbb{Z})\,(1,0)+R$ and $\mathtt{Dh}_n(R)/R$ is canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

(2) Note that a general version of the Dorroh's construction (previous Case 2) was considered in 1974 by Shores [18, Definition 6.3] for constructing examples of local commutative rings with arbitrarily large Loewy length. We are indebted to L. Salce for pointing out to us that the amalgamated duplication of a ring along an ideal [6] can also be viewed as a special case of Shores construction (cf. also [17]). Moreover, before Shores, Corner in 1969 [4], for studying endomorphisms rings of Abelian groups, considered a similar construction called "split extension of a ring by an ideal".

A natural situation in which we can apply the previous general construction (Lemma 2.1) is the following. Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Note that f induces on J a natural structure of A–module by setting $a \cdot j := f(a)j$, for all $a \in A$ and $j \in J$. Then, we can consider $A\dot{\oplus}J$.

The following properties, except (2) that is easy to verify, follow from Lemma 2.1.

Lemma 2.3. *With the notation introduced above, we have:*

(1) $A\dot{\oplus}J$ *is a ring.*

(2) *The map* $f^{\bowtie} : A\dot{\oplus}J \to A \times B$, *defined by* $(a,j) \mapsto (a, f(a)+j)$ *for all* $a \in A$ *and* $j \in J$, *is an injective ring homomorphism.*

(3) *The map $\iota_A : A \to A\dot{\oplus}J$ (respectively, $\iota_J : J \to A\dot{\oplus}J$), defined by $a \mapsto (a,0)$ for all $a \in A$ (respectively, by $j \mapsto (0,j)$ for all $j \in J$), is an injective ring homomorphism (respectively, an injective A–module homomorphism). If we identify A with $\iota_A(A)$ (respectively, J with $\iota_J(J)$), then the ring $A\dot{\oplus}J$ coincides with $A+J$.*

(4) *Let $p_A : A\dot{\oplus}J \to A$ be the canonical projection (defined by $(a,j) \mapsto a$ for all $a \in A$ and $j \in J$), then the following is a split exact sequence of A–modules:*

$$0 \to J \xrightarrow{\iota_J} A\dot{\oplus}J \xrightarrow{p_A} A \to 0\,. \qquad \square$$

We set

$$A\bowtie^f J := f^{\bowtie}(A\dot{\oplus}J), \quad \Gamma(f) := \{(a, f(a)) \mid a \in A\}.$$

Clearly, $\Gamma(f) \subseteq A\bowtie^f J$ and they are subrings of $A \times B$. The motivation for replacing $A\dot{\oplus}J$ with its canonical image $A\bowtie^f J$ inside $A \times B$ (under $f^{\bowtie}$) is related to the fact that the multiplicative structure defined in $A\dot{\oplus}J$, which looks somewhat "artificial", becomes the restriction to $A\bowtie^f J$ of the natural multiplication defined componentwise in the direct product $A \times B$. The ring $A\bowtie^f J$ will be called *the amalgamation of A with B along J, with respect to $f : A \to B$*.

Example 2.4. *The amalgamated duplication of a ring.*
A particular case of the construction introduced above is the amalgamated duplication of a ring [6]. Let A be a commutative ring with unity, and let E be an A–submodule of the total ring of fractions $\mathrm{Tot}(A)$ of A such that $E \cdot E \subseteq E$. In this case, E is an ideal in the subring $B := (E : E)$ $(:= \{z \in \mathrm{Tot}(A) \mid zE \subseteq E\})$ of $\mathrm{Tot}(A)$. If $\iota : A \to B$ is the natural embedding, then $A\bowtie^{\iota} E$ coincides with $A \bowtie E$, the amalgamated duplication of A along E, as defined in [6]. A particular and relevant case is when $E := I$ is an ideal in A. In this case, we can take $B := A$, we can consider the identity map $\mathrm{id} := \mathrm{id}_A : A \to A$ and we have that $A \bowtie I$, the amalgamated duplication of A along the ideal I, coincides with $A\bowtie^{\mathrm{id}} I$, that we will call also *the simple amalgamation of A along I* (instead of the amalgamation of A along I, with respect to id_A).

Example 2.5. *The constructions $A + \boldsymbol{X}B[\boldsymbol{X}]$ and $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$.*
Let $A \subset B$ be an extension of commutative rings and $\boldsymbol{X} := \{X_1, X_2, \ldots, X_n\}$ a finite set of indeterminates over B. In the polynomial ring $B[\mathbf{X}]$, we can consider the following subring

$$A + \boldsymbol{X}B[\boldsymbol{X}] := \{h \in B[\boldsymbol{X}] \mid h(\mathbf{0}) \in A\}\,,$$

where $\mathbf{0}$ is the $n-$tuple whose components are 0. This is a particular case of the general construction introduced above. In fact, if $\sigma' : A \hookrightarrow B[\boldsymbol{X}]$ is the natural embedding and $J' := \boldsymbol{X}B[\boldsymbol{X}]$, then it is easy to check that $A\bowtie^{\sigma'} J'$ is isomorphic to $A + \boldsymbol{X}B[\boldsymbol{X}]$ (see also the following Proposition 5.1(3)).

Similarly, the subring $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!] := \{h \in B[\![\boldsymbol{X}]\!] \mid h(\mathbf{0}) \in A\}$ of the ring of power series $B[\![\boldsymbol{X}]\!]$ is isomorphic to $A\bowtie^{\sigma''} J''$, where $\sigma'' : A \hookrightarrow B[\![\boldsymbol{X}]\!]$ is the natural embedding and $J'' := \boldsymbol{X}B[\![\boldsymbol{X}]\!]$.

Example 2.6. *The $D + M$ construction.*
Let M be a maximal ideal of a ring (usually, an integral domain) T and let D be a subring of T such that $M \cap D = (0)$. The ring $D + M := \{x + m \mid x \in D,\ m \in M\}$ is canonically isomorphic to $D \bowtie^{\iota} M$, where $\iota : D \hookrightarrow T$ is the natural embedding.

More generally, let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a subset of the set of the maximal ideals of T, such that $M_\lambda \cap D = (0)$ for all $\lambda \in \Lambda$, and set $J := \bigcap_{\lambda\in\Lambda} M_\lambda$. The ring $D + J := \{x + j \mid x \in D,\ j \in J\}$ is canonically isomorphic to $D \bowtie^{\iota} J$. In particular, if $D := K$ is a field contained in T and $J := \mathrm{Jac}(T)$ is the Jacobson ideal of (the K–algebra) T, then $K + \mathrm{Jac}(T)$ is canonically isomorphic to $K \bowtie^{\iota} \mathrm{Jac}(T)$, where $\iota : K \hookrightarrow T$ is the natural embedding.

Example 2.7. *The CPI–extensions (in the sense of Boisen-Sheldon [3]).*
Let A be a ring and P be a prime ideal of A. Let $\boldsymbol{k}(P)$ be the residue field of the localization A_P and denote by ψ_P (or simply, by ψ) the canonical surjective ring homomorphism $A_P \longrightarrow \boldsymbol{k}(P)$. It is wellknown that $\boldsymbol{k}(P)$ is canonically isomorphic to the quotient field of A/P, so we can identify A/P with its canonical image into $\boldsymbol{k}(P)$. Then the subring $\boldsymbol{C}(A, P) := \psi^{-1}(A/P)$ of A_P is called the *CPI–extension of A with respect to P*. It is immediately seen that, if we denote by λ_P (or, simply, by λ) the localization homomorphism $A \longrightarrow A_P$, then $\boldsymbol{C}(A, P)$ coincides with the ring $\lambda(A) + PA_P$. On the other hand, if $J := PA_P$, we can consider $A \bowtie^{\lambda} J$ and we have the canonical projection $A \bowtie^{\lambda} J \to \lambda(A) + PA_P$, defined by $(a, \lambda(a) + j) \mapsto \lambda(a) + j$, where $a \in A$ and $j \in PA_P$. It follows that $\boldsymbol{C}(A, P)$ is canonically isomorphic to $(A \bowtie^{\lambda} PA_P)/(P \times \{0\})$ (Proposition 5.1(3)).

More generally, let I be an ideal of A and let S_I be the set of the elements $s \in A$ such that $s + I$ is a regular element of A/I. Obviously, S_I is a multiplicative subset of A and if $\overline{S_I}$ is its canonical projection onto A/I, then $\mathrm{Tot}(A/I) = (\overline{S_I})^{-1}(A/I)$. Let $\varphi_I : S^{-1}A \longrightarrow \mathrm{Tot}(A/I)$ be the canonical surjective ring homomorphism defined by $\varphi_I(as^{-1}) := (a + I)(s + I)^{-1}$, for all $a \in A$ and $s \in S$. Then, the subring $\boldsymbol{C}(A, I) := \varphi_I^{-1}(A/I)$ of $S_I^{-1}A$ is called the *CPI–extension of A with respect to I*. If $\lambda_I : A \longrightarrow S_I^{-1}A$ is the localization homomorphism, then it is easy to see that $\boldsymbol{C}(A, I)$ coincides with the ring $\lambda_I(A) + S_I^{-1}I$. It will follow by Proposition 5.1(3) that, if we consider the ideal $J := S_I^{-1}I$ of $S_I^{-1}A$, then $\boldsymbol{C}(A, I)$ is canonically isomorphic to $(A \bowtie^{\lambda_I} J)/(\lambda_I^{-1}(J) \times \{0\})$.

Remark 2.8. *Nagata's idealization.*
Let A be a commutative ring and $\mathcal{M}$ a A–module. We recall that, in 1955, Nagata introduced the ring extension of A called *the idealization of $\mathcal{M}$ in A*, denoted here by $A \ltimes \mathcal{M}$, as the A–module $A \oplus \mathcal{M}$ endowed with a multiplicative structure defined by:

$$(a, x)(a', x') := (aa', ax' + a'x), \quad \text{for all } a, a' \in A \text{ and } x, x' \in \mathcal{M}$$

(cf. [15], Nagata's book [16, page 2], and Huckaba's book [13, Chapter VI, Section 25]). The idealization $A \ltimes \mathcal{M}$ is a ring, such that the canonical embedding $\iota_A : A \hookrightarrow A \ltimes \mathcal{M}$ (defined by $a \mapsto (a, 0)$, for all $a \in A$) induces a subring $A^{\ltimes}$ $(:= \iota_A(A))$ of $A \ltimes \mathcal{M}$ isomorphic to A and the embedding $\iota_{\mathcal{M}} : \mathcal{M} \hookrightarrow A \ltimes \mathcal{M}$ (defined by $x \mapsto (0, x)$,

for all $x \in \mathcal{M}$) determines an ideal $\mathcal{M}^{\ltimes}$ ($:= \iota_{\mathcal{M}}(\mathcal{M})$) in $A \ltimes \mathcal{M}$ (isomorphic, as an A–module, to $\mathcal{M}$), which is nilpotent of index 2 (i.e. $\mathcal{M}^{\ltimes} \cdot \mathcal{M}^{\ltimes} = 0$).

For the sake of simplicity, we will identify $\mathcal{M}$ with $\mathcal{M}^{\ltimes}$ and A with $A^{\ltimes}$. If $p_A : A \ltimes \mathcal{M} \to A$ is the canonical projection (defined by $(a, x) \mapsto a$, for all $a \in A$ and $x \in \mathcal{M}$), then

$$0 \to \mathcal{M} \xrightarrow{\iota_{\mathcal{M}}} A \ltimes \mathcal{M} \xrightarrow{p_A} A \to 0$$

is a spitting exact sequence of A–modules. (Note that the idealization $A \ltimes \mathcal{M}$ is also called in [11] *the trivial extension of A by $\mathcal{M}$*.)

We can apply the construction of Lemma 2.1 by taking $\mathcal{R} := \mathcal{M}$, where $\mathcal{M}$ is an A–module, and considering $\mathcal{M}$ as a (commutative) ring without identity, endowed with a trivial multiplication (defined by $x \cdot y := 0$ for all $x, y \in \mathcal{M}$). In this way, we have that the Nagata's idealization is a particular case of the construction considered in Lemma 2.1. Therefore, the Nagata's idealization can be interpreted as a particular case of the general amalgamation construction. Let $B := A \ltimes \mathcal{M}$ and $\iota\ (= \iota_A) : A \hookrightarrow B$ be the canonical embedding. After identifying $\mathcal{M}$ with $\mathcal{M}^{\ltimes}$, $\mathcal{M}$ becomes an ideal of B. It is now straighforward that $A \ltimes \mathcal{M}$ coincides with the amalgamation $A \bowtie^{\iota} \mathcal{M}$.

Although this, the Nagata's idealization and the constructions of the type $A \bowtie^{f} J$ can be very different from an algebraic point of view. In fact, for example, if $\mathcal{M}$ is a nonzero A–module, the ring $A \ltimes \mathcal{M}$ is always not reduced (the element $(0, x)$ is nilpotent, for all $x \in \mathcal{M}$), but the amalgamation $A \bowtie^{f} J$ can be an integral domain (see Example 2.6 and Proposition 5.2).

3 Iteration of the construction $A \bowtie^{f} J$

In the following all rings will always be commutative with identity, and every ring homomorphism will send 1 to 1.

If A is a ring and I is an ideal of A, we can consider the amalgamated duplication of the ring A along its ideal I (= the simple amalgamation of A along I), i.e., $A \bowtie I := \{(a, a+i) \mid a \in A,\ i \in I\}$ (Example 2.4). For the sake of simplicity, set $A' := A \bowtie I$. It is immediately seen that $I' := \{0\} \times I$ is an ideal of A', and thus we can consider again the simple amalgamation of A' along I', i.e., the ring $A'' := A' \bowtie I'$ $(= (A \bowtie I) \bowtie (\{0\} \times I))$. It is easy to check that the ring A'' may not be considered as a simple amalgamation of A along one of its ideals. However, we can show that A'' can be interpreted as an amalgamation of algebras, giving in this way an answer to a problem posed by B. Olberding in 2006 at Padova's Conference in honour of L. Salce.

We start by showing that it is possible to iterate the amalgamation of algebras and the result is still an amalgamation of algebras.

More precisely, let $f : A \to B$ be a ring homomorphism and J an ideal of B. Since $J'^{f} := \{0\} \times J$ is an ideal of the ring $A'^{f} := A \bowtie^{f} J$, we can consider the simple amalgamation of A'^{f} along J'^{f}, i.e., $A''^{f} := A'^{f} \bowtie J'^{f}$ (which coincides with $A'^{f} \bowtie^{\mathrm{id}} J'^{f}$, where $\mathrm{id} := \mathrm{id}_{A'^{f}}$ is the identity mapping of A'^{f}). On the other hand, we can consider the mapping $f^{(2)} : A \to B^{(2)} := B \times B$, defined by $a \mapsto (f(a), f(a))$ for all $a \in A$. Since $J^{(2)} := J \times J$ is an ideal of the ring $B^{(2)}$, we can consider

the amalgamation $A \bowtie^{f^{(2)}} J^{(2)}$. Then, the mapping $A''^{f} \to A \bowtie^{f^{(2)}} J^{(2)}$, defined by $((a, f(a)+j_1), (a, f(a)+j_1)+(0, j_2)) \mapsto (a, (f(a), f(a))+(j_1, j_1+j_2))$ for all $a \in A$ and $j_1, j_2 \in J$, is a ring isomorphism, having as inverse map the map $A \bowtie^{f^{(2)}} J^{(2)} \to A''^{f}$, defined by $(a, (f(a)+j_1, f(a)+j_2)) \mapsto ((a, f(a)+j_1), (a, f(a)+j_1)+(0, j_2-j_1))$ for all $a \in A$ and $j_1, j_2 \in J$. We will denote by $A \bowtie^{2,f} J$ or, simply, $A^{(2,f)}$ (if no confusion can arise) the ring $A \bowtie^{f^{(2)}} J^{(2)}$, that we will call the *2-amalgamation of the A–algebra B along J (with respect to f)*.

For $n \geq 2$, we define the *n-amalgamation of the $A-$algebra B along J (with respect to f)* by setting

$$A \bowtie^{n,f} J := A^{(n,f)} := A \bowtie^{f^{(n)}} J^{(n)},$$

where $f^{(n)} : A \to B^{(n)} := B \times B \times ... \times B$ (n–times) is the diagonal homomorphism associated to f and $J^{(n)} := J \times J \times ... \times J$ (n–times). Therefore,

$$A \bowtie^{n,f} J = \{(a, (f(a), f(a), ..., f(a)) + (j_1, j_2, ..., j_n)) \mid a \in A,\ j_1, j_2, ..., j_n \in J\}.$$

Proposition 3.1. *Let $f : A \to B$ be a ring homomorphism and J an ideal of B. Then $A \bowtie^{n,f} J$ is canonically isomorphic to the simple amalgamation $A^{(n-1,f)} \bowtie J^{(n-1,f)}$ ($= A^{(n-1,f)} \bowtie^{\mathrm{id}} J^{(n-1,f)}$), where $J^{(n-1,f)}$ is the canonical isomorphic image of J inside $A^{(n-1,f)}$ and $\mathrm{id} := \mathrm{id}_{A^{(n-1,f)}}$ is the identity mapping of $A^{(n-1,f)}$.*

Proof. The proof can be given by induction on $n \geq 2$. For the sake of simplicity, we only consider here the inductive step from $n = 2$ to $n + 1$ ($= 3$). It is straightforward that the mapping $A \bowtie^{3,f} J \to A''^{f} \bowtie J''^{f}$, defined by $(a, (f(a), f(a), f(a)) + (j_1, j_2, j_3)) \mapsto (a'', a'' + j'')$, where $a'' := ((a, f(a) + j_1), (a, f(a) + j_1) + (0, j_2 - j_1)) \in A''^{f}$ and $j'' := ((0, 0), (0, j_3 - j_2)) \in J''^{f}$, for all $a \in A$ and $j_1, j_2, j_3 \in J$ establishes a canonical ring isomorphism. □

In particular, let A be a ring and I an ideal of A, the simple amalgamation of $A' := A \bowtie I$ along $I' := \{0\} \times I$, that is $A'' := A' \bowtie I'$, is canonically isomorphic to the 2-amalgamation $A \bowtie^{2,\mathrm{id}} I = \{(a, (a, a) + (i_1, i_2)) \mid a \in A,\ i_1, i_2 \in I\}$.

Example 3.2. We can apply the previous (iterated) construction to curve singularities. Let A be the ring of an algebroid curve with h branches (i.e., A is a one-dimensional reduced ring of the form $K[\![X_1, X_2, \ldots, X_r]\!] / \bigcap_{i=1}^{h} P_i$, where K is an algebraically closed field, $X_1, X_2, \ldots, X_r$ are indeterminates over K and P_i is an height $r-1$ prime ideal of $K[\![X_1, X_2, \ldots, X_r]\!]$, for $1 \leq i \leq r$). If I is a regular and proper ideal of A, then, with an argument similar to that used in the proof of [5, Theorem 14] (where the case of a simple amalgamation of the ring of the given algebroid curve is investigated), it can be shown that $A \bowtie^{n} I$ is the ring of an algebroid curve with $(n+1)h$ branches; moreover, for each branch of A, there are exactly $n+1$ branches of $A \bowtie^{n} I$ isomorphic to it.

4 Pullback constructions

Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B. In the remaining part of the paper, we intend to investigate the algebraic properties of the ring $A \bowtie^{f} J$, in

relation with those of A, B, J and f. One important tool we can use for this purpose is the fact that the ring $A\bowtie^f J$ can be represented as a pullback (see next Proposition 4.2). On the other hand, we will provide a characterization of those pullbacks that give rise to amalgamated algebras (see next Proposition 4.7). After proving these facts, we will make some pertinent remarks useful for the subsequent investigation on amalgamated algebras.

Definition 4.1. We recall that, if $\alpha : A \to C,\ \beta : B \to C$ are ring homomorphisms, the subring $D := \alpha \times_C \beta := \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$ of $A \times B$ is called the *pullback* (or *fiber product*) of α and β.

The fact tat D is a pullback can also be described by saying that the triplet (D, p_A, p_B) is a solution of the universal problem of rendering commutative the diagram built on α and β

$$\begin{array}{ccc} D & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\beta} & C \end{array}$$

where p_A (respectively, p_B) is the restriction to $\alpha \times_C \beta$ of the projection of $A \times B$ onto A (respectively, B).

Proposition 4.2. *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. If $\pi : B \to B/J$ is the canonical projection and $\check{f} := \pi \circ f$, then $A\bowtie^f J = \check{f} \times_{B/J} \pi$.*

Proof. The statement follows easily from the definitions. □

Remark 4.3. Notice that we have many other ways to describe the ring $A\bowtie^f J$ as a pullback. In fact, if $C := A \times B/J$ and $u : A \to C,\ v : A \times B \to C$ are the canonical ring homomorphisms defined by $u(a) := (a, f(a) + J), v((a, b)) := (a, b + J)$, for every $(a, b) \in A\times B$, it is straightforward to show that $A\bowtie^f J$ is canonically isomorphic to $u \times_C v$. On the other hand, if $I := f^{-1}(J)$, $\breve{u} : A/I \to A/I \times B/J$ and $\breve{v} : A \times B \to A/I \times B/J$ are the natural ring homomorphisms induced by u and v, respectively, then $A\bowtie^f J$ is also canonically isomorphic to the pullback of $\breve{u}$ and $\breve{v}$.

The next goal is to show that the rings of the form $A\bowtie^f J$, for some ring homomorphism $f : A \to B$ and some ideal J of B, determine a distinguished subclass of the class of all fiber products.

Proposition 4.4. *Let A, B, C, α, β as in Definition 4.1, and let $f : A \to B$ a ring homomorphism. Then the following conditions are equivalent.*

(i) *There exist an ideal J of B such that $A\bowtie^f J$ is the fiber product of α and β.*

(ii) *α is the composition $\beta \circ f$.*

If the previous conditions hold, then $J = \mathrm{Ker}(\beta)$.

Proof. Assume condition (i) holds, and let a be an element of A. Then $(a, f(a)) \in A\bowtie^f J$ and, by assumption, we have $\alpha(a) = \beta(f(a))$. This prove condition (ii).

Conversely, assume that $\alpha = \beta \circ f$. We want to show that the ring $A \bowtie^f \text{Ker}(\beta)$ is the fiber product of α and β. The inclusion $A \bowtie^f \text{Ker}(\beta) \subseteq \alpha \times_C \beta$ is clear. On the other hand, let $(a, b) \in \alpha \times_C \beta$. By assumption, we have $\beta(b) = \alpha(a) = \beta(f(a))$. This shows that $b - f(a) \in \text{Ker}(\beta)$, and thus $(a, b) = (a, f(a) + k)$, for some $k \in \text{Ker}(\beta)$. Then $A \bowtie^f \text{Ker}(\beta) = \alpha \times_C \beta$ and condition (i) is true.

The last statement of the proposition is straightforward. □

In the previous proposition we assume the existence of the ring homomorphism f. The next step is to give a condition for the existence of f. We start by recalling that a ring homomorphism $r : B \to A$ is called *a ring retraction* if there exists a ring homomorphism $\iota : A \to B$, such that $r \circ \iota = \text{id}_A$. In this situation, ι is necessarily injective, r is necessarily surjective, and A is called a *retract of* B.

Example 4.5. If $r : B \to A$ is a ring retraction and $\iota : A \hookrightarrow B$ is a ring embedding such that $r \circ \iota = \text{id}_A$, then B is naturally isomorphic to $A \bowtie^\iota \text{Ker}(r)$. This is a consequence of the facts, easy to verify, that $B = \iota(A) + \text{Ker}(r)$ and that $\iota^{-1}(\text{Ker}(r)) = \{0\}$ (for more details see next Proposition 5.1(3)).

Remark 4.6. Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then A is a retract of $A\bowtie^f J$. More precisely, $\pi_A : A\bowtie^f J \to A$, $(a, f(a), j) \mapsto a$, is a retraction, since the map $\iota : A \to A\bowtie^f J$, $a \mapsto (a, f(a))$, is a ring embedding such that $\pi_A \circ \iota = \text{id}_A$.

Proposition 4.7. *Let $A, B, C, \alpha, \beta, p_A, p_B$ be as in Definition 4.1. Then, the following conditions are equivalent.*

(i) *$p_A : \alpha \times_C \beta \to A$ is a ring retraction.*

(ii) *There exist an ideal J of B and a ring homomorphism $f : A \to B$ such that $\alpha \times_C \beta = A\bowtie^f J$.*

Proof. Set $D := \alpha \times_C \beta$. Assume that condition (i) holds and let $\iota : A \hookrightarrow D$ be a ring embedding such that $p_A \circ \iota = \text{id}_A$. If we consider the ring homomorphism $f := p_B \circ \iota : A \to B$, then, by using the definition of a pullback, we have $\beta \circ f = \beta \circ p_B \circ \iota = \alpha \circ p_A \circ \iota = \alpha \circ \text{id}_A = \alpha$. Then, condition (ii) follows by applying Proposition 4.4. Conversely, let $f : A \to B$ be a ring homomorphism such that $D = A\bowtie^f J$, for some ideal J of B. By Remark 4.6, the projection of $A\bowtie^f J$ onto A is a ring retraction. □

Remark 4.8. Let $f,\ g : A \to B$ be two ring homomorphisms and J be an ideal of B. It can happen that $A\bowtie^f J = A \bowtie^g J$, with $f \neq g$. In fact, it is easily seen that $A\bowtie^f J = A\bowtie^g J$ if and only if $f(a) - g(a) \in J$, for every $a \in A$.
For example, let $f, g : A[X] \to A[X]$ be the ring homomorphisms defined by $f(X) := X^2,\ f(a) := a,\ g(X) := X^3,\ g(a) := a$, for all $a \in A$, and set $J := XA[X]$. Then $f \neq g$, but $A[X] \bowtie^f J = A[X] \bowtie^g J$, since $f(p) - g(p) \in J$, for all $p \in A[X]$.

The next goal is to give some sufficient conditions for a pullback to be reduced. Given a ring A, we denote by $\mathrm{Nilp}(A)$ the ideal of all nilpotent elements of A.

Proposition 4.9. *With the notation of Definition 4.1, we have:*

(1) *If* $D\ (= \alpha \times_{C} \beta)$ *is reduced, then*

$$\mathrm{Nilp}(A) \cap \mathrm{Ker}(\alpha) = \{0\} \ \textit{and}\ \mathrm{Nilp}(B) \cap \mathrm{Ker}(\beta) = \{0\}.$$

(2) *If at least one of the following conditions holds*

(a) A *is reduced and* $\mathrm{Nilp}(B) \cap \mathrm{Ker}(\beta) = \{0\}$,

(b) B *is reduced and* $\mathrm{Nilp}(A) \cap \mathrm{Ker}(\alpha) = \{0\}$,

then D *is reduced.*

Proof. (1) Assume D reduced. By simmetry, it sufficies to show that $\mathrm{Nilp}(A) \cap \mathrm{Ker}(\alpha) = \{0\}$. If $a \in \mathrm{Nilp}(A) \cap \mathrm{Ker}(\alpha)$, then $(a, 0)$ is a nilpotent element of D, and thus $a = 0$.

(2) By the simmetry of conditions (a) and (b), it is enough to show that, if condition (a) holds, then D is reduced. Let (a, b) be a nilpotent element of D. Then $a = 0$, since $a \in \mathrm{Nilp}(A)$ and A is reduced. Thus we have $(a, b) = (0, b) \in \mathrm{Nilp}(D)$, hence $b \in \mathrm{Nilp}(B) \cap \mathrm{Ker}(\beta) = \{0\}$. □

We study next the Noetherianity of a ring arising from a pullback construction as in Definition 4.1.

Proposition 4.10. *With the notation of Definition 4.1, the following conditions are equivalent.*

(i) $D\ (= \alpha \times_{C} \beta)$ *is a Noetherian ring.*

(ii) $\mathrm{Ker}(\beta)$ *is a Noetherian* D*–module (with the* D*–module structure naturally induced by* p_{B}*) and* $p_{A}(D)$ *is a Noetherian ring.*

Proof. It is easy to see that $\mathrm{Ker}(p_{A}) = \{0\} \times \mathrm{Ker}(\beta)$. Thus, we have the following short exact sequence of D–modules

$$0 \longrightarrow \mathrm{Ker}(\beta) \xrightarrow{i} D \xrightarrow{p_{A}} p_{A}(D) \longrightarrow 0,$$

where i is the natural D–module embedding (defined by $x \mapsto (0, x)$ for all $x \in \mathrm{Ker}(\beta)$). By [2, Proposition (6.3)], D is a Noetherian ring if and only if $\mathrm{Ker}(\beta)$ and $p_{A}(D)$ are Noetherian as D–modules. The statement now follows immediately, since the D–submodules of $p_{A}(D)$ are exactly the ideals of the ring $p_{A}(D)$. □

Remark 4.11. Note that, in Proposition 4.10, we did not require β to be surjective. However, if β is surjective, then p_A is also surjective and so $p_A(D) = A$. Therefore, in this case, D is a Noetherian ring if and only if A is a Noetherian ring and $\mathrm{Ker}(\beta)$ is a Noetherian D–module.

5 The ring $A \bowtie^f J$: some basic algebraic properties

We start with some straightforward consequences of the definition of amalgamated algebra $A \bowtie^f J$.

Proposition 5.1. *Let $f : A \to B$ be a ring homomorphism, J an ideal of B and let $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A,\ j \in J\}$ be as in Section 2.*

(1) *Let $\iota := \iota_{A,f,J} : A \to A \bowtie^f J$ be the natural the ring homomorphism defined by $\iota(a) := (a, f(a))$, for all $a \in A$. Then ι is an embedding, making $A \bowtie^f J$ a ring extension of A (with $\iota(A) = \Gamma(f)$ $(:= \{(a, f(a)) \mid a \in A\}$ subring of $A \bowtie^f J$).*

(2) *Let I be an ideal of A and set $I \bowtie^f J := \{(i, f(i)+j) \mid i \in I, j \in J\}$. Then $I \bowtie^f J$ is an ideal of $A \bowtie^f J$, the composition of canonical homomorphisms $A \overset{\iota}{\hookrightarrow} A \bowtie^f J \twoheadrightarrow A \bowtie^f J / I \bowtie^f J$ is a surjective ring homomorphism and its kernel coincides with I. Hence, we have the following canonical isomorphism:*

$$\frac{A \bowtie^f J}{I \bowtie^f J} \cong \frac{A}{I}\,.$$

(3) *Let $p_{_A} : A \bowtie^f J \to A$ and $p_{_B} : A \bowtie^f J \to B$ be the natural projections of $A \bowtie^f J \subseteq A \times B$ into A and B, respectively. Then $p_{_A}$ is surjective and $\mathrm{Ker}(p_{_A}) = \{0\} \times J$. Moreover, $p_{_B}(A \bowtie^f J) = f(A) + J$ and $\mathrm{Ker}(p_{_B}) = f^{-1}(J) \times \{0\}$. Hence, the following canonical isomorphisms hold:*

$$\frac{A \bowtie^f J}{(\{0\} \times J)} \cong A \quad \text{and} \quad \frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(A) + J\,.$$

(4) *Let $\gamma : A \bowtie^f J \to (f(A) + J)/J$ be the natural ring homomorphism, defined by $(a, f(a) + j) \mapsto f(a) + J$. Then γ is surjective and $\mathrm{Ker}(\gamma) = f^{-1}(J) \times J$. Thus, there exists a natural isomorphism*

$$\frac{A \bowtie^f J}{f^{-1}(J) \times J} \cong \frac{f(A) + J}{J}\,.$$

In particular, when f is surjective we have

$$\frac{A \bowtie^f J}{f^{-1}(J) \times J} \cong \frac{B}{J}\,.$$

□

The ring $B_\diamond := f(A) + J$ (which is a subring of B) has an important role in the construction $A \bowtie^f J$. For instance, if $f^{-1}(J) = \{0\}$, we have $A \bowtie^f J \cong B_\diamond$ (Proposition 5.1(3)). Moreover, in general, J is an ideal also in $B_\diamond$ and, if we denote by $f_\diamond : A \to B_\diamond$ the ring homomorphism induced from f, then $A \bowtie^{f_\diamond} J = A \bowtie^f J$. The next result shows one more aspect of the essential role of the ring $B_\diamond$ for the construction $A \bowtie^f J$.

Proposition 5.2. *With the notation of Proposition 5.1, assume $J \neq \{0\}$. Then, the following conditions are equivalent.*

(i) *$A\bowtie^f J$ is an integral domain.*

(ii) *$f(A)+J$ is an integral domain and $f^{-1}(J)=\{0\}$.*

In particular, if B is an integral domain and $f^{-1}(J)=\{0\}$, then $A\bowtie^f J$ is an integral domain.

Proof. (ii)⇒(i) is obvious, since $f^{-1}(J)=\{0\}$ implies that $A\bowtie^f J\cong f(A)+J$ (Proposition 5.1(3)).

Assume that condition (i) holds. If there exists an element $a\in A\backslash\{0\}$ such that $f(a)\in J$, then $(a,0)\in(A\bowtie^f J)\backslash\{(0,0)\}$. Hence, if j is a nonzero element of J, we have $(a,0)(0,j)=(0,0)$, a contradiction. Thus $f^{-1}(J)=\{0\}$. In this case, as observed above, $A\bowtie^f J\cong f(A)+J$ (Proposition 5.1(3)), so $f(A)+J$ is an integral domain. □

Remark 5.3. (1) Note that, if $A\bowtie^f J$ is an integral domain, then A is also an integral domain, by Proposition 5.1(1).

(2) Let $B=A$, $f=\mathrm{id}_A$ and $J=I$ be an ideal of A. In this situation, $A\bowtie^{\mathrm{id}_A} I$ (the simple amalgamation of A along I) coincides with the amalgamated duplication of A along I (Example 2.4) and it is never an integral domain, unless $I=\{0\}$ and A is an integral domain.

Now, we characterize when the amalgamated algebra $A\bowtie^f J$ is a reduced ring.

Proposition 5.4. *We preserve the notation of Proposition 5.1. The following conditions are equivalent.*

(i) *$A\bowtie^f J$ is a reduced ring.*

(ii) *A is a reduced ring and* $\mathrm{Nilp}(B)\cap J=\{0\}$.

In particular, if A and B are reduced, then $A\bowtie^f J$ is reduced; conversely, if J is a radical ideal of B and $A\bowtie^f J$ is reduced, then B (and A) is reduced.

Proof. From Proposition 4.9(2, a) we deduce easily that (ii)⇒(i), after noting that, with the notation of Proposition 4.2, in this case $\mathrm{Ker}(\pi)=J$.

(i)⇒(ii) By Proposition 4.9(1) and the previous equality, it is enough to show that if $A\bowtie^f J$ is reduced, then A is reduced. This is trivial because, if $a\in\mathrm{Nilp}(A)$, then $(a,f(a))\in\mathrm{Nilp}(A\bowtie^f J)$.

Finally, the first part of the last statement is straightforward. As for the second part, we have $\{0\}=\mathrm{Nilp}(B)\cap J=\mathrm{Nilp}(B)$ (since J is radical, and so $J\supseteq\mathrm{Nilp}(B)$). Hence B is reduced. □

Remark 5.5. (1) Note that, from the previous result, when $B=A$, $f=\mathrm{id}_A$ $(=\mathrm{id})$ and $J=I$ is an ideal of A, we reobtain easily that $A\bowtie I$ $(=A\bowtie^{\mathrm{id}} I)$ is a reduced ring if and only if A is a reduced ring [7, Proposition 2.1].

(2) The previous proposition implies that the property of being reduced for $A\bowtie^f J$ is independent of the nature of f.

(3) If A and $f(A)+J$ are reduced rings, then $A\bowtie^f J$ is a reduced ring, by Proposition 5.4. But the converse is not true in general. As a matter of fact, let $A:=\mathbb{Z}$, $B:=$

$\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})$, $f : A \to B$ be the ring homomorphism such that $f(n) = (n, [n]_4)$, for every $n \in \mathbb{Z}$ (where $[n]_4$ denotes the class of n modulo 4). If we set $J := \mathbb{Z} \times \{[0]_4\}$, then $J \cap \mathrm{Nilp}(B) = \{0\}$, and thus $A \bowtie^f J$ is a reduced ring, but $(0, [2]_4) = (2, [2]_4) + (-2, [0]_4)$ is a nonzero nilpotent element of $f(\mathbb{Z}) + J$.

The next proposition provides an answer to the question of when $A \bowtie^f J$ is a Noetherian ring.

Proposition 5.6. *With the notation of Proposition 5.1, the following conditions are equivalent.*

(i) $A \bowtie^f J$ *is a Noetherian ring.*

(ii) A *and* $f(A) + J$ *are Noetherian rings.*

Proof. (ii)⇒(i). Recall that $A \bowtie^f J$ is the fiber product of the ring homomorphism $\check{f} : A \to B/J$ (defined by $a \mapsto f(a)+J$) and of the canonical projection $\pi : B \to B/J$. Since the projection $p_A : A \bowtie^f J \to A$ is surjective (Proposition 5.1(3)) and A is a Noetherian ring, by Proposition 4.10, it sufficies to show that $J(= \mathrm{Ker}(\pi))$, with the structure of $A \bowtie^f J-$module induced by p_B, is Noetherian. But this fact is easy, since every $A \bowtie^f J-$submodule of J is an ideal of the Noetherian ring $f(A) + J$.

(i)⇒(ii) is a straighforward consequence of Proposition 5.1(3). □

Note that, from the previous result, when $B = A$, $f = \mathrm{id}_A$ $(= \mathrm{id})$ and $J = I$ is an ideal of A, we reobtain easily that $A \bowtie I$ $(= A \bowtie^{\mathrm{id}} I)$ is a Noetherian ring if and only if A is a Noetherian ring [6, Corollary 2.11].

However, the previous proposition has a moderate interest because the Noetherianity of $A \bowtie^f J$ is not directly related to the data (i.e., A, B, f and J), but to the ring $B_\diamond = f(A) + J$ which is canonically isomorphic $A \bowtie^f J$, if $f^{-1}(J) = \{0\}$ (Proposition 5.1(3)). Therefore, in order to obtain more useful criteria for the Noetherianity of $A \bowtie^f J$, we specialize Proposition 5.6 in some relevant cases.

Proposition 5.7. *With the notation of Proposition 5.1, assume that at least one of the following conditions holds:*

(a) J *is a finitely generated* A*–module (with the structure naturally induced by* f*).*

(b) J *is a Noetherian* A*–module (with the structure naturally induced by* f*).*

(c) $f(A) + J$ *is Noetherian as* A*–module (with the structure naturally induced by* f*).*

(d) f *is a finite homomorphism.*

Then $A \bowtie^f J$ *is Noetherian if and only if* A *is Noetherian. In particular, if* A *is a Noetherian ring and* B *is a Noetherian* A*–module (e.g., if* f *is a finite homomorphism [2, Proposition 6.5]), then* $A \bowtie^f J$ *is a Noetherian ring for all ideal* J *of* B*.*

Proof. Clearly, without any extra assumption, if $A \bowtie^f J$ is a Noetherian ring, then A is a Noetherian ring, since it is isomorphic to $A \bowtie^f J/(\{0\} \times J)$ (Proposition 5.1(3)).

Conversely, assume that A is a Noetherian ring. In this case, it is straighforward to verify that conditions (a), (b), and (c) are equivalent [2, Propositions 6.2, 6.3, and

6.5]. Moreover (d) implies (a), since J is an A–submodule of B, and B is a Noetherian A–module under condition (d) [2, Proposition 6.5].

Using the previous observations, it is enough to show that $A\bowtie^f J$ is Noetherian if A is Noetherian and condition (c) holds. If $f(A)+J$ is Noetherian as an A–module, then $f(A)+J$ is a Noetherian ring (every ideal of $f(A)+J$ is an A–submodule of $f(A)+J$). The conclusion follows from Proposition 5.6((ii)⇒(i)).

The last statement is a consequence of the first part and of the fact that, if B is a Noetherian A–module, then (a) holds [2, Proposition 6.2]. □

Proposition 5.8. *We preserve the notation of Propositions 5.1 and 4.2. If B is a Noetherian ring and the ring homomorphism $\breve{f} : A \to B/J$ is finite, then $A\bowtie^f J$ is a Noetherian ring if and only if A is a Noetherian ring.*

Proof. If $A\bowtie^f J$ is Noetherian we already know that A is Noetherian. Hence, we only need to show that if A and B are Noetherian rings and $\breve{f}$ is finite then $A\bowtie^f J$ is Noetherian. But this fact follows immediately from [10, Proposition 1.8]. □

As a consequence of the previous proposition, we can characterize when rings of the form $A+XB[X]$ and $A+XB[\![X]\!]$ are Noetherian. Note that S. Hizem and A. Benhissi [12] have already given a characterization of the Noetherianity of the power series rings of the form $A+XB[\![X]\!]$. The next corollary provides a simple proof of Hizem and Benhissi's Theorem and shows that a similar characterization holds for the polynomial case (in several indeterminates). At the Fez Conference in June 2008, S. Hizem has announced to have proven a similar result in the polynomial ring case with a totally different approach.

Corollary 5.9. *Let $A \subseteq B$ be a ring extension and $\boldsymbol{X} := \{X_1, \ldots, X_n\}$ a finite set of indeterminates over B. Then the following conditions are equivalent.*

(i) *$A + \boldsymbol{X}B[\boldsymbol{X}]$ is a Noetherian ring.*

(ii) *$A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$ is a Noetherian ring.*

(iii) *A is a Noetherian ring and $A \subseteq B$ is a finite ring extension.*

Proof. (iii)⇒(i, ii). With the notations of Example 2.5, recall that $A + \boldsymbol{X}B[\boldsymbol{X}]$ is isomorphic to $A \bowtie^{\sigma'} \boldsymbol{X}B[\boldsymbol{X}]$ (and $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$ is isomorphic to $A \bowtie^{\sigma''} \boldsymbol{X}B[\![\boldsymbol{X}]\!]$). Since we have the following canonical isomorphisms

$$\frac{B[\boldsymbol{X}]}{\boldsymbol{X}B[\boldsymbol{X}]} \cong B \cong \frac{B[\![\boldsymbol{X}]\!]}{\boldsymbol{X}B[\![\boldsymbol{X}]\!]},$$

in the present situation, the homomophism $\breve{\sigma}' : A \hookrightarrow B[\boldsymbol{X}]/\boldsymbol{X}B[\boldsymbol{X}]$ (or, $\breve{\sigma}'' : A \hookrightarrow B[\![\boldsymbol{X}]\!]/\boldsymbol{X}B[\![\boldsymbol{X}]\!]$) is finite. Hence, statements (i) and (ii) follow easily from Proposition 5.8.

(i) (or, (ii)) ⇒ (iii). Assume that $A+\boldsymbol{X}B[\boldsymbol{X}]$ (or, $A+\boldsymbol{X}B[\![\boldsymbol{X}]\!]$) is a Noetherian ring. By Proposition 5.6, or by the isomorphism $(A + \boldsymbol{X}B[\boldsymbol{X}])/\boldsymbol{X}B[\boldsymbol{X}] \cong A$ (respectively $(A+\boldsymbol{X}B[\![\boldsymbol{X}]\!])/\boldsymbol{X}B[\![\boldsymbol{X}]\!] \cong A$), we deduce that A is also a Noetherian ring. Moreover,

by assumption, the ideal I of $A + \boldsymbol{X}B[\boldsymbol{X}]$ (respectively, of $A + \boldsymbol{X}B[\![\boldsymbol{X}]\!]$) generated by the set $\{bX_k \mid b \in B, 1 \leq k \leq n\}$ is finitely generated. Hence $I = (f_1, f_2, \ldots, f_m)$, for some $f_1, f_2, \ldots, f_m \in I$. Let $\{b_{jk} \mid 1 \leq k \leq n\}$ be the set of coefficients of linear monomials of the polynomial (respectively, power series) f_j, $1 \leq j \leq m$. It is easy to verify that $\{b_{jk} \mid 1 \leq j \leq m, 1 \leq k \leq n\}$ generates B as A–module; thus $A \subseteq B$ is a finite ring extension. □

Remark 5.10. Let $A \subseteq B$ be a ring extension, and let X be an indeterminate over B. Note that the ideal $J' = XB[X]$ of $B[X]$ is never finitely generated as an A–module (with the structure induced by the inclusion $\sigma' : A \hookrightarrow B[X]$). As a matter of fact, assume that $\{g_1, g_2, \ldots, g_r\}$ $(\subset B[X])$ is a set of generators of J' as $A-$module and set $N := \max\{\deg(g_i) \mid i = 1, 2, \ldots, r\}$. Clearly, we have $X^{N+1} \in J' \setminus \sum_{i=1}^{r} Ag_i$, which is a contradiction. Therefore, the previous observation shows that the Noetherianity of the ring $A \bowtie^f J$ does not imply that J is finitely generated as an A–module (with the structure induced by f); for instance $\mathbb{R} + X\mathbb{C}[X]$ $(\cong \mathbb{R} \bowtie^{\sigma'} X\mathbb{C}[X]$, where $\sigma' : \mathbb{R} \hookrightarrow \mathbb{C}[X]$ is the natural embedding) is a Noetherian ring (Proposition 5.9), but $X\mathbb{C}[X]$ is not finitely generated as an $\mathbb{R}$–vector space. This fact shows that condition (a) (or, equivalently, (b) or (c)) of Proposition 5.7 is not necessary for the Noetherianity of $A \bowtie^f J$.

Example 5.11. Let $A \subseteq B$ be a ring extension, J an ideal of B and $\boldsymbol{X} := \{X_1, \ldots, X_r\}$ a finite set of intederminates over B. We set $B' := B[\boldsymbol{X}]$, $J' := \boldsymbol{X}J[\boldsymbol{X}]$ and we denote by σ' the canonical embedding of A into B'. By a routine argument, it is easy to see that the ring $A \bowtie^{\sigma'} J'$ is naturally isomorphic to the ring $A + \boldsymbol{X}J[\boldsymbol{X}]$. Now, we want to show that, in this case, we can characterize the Noetherianity of the ring $A + \boldsymbol{X}J[\boldsymbol{X}]$, without assuming a finiteness condition on the inclusion $A \subseteq B$ (as in Corollary 5.9 (iii)) or on the inclusion $A + \boldsymbol{X}J[\boldsymbol{X}] \subseteq B[\boldsymbol{X}]$. More precisely, *the following conditions are equivalent.*

(i) $A + \boldsymbol{X}J[\boldsymbol{X}]$ *is a Noetherian ring.*

(ii) A *is a Noetherian ring,* J *is an idempotent ideal of* B *and it is finitely generated as an* A*–module.*

(i)⇒(ii). Assume that $R := A + \boldsymbol{X}J[\boldsymbol{X}] = A + J'$ is a Noetherian ring. Then, clearly, A is Noetherian, since A is canonically isomorphic to R/J'. Now, consider the ideal L of R generated by the set of linear monomials $\{bX_i \mid 1 \leq i \leq r,\ b \in J\}$. By assumption, we can find $\ell_1, \ell_2, \ldots, \ell_t \in L$ such that $L = \sum_{k=1}^{t} \ell_k R$. Note that $\ell_k(0, 0, \ldots, 0) = 0$, for all k, $1 \leq k \leq t$. If we denote by b_k the coefficient of the monomial X_1 in the polynomial ℓ_k, then it is easy to see that $\{b_1, b_2, \ldots, b_t\}$ is a set of generators of J as an A–module.

The next step is to show that J is an idempotent ideal of B. By assumption, J' is a finitely generated ideal of R. Let

$$g_h := \sum_{j_1+\ldots+j_r=1}^{m_h} c_{h,j_1\ldots j_r} X_1^{j_1} \cdots X_r^{j_r}, \text{ with } h = 1, 2, \ldots, s,$$

be a finite set of generators of J' in R. Set $\overline{j_1} := \max\{j_1 \mid c_{h,j_1 0\dots 0} \neq 0, \text{ for } 1 \leq h \leq s\}$. Take now an arbitrary element $b \in J$ and consider the monomial $bX_1^{\overline{j_1}+1} \in J'$. Clearly, we have

$$bX_1^{\overline{j_1}+1} = \sum_{h=1}^{s} f_h g_h, \text{ with } f_h := \sum_{e_1+\dots+e_r=0}^{n_h} d_{h,e_1\dots e_r} X_1^{e_1}\cdots X_r^{e_r} \in R\,.$$

Therefore,

$$b = \sum_{h=1}^{s} \sum_{j_1+e_1=\overline{j_1}+1} c_{h,j_1 0\dots 0} d_{h,e_1 0\dots 0}\,.$$

Since $j_1 < \overline{j_1}+1$, we have necessarily that $e_1 \geq 1$. Henceforth f_h belongs to J' and so $d_{h,e_1 0\dots 0} \in J$, for all h, $1 \leq h \leq s$. This proves that $b \in J^2$.

(ii)⇒(i). In this situation, by Nakayama's lemma, we easily deduce that $J = eB$, for some idempotent element $e \in J$. Let $\{b_1, \dots, b_s\}$ be a set of generators of J as an A–module, i.e., $J = eB = \sum_{1\leq h\leq s} b_h A$. We consider a new set of indeterminates over B (and A) and precisely $\boldsymbol{Y} := \{Y_{ih} \mid 1 \leq i \leq r, 1 \leq h \leq s\}$. We can define a map $\varphi : A[\boldsymbol{X}, \boldsymbol{Y}] \to B[\boldsymbol{X}]$ by setting $\varphi(X_i) := eX_i$, and $\varphi(Y_{ih}) := b_h X_i$, for all $i = 1, \dots, r,\ h = 1, \dots, s$. It is easy to see that φ is a ring homomorphism and $\mathrm{Im}(\varphi) \subseteq R\ (= A + \boldsymbol{X}J[\boldsymbol{X}])$. Conversely, let

$$f := a + \sum_{i=1}^{r} \left(\sum_{e_{i_1}+\dots+e_{i_r}=0}^{n_i} c_{i,e_{i_1}\dots e_{i_r}} X_1^{e_{i_1}} \cdots X_r^{e_{i_r}} \right) X_i \in R \text{ (and so } c_{i,e_{i_1}\dots e_{i_r}} \in J)\,.$$

Since $J = \sum_{1\leq h\leq s} b_h A$, then for all $i = 1, \dots, r$ and $e_{i_1}, \dots, e_{i_r}$, with $e_{i_1} + \dots + e_{i_r} \in \{0, \dots, n_i\}$, we can find elements $a_{i,e_{i_1}\dots e_{i_r},h} \in A$, with $1 \leq h \leq s$, such that $c_{i,e_{i_1}\dots e_{i_r}} = \sum_{h=1}^{s} a_{i,e_{i_1}\dots e_{i_r},h} b_h$. Consider the polynomial

$$g := a + \sum_{i=1}^{r} \sum_{h=1}^{s} \sum_{e_{i_1}+\dots+e_{i_r}=0}^{n_i} a_{i,e_{i_1}\dots e_{i_r},h} X_1^{e_{i_1}} \cdots X_r^{e_{i_r}} Y_{ih} \in A[\boldsymbol{X}, \boldsymbol{Y}].$$

It is straightforward to see that $\varphi(g) = f$ and so $\mathrm{Im}(\varphi) = R$. By Hilbert Basis Theorem, we conclude easily that R is Noetherian.

Remark 5.12. We preserve the notation of Example 5.11.

(1) Note that in the previous example, when $J = B$, we reobtain Corollary 5.9 ((i)⇔(iii)). If $B = A$ and I is an ideal of of A, then we simply have that $A + \boldsymbol{X}I[\boldsymbol{X}]$ *is a Noetherian ring if and only if* A *is a Noetherian ring and* I *is an idempotent ideal of* A. Note the previous two cases were studied as separate cases by S. Hizem, who announced similar results in her talk at the Fez Conference in June 2008, presenting an ample and systematic study of the transfer of various finiteness conditions in the constructions $A + \boldsymbol{X}I[\boldsymbol{X}]$ and $A + \boldsymbol{X}B[\boldsymbol{X}]$.

(2) The Noetherianity of B it is not a necessary condition for the Noetherianity of the ring $A + \boldsymbol{X}J[\boldsymbol{X}]$. For instance, take A any field, B the product of infinitely many copies of A, so that we can consider A as a subring of B, via the diagonal ring embedding $a \mapsto (a, a, \ldots)$, $a \in A$. Set $J := (1, 0, \ldots)B$. Then J is an idempotent ideal of B and, at the same time, a cyclic A-module. Thus, by Example 5.11, $A + \boldsymbol{X}J[\boldsymbol{X}]$ is a Noetherian ring. Obviously, B is not Noetherian.

(3) Note that, if $A + \boldsymbol{X}J[\boldsymbol{X}]$ is Noetherian and B is not Noetherian, then $A \subseteq B$ and $A + \boldsymbol{X}J[\boldsymbol{X}] \subseteq B[\boldsymbol{X}]$ are necessarily not finite. Moreover, it is easy to see that $A + \boldsymbol{X}J[\boldsymbol{X}] \subseteq B[\boldsymbol{X}]$ is a finite extension if and only if the canonical homomorphism $A \hookrightarrow B[\boldsymbol{X}]/(\boldsymbol{X}J[\boldsymbol{X}])$ is finite. Finally, it can be shown that last condition holds if and only if $J = B$ and $A \subseteq B$ is finite.

References

[1] D. D. Anderson, Commutative rings, in "Multiplicative Ideal Theory in Commutative Algebra: A tribute to the work of Robert Gilmer" (Jim Brewer, Sarah Glaz, William Heinzer, and Bruce Olberding Editors),1–20, Springer, New York, 2006.

[2] M. F. Atiyah and I. G. MacDonald, Introduction to commutative algebra, Addison-Wesley, Reading, 1969.

[3] M. B. Boisen and P. B. Sheldon, CPI–extension: overrings of integral domains with special prime spectrum, *Canad. J. Math.* **29** (1977), 722–737.

[4] A.L.S. Corner, On endomorphism rings of primary abelian groups, *Quart. J. Math. Oxford* **20** (1969), 277–296.

[5] M. D'Anna, A construction of Gorenstein rings, *J. Algebra* **306** (2006), 507–519.

[6] M. D'Anna and M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, *J. Algebra Appl.* **6** (2007), 443–459.

[7] M. D'Anna and M. Fontana, The amalgamated duplication of a ring along a multiplicative-canonical ideal, *Arkiv Mat.* **45** (2007), 241–252.

[8] J. L. Dorroh, Concerning adjunctions to algebras, *Bull. Amer. Math. Soc.* **38** (1932), 85–88.

[9] J. L. Dorroh, Concerning the direct product of algebras, *Ann. Math.* **36** (1935), 882–885.

[10] M. Fontana, Topologically defined classes of commutative rings, *Ann. Mat. Pura Appl.* **123** (1980), 331–355.

[11] R. Fossum, Commutative extensions by canonical modules are Gorenstein rings, *Proc. Am. Math. Soc.* **40** (1973), 395–400.

[12] S. Hizem and A. Benhissi, When is $A + XB[\![X]\!]$ Noetherian?, *C. R. Acad. Sci. Paris*, Ser. **I 340** (2005), 5–7.

[13] J. Huckaba, Commutative rings with zero divisors, M. Dekker, New York, 1988.

[14] N. Jacobson, Basic Algebra I, W H Freeman & Co, San Francisco, 1985.

[15] M. Nagata, The theory of multiplicity in general local rings, *Proc. Intern. Symp. Tokyo-Nikko* 1955, Sci. Council of Japan, Tokyo 1956, 191-226.

[16] M. Nagata, Local Rings, Interscience, New York, 1962.

[17] L. Salce, Transfinite self-idealization and commutative rings of triangular matrices, preprint, 2008.

[18] T.S. Shores, Loewy series of modules, *J. reine angew. Math.* **265** (1974), 183–200.

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