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Prüfer $\star$-multiplication domains and $\star$-coherence. (English summary)
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In this article the authors introduce the notion of a $\star$-domain, where $\star$ is a semistar operation, and use this notion to characterize Prüfer $\star$-multiplication domains in terms of $\star$-domains satisfying various coherent-like conditions involving the $\star$ operation. A semistar operation on a domain D is a map $\star$ from the set of all nonzero D-submodules of the quotient field K of D into itself such that for all such submodules E and F, $(xE)^\star = xE^\star$ whenever x is a nonzero element of K; $E \subseteq F$ implies $E^\star \subseteq F^\star$; $E \subseteq E^\star$; and $(E^\star)^\star = E^\star$. The domain D is a $\star$-domain if $(II^{-1})^\star = D^\star$ for each nonzero finitely generated D-submodule I of K. The domain D is a Prüfer $\star$-multiplication domain if for each nonzero finitely generated ideal I of D, $D^\star$ is the union of the D-modules $F^\star$, where F is a finitely generated D-submodule of II^{-1}. Thus, when $\star$ is the classical v-operation, a $\star$-domain is simply a v-domain, and a Prüfer $\star$-multiplication domain is a Prüfer v-multiplication domain.

The authors give a detailed and interesting analysis of $\star$-domains and Prüfer $\star$-multiplication domains. They observe in Proposition 1 that a Prüfer $\star$-multiplication domain is always a $\star$-domain, and they show that in order for the converse to hold, some form of coherency is needed. In particular, they introduce several new notions of coherent-like conditions via the $\star$-operation. To mention one such example: A domain D is $\star$-extracoherent if for all finitely generated nonzero D-submodules E and F of K, there exists a finitely generated D-submodule J of $E \cap F$ such that $J^\star = E^\star \cap F^\star$. (Taking $\star$ to be the d-operation, one obtains the usual notion of a coherent domain.) Then a domain D is a Prüfer $\star$-multiplication domain if and only if D is a $\star$-extracoherent $\star$-domain (Theorem 2). Several other natural $\star$-versions of coherency are also shown to yield similar characterizations.

A final section of the paper examines the relationship between the classes of Prüfer $\star$-multiplication domains and $H(\star)$-domains, those domains such that each nonzero ideal I of D with $I^\star = D^\star$ has a finitely generated subideal J that satisfies $J^\star = D^\star$.

Reviewed by *Bruce Olberding*