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Classes of Commutative Rings Characterized by Going-Up and Going-Down Behavior.

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1. Introduction and notation.

As chronicled in [7], the past decade has witnessed considerable interest in various classes of commutative integral domains A for which particular types of overring extensions $A \subset B$ are assumed to satisfy the going-down (GD) property, often for homological or topological reasons. The present note represents the first contribution to similar studies for arbitrary commutative rings A for which various homomorphisms $A \to B$ are subjected to either GD or GU (going-up) restrictions. The homomorphisms of particular interest are the canonical ones, $f_P: A \to A_P$ and $g_P: A \to A/P$ arising when $P \in \operatorname{Spec}(A)$. Although f_P (resp., g_P) always satisfies GD (resp., GU), it is evidently restrictive to require that f_P (resp., g_P) satisfy GU (resp., GD). By varying the kinds of prime ideals P for which such restrictions obtain, we characterize rings A of (Krull) dimension zero (Proposition 2.1); von Neumann regular rings (Corollary 2.2); certain rings of dimension at most 1 (Proposition 2.3); rings whose spectra are T_{YS}-spaces in the sense of [2] (Proposition 2.4); and those pm-rings, in the sense of [6], whose spectra are T_F-spaces in the sense of [2] (Proposition 2.5).

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The relations between the above sorts of rings are detailed in Remark 2.6.

Section 3 is devoted to a deeper study of the rings in Proposition 2.4, specifically the extent to which they differ from products of 1-dimensional integral domains. Their characterization in Proposition 3.1 includes a condition whose delicate nature is reflected by the analysis of three examples in Remark 3.3. In parts (*a*) and (*b*) of the latter result, familiar subdirect products of 1-dimensional integral domains and the construction in [14, section 4] are seen to lead to Baer rings. Indeed, any Baer ring of dimension at most 1 satisfies the conditions in Proposition 2.4. However, Remark 3.3 (*c*) presents a new construction —in essence a modification, in the pullback spirit of [8], of the topological method in [14]— which is of independent interest, as it produces a Proposition 2.4—type of ring, necessarily 1-dimensional, which is *not* a Baer ring.

Throughout, all rings are assumed commutative, ring-homomorphisms are unital, the maps f_P and g_P are as above, dim denotes Krull dimension, ht denotes height, and A_{red} denotes the reduced ring canonically associated to a given ring A.

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2. Characterizations of certain low-dimensional rings.

Our first result shows that 0-dimensional rings result when the maps f_P (resp., g_P) are constrained to satisfy GU (resp., GD). The later results in this section will be motivated by relaxing various of the equivalent conditions also given in Proposition 2.1.

PROPOSITION 2.1. *For a ring A, the following are equivalent*:

(1) $\dim(A) = 0$,

(2) *each ring-homomorphism $f\colon A \to B$ with domain A satisfies* GD,

(3) g_P *satisfies* GD, *for each* $P \in \operatorname{Spec}(A)$,

(4) g_P *satisfies* GD, *for each nonminimal* $P \in \operatorname{Spec}(A)$,

(5) *each ring-homorphism $f\colon A \to B$ with domain A satisfies* GU,

(6) f_P *satisfies* GU, *for each* $P \in \operatorname{Spec}(A)$,

(7) f_P *satisfies* GU, *for each nonmaximal* $P \in \operatorname{Spec}(A)$,

(8) g_P *is flat, for each nonminimal* $P \in \operatorname{Spec}(A)$.

PROOF. If $\dim(A) = 0$, it is evident that any ring-homomorphism $A \to B$ must satisfy both GD and GU; i.e., (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (5). Moreover, (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) $\Rightarrow$ (7) trivially; and (1) $\Rightarrow$ (8) vacuously. Since flat maps always satisfy GD (cf. [7, Theorem 3.5]), we also have (8) $\Rightarrow$ (4), and so it remains only to establish the implications (4) $\Rightarrow$ (1) and (7) $\Rightarrow$ (1).

We shall prove the contrapositive of (4) $\Rightarrow$ (1). Indeed, if $P \subset Q$ are distict prime ideals of A, observe that g_Q does not satisfy GD, since no prime W of A/Q can satisfy (both $W \subset Q/Q$ and) $g_Q^{-1}(W) = P$.

Finally, if (7) holds and (1) fails, select distinct prime ideals $P \subset Q$ of A, and let h be the composite of the going-up maps $f_P\colon A \to A_P$ and $A_P \twoheadrightarrow A_P/PA_P$. Then, $h^{-1}(0) = P$, although no prime ideal W of A_P/PA_P can satisfy (both $0 \subset W$ and) $h^{-1}(W) = Q$, contradicting the fact that h satisfies *GU*. This completes the proof.

Much of [7], including the very definition of a going-down domain [7, pp. 272-273], was motivated by the result that flat maps satisfy GD (a result that was also used in the preceding proof). Accordingly, in view of conditions (3) and (8) in Proposition 2.1, it is natural to ask which (0-dimensional) rings sustain flat g_P's. (Of course, f_P is always flat). Corollary 2.2 gives the answer. It can be viewed as an extension of the result that a ring A is von Neumann regular if and only if each A-module is A-flat ([3, Theorem 1] and [9, Theorem 5]; cf. [4, Exercice 17, p. 64]).

COROLLARY 2.2. *A ring A is von Neumann regular if, and only if, g_P is flat for each $P \in \operatorname{Spec}(A)$.*

PROOF. It is well known that A is von Neumann regular if and only if both $\dim(A) = 0$ and A is reduced (cf. [4, Exercice 16 (*d*), p. 173]). By Proposition 2.1 and the above remarks, we need only to show that A is reduced whenever each g_P is flat. If this fails, select a nonzero element $b \in A$ such that $b^2 = 0$, and set $I = \{a \in A : ab = 0\}$. As $1 \notin I$, we may choose $Q \in \operatorname{Spec}(A)$ such that $I \subset Q$. Since A/Q is A-flat by hypothesis, we have a monomorphism $h\colon Ab \otimes_A A/Q \to A/Q$, given by $h(a_1 b \otimes a_2 + Q) = a_1 a_2 b + Q$ for all $a_1, a_2 \in A$. Evidently, the prime Q contains b since $b \in I$, and so h is the zero-map, whence

$$0 = Ab \otimes_A A/Q \cong A/I \otimes_A A/Q \cong A/(I + Q)\ .$$

Thus, $I + Q = A$, although $I + Q = Q$, the desired contradiction, to complete the proof.

In view of Corollary 2.2, Proposition 2.3 is motivated by subjecting fewer g_P's to the assumption of flatness. By further weakening such flatness to assumptions of GD, one then motivates the study of the rings in Proposition 2.4.

PROPOSITION 2.3. *For a ring A, the following are equivalent:*

(1) *g_P is flat, for each nonmaximal $P \in \operatorname{Spec}(A)$;*

(2) *For each maximal ideal P of A, either* $\operatorname{ht}(P) = 0$ *or A_P is a* 1*-dimensional integral domain.*

PROOF. Since (1) and (2) each hold in case $\dim(A) = 0$, we may assume that $\dim(A) > 0$. By standard localization-globalization theory (cf. [4, Proposition 11, p. 91; Corollaire, p. 116]), we may further reduce to the case A quasilocal, with maximal ideal P. It now suffices to show, for each nonmaximal prime ideal Q of A, that g_Q is flat if and only if $Q = 0$.

It will be convenient to introduce the following notation. For $b \in A$, set $J_b = \{a \in A: ab = 0\}$ and $K_b = \{a \in A: ab \in Q\}$.

Now, g_Q is flat if and only if the induced map $I \otimes_A A/Q \to A/Q$ is a monomorphism for each ideal I of A. Since $I \otimes_A A/Q \cong I/QI$, the typical element of $I \otimes_A A/Q$ is a indecomposable tensor $b \otimes a + Q$ (for some $b \in I$, $a \in A$); and so g_Q is flat if and only if $Ab \otimes_A A/Q \to A/Q$ is a monomorphism for each $b \in A$. By using standard isomorphisms as in the proof of Corollary 2.2, we see that the latter condition is equivalent to: $A/(J_b + Q) \to A/Q$ is a monomorphism for each $b \in A$. As the map in question sends the typical coset $a + J_b + Q$ to $ab + Q$, we see that the preceding condition is itself equivalent to: $K_b \subset J_b + Q$ for each $b \in A$. However, $K_b = Q$ if $b \in A - Q$ (since Q is prime); and $K_b = A$ if $b \in Q$. Thus, g_Q is flat if and only if $A \subset J_b + Q$ for each $b \in Q$. Since $Q \subset P$, this last condition is evidently equivalent to the requirement that $J_b = A$ for each $b \in Q$, i.e. to $Q = 0$, which completes the proof.

PROPOSITION 2.4. *For a ring A, the following are equivalent:*

(1) *g_P satisfies* GD, *for each nonmaximal $P \in \operatorname{Spec}(A)$;*

(2) $\dim(A) \leqslant 1$ *and each prime ideal of A contains a unique minimal prime ideal.*

PROOF. Assume (2), and let $P \in \operatorname{Spec}(A)$ be nonmaximal. If g_P is to be tested for GD, we may take distinct prime ideals $Q_1 \subset Q_2$ of A, and seek a prime ideal W of A/P satisfying both $W \subset Q_2/P$ and $g_P^{-1}(W) = Q_1$. Now, Q_2 is a maximal ideal since $\dim(A) \leqslant 1$, and so $Q_2 \neq P$. Therefore, P and Q_1 are each minimal primes contained in Q_2, whence $P = Q_1$ by hypothesis. Then, $W = 0$ satisfies the required conditions, thus proving (2) $\Rightarrow$ (1).

Conversely, assume (1). To see that $\dim(A) \leqslant 1$, notice that the existence of a chain $P_1 \subset P \subset P_2$ of distinct prime ideals in A would ensure that g_P does not satisfy GD, since no prime W of A/P can (be within P_2/P and) satisfy $g_P^{-1}(W) = P_1$. Hence, if a prime Q of A were to contain distinct minimal primes Q_1 and Q_2, observe that A/Q_1 would have no prime V satisfying both $V \subset Q/Q_1$ and $g_{Q_1}^{-1}(V) = Q_2$, contrary to the hypothesis that g_{Q_1} satisfies GD. Thus, (1) $\Rightarrow$ (2), and the proof is complete.

Consideration of condition (1) in Proposition 2.4 might just as well have been suggested by modifying the types of g_P's for which GD is required in condition (4) of Proposition 2.1. A similar modification of the kinds of f_P's for which GU is specified in condition (7) of Proposition 2.1 leads naturally to condition (1) in the next result.

PROPOSITION 2.5. *For a ring A, the following are equivalent*:

(1) *f_P satisfies* GU, *for each nonminimal $P \in \operatorname{Spec}(A)$*,

(2) $\dim(A) \leqslant 1$ *and each prime ideal of A is contained in a unique maximal ideal.*

PROOF. Assume (2), and let $P \in \operatorname{Spec}(A)$ be nonminimal. If f_P is to be tested for GU, we may take distinct prime ideals $Q_1 \subset Q_2$ of A, and seek a prime ideal W of A_P satisfying both $Q_1 A_P \subset W$ and $f_P^{-1}(W) = Q_2$. Now, P, being nonminimal, must be maximal since $\dim(A) \leqslant 1$. Similarly, Q_2 is also maximal. Therefore, P and Q_2 are each maximal ideals containing Q_1, whence $P = Q_2$ by hypothesis. Then, $W = PA_P$ satisfies the required conditions, thus proving (2) $\Rightarrow$ (1).

We omit the proof that (1) $\Rightarrow$ (2), as it also parallels the corresponding portion of the proof of Proposition 2.4.

We close this section by describing the relations between the above types of rings.

REMARK 2.6. (*a*) First, although the above results were motivated by GD- and GU-theoretic considerations, it is interesting to note that the most of rings involved may be characterized using topological notions. Indeed, let A be a ring, and let $X = \mathrm{Spec}(A)$ equipped with the usual Zariski topology (cf. [4, p. 125]). One has the following dictionary:

A satisfies the conditions in Proposition ...	*if and only if*	*the topological space X is ...*
2.1		T_1
2.4		T_{YS}
2.5		T_F and normal

(For the definitions of T_{YS}- and T_F-spaces, see [2, pp. 30-31].) In fact, the first assertion is trivial, the second is elementary (cf. [5, Prop. 16]), and the third is a consequence of the following citations. Each prime ideal of A is contained in a unique maximal ideal if and only if X is a normal space [6, Theorem 1.2]; and $\dim(A) \leqslant 1$ if and only if X is a T_F-space [5, Prop. 6].

(*b*) It will be convenient to let A_i denote the statement « the ring A satisfies the conditions in Proposition (or Corollary) 2.i ». We may take as evident the implications $A_1 \Rightarrow A_5$ and $A_2 \Rightarrow A_1 \Rightarrow$ $\Rightarrow A_3 \Rightarrow A_4$. The next (five) examples serve to show that no further implications obtain among the A_j's.

$A_1 \not\Rightarrow A_2$: Consider any Artinian local ring which is not a field such as $K[T]/(T^2)$, where T is an indeterminate over the field K.

$A_4 \not\Rightarrow A_3$: It is enough to produce a 1-dimensional quasilocal ring which is not an integral domain but has a unique minimal prime ideal. To this end, let B be a ring of the type considered in the preceding paragraph, let P be the maximal ideal of B, let Y be an indeterminate over B, and then note that $B[Y]_{(P,Y)}$ satisfies the required conditions (cf. [11, Theorem 38]).

$A_5 \not\Rightarrow A_4$: It is enough to produce a 1-dimensional quasilocal ring with precisely three prime ideals. There are several ways to find such a ring. Here are three such. Appeal to [13, Theorem 2.10]; or consider the pullback $R \times_k R$, where R is a 1-dimensional quasi-local ring with residue field k (cf. [8, Theorem 1.4]); or let U, V be algebraically indipendent indeterminates over a field F, let $S = F[U, V]/(UV) =$ $= F[u, v]$ and consider $S_{(u,v)}$.

$A_3 \not\Rightarrow A_5$: It is enough to consider any 1-dimensional integral domain, which is not quasilocal (for example, $\mathbb{Z}$).

$A_3 \not\Rightarrow A_1$: Consider any 1-dimensional integral domain (for example, $\mathbb{Z}$).

(*c*) Since Spec (A_{red}) = Spec (A) as topological spaces for any ring A, it is of some interest to ask questions analogous to those in (*b*) for reduced rings. Certainly, the final three examples in (*b*) persist, since the presented rings are reduced. However, the first example in (*b*) cannot carry over, as we recalled earlier that any 0-dimensional reduced ring must be von Neumann regular. Most interestingly, the second example in (*b*) also fails to carry over. Indeed, if A is reduced ring, then $A_4 \Rightarrow A_3$.

To prove the preceding assertion, let A be a 1-dimensional reduced ring, let $P \in \text{Spec}(A)$ have height 1, and let Q be the unique minimal prime contained in P. Since A_P is reduced, $0 = QA_P \in \text{Spec}(A_P)$, so that A_P is a 1-dimensional integral domain, as desired.

The main purpose of section 3 will be a deeper analysis of the rings satisfying the conditions in Proposition 2.4. Because of the preceding remarks, reduced rings will figure prominently in section 3.

(*d*) We next record the facts underlying the above proofs of Proposition 2.4, Proposition 2.5, and the implications (4) $\Rightarrow$ (1) and (7) $\Rightarrow$ (1) in Proposition 2.1. If $P \in \text{Spec}(A)$, then g_P satisfies GD (resp. f_P satisfies GU) if and only if, whenever prime ideals Q and W of A are such that W contains (resp., is contained in) both P and Q, it follows that $P \subset Q$ (resp., $Q \subset P$). These assertions may be proved by reasoning as above, using the nature of the primes in A/P (resp., A_P). The interested reader may also wish to check how these assertions lead to proofs of the above-cited results.

(*e*) Professor Akira Ooishi has communicated to us the following sharpening of Corollary 2.2. A ring A is von Neumann regular if (and only if) g_M is flat for each maximal ideal M of A. (One proof of this uses the fact that each g_M satisfies GD, as characterized in (*d*) above, to conclude that $\dim(A) = 0$, and then follows the lines of the earlier proof of Corollary 2.2, taking Q maximal in that proof). Professor Ooishi has also noted that both Corollary 2.2 and Proposition 2.3 are consequences of the following observation, of independent interest. If an ideal I of a ring A is contained in the Jacobson radical of A and if A/I is A-flat, then $I = 0$.

3. Subdirect product and Baer rings.

In case A is an integral domain, the conditions in Proposition 2.1 and Corollary 2.2 each reduce to A a field; in each of Proposition 2.3 and Proposition 2.4, to $\dim(A) \leqslant 1$; and in Proposition 2.5, to A quasilocal such that $\dim(A) \leqslant 1$. Accordingly, it is natural to ask to what extent arbitrary reduced rings of the earlier types resemble products of such integral domains. This section begins by answering this question for the class of rings characterized in Proposition 2.4. The later work in this section will assume some familiarity with Baer rings (as in, for example, [1]) and the construction of Lewis-Ohm in [14, section 4].

PROPOSITION 3.1. *For a ring A, the following two conditions are equivalent*:

(1) *A satisfies the conditions in Proposition 2.4, i.e., $\dim(A) \leqslant 1$ and each prime ideal of A contains a unique minimal prime ideal,*

(2) *A_{red} is a subdirect product of a family $\{B_i : i \in I\}$ of integral domains B_i, such that*:

(2a) *$\dim(B_i) \leqslant 1$ for each $i \in I$, and*

(2b) *for each $P \in \operatorname{Spec}(A)$, there exists a unique index $i = i(P) \in I$ such that there is an A-algebra isomorphism $A/P \otimes_A B_i \cong A/P$ and $A/P \otimes_A B_j = 0$ whenever $i \neq j \in I$.*

PROOF. (2) $\Rightarrow$ (1): Assume (2). Set $B = \prod_{j \in I} B_j$. For each $i \in I$, let P_i be the kernel of the surjective map obtained by composing the inclusion map $A_{\mathrm{red}} \hookrightarrow B$ with the canonical projection $B \twoheadrightarrow B_i$. Since $A_{\mathrm{red}}/P_i \cong B_i$ and $P_i = Q_i/\sqrt{A}$ for a uniquely determined $Q_i \in \operatorname{Spec}(A)$, it follows that $A/Q_i \cong B_i$. We claim that $\{Q_i : i \in I\}$ is the set of minimal prime ideals of A.

Let $P \in \operatorname{Spec}(A)$. For each $k \in I$, we have $A/P \otimes_A B_k \cong A/(P + Q_k)$. Thus, by (2b), there exists a unique $i = i(P) \in I$ such that $P + Q_i \neq A$; and, moreover, one sees easily from the A-algebra isomorphism $A/(P + Q_i) \cong A/P$ that $Q_i \subset P$. In particular, $Q_k \neq Q_t$ whenever k and t are distinct elements of I. We may conclude that each minimal prime of A takes the form Q_k.

Conversely, to show that each Q_k is a minimal prime, suppose not, and consider distinct primes $P \subset Q_k$ in A. Selecting $i = i(P)$ as in the preceding paragraph leads to $Q_i \subsetneq Q_k$ and $P + Q_k \neq A \neq P + Q_i$, contradicting the uniqueness of $i(P)$. These comments serve to establish the earlier claim, and actually set up a bijection between I and the collection of minimal primes of A.

Finally, (1) now follows from the observation that

$$\dim(A) = \sup\{\dim(A/Q_i) \colon i \in I\}$$

since, by (2*a*), we have $\dim(A/Q_i) \leqslant 1$.

(1) $\Rightarrow$ (2): The preceding argument shows the way we must go. Let $\{Q_i \colon i \in I\}$ be an indexing for the collection of minimal primes of A, each such prime corresponding to a unique index. Set $B_i = A/Q_i$ and $B = \prod B_i$. The canonical map $A_{\text{red}} \to B$ is an injection since $\cap Q_i = \sqrt{A}$, and so A_{red} is evidently a subdirect product of $\{B_i \colon i \in I\}$. As it is now straightforward to use the above ideas in order to infer (2*a*) and (2*b*) from (1), we omit the details, completing the proof.

The preceding proof shows that the relevant subdirect product structure is essentially determined by A. The next result treats some additional aspects of that structure.

COROLLARY 3.2. *With the assumptions and notations of Proposition* 3.1, *we have the following*:

(*a*) *For each* $i \in I$, $B_i \otimes_A B_i \cong B_i$. *If* i *and* j *are distinct elements of* I, *then* $B_i \otimes_A B_j = 0$.

(*b*) *If* $h_i \colon A \twoheadrightarrow B_i$ *denotes the canonical surjection for each* $i \in I$, *then* Spec(A) *is the disjoint union of* $\{h_i^{-1}(\text{Spec}(B_i)) \colon i \in I\}$.

(*c*) *The map* $A \to B$, *obtained by composing the canonical surjection* $A \twoheadrightarrow A_{\text{red}}$ *with the inclusion* $A_{\text{red}} \hookrightarrow B$, *satisfies the lying-over property.*

PROOF. In view of the foregoing material, the details for (*a*) and (*b*) may safely be omitted. As for (*c*), let f denote the relevant map $A \to B$. Our task is to show that $f^{-1} \colon \text{Spec}(B) \to \text{Spec}(A)$ is surjective. To this end, consider $P \in \text{Spec}(A)$. Let $i = i(P)$, so that $Q_i \subset P$. For ease of notation, we may assume I well-ordered, with least element i. Let h_i be as in the statement of (*b*), let $P^* =$

$h_i(P) \in \mathrm{Spec}\,(B_i)$ and set $P' = P^* \times \prod_{j>i} B_j \in \mathrm{Spec}\,(B)$. We claim that $f^{-1}(P') = P$.

Indeed, it is immediate from the definitions of P^* and P' that $f(P) \subset P'$, that is, $P \subset f^{-1}(P')$. For the reverse inclusion, let $x \in f^{-1}(P')$. Then $f(x)$ may be viewed as a pair (y, z), with $y \in P^*$ and $z \in \prod_{j>i} B_j$. By the construction of P^*, we have $y = h_i(v)$, for some $v \in P$. However, $y = h_i(x)$ by the construction of f, and so $x - v \in \ker(h_i) = Q_i$, whence $x \in P + Q_i = P$, completing the proof.

It is convenient next to recall the following definitions and related facts. Let A be a ring. $\mathrm{Min}\,(A)$ denotes the set of minimal prime ideals of A, equipped with the subspace topology inherited from the Zariski topology on $\mathrm{Spec}\,(A)$. It is well known that $\mathrm{Min}\,(A)$ is a Hausdorff space (cf. [1, p. 80]), but not necessarily compact. A is said to be a *Baer ring* if the annihilator ideal of each element of A is generated by an idempotent; i.e., if each principal ideal of A is A-projective. Any (von Neumann) regular ring is a Baer ring. Any Baer ring is reduced (cf. [12, Lemma 7.1], [1, Remark 1, p. 83]). Moreover, any Baer ring of dimension at most 1 satisfies the conditions in Proposition 2.4, since Kist [12, Theorem 9.5] asserts that, if A is a Baer ring, then each prime ideal of A contains a unique minimal prime. Speed [15, Theorem 2.2] establishes the converse in the presence of an extra condition phrased in terms of annihilators. A most helpful characterization for our purposes appears in Artico-Marconi [1, Theorem, p. 83] (cf. also [16, Proposition 3.4]): *A is a Baer ring if and only if A is reduced,* $\mathrm{Min}\,(A)$ *is compact, and each prime ideal of A contains a unique minimal prime.* Finally, we recall that any product of integral domains must be a Baer ring [15, p. 259]; and [15, Theorem 4.3] characterizes those Baer rings which are expressible as products of integral domains.

Since a ring A satisfies the conditions in Proposition 2.4 if and only if A_{red} satisfies those conditions, the preceding observations lead naturally to the following question. If A is a reduced ring satisfying the conditions of Proposition 2.4 (equivalently, of Proposition 2.3), must A be a Baer ring? As Remark 3.3 (c) will show, the answer is negative in general, but we shall encounter some interesting examples along the way.

REMARK 3.3 (a). To indicate the difficulty facing us (and the adundance of Baer rings), we next sketch a result implying that the subring of $\prod \mathbb{Z}_{p\mathbb{Z}}$ generated by $\oplus \mathbb{Z}_{p\mathbb{Z}}$ is a Baer ring. To wit: if $D = \prod D_i$ is a product of family of integral domains D_i each of characteristic zero and if A is the (subdirect product) subring of D generated by $K = \oplus D_i$, then A is a Baer ring.

For the proof, we may take I infinite (lest $A = D$, a Baer ring by earlier remarks). Solely for the ease of notation, take $I = \{1, 2, 3, ...\}$. Observe that the typical element of A is $b = n + k$, where $n \in \mathbb{Z}$ induces $n = (n, n, ...) \in D$ and $k = (k_1, k_2, ..., k_t, 0, 0, ...\} \in K$. Consider the annihilator $J = \{a \in A: ab = 0\}$. We shall show that J is generated by an idempotent.

There are two cases to consider. First, suppose that $n \neq 0$. Define $e = (e_i) \in D$ by $e_i = 1$ if $1 \leqslant i \leqslant t$ and $n + k_i = 0$; $e_i = 0$ otherwise. Evidently, $e = e^2 \in J$. To show that $J = Ae$, note first that $J = \{d = (d_i) \in D$: $d_i = 0$ if either $1 \leqslant i \leqslant t$ and $n + k_i \neq 0$ or $t < i\}$. Thus each $d \in J$ satisfies $d = de \in Ae$.

In the remaining case, $n = 0$, so that $b = k \in K$. Define $e = (e_i) \in D$ by $e_i = 0$ if $1 \leqslant i \leqslant t$ and $k_i \neq 0$; $e_i = 1$ otherwise. Evidently, $e = e^2 \in J$. Note that $J = J_1 \cup J_2$, where

$$J_1 = \{r = (r_i) \in K\colon\ r_i = 0, \text{ whenever } 1 \leqslant i \leqslant t \text{ and } k_i \neq 0\}$$

and

$$J_2 = \{m + p \in A\colon\ 0 \neq m = (m, m, ...),\ p = (p_i) \in K,$$
$$p_i = -m \text{ whenever } 1 \leqslant i \leqslant t \text{ and } k_i \neq 0\}\,.$$

This explicit description of J now makes it easy to check that $d = de$ for each $d \in J$, whence $J = Ae$, completing the proof.

(*b*) Since subdirect products of the sort in (*a*) are Baer rings and it is difficult in any event to compute the dimension of such a subdirect product, we turn next to a construction whose dimension *is* easy to prearrange. Let I' (resp., I) be the set of positive (resp., nonnegative) integers. Consider mutually disjoint topological spaces $X_0 = \{x_0\}$, $X_1 = \{p_1, m_1\}$, $X_2 = \{p_2, m_2\}$, ..., where $\{m_i\}$ is the only nontrivial closed subset of X_i. Let T be an indeterminate over a field F. Set $R_0 = F$ and $R_i = F[T]_{(T)}$ for each $i \in I'$. Of course, we have homeomorphisms Spec $(R_i) \simeq X_i$ and F-algebra homomorphisms $f_i\colon F \to R_i$ for each $i \in I$. Following Lewis-Ohm [14, p. 825], we consider the subdirect product $R = \{r = (r_0, r_1, r_2, ...) \in \prod R_i\colon\ r_i = f_i(r_0)$ for all but finitely many $i \in I'\}$. Lewis-Ohm have shown that Spec (R)

is order-isomorphic to the ordered disjoint union of the sets X_i, $i \in I$ (and, by the proof of [14, Theorem 4.2], that R is a Bézout ring). Thus R is a reduced (1-dimensional) ring satisfying the conditions of Proposition 2.4. We claim that R is in fact a Baer ring.

By the earlier remarks, it is enough to show that $\mathrm{Min}(R)$ is compact. As in [14], it is convenient to identify X_i with $\{P \in \mathrm{Spec}(R)\colon A_i \subset P\}$, where $A_i = \{r = (r_0, r_1, r_2, \dots) \in R\colon r_i = 0\}$. Then $\mathrm{Min}(R) = \{x_0, p_1, p_2, \dots\}$. For the moment, fix $i \in I'$. Evidently, $A_i = Re_i$, where $e_i = (1, 1, \dots, 1, 0, 1, \dots)$ is the idempotent of R whose only zero entry is in the i-th position. Thus, $X_i = \{P \in \mathrm{Spec}(R)\colon 1 - e_i \notin P\}$ is a clopen subset of $\mathrm{Spec}(R)$. Since $R/A_i = R_i$, the singleton set $\{p_i\}$ is open in X_i, and so the denumerable set $Y = \{p_1, p_2, \dots\}$ is a discrete subspace of $\mathrm{Spec}(R)$. We claim that $\mathrm{Min}(R)$ is a one point compactification of Y.

Since $\mathrm{Min}(R)$ is Hausdorff, any set of the form $\mathrm{Min}(R) - \{p_{i_1}, p_{i_2}, \dots, p_{i_n}\}$, where $1 \leqslant i_1 < \dots < i_n$, is open in $\mathrm{Min}(R)$. To complete the proof, it is enough to show that each proper open subset of $\mathrm{Min}(R)$ which contains x_0, must assume such a form. This however, is a direct consequence of the conditions, found by Lewis-Ohm [14, p. 827], and satisfied by each *closed* subset of $\mathrm{Spec}(R)$ which does *not* contain x_0.

(*c*) As promised, we shall next construct a 1-dimensional reduced ring S, in which each prime ideal contains a unique minimal prime, such that S is not a Baer ring. In fact, $\mathrm{Min}(S)$ will be shown to be a denumerable discrete space (and, hence, not compact). The reader may wish to contrast the following construction with the Gillman-Jerison technique (noted in [12, Theorem 9.4], [15, p. 258], [1, Remark 3]): if X is a compact F-space which is not basically disconnected, then the ring $C(X, \mathbb{R})$ of continuous real-valued functions on X is not a Baer ring, although each of its primes does contain a unique minimal prime.

To construct S, we start with the ring R constructed in (*b*). By a minor abuse of notation, let x_0 and m_1 denote the prime ideal of R_0 and the maximal ideal of R_1, respectively. Let v denote the surjective composite

$$R \twoheadrightarrow R_0 \times R_1 \twoheadrightarrow R_0/x_0 \times R_1/m_1 \xrightarrow{\cong} F \times F$$

and let $u\colon F \hookrightarrow F \times F$ be the diagonal homomorphism. Set

$$S = F \times_{F \times F} R\ ;$$

that is, S is the pullback of the diagram

$$\begin{array}{ccc} & & F \\ & & \downarrow u \\ R & \xrightarrow{v} & F \times F \end{array}$$

More concretely,
$S = \{r = (r_0, r_1, r_2, \dots) \in R\colon\ r_0 = r_1 + m_1 \in R_1/m_1 = F\}$.

By virtue of [8, Theorem 1.4]. Spec (S) is homeomorphic to the space obtained by attaching Spec (F) to Spec (R), over the closed set Spec ($F \times F$), by the continuous map Spec ($F \times F$) $\to$ Spec (F). In fact, $Q = \ker(v) = A_0 \cap m_1 \subset R$ may be viewed as a prime ideal of S (also satisfying $Q = Q \cap S = A_0 \cap S = m_1 \cap S$) sustaining a scheme-theoretic isomorphism

$$\text{Spec}\,(S) - \{Q\} \cong \text{Spec}\,(R) - \{x_0, m_1\}\,,$$

by [8, Theorem 1.4(d)]. This restricts to both a homeomorphism h of the underlying topological spaces and an order-isomorphism g of the corresponding posets (cf. [8, Corollary 1.5 (3)]). It is straightforward to verify that the only prime ideal of S, besides itself, that Q is comparable with « is » p_1; and that $p_1 \subsetneq Q$, essentially because $m_1 \neq p_1$. Accordingly, g may be extended so that, *qua* poset, Spec (S) is just order-isomorphic to the ordered disjoint union of the sets X_i, $i \in I'$. (In other words, the passage from R to S affects poset structure by identifying x_0 with m_1, as a ht 1 element containing no ht 0 element besides p_1). Thus, all assertions except for the topological nature of Min (S) are evident. However, the above information about posets gives Min (S) $\cong \{p_1, p_2, p_3, \dots\} = Y$ as sets. There is such a bijection which is actually a homeomorphism: simply, restrict h. Since it was shown in (b) that Y has the discrete topology, the proof is complete.

(d) One should note that purely algebraic arguments suffice to show that the ring S in (c) is not a Baer ring. (Althout one thereby eliminates consideration of h and Y, the principal discussion, including g, is still needed to show that S satisfies the conditions of Proposition 2.4). One such algebraic method is to verify that S does not satisfy the conditions of Irlbeck [10, Theorem 1] characterizing Baer rings. We shall choose instead the more direct path of producing an element $s \in S$ whose annihilator is not generated by an idempotent. To this end, note first that the idempotents of S are of but two types: $(0, 0, r_2, r_3, \ldots, r_n, 0, 0, \ldots)$, where $r_i \in \{0, 1\}$ whenever $2 \leqslant i \leqslant n$; and $(1, 1, r_2, r_3, \ldots, r_n, 1, 1, \ldots)$, where $r_i \in \{0, 1\}$ whenever $2 \leqslant i \leqslant n$. It is easy to see that the element $s = (0, T, 0, 0, \ldots)$ has the asserted property.

(*e*) Finally, it may be of some interest to note the following topological sufficient condition for a Baer ring. If the reduced ring A satisfies the condition in Proposition 2.4, if $B = \prod B_i$ is the canonically associated product, as in Proposition 3.1 and Corollary 3.2, and if the induced function $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ maps $\mathrm{Min}(B)$ into $\mathrm{Min}(A)$, then A is a Baer ring.

To sketch the proof, one first uses the description $B_i = A/Q_i$ to show that $\mathrm{Min}(A)$ is contained in the image of $\mathrm{Min}(B)$ and, hence, by hypothesis, *is* that image. Next, $\mathrm{Min}(B)$ is compact, since B is a product of integral domains. (We are indebted to W. Brandal for his motivating observation that the minimal spectrum of a product of denumerably many integral domains is homeomorphic to the Stone-Čech compactification of the positive integers). Thus, $\mathrm{Min}(A)$, being the Hausdorff continuous image of a compact space, is itself compact. An application of the criterion of Artico-Marconi completes the proof.

In closing, we suggest that future studies of rings A along these lines might well proceed by imputing GD to various *injective* homomorphisms with domain A. It is not clear at present how to choose an interesting universe for codomains of such injections (as in [7, (4.1)] for the case of integral domains). It is to be hoped that ensuing studies will treat rings of arbitrary dimension.

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