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SOME RESULTS ON THE WEAK NORMALIZATION OF AN INTEGRAL DOMAIN

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1. Introduction. This paper is motivated by the work of Yanagihara [16] on ${}_{B}^{*}A$, the weak normalization relative to an integral extension $A \subset B$ of commutative rings. For simplicity, we consider the special case in which A is a (commutative integral) domain R and $B = R'$, the integral closure of R. A particular focus is on the case in which R is weakly normal, in the sense that $R = {}_{R'}^{*}(R)$.

It seems natural to study weak normality in terms of related properties that are better understood. In this regard, recall that for domains

root closed ⇨ weakly normal ⇨ seminormal,

with none of these implications being reversible in general. It will be convenient to say that a domain R *satisfies the Yanagihara conditions* if the following holds for each $P \in \mathrm{Spec}(R)$: if $\mathrm{ch}(R/P) = 0$, then R_P is seminormal ; and if $\mathrm{ch}(R/P) = p > 0$, then R_P is p-closed. It was shown in [16, second Corollary on page 653] that if R satisfies the Yanagihara conditions, then R is weakly normal. However, by applying the $D+M$ construction to the example in [16, Remark 2], we see in Example 2.1(b) that a weakly normal domain of (Krull) dimension ≥ 3 need not satisfy the Yanagihara conditions. In fact, we show in Example 2.1(a) that the same conclusion holds in dimension 2, by changing the polynomial ring in Yanagihara's example to a Nagata ring. Nevertheless, we show that the Yanagihara conditions *do* characterize weak normality for certain types of domains : those of dimension ≤ 1 (see Proposition 2.2) and pseudo-valuation domains in the sense of [13] (see Proposition 2.3).

Our contribution in section 3 relates to the following result of Yanagihara [16, Theorem 1] (see also Itoh [14]). A domain R, with quotient field K, is weakly normal if and only if R is seminormal and satisfies the following additional condition : if $u \in K$ and p is a prime number such that u^p and pu are in R, then $u \in R$. Section 3 effects a modest sharpening of this charac-

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terization (see Proposition 3.7(4)) in the spirit of what we called the Yanagihara conditions, by considering separately the primes P of R with R/P of characteristic zero or of positive characteristic. Related to this work are two "decompositions" of the weak normalization $R^*(= {}^*_{R'}(R))$ for any domain R: see (3.6), (3.11).

The rings discussed in Example 2.1(a), Proposition 2.2 and Proposition 2.3 (but not those in Example 2.1(b)) are all going-down domains, in the sense of [4]. In fact, weak normality has figured earlier in our work on universally going-down domains (definition recalled below), principally in connection with the result [8, Corollary 2.3] that a domain R is a Prüfer domain if and only if R is an integrally closed universally going-down domain. In section 4, this is sharpened in several ways. First, it is noted in Proposition 4.1 that a domain R is a universally going-down domain if and only if R^* is a Prüfer domain. Secondly, by using our extension of the Yanagihara-Itoh criterion (from Proposition 3.7), Corollary 4.2(4) characterizes Prüfer domains as a certain type of seminormal universally going-down domain. (This is the spirit of Angermüller [3, Theorem 1], who showed that certain one-dimensional root closed domains must be integrally closed. Note, however, that a root closed going-down domain need not be integrally closed: cf. [10, Exercise 6, page 184], [5, Remark 2.7(c)].) Section 4 also includes proofs that the classes of weakly normal going-down domains and of universally going-down domains are stable under formation of factor domains: see Propositions 4.5 and 4.7.

Throughout, we assume familiarity with the material in [16], [14] on weak normalization and in [5] on going-down domains and divided primes. Here, we recall from [2], [15] only the characterizations of weak (resp., semi-)normalization of a domain: R^* (resp., R^+) is the largest integral overring T of R such that $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is a bijection and the residue class field extensions induced by $R \subset T$ are all purely inseparable (resp., isomorphisms). For additional background or points of view, the interested reader may consult [11] or the references listed in [16].

2. On the Yanagihara conditions. The effect of Example 2.1 will be to show that a weakly normal domain R need not satisfy the Yanagihara conditions if $\dim(R) \geq 2$. However, we shall show that these conditions *do* characterize weak normality if either $\dim(R) \leq 1$ or R is a pseudo-valuation domain (see Propositions 2.2 and 2.3). It is interesting to note that all the rings figuring in these results are going-down domains. (Recall from [4] that a domain R is called a going-down domain if $R \subset T$ satisfies the going-down property for each overring T of R.) It will be helpful to recall the result [4, Theorem 2.2] that if R is a going-down domain, then Spec(R), as a poset under inclusion, is a tree.

Example 2.1. (a) *Let n be either ∞ or a positive integer greater than 1. Let p be a prime. Then there exists an n-dimensional quasilocal weakly normal going-down domain (R, N) such that* $\mathrm{ch}(R/N) = p$ *and R is not p-closed. In particular, R does not satisfy the Yanagihara conditions.*

To construct a suitable R, we begin with the Nagata ring $A = \mathbf{Z}_{p\mathbf{Z}}(X^p)$. (By definition [10, page 410], $A = \mathbf{Z}_{p\mathbf{Z}}[X^p]_{(p)}$.) Note that A is a one-dimensional valuation domain (cf. [10, Theorem 33.4]), and thus is a going-down domain. Next, take an $(n-1)$-dimensional valuation domain (V, M) of the form $V = \mathbf{Q}(X) + M$. (As usual, we adopt the conventions that $\infty - 1 = \infty = \infty + 1$.) We shall show that $R = A + M$ has the asserted properties.

Standard facts about the $D+M$ construction (as in [10]) reveal R is quasilocal and n-dimensional. By [9, Corollary], R is also a going-down domain. Moreover, the maximal ideal of R is $N = pA + M$, so that $R/N \cong A/pA \cong \mathbf{F}_p(X^p)$, which has characteristic p. Notice also that X is in the quotient field of R, $X^p \in R$, and $X \notin R$ (since $X \notin A$). Hence, R is not p-closed.

It remains only to show that R is weakly normal. This can be done by applying the criterion in [16, Theorem 1]. First, note that R is seminormal since A is seminormal. Next, suppose that u in the quotient field of R satisfies u^q, $qu \in R$ for some prime q. As V is q-closed, $u \in V$. Without loss of generality, $u \in \mathbf{Q}(X)$. If $q \neq p$, then $q^{-1} \in A \subset R$, so that $u = q^{-1}(qu) \in R^2 = R$, as desired. Thus, we may suppose $q = p$. Now, since

$$u^p \in A \subset \mathbf{Z}_{p\mathbf{Z}}[X]_{(p)} = \mathbf{Z}_{p\mathbf{Z}}(X)$$

and $\mathbf{Z}_{p\mathbf{Z}}(X)$ is integrally closed, it follows that $u \in \mathbf{Z}_{p\mathbf{Z}}(X)$. Moreover, since $pu \in A$, we have $u \in Ap^{-1}$. To show $u \in A$ (and hence $u \in R$), it suffices to prove

$$Ap^{-1} \cap \mathbf{Z}_{p\mathbf{Z}}(X) \subset A$$

or, equivalently, that $A \cap p\mathbf{Z}_{p\mathbf{Z}}(X) \subset pA$. If this were to fail, $1 \in p\mathbf{Z}_{p\mathbf{Z}}(X)$, since pA is the unique maximal ideal of A; but then 1 would be in the maximal ideal of $\mathbf{Z}_{p\mathbf{Z}}(X)$. This (desired) contradiction gives $u \in R$, and so R is weakly normal. □

(b) By applying the $D+M$ construction directly to the extension $\mathbf{Z}[X^p] \subset \mathbf{Z}[X]$ considered by Yanagihara in [16, Remark 2], we obtain only some of the properties of the example in (a). For instance, the two-dimensional case is not addressed, since $\dim(\mathbf{Z}[X^p]+M) = \dim(\mathbf{Z}[X^p])+\dim(V) \geq 2+1 = 3$. Moreover, $\mathbf{Z}[X^p]+M$ is not a going-down domain (because, for instance, its spectrum is not a tree).

Each domain of dimension at most 1 is a going-down domain. We show next that, in contrast with Example 2.1, the Yanagihara conditions characterize weak normality in the one-dimensional case.

Proposition 2.2. *For a domain R such that* $\dim(R) \leq 1$, *the following conditions are equivalent :*

(1) *R is weakly normal ;*

(2) *R satisfies the Yanagihara conditions.*

Proof. (2) ⇨ (1) : As mentioned earlier, this is a special case of [16, second Corollary on page 653].

(1) ⇨ (2) : Assume (1). By [16, Proposition 2], each localization of R is weakly normal. Moroever, (2) is preserved by localization (a fact which is especially obvious when $\dim(R) \leq 1$). Thus, we may assume that R is quasilocal, say with maximal ideal M. Since fields are trivially seminormal and p-closed, we may assume $P = M \neq 0$. Since weakly normal implies seminormal, [16, Proposition 2] reduces our task to proving that if $\mathrm{ch}(R/M) = p > 0$, then R is p-closed.

Deny, and consider $u \in R' \backslash R$ such that $u^p \in R$. Since R is weakly normal, [16, Theorem 1] yields $pu \notin R$. By an easy induction, $p^n u \notin R$ for each positive integer n. (For the induction step, consider $p^{n+1}u = p(p^n u)$ and note that $(p^n u)^p \in R$.) Next, write u as a fraction, $u = ab^{-1}$, with $a, b \in R\backslash\{0\}$. As $u \notin R$, $b \in M$. Since R is one-dimensional quasilocal, $\mathrm{rad}_R(Rb) = M$. In addition, $p \in M$ since $\mathrm{ch}(R/M) = p$. Hence, $p \in \mathrm{rad}_R(Rb)$; i.e., $p^n = rb$ for some $n \geq 1$ and $r \in R$. It follows that $p^n u = rbu = ra \in R$, the desired contradiction. □

Despite Example 2.1, we show next that the Yanagihara conditions characterize weak normality for a special type of seminormal going-down domain, the pseudo-valuation domain (PVD) in the sense of [13]. Note, by [13, Example 2.1] that a PVD can have any Krull dimension. By definition, a domain R is a PVD if R has a ("canonically associated") valuation overring

V such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ as sets. A useful characterization [1, Proposition 2.6] of a PVD, R, with canonically associated valuation overring (V, M) is this : $R = V \times_{V/M} F$, where F is a subfield (necessarily R/M) of V/M. Another useful characterization [13, Theorems 1.4 and 2.7] states that a quasilocal domain (R, M) is a PVD if and only if M is a "strongly prime" ideal (in the sense that $xy \in M$ with x, y in the quotient field of R implies that either x or y is in M).

Proposition 2.3. *Let (R, M) be a PVD with canonically associated valuation overring V. Set $F = R/M$ and $k = V/M$. Then the following conditions are equivalent :*

(1) *R is weakly normal ;*

(2) *If $\mathrm{ch}(F) = p > 0$, then R is p-closed ;*

(3) *R satisfies the Yanagihara conditions ;*

(4) *If $v \in k \backslash F$, then v is not purely inseparable over F.*

Proof. (1) ⇒ (4) : Deny. Choose $v \in k \backslash F$ such that v is purely inseparable (and hence algebraic) over F. Hence, v is not separable over F. Thus, $p = \mathrm{ch}(F) > 0$, and $v^{p^n} \in F$ for some $n \geq 1$. If φ denotes the canonical surjection $V \to k$, consider $A = \varphi^{-1}(F(v))$. Then $A = V \times_{V/M} F(v)$ is a PVD with canonically associated valuation overring V. Thus, $\mathrm{Spec}(A) = \mathrm{Spec}(V) = \mathrm{Spec}(R)$. Note that the field extension $R/M \subset A/M$ is just $F \subset F(v)$, which is purely inseparable. (Since R is weakly normal in A and F is not weakly normal in $F(v)$, [16, Proposition 3] leads to a contradiction. We continue with another proof.) If $P \in \mathrm{Spec}(R)$ is nonmaximal, then $R_P = V_P = A_P$ by [13, Proposition 2.6], and so the field extension induced by $R/P \subset A/P$ is an isomorphism (hence, purely inseparable). We have shown that $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is a bijection inducing purely inseparable residue field extensions. Hence, $A \subset R^* = R$, whence $F(v) = \varphi(A) \subset \varphi(R) = k$, contrary to the choice of v.

(4) ⇒ (1) : Assume (4), and again let $\varphi : V \to k$ denote the canonical surjection. Let $A = R^*$. Since $R \subset A \subset R' \subset V$, it follows via integrality that M is also a maximal ideal of A. Hence, $F = R/M \subset A/M$ is a purely inseparable subextension of $F \subset k$. By (4), $A/M = F$, and so $A = \varphi^{-1}(A/M) = \varphi^{-1}(F) = R$. Thus, $R^* = R$, yielding (1).

(2) ⇒ (3) : This follows from the facts that if $P \in \mathrm{Spec}(R)$ is nonmaximal, then R_P is a valuation domain (hence seminormal and p-closed for all p) ; and that $R = R_M$ is seminormal.

(3) ⇒ (1) : This is another case of [16, second Corollary on page 653].

(1) ⇒ (2) : Assume (1) and consider u in the quotient field of R such that $u^p \in R$, with $p = \mathrm{ch}(F) > 0$. Since $p \in M$, we have $pu^p \in M$, and so $(pu)^p = p^{p-1}(pu^p) \in M$. Now, since R is a PVD, M is a strongly prime ideal of R. Hence, $pu \in M \subset R$. Thus, by (1) and the criterion in [16, Theorem 1], $u \in R$. Hence, R is p-closed. □

The proof of (1) ⇒ (2) in Proposition 2.3 also establishes the following result.

Corollary 2.4. *Let P be a strongly prime ideal of a domain R such that* $\mathrm{ch}(R/P) = p > 0$. *Then R is weakly normal (if and) only if R is p-closed.*

Remark 2.5. (a) The "strongly prime" hypothesis in Corollary 2.4 is (sufficient but) not necessary. In other words, there exists a p-closed (and weakly normal) domain R with $P \in \mathrm{Spec}(R)$ such that $\mathrm{ch}(R/P) = p$ and P is not a strongly prime ideal of R. To illustrate this, consider $R = \mathbf{F}_p[X, Y]_{(X,Y)}$ and let P be its maximal ideal. (Since this R is Noetherian and two-dimensional, [13, Proposition 3.2] shows that R is not a PVD, and so P is not strongly prime.)

(b) Corollary 2.4 can be used to give an amusing proof that the maximal ideal of the ring $R = \mathbf{Z}_{p\mathbf{Z}}(X^p) + M$ (considered in Example 2.1(a)) is not strongly prime. Notice that although M, the height 1 prime of R, is strongly prime and R is weakly normal, one cannot infer this latter fact from Corollary 2.4 since $R/M \cong \mathbf{Z}_{p\mathbf{Z}}(X^p)$ has characteristic zero.

(c) Since weak normality is a local property [16, Theorem 2], Proposition 2.3 may be used to characterize weak normality for the LPVD's introduced in [6]. We leave the details to the reader.

3. A decomposition of the weak normalization. The first result of this section sharpens both conditions in the Yanagihara-Itoh characterization [16, Theorem 1] of weak normality. Other characterizations will involve "decomposing" a weak normalization as a suitable intersection of overrings. It will be convenient to *fix notation* throughout this section as follows. R will denote a domain with quotient field K. If $P \in \mathrm{Spec}(R)$, the corresponding prime ideals of R^+ and R^* will be denoted by P^+ and P^* respectively. Since weak normalization commutes with localization [16, first Corollary on page 653], $(R_P)^* = R^*_P (= R^*_{R \setminus P}) = R^*_{P^*}$ for each $P \in \mathrm{Spec}(R)$; similarly,

$(R_P)^+ = (R^+)_{P^+}$. In addition, p and q will denote positive prime numbers ; and $J(-)$ will denote Jacobson radical.

For each p, we define

$$\begin{aligned} T^+(p) &= T_R^+(p) \\ &= \cap\{R_P + J(R'_P) : \text{ there exist } P \subset P_1 \\ &\qquad \text{in } \operatorname{Spec}(R) \text{ with } \operatorname{ch}(R/P_1) = p\}. \end{aligned}$$

Now, for each $P_1 \in \operatorname{Spec}(R)$, it follows from the definition of seminormalization that

$$\begin{aligned} R^+{}_{P_1^+} = (R_{P_1})^+ &= \cap\{(R_{P_1})_{PR_{P_1}} + J((R'_{P_1})_{PR_{P_1}}) : P \subset P_1 \text{ in } \operatorname{Spec}(R)\} \\ &= \cap\{R_P + J(R'_P) : P \subset P_1 \text{ in } \operatorname{Spec}(R)\}. \end{aligned}$$

Thus, we have

$$(3.1) \qquad T^+(p) = \cap\{R^+{}_{P_1^+} : P_1 \in \operatorname{Spec}(R) \text{ and } \operatorname{ch}(R/P_1) = p\}.$$

Next, defining $S^+(p) = S_R^+(p) = \cap\{T^+(q) : q \neq p\}$, we find that (3.1) yields

$$(3.2) \qquad \begin{aligned} S^+(p) = \cap\{R^+{}_{P_1^+} : \ & P_1 \in \operatorname{Spec}(R) \text{ and} \\ & \operatorname{ch}(R/P_1) \text{ is neither } 0 \text{ nor } p\}. \end{aligned}$$

Next, defining $T^+(0) = \cap\{R^+{}_{P^+} : P \in \operatorname{Spec}(R) \text{ and } \operatorname{ch}(R/P) = 0\}$, we have via the principle of globalization:

$$(3.3) \qquad R^+ = T^+(p) \cap S^+(p) \cap T^+(0) \text{ for each } p.$$

We next arrange a similar decomposition of R^*. For each p, we define $T^*(p) = T_R^*(p) = \{u \in K :$ for each $P \subset P_1$ in $\operatorname{Spec}(R)$ with $\operatorname{ch}(R/P_1) = p$, there exists $n \geq 1$ such that $u^{e^n} \in R_P + J(R'_P)\}$, where

$$e = e_P = \begin{cases} p \text{ if } \operatorname{ch}(R/P) = p \\ 1 \text{ if } \operatorname{ch}(R/P) = 0. \end{cases}$$

Now, if $P_1 \in \operatorname{Spec}(R)$ with $\operatorname{ch}(R/P_1) = p$, it follows from the definition of weak normalization that $R^*{}_{P_1^*} = (R_{P_1})^* = \{u \in K :$ for each $P \subset P_1 \in \operatorname{Spec}(R)$, there exists $n \geq 1$ such that $e = e_P$ satisfies $u^{e^n} \in R_P + J(R'_P)\}$. Thus, we have

$$(3.4) \qquad T^*(p) = \cap\{R^*{}_{P_1^*} : P_1 \in \operatorname{Spec}(R) \text{ and } \operatorname{ch}(R/P_1) = p\}.$$

Next, defining $S^*(p) = S_R^*(p) = \cap\{T^*(q) : q \neq p\}$, we find via (3.4) that

(3.5) $S^*(p) = \cap\{R^*_{P_1^*}: P_1 \in \mathrm{Spec}(R)$ and $\mathrm{ch}(R/P_1)$ is neither 0 nor $p\}$.

Next, define $T^*(0) = T^+(0)$, and note that $T^*(0) = \cap\{R^*_{P^*}: P \in \mathrm{Spec}(R)$ and $\mathrm{ch}(R/P) = 0\}$. Thus, we have, from (3.4), (3.5) and the principle of globalization, the desired decomposition of R^* :

(3.6) $R^* = T^*(p) \cap S^*(p) \cap T^*(0)$ for each p.

We may now give our improvements of the Yanagihara-Itoh characterization. (Notice how condition (4) sharpens both parts of (5) below.)

Proposition 3.7. *For a domain R with quotient field K, the following five conditions are equivalent :*

(1) *R is weakly normal.*

(2) (a) *R_P is seminormal for each $P \in \mathrm{Spec}(R)$ with $\mathrm{ch}(R/P) = 0$.*
(b) *There exists p such that $T^*(p) \cap S^*(p) \subset \cap\{R_P: P \in \mathrm{Spec}(R)$ and $\mathrm{ch}(R/P) \neq 0\}$.*

(3) (a) *R_P is seminormal for each $P \in \mathrm{Spec}(R)$ with $\mathrm{ch}(R/P) = 0$.*
(b) *For all p, $T^*(p) \cap S^*(p) \subset \cap\{R_P: P \in \mathrm{Spec}(R)$ and $\mathrm{ch}(R/P) \neq 0\}$.*

(4) (a) *R_P is seminormal for each $P \in \mathrm{Spec}(R)$ with $\mathrm{ch}(R/P) = 0$.*
(b) *If $P \in \mathrm{Spec}(R)$ with $\mathrm{ch}(R/P) = p$ and $u \in K$ satisfies u^p, $pu \in R_P$, then $u \in R_P$.*

(5) (a) *R is seminormal.*
(b) *If p is a prime number and $u \in K$ satisfies u^p, $pu \in R$, then $u \in R$.*

Proof. (1) ⇒ (3) : Assume (1). Then (3a) follows since weak normality implies seminormality and localization preserves seminormality. As for (3b), one need only apply (3.4) and (3.5), since (1) assures that $R^*_{P^*} = (R_P)^* = R_P$ for each $P \in \mathrm{Spec}(R)$.

(3) ⇒ (2) : Trivial.

(2) ⇒ (1) : Assume (2). Since $R^+_{P^+} = (R_P)^+ = R_P$ whenever $\mathrm{ch}(R/P) = 0$, (3.6) leads to

$$R^* = T^*(p) \cap S^*(p) \cap T^*(0) \subset \cap\{R_P: P \in \mathrm{Spec}(R)\} = R,$$

whence $R^* = R$, thus yielding (1).

(4) ⇒ (1) : This follows as in the second half of the proof of [16, Theorem 1] once it is shown that (4) implies R is seminormal. (An earlier

draft omitted this detail. Its inclusion here was suggested by ideas in correspondence from Professor Yanagihara.)

Assume (4). Suppose first that R contains a field k. If $\mathrm{ch}(k) = 0$, then (4a) yields that R_P is seminormal for each $P \in \mathrm{Spec}(R)$, and hence so is $\cap R_P = R$. If $\mathrm{ch}(k) > 0$, then (4b) and [16, Corollary to Theorem 2] yield that R is weakly normal (and hence seminormal).

In the remaining case, $R \supset \mathbf{Z}$(and $R \not\supset Q$). As $T = R_{\mathbf{Z}\setminus\{0\}}$ inherits (4) from R, the previous case shows that T is seminormal. Thus, given $u \in K$ with u^2 and u^3 in R, we have $u \in T$. Write $nu \in R$, with prime-power factorization $n = \prod_{i=1}^{s} p_i^{e_i}$. We shall show $u \in R_P$ for each $P \in \mathrm{Spec}(R)$.

If $\mathrm{ch}(R/P) = 0$, then $P \cap (\mathbf{Z}\setminus\{0\}) = \phi$, so that R_P is a ring of fractions of T; thus, R_P is seminormal and $u \in R_P$. Hence, we may assume $\mathrm{ch}(R/P) = p > 0$. In particular, $p \in P$, and so $p_i \notin P$ if $p_i \neq p$. If $p \neq p_i$ for all i, then n is a unit of R_P, so that $u = n^{-1}(nu) \in R_P$. Without loss of generality, $p = p_1$. Then $v = up^{-1}$ is such that v^p and pv are in $R \subset R_P$; it follows from (4b) that $p_1^{e_1-1}p_2^{e_2}\dots p_s^{e_s}u = v \in R_P$. By iteration, $mu \in R_P$, where $m = p_2^{e_2}\dots p_s^{e_s}$. As m is a unit of R_P, $u = m^{-1}(mu) \in R_P$, as desired.

(1) ⇨ (5) : This follows from [16, Theorem 1, (i) ⇨ (ii)].

(5) ⇨ (4) : Since localization preserves seminormality, it suffices to show that (5b) implies (4b). Consider $P \in \mathrm{Spec}(R)$ and $u \in K$ with $\mathrm{ch}(R/P) = p$, $u^p \in R_P$ and $pu \in R_P$. Pick $z \in R\setminus P$ such that zu^p, $zpu \in R$. Then $(zu)^p \in R$ also, and so (5b) gives $zu \in R \subset R_P$. As $z^{-1} \in R_P$, we have $u = z^{-1}(zu) \in R_P$. □

Lastly, we shall show that the Yanagihara-Itoh restriction on u^p, pu in (5b) above is related to another decomposition of R^*. The next two definitions are relevant. For each p, let $T_1^*(p) = \{u \in K :$ for each P_1 in $\mathrm{Spec}(R)$ with $\mathrm{ch}(R/P_1) = p$, there exists $n \geq 1$ such that $u^{p^n} \in R_{P_1} + J(R'_{P_1})\}$; and let $S_1^*(p) = \cap\{T_1^*(q) : q \neq p\}$. These concepts are related to the earlier material in the next result.

Proposition 3.8. *Let $u \in K$ and let p be a prime number. Then :*

(a) *If $u^p \in T^*(p)$, then $u \in T_1^*(p)$.*

(b) *If $pu \in S^*(p)$, then $u \in S_1^*(p)$.*

Proof. (a) Consider $P_1 \in \mathrm{Spec}(R)$ with $\mathrm{ch}(R/P_1) = p$. By hypothesis and (3.4), $u^p \in R^*_{P_1^*}$. Using the above description of $R^*_{P_1^*}$, we have $n \geq 1$ such that $u^{p^{n+1}} = (u^p)^{p^n} \in R_{P_1} + J(R'_{P_1})$. Hence, $u \in T_1^*(p)$.

(b) Consider $Q_1 \in \mathrm{Spec}(R)$ with $\mathrm{ch}(R/Q_1) = q \neq p$. As $q \in Q_1$, $p \notin$

Q_1 (otherwise, $1 \in Q_1$, a contradiction). Thus, $p^{-1} \in R_{Q_1} \subset R^*_{Q_1^*}$. It follows via (3.5) that $u = (p^{-1})pu \in R^*_{Q_1^*}$. Hence, $u^p \in R^*_{Q_1^*}$. By (3.4) and (a), $u \in T_1{}^*(q)$ for all $q \neq p$. Hence, $u \in S_1{}^*(p)$. □

We next fit $T_1{}^*(p)$, $S_1{}^*(p)$ into another decomposition of R^*. First, notice from Proposition 2.8 or the definitions that

(3.9) $$T^*(p) \subset T_1{}^*(p) \text{ and } S^*(p) \subset S_1{}^*(p) \text{ for each } p.$$

Next, define $T_1{}^*(0) = \cap\{R_P + J(R'_P) : P \in \mathrm{Spec}(R) \text{ and } \mathrm{ch}(R/P) = 0\}$. By the above, it is evident that $T_1{}^*(0) = \cap\{R^+{}_{P^+} : P \in \mathrm{Spec}(R)$ and $\mathrm{ch}(R/P) = 0\}$. Hence, it follows from the definition of $T^*(0) = T^+(0)$ that

(3.10) $$T_1{}^*(0) = T^*(0).$$

Moreover, it follows from the definition of weak normalization that

(3.11) $$R^* = T_1{}^*(p) \cap S_1{}^*(p) \cap T_1{}^*(0) \text{ for each } p.$$

We leave it to the reader to develop a similar decomposition of R^+.

4. Weak normality and universally going-down. We turn next to connections with universally going-down domains. Let R be a domain. As in [8], R is said to be a universally going-down domain in case $S \to S \otimes_R T$ satisfies going-down for each domain T containing R and each homomorphism $R \to S$ of commutative rings. Equivalently, by [8, Theorem 2.6] and [7, Corollary 2.3], R is a universally going-down domain in case the inclusion $R[X_1, \ldots, X_n] \subset T[X_1, \ldots, X_n]$ satisfies going-down for each overring T of R and each finite set $\{X_1, \ldots, X_n\}$ of algebraically independent indeterminates over R. Of course, each universally going-down domain is a going-down domain, but the converse is false (cf. [8, Remark 2.5(b)]). Arbitrary Prüfer domains are the most natural examples of universally going-down domains. (If R is Prüfer and T a domain containing R, observe that the inclusion $R \to T$ is flat, and hence satisfies going-down. Since flatness is a universal property, $R \to T$ is thus a universally going-down homomorphism in the sense of [12], [7].) In fact, [8, Corollary 2.3] established that R is a Prüfer domain if (and only if) R is an integrally closed universally going-down domain. We next give some useful characterizations of universally going-down domains.

Proposition 4.1. *For a domain R, the following conditions are equivalent :*

(1) *R is a universally going-down domain ;*

(2) *R^+ is a universally going-down domain ;*

(3) *R^* is a universally going-down domain ;*

(4) *R^* is a Prüfer domain.*

Proof. (1) $\Leftrightarrow$ (4) : This amounts to a restatement of the main result in [8]. Indeed, [8, Theorem 2.4] shows that (1) is equivalent to "R' is a Prüfer domain and $R' = R^*$." Accordingly, one need only observe that if R^* *is* a Prüfer domain, then $R' = R^*$. For this, just note that $R \subset R^* \subset R'$ in general and recall that Prüfer domains are integrally closed.

(2) $\Leftrightarrow$ (4) : The above characterizations of weak (resp., semi-)normalization make it clear that $(R^+)^* = R^*$. Applying (1) $\Leftrightarrow$ (4) to R^+ instead of R, we have (2) $\Leftrightarrow$ (4).

(3) $\Leftrightarrow$ (4) : Since a composite of purely inseparable field extensions is purely inseparable, it is clear that $(R^*)^* = R^*$. Applying (1) $\Leftrightarrow$ (4) to R^* instead of R, we have (3) $\Leftrightarrow$ (4). □

Corollary 4.2. *For a domain R, the following conditions are equivalent:*

(1) *R is a Prüfer domain ;*

(2) *R is a root closed universally going-down domain ;*

(3) *R is a weakly normal universally going-down domain ;*

(4) *R is a seminormal universally going-down domain. If u in the quotient field of R satisfies u^p, $pu \in R$ for some prime p, then $u \in \bigcap\{R_P : P \in$* $\mathrm{Spec}(R)$, $\mathrm{ch}(R/P) = p\}$.

Proof. Prüfer domain $\Rightarrow$ root closed domain $\Rightarrow$ weakly normal domain. Hence, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Moreover, Proposition 3.7 gives (3) $\Leftrightarrow$ (4) ; and Proposition 4.1 [(1) $\Leftrightarrow$ (4)] gives (3) $\Leftrightarrow$ (1). □

We next make matters a bit more precise in case of positive characteristic. First recall ([14], [16]) that a domain R of positive characteristic p is weakly normal if and only if R is p-closed.

Corollary 4.3. *Let R be a domain. Then :*

(a) *R^+ is a Prüfer domain if and only if R is a universally going-down domain such that $R^+ = R^*$.*

(b) *Suppose that* $\mathrm{ch}(R) = p > 0$. *Then R^+ is a Prüfer domain if and only if R is a universally going-down domain such that R^+ is p-closed.*

(c) *Suppose that* $\mathrm{ch}(R) = p > 0$. *Then R is a Prüfer domain if and* *only if R is a p-closed universally going-down domain.*

Proof. (a) Observe that R^* is an integral overring of R^+. As each overring of a Prüfer domain is Prüfer and hence integrally closed, we see that R^+ is Prüfer if and only if R^* is Prüfer and $R^+ = R^*$. An application of Proposition 4.1 [(1) $\Leftrightarrow$ (4)] yields (a).

(b) and (c) : In view of Proposition 4.1 [(1) $\Leftrightarrow$ (2)], applying (c) to R^+ instead of R yields (b). Thus, it suffices to prove (c). The "only if" assertion follows from earlier comments. For the converse, apply Corollary 4.2 [(3) $\Rightarrow$ (1)] and the comment preceding the statement of this corollary. □

Remark 4.4. (a) The condition "$R^+ = R^*$" in Corollary 4.3(a) cannot be deleted. Indeed, [8, Remark 2.5(a)] shows for each d, $1 \leq d \leq \infty$, and each prime p, there exists a d-dimensional seminormal universally going-down domain R of characteristic p such that $R(= R^+)$ is not a Prüfer domain. This same example shows that "p-closed" cannot be weakened to "seminormal" in Corollary 4.3(b), (c).

(b) For convenience, let us say that a domain R satisfies (*) in case the extension $R \subset S$ is mated (in the sense of [4]) for each overring S of R. By [4, Proposition 3.6], R is a Prüfer domain if and only if R is an integrally closed domain satifying (*). Moreover, it was shown in [8, Proposition 2.2(b)] that each universally going-down domain satisfies (*). The converse, however, is false. Indeed, [5, Remark 2.7(c)] shows for each d, $1 \leq d \leq \infty$, there exists a d-dimensional (quasilocal) root-closed (going-down) domain R of characteristic 0 such that R satisfies (*) and R is not a Prüfer domain. Somewhat as a consolation, we note that each of these rings R is weakly normal.

Our final results are motivated by Corollary 4.2 [(1) $\Leftrightarrow$ (3)] and the fact that any factor domain of a Prüfer domain is a Prüfer domain.

Proposition 4.5. *If R is a weakly normal going-down domain and $P \in$ Spec(R), then R/P is a weakly normal going-down domain.*

Proof. By [5, Remark 2.11], R/P is a going-down domain. As for weak normality, it is enough to consider $(R/P)_{M/P} \cong R_M/PR_M$ for the maximal ideals M containing P. Now, R_M is a quasilocal weakly normal (hence seminormal) going-down domain. Thus, by [5, Corollary 2.6], $A = R_M$ is a divided domain ; i.e., $QA_Q = Q$ for all $Q \in$ Spec(A). Consequently, the assertion follows from the following easy consequence of the Yanagihara-Itoh characterization of weak normality [16, Theorem 1]. If B is a weakly normal domain and $I = IB_I \in \mathrm{Spec}(B)$, then B/I is weakly normal. □

Remark 4.6. It is easy to see that Proposition 4.5 fails without the "going-down" hypothesis. Consider, for instance, $R = \mathbf{F}_2[X, Y]$ and $P = (X^2 - Y^3)$. Since R is integrally closed, R is weakly normal. However, R/P is not weakly normal since it is not 2-closed : $x = X+P$ and $y = Y+P$ satisfy $(xy^{-1})^2 = y \in R/P$ although $xy^{-1} \notin R/P$. (Of course, as Proposition 4.5 requires, this R is not a going-down domain. This is also evident directly since $\mathrm{Spec}(R)$ is not a tree.) □

Proposition 4.7 is the "universal" analogue of a stability result on the class of going-down domains [5, Remarks 2.11 and 3.2(a), (b)].

Proposition 4.7. *If R is a universally going-down domain and $P \in$ $\mathrm{Spec}(R)$, then R/P is also a universally going-down domain.*

Proof. Let $A = R/P$. We must show that the inclusion map $A \to T$ is a universally going-down homomorphism for each overring T of A. Put $S = R + PR_P$ and $Q = PR_P$. By standard homomorphism theorems, $S/Q \cong A$ and $T = B/Q$ for a suitable domain B satisfying $S \subset B \subset R_P$. Moreover, $S_Q = R_P$ and $Q = QS_Q$. As S inherits the property of being a universally going-down domain from R [8, Proposition 2.2(a)], we may abuse notation, identifying R with S and P with Q. In particular, we have $P = PR_P$.

Now, since B is an overring of R, the hypothesis on R yields that the inclusion map $R \to B$ is a universally going-down homomorphism. Hence $A \to A \otimes_R B$ is also a universally going-down homomorphism. It will therefore suffice to prove that $A \otimes_R B$ is canonically isomorphic to T. For this, observe first that

$$P \subset PB \subset PR_P = P,$$

whence $PB = P$. It follows that

$$A \otimes_R B = R/P \otimes_R B \cong B/PB = B/P = T. \quad \square$$

Remark 4.8. (a) Let R be a universally going-down domain. Not every domain containing R is a (universally) going-down domain : consider, for instance, $R[X, Y]$ (whose spectrum is not even a tree). However, by [8, Proposition 2.2(a)], each overring of R is a universally going-down domain. Thus, by Proposition 4.7, if $P \in \mathrm{Spec}(R)$ (and R is a universally going-down domain), then each overring of R/P is a universally going-down domain.

(b) The following result is in the spirit of (a). Let $R \subset T$ be an integral extension of domains such that R is a universally going-down domain and T is the weak normalization of R in T. (This last condition just means that ${}_{*T}R = T$.) Then T is also a universally going-down domain.

The proof follows easily by considering the tower

$$R[X_1, \ldots, X_n] \subset T[X_1, \ldots, X_n] \subset D[X_1, \ldots, X_n]$$

for each domain D containing T and each positive integer n. Indeed, if we call this tower $A \subset B \subset C$, the key point to notice is that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an order-isomorphism (since weak normalization is a universal homeomorphism [2]). Hence, since $A \subset C$ satisfies going-down, so does $B \subset C$.

(c) The assertion in (b) fails without the "weak normalization" hypothesis. Indeed, consider $R = \mathbf{Z} \subset \mathbf{Z}[3\sqrt{2}\,] = T$. This is an integral extension and R (being Prüfer) is a universally going-down domain. However, T is not a universally going-down domain since $T^* = T^+ = T \subsetneq T' = \mathbf{Z}[\sqrt{2}\,]$ (cf. Corollary 4.3(a)).

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