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UNIVERSALLY INCOMPARABLE RING-HOMOMORPHISMS

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A homomorphism $f : R \to T$ of (commutative) rings is said to be universally incomparable in case each base change $R \to S$ induces an incomparable map $S \to S \otimes_R T$. The most natural examples of universally incomparable homomorphisms are the integral maps and radiciel maps. It is proved that a homomorphism $f : R \to T$ is universally incomparable if and only if f is an incomparable map which induces algebraic field extensions of fibres, $k(f^{-1}(Q)) \to k(Q)$, for each prime ideal Q of T. In several cases (f algebra-finite, T generated as R-algebra by primitive elements, T an overring of a one-dimensional Noetherian domain R), each universally incomparable map is shown to factor as a composite of an integral map and a special kind of radiciel.

1. Introduction

Considerable attention has been paid over the years to the properties of lying-over, going-up, going-down, and incomparability concerning the behavior of prime ideals relative to homomorphisms, especially inclusion maps, of commutative rings (*cf.* [12, page 28]). It has seemed natural to

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consider as well the corresponding universal properties, and most of these have been characterized. (As usual, if P is a property of homomorphisms, then $R \to T$ is said to be *universally* P in case, for each change of base $R \to S$, the induced homomorphism $S \to S \otimes_R T$ satisfies P.) Indeed, universally lying-over is equivalent to lying-over [11, Proposition 3.6.1 (ii), page 244]; universally going-up is equivalent to integral [2, Lemma, page 160]; and universally going-down has been closely linked in [4] with the notion of weak normalization, in the sense of Andreotti and Bombieri. Our purpose here is to complete this circle of ideas by studying the universally incomparable homomorphisms. These maps are characterized first in Theorem 2.2 in ways that are, perhaps surprisingly, not wholly analogous to characterizations of related classes of homomorphisms in [4], [13].

The most natural examples of universally incomparable homomorphisms are the integral maps and the radiciel maps. (As in [11], a homomorphism $f : R \to T$ is said to be *radiciel* in case the induced function $f^* : \text{Spec}(T) \to \text{Spec}(R)$ is an injection and whenever $f^*(q) = p$, the resulting field extension $R_p/pR_p \to T_q/qT_q$ is purely inseparable. As radiciel is equivalent to universally radiciel [11, Proposition 3.7.1, page 246], it evidently implies universally incomparable.) Since any composition of incomparable maps is incomparable and since tensor product commutes with direct limit, any composition of universally incomparable maps is universally incomparable. A sharp converse is available in the algebra-finite case, for then any universally incomparable map is the composite of an integral map with a special type of radiciel homomorphism (Remark 2.12 (b)). Moreover, each incomparable map is universally incomparable in the algebra-finite case (Corollary 2.7), but not in general (Remark 2.3). Factoring results of the above sort are also obtained for two types of maps $R \to T$ which need not be algebra-finite. In the first of these (Corollary 2.8, Remark 2.12 (c)), T is assumed to be generated as an R-algebra by algebraic elements of a particular kind. The second (Proposition 2.13) addresses a Noetherian context and leads to a geometric example (Example 2.14) illustrating that universally incomparable overring extensions form a strictly larger class than the universally going-down overring extensions characterized in [4].

Throughout, all rings are assumed commutative, with unit; and all ring-homomorphisms are assumed unital. In addition, $X = X_1, \ldots, X_n$ denote commuting algebraically independent indeterminates over the appropriate rings. If $f : R \to T$ is a ring-homomorphism, then f_n denotes the induced homomorphism $R[X_1, \ldots, X_n] \to T[X_1, \ldots, X_n]$; and if p is a prime of R, then $k_R(p)$ denotes R_p/pR_p and T_p denotes $T_{f(R\setminus p)}$ $(\cong T \otimes_R R_p)$. Any unexplained material is standard, as in [9] and [12].

2. Results

Before characterizing universally incomparable homomorphisms, we begin by adapting some material from [13]. Let A be a commutative ring and let $p \in \mathrm{Spec}(A)$. The prime ideal $pA[X]$ of the polynomial ring $A[X]$ will be denoted by p^* . For each monic irreducible $\alpha \in k_A(p)[X]$, the *upper of* p corresponding to α is

$$\langle p, \alpha\rangle = \{h \in A[X] \mid \text{the canonical image of } h \text{ in } (A/p)[X] \text{ is divisible by } \alpha\} .$$

The following result was obtained by McAdam [13, Theorems 1 and 2] for the case of inclusion maps of domains, but the methods of [13] carry over directly, and so we omit the proof.

LEMMA 2.1. *Let* R *be a ring and* p *a prime ideal of* R . *Then:*

(a) Let $j : R \to R[X]$ *be the canonical inclusion map. Then the prime ideals* P *of* $R[X]$ *such that* $j^{-1}(P) = p$ *are of two types:* p^* *and the uppers of* p . *Moreover,* p *is properly contained in each upper of* p ; *and if* α, γ *are distinct monic irreducible polynomials in* $k_R(p)[X]$, *then* $\langle p, \alpha\rangle$ *and* $\langle p, \gamma\rangle$ *are incomparable.*

(b) Let $f : R \to T$ *be a ring-homomorphism, with* $f_1 : R[X] \to T[X]$ *the induced homomorphism. Then for each monic irreducible* $\alpha \in k_R(p)[X]$, *the primes* Q *of* $T[X]$ *such that* $f_1^{-1}(Q) = \langle p, \alpha\rangle$ *are the uppers of the form* $\langle q, \beta\rangle$, *where* $q \in \mathrm{Spec}(T)$, $f^{-1}(q) = p$, *and the canonical inclusion* $k_R(p)[X] \to k_T(q)[X]$ *sends* α *to a polynomial which is*

divisible by β . *Moreover, the primes* Q *of* $T[X]$ *such that* $f_1^{-1}(Q) = p^*$ *are of two types:* q^* *and* $\langle q, \delta \rangle$, *where* $q \in \mathrm{Spec}(T)$, $f^{-1}(q) = p$, *and no monic irreducible polynomial in* $k_R(p)[X]$ *is divisible by* δ .

The following definition will be of fundamental importance. A ring-homomorphism $f : R \to T$ is called *residually algebraic* if, for each $q \in \mathrm{Spec}(T)$ and $p = f^{-1}(q)$, the induced field extension $k_R(p) \to k_T(q)$ is algebraic. The role of this property in the following characterization of "universally incomparable" should be contrasted with the part played by purely inseparable requirements in characterizations of radiciel and various related properties [4, Theorems 2.1 and 2.5] (*cf.* also [13, Theorems 3 and B]).

THEOREM 2.2. *For a ring-homomorphism* $f : R \to T$, *the following conditions are equivalent:*

(i) f *is universally incomparable;*

(ii) f *is incomparable and residually algebraic;*

(iii) *the inclusion map* $f(R) \to T$ *is universally incomparable;*

(iv) *there exists* $n \geq 1$ *such that* f_n *is incomparable;*

(v) f_1 *is incomparable;*

(vi) *for each* $n \geq 0$, f_n *is incomparable;*

(vii) *for each* $n \geq 0$, f_n *is incomparable and residually algebraic.*

Proof. It is evident that *(vii)* ⇒ *(vi)* ⇒ *(v)* ⇒ *(iv)*.

(i) ⇒ *(vi)*. Trivial since $R[X_1, \ldots, X_n] \otimes_R T \cong T[X_1, \ldots, X_n]$.

(vi) ⇒ *(i)*. It is known that direct limit preserves incomparability [6, Proposition 2.3]. Consequently, the criterion in [4, Proposition 2.2] reduces the present assertion to the easy observation that if f is incomparable, so is the induced map $R/J \to T/JT$ for each ideal J of R .

(i) $\Leftrightarrow$ *(iii)*. f_n is incomparable if and only if the inclusion map $f(R)[X_1, \ldots, X_n] \to T[X_1, \ldots, X_n]$ is incomparable, because $f(R)[X_1, \ldots, X_n]$ is the image of f_n. The present assertion therefore follows since we have already seen that *(i)* $\Leftrightarrow$ *(vi)*.

The remainder of the proof will sketch how to modify various arguments of McAdam [13], originally used to treat (what [4, Theorem 2.5] showed to be) universally unibranched inclusion maps of domains.

(v) $\Rightarrow$ *(ii)*. Assume *(v)*. That f is incomparable now follows from Lemma 2.1 *(a)* and the fact that $f^{-1}(q) = p$ entails $f_1^{-1}(q^*) = p^*$. In view of Lemma 2.1 *(b)*, we can show that f is residually algebraic by reasoning as in [13, page 709, lines 4-8].

(ii) $\Rightarrow$ *(v)*. Assume *(ii)*. If *(v)* fails, there exist distinct primes $Q_1 \subset Q_2$ of $T[X]$ such that $f_1^{-1}(Q_1) = f_1^{-1}(Q_2) = P \in \mathrm{Spec}(R[X])$. If $P = \langle p, \alpha \rangle$ then Lemma 2.1 gives $Q_i = \langle q_i, \gamma_i \rangle$, where $f^{-1}(q_i) = p$ and $q_1 \neq q_2$. As Q_i lies over q_i, it follows that $q_1 \subset q_2$, contradicting incomparability of f. The remaining case, in which P assumes the form p^*, may be treated as in [13, page 709, lines 14-18].

(iv) $\Rightarrow$ *(v)*. Since $(f_{n-1})_1 = f_n$, the present assertion follows from the first observation in the above proof that *(v)* $\Rightarrow$ *(ii)*.

(vi) $\Rightarrow$ *(vii)*. Since $(f_n)_1 = f_{n+1}$, the present assertion follows since we have already seen that *(v)* $\Rightarrow$ *(ii)*.

(v) $\Rightarrow$ *(vi)*. Since $f_{n+2} = (f_n)_2$ and since *(v)* $\Leftrightarrow$ *(ii)*, it is enough to show that *(ii)* implies f_1 is residually algebraic. Accordingly, assume *(ii)*, and consider $Q \in \mathrm{Spec}(T[X])$, with $P = f_1^{-1}(Q)$. If $Q = q^*$ then $P = p^*$ with $p = f^{-1}(q)$, and the asserted algebraicity of $k_{R[X]}(P) = k_R(p)(X) \to k_{T[X]}(Q) = k_T(q)(X)$ follows from the assumed algebraicity of $k_R(p) \to k_T(q)$. In view of Lemma 2.1 *(b)*, the residually algebraic nature of f readily assures that the only remaining case is

$Q = \langle q, \beta \rangle$, $P = \langle p, \alpha \rangle$. By taking suitable localizations of factor-rings, precisely as in [13, page 710], we need only prove the following statement. If $K_1 \subset K_2$ is an algebraic field extension and P_i is a maximal ideal of $K_i[X]$, then the field extension $K_1[X]/P_1 \to K_2[X]/P_2$ is algebraic. This, however, is evident since algebraicity is transitive. The proof is complete.

REMARK 2.3. An incomparable ring-homomorphism need not be universally incomparable. Surely the simplest example of this is a transcendental field extension $k \to k(Y)$, for it is evidently not residually algebraic. Of course, $k(Y)$ is not algebra-finite over k (by, for instance, Hilbert's Nullstellensatz), thus suggesting the positive result in Corollary 2.7 below.

The following definition will be helpful. If $f : R \to T$ is a ring-homomorphism and $p \in \mathrm{Spec}(R)$, then $T(p) = T(p;\, f)$ will denote $T \otimes_R k_R(p)$ $[\cong T_p/pT_p]$. Notice that $\mathrm{Spec}(T(p))$ is isomorphic, *qua* topological space or partially ordered set, with

$$\{q \in \mathrm{Spec}(T) \mid f^{-1}(q) = p\} .$$

When the latter set is empty, $T(p)$ is the zero-ring, whose dimension is conventionally taken as -1 .

The $T(p)$ notation leads to the following characterization of incomparability. In view of the above comments, its proof may be left to the reader.

LEMMA 2.4. *For a ring-homomorphism $f : R \to T$, the following conditions are equivalent:*

(i) *f is incomparable;*

(ii) *for each $p \in \mathrm{Spec}(R)$ such that $T(p) \neq 0$, the canonical homomorphism $k_R(p) \to T(p)$ is incomparable;*

(iii) *for each $p \in \mathrm{Spec}(R)$, $\dim(T(p)) \leq 0$;*

(iv) *for each $p \in \mathrm{Spec}(R)$ such that $T(p) \neq 0$, the induced map $R_p \to T_p$ is incomparable.*

In view of Theorem 2.2 [*(i)* $\Leftrightarrow$ *(ii)*], we obtain characterizations of "universally incomparable" by augmenting each of the conditions in Lemma 2.4 with "for each $p \in \mathrm{Spec}(R)$ such that $T(p) \neq 0$, the canonical ring-homomorphism $k_R(p) \to T(p)$ is residually algebraic". Next, we tend to the zero-dimensional case, by showing that universally incomparable, in that context, implies integral.

PROPOSITION 2.5. *Let $f : R \to T$ be a ring-homomorphism. Then:*

(a) Let R, T each be zero-dimensional rings. Then f is integral if (and only if) f is residually algebraic.

(b) Let f appear in a pullback diagram

$$\begin{array}{ccc} R & \xrightarrow{\ f\ } & T \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\ C & \xrightarrow[\ g\ ]{} & D \end{array}$$

in which C, D are zero-dimensional rings, g is an inclusion, T is quasilocal, and v is surjective. Then f is integral if (and only if) f is residually algebraic.

Proof. *(a)* By passing to $f(R) \to T$, we may assume that f is an inclusion map. If the result fails, select $b \in T$ such that b is not integral over R. Then

$$S = \left\{b^n + r_1 b^{n-1} + \ldots + r_{n-1} b + r_n \mid r_i \in R \text{ for each } i;\ n \geq 1\right\} \cup \{1\}$$

is a multiplicative subset of T which does not contain 0. Accordingly (*cf.* [12, Theorem 1]), T has a prime ideal N which is disjoint from S. Let $M = N \cap R$. As N and M are maximal ideals in T and R, respectively, algebraicity of the field extension $R/M \to T/N$ produces an equation

$$(b+N)^n + (r_1+N)(b+N)^{n-1} + \ldots + r_n+N = 0 \in T/N$$

for suitable $r_i \in R$, $n \geq 1$. But then

$$b^n + r_1 b^{n-1} + \ldots + r_n \in N \cap S = \emptyset ,$$

the desired contradiction.

(b) It is well-known (and easy to see) that $I = \ker(v)$ coincides with $\ker(w)$, so that D (respectively, C) may be identified with T/I (respectively, R/I). One readily verifies then that g inherits the residually algebraic property from f. Therefore, by *(a)*, g is integral. Applying [8, Proposition 2.2 (10)] to the above pullback diagram or calculating directly with the help of the above identifications, one shows that f is integral, completing the proof.

We pause to record an application of Proposition 2.5 *(b)*. Let $f : R \to T$ be an inclusion map of rings such that $\text{Spec}(R) = \text{Spec}(T)$ as sets (or, equivalently, as topological spaces). Then f is integral if (and only if) f is residually algebraic. (For the proof, we may assume $R \neq T$. Then R is easily seen to be quasilocal, say with maximal ideal M, and we may apply Proposition 2.5 *(b)* to the pullback description $R = T \times_{T/M} R/M$.)

It is interesting to note that "$\text{Spec}(R) = \text{Spec}(T)$ as *sets*" is not a universal property. (Contrast the situation for *schemes*!) Indeed, let K/F be a transcendental extension of distinct fields, $B = K[[X]] = K + M$ (with $M = XB$), and $A = F + M$. It is well-known (*cf.* [9, Exercise 12, page 202]) that $\text{Spec}(A) = \{0, M\} = \text{Spec}(B)$. However $k_A(M) \to k_B(M)$ is just $F \to K$, which is not algebraic. Thus, by Theorem 2.2 [*(i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(v)*], the inclusion map $A[Y] \to B[Y]$ is not incomparable; *a fortiori*, $\text{Spec}(A[Y]) \neq \text{Spec}(B[Y])$.

We next consider the algebra-finite context.

PROPOSITION 2.6. *Let $f : R \to T$ be an incomparable ring-homomorphism relative to which T is algebra-finite over R and R is integrally closed in T. Then f is radiciel. Moreover, if $q \in \text{Spec}(T)$ and $p = f^{-1}(q)$, then the canonical field extension $k_R(p) \to k_T(q)$ is an isomorphism.*

Proof. By passing to $f(R) \to T$, we may assume that f is an inclusion map. Let q, p be as in the statement. Note that R_p is integrally closed in T_p; by incomparability, qT_p is isolated in its fibre (over pR_p); and T_p is algebra-finite over R_p. Thus by Zariski's Main Theorem (as in, for instance, [7]), there exists an element

$g \in R_p \backslash pR_p$ such that $(T_p)_g = (R_p)_g$. Since g is a unit of R_p (and of T_p), this entails $T_p = R_p$, whence $qT_p = pR_p$. If $j : T \to T_p$ is the canonical homomorphism, it follows that $j^{-1}(pR_p) = q$. Hence the canonical function $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is an injection.

It suffices to prove the final assertion. For q, p as above, the fact that $T_p = R_p$ combines with [9, Corollary 5.2] to yield $T_p \cong T_q$. Thus we obtain the canonical identifications

$$k_T(q) = T_q/qT_q = T_p/qT_p = R_p/pR_p = k_R(p) \ ,$$

completing the proof.

COROLLARY 2.7. *Let $f : R \to T$ be a ring-homomorphism relative to which T is algebra-finite over R . Then f is universally incomparable if (and only if) f is incomparable.*

Proof. Let S be the integral closure of R in T . Then $f = hg$, for the canonical homomorphisms $g : R \to S$ and $h : S \to T$. By integrality, g is universally incomparable, and so it suffices to show that h is universally incomparable. However this, in turn, follows from Proposition 2.6 since h inherits incomparability from f . The proof is complete.

The preceding ideas permit us next to give another class of examples of universally incomparable homomorphisms. Following [3], we shall say that an element b of a commutative R-algebra T is *primitive over* R in case $g(b) = 0$ for some $g \in R[X]$ with at least one coefficient equal to 1 .

COROLLARY 2.8. *Let $f : R \to T$ be a ring-homomorphism relative to which T may be generated as an R-algebra by a set $S = \{b_i\}$, where each b_i is primitive over R . Then f is universally incomparable. Moreover, if R is integrally closed in T , then f is radiciel.*

Proof. By Theorem 2.2 [*(iii)* ⇒ *(i)*], we may assume that f is an inclusion map. In addition, we may assume that T is algebra-finite over R . (The points involved are that $T = \varinjlim R[b_1, \ldots, b_n]$ where $\{b_1, \ldots, b_n\}$ ranges over the finite subsets of S ; and direct limit preserves (universal) incomparability [6, Proposition 2.3].) Thus, we can suppose that $S = \{b_1, \ldots, b_n\}$, so that f "factors" as

$$R \subset R[b_1] \subset R[b_1, b_2] \subset \ldots \subset R[b_1, b_2, \ldots, b_{n-1}] \subset R[b_1, \ldots, b_n] = T .$$

However for each $i \geq 0$, b_{i+1} is primitive over $R[b_1, \ldots, b_i]$, and so [3, Theorem] assures that the extension $R[b_1, \ldots, b_i] \subset R[b_1, \ldots, b_{i+1}]$ is incomparable. By considering the displayed tower, we thus see that f is incomparable, and so an application of Corollary 2.7 establishes the first assertion. Using the fact that direct limit preserves radiciel [4, Lemma 2.4 (b)], and appealing to Proposition 2.6 instead of Corollary 2.7, one may fashion a parallel proof of the second assertion.

REMARK 2.9. The hypothesis of Corollary 2.8 was suggested by a characterization of integrality in [3, Remark 8 (c)]. We next record a nonintegral application of Corollary 2.8: if T is an overring of a Prüfer domain R , then the inclusion map $f : R \to T$ is radiciel. Indeed, each element of T is primitive over R (by either [10, Theorem 2] or [3, Corollary 5]) and R is integrally closed (in T), so that the second assertion in Corollary 2.8 yields the desired conclusion.

Another proof of the result in the preceding paragraph uses ideas that will reappear in the proof of Proposition 2.10. To wit: since f is flat, it is a universally going-down overring extension, and hence is radiciel by [4, Theorem 3.17 and Corollary 3.12 (b)].

Let P be a property of (some) ring-homomorphisms. Following [5], we shall say that a domain R is a *P-domain* in case the inclusion map $R \to T$ satisfies P for each overring T of R .

PROPOSITION 2.10. *For an integrally closed domain R , the following conditions are equivalent:*

(i) R is a universally incomparable-domain;

(ii) R is a radiciel-domain;

(iii) R is a universally going-down-domain;

(iv) R is a Prüfer domain.

Proof. *(ii) ⇒ (i)*. Trivial.

(iii) ⇔ (iv). This is [5, Corollary 2.3].

(i) ⇒ *(iv)*. It is well-known that an integrally closed incomparable-domain must be a Prüfer domain (*cf*. [9, Theorem 26.2]).

(iii) ⇒ *(ii)*. Assume *(iii)*, and consider the inclusion map $f : R \to T$ for an overring T of R. Since f is a universally going-down overring map, [4, Theorem 3.17] implies that f satisfies the UGD property and so, by [4, Corollary 3.12 (b)], f is radiciel, completing the proof.

REMARK 2.11. One may, in the spirit of [5], proceed to develop a theory for universally incomparable-domains. Typical results state that a domain R is a universally incomparable-domain if and only if each localization R_M is; and, as above, each universally going-down-domain is a universally incomparable-domain. However, such domains seem less fruitful than the universally going-down domains, since they fail to sustain the analogue of [5, Theorem 2.6]. Indeed, there is no (universally incomparable-) domain R for which the inclusion map $R \to T$ is (universally) incomparable for *each* domain T containing R : consider $T = R[X]$!

We next formalize a concept which has appeared implicitly in Proposition 2.6 and Corollary 2.8. A (typically injective) ring-homomorphism $f : R \to T$ will be called an *essential-identity* (and R, T will be called *essentially equal*) if, for each $p \in \operatorname{Spec}(R)$ such that $T(p) \neq 0$, the induced map $R_p \to T_p$ is an isomorphism. The next remark collects some relevant material.

REMARK 2.12. (a) Let f be an essential-identity. Then Lemma 2.4 [*(iv)* ⇒ *(i)*] implies that f is incomparable. In fact, f is universally incomparable since, by the latter part of the proof of Proposition 2.6, f is actually radiciel.

(b) By (a), the combined effect of Proposition 2.6 and Corollary 2.7 is the following assertion. Let $f : R \to T$ be an incomparable ring-homomorphism relative to which T is algebra-finite over R . Let S be the integral closure of R in T , and consider the canonical homomorphisms $g : R \to S$ and $h : S \to T$. Then $f = hg$ is universally incomparable since g is integral and h is an essential-identity.

(c) It is easy to verify (using [11, Propositions 6.1.2-6.1.6, pages 128-130]) that direct limit preserves the essential-identity property. Thus, by (b), the second assertion of Corollary 2.8 may be strengthened, in case f is an injection, to say that R and T are essentially equal.

(d) The characterization of flat overrings in [14, Theorem 1] readily leads to the following result. Let T be an overring of a domain R, with $f : R \to T$ the inclusion map. If T is R-flat, then f is an essential-identity; the converse holds if $\dim(R) = 1$.

In the absence of finite-type hypotheses, a factoring result in the spirit of Remark 2.12 (b) seems unavailable. However, we do have the following important special case.

PROPOSITION 2.13. *Let T be an overring of a one-dimensional Noetherian domain R, with $f : R \to T$ the inclusion map. Then f is universally incomparable. Indeed, if S denotes the integral closure of R in T, with $g : R \to S$ and $h : S \to T$ the canonical maps, then $f = hg$, g is integral, and h is flat. (Hence h is also an essential-identity and radiciel.)*

Proof. The first assertion may be seen directly since the Krull-Akizuki Theorem, in the version given in [1, Proposition 5, page 500], readily implies that f satisfies condition *(ii)* in the statement of Theorem 2.2.

We may therefore assume that $R = S$ is integrally closed in T. By Remark 2.12(a) and (d), it is enough to prove that f is an essential-identity. Consider $M \in \mathrm{Spec}(R)$ such that $T(M) \neq 0$; we must show that $R_M \to T_M$ is an isomorphism. Without loss of generality, $M \neq 0$. By *abus de langage*, we may further assume that (R, M) is quasi-local, and need to show that f is surjective.

It is enough to prove that R contains each element $u \in T$. Set $A = R[u]$. Choose a nonzero prime N of T. (Such exists since $T(M) \neq 0$.) By the remarks made two paragraphs ago, $N \cap A$ is isolated in its fibre (above M). Then, just as in the proof of Proposition 2.6, Zariski's Main Theorem leads to $A = R$. This completes the proof.

One consequence of Proposition 2.13 is that each one-dimensional Noetherian domain R is a universally incomparable-domain. In view of Remarks 2.11 and 2.9, it is important to note that such R need not be a *universally* going-down-domain. In closing, we shall illustrate this fact with a geometric example.

EXAMPLE 2.14. Let R be the local ring at the origin for the nodal curve $y^2 = x^3 + x^2$. In other words, R is the localization of $\mathbf{C}[X, Y]/(Y^2-X^3-X^2)$ at the canonical image of (X, Y) . Certainly, R is a one-dimensional local (Noetherian) domain, say with maximal ideal M . Let R' be the integral closure of R ; of course, R' has but two maximal ideals, say N_1 and N_2 . As R is seminormal with residue fields of characteristic zero, R is the weak normalization (in the sense of Andreotti-Bombieri) of R in R' . So R' is not that weak normalization, whence [5, Theorem 2.4] assures that R is not a universally-going-down domain. Accordingly, by [5, Theorem 2.6], R has a valuation overring T such that the inclusion map $f : R \to T$ is not universally going-down. However, $T = (R')_{N_i}$ and, as predicted by Proposition 2.13, f factors as the composite of the integral map $R \to R'$ and the radiciel (essential-identity, flat) map $R' \to T$.

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