SEMINORMAL RINGS GENERATED BY ALGEBRAIC INTEGERS

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§1. *Introduction.* For each algebraic integer α, let $\mathbb{Z}_\alpha$ denote the ring of integers of the algebraic number field $\mathbb{Q}(\alpha)$. There has been continuing interest in finding ring-theoretic conditions characterizing when $\mathbb{Z}_\alpha$ coincides with its subring $\mathbb{Z}[\alpha]$ (*cf.* [**15**, **18**, **1**, **13**, **12**]). One way to extend such work is to consider the intermediate ring $\mathbb{Z}[\alpha]^+$, the seminormalization (in the sense of [**17**]) of $\mathbb{Z}[\alpha]$ in $\mathbb{Z}_\alpha$. Indeed, if we let I_α denote the conductor $(\mathbb{Z}[\alpha]: \mathbb{Z}_\alpha)$, then it is easy to see (*cf.* Proposition 3.1) that $\mathbb{Z}[\alpha] = \mathbb{Z}_\alpha$, if, and only if, $\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$ and I_α is a radical ideal of $\mathbb{Z}_\alpha$. The condition $\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$ seems worthy of separate attention in view of recent results (*cf.* [**3**]) that seminormal rings generated by algebraic integers are "often" automatically of the form $\mathbb{Z}_\alpha$. We show in Proposition 3.3 that the condition $\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$ is equivalent to several universal properties, including notably that the canonical closed surjection $\mathrm{Spec}\,(\mathbb{Z}_\alpha) \to \mathrm{Spec}\,(\mathbb{Z}[\alpha])$ be universally open, be universally going-down, or be a universal homeomorphism.

Quadratic algebraic number fields present a situation in which the condition $\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$ may be characterized in terms of elementary number theory. Recall that each quadratic number field K is uniquely of the form $\mathbb{Q}(\sqrt{d})$, for a suitable square-free integer d. As usual, let

$$\omega_d = \begin{cases} \dfrac{1+\sqrt{d}}{2}, & \text{if} \quad d \equiv 1 \pmod 4, \\ \sqrt{d}, & \text{if} \quad d \equiv 2, 3 \pmod 4. \end{cases}$$

It is well-known that the ring of integers of K (that is, the maximal order of K) is $\mathbb{Z}[\omega_d]$, which is a free abelian group on the basis $\{1, \omega_d\}$. Each non-maximal order of K is uniquely of the form $\mathbb{Z}[n\omega_d]$, for a suitable integer $n \geq 2$. For $\alpha = n\omega_d$, the property $\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$ is just $\mathbb{Z}[n\omega_d]^+ = \mathbb{Z}[\omega_d]$, and this is characterized in Theorem 3.4 *via* divisibility and ramification conditions; alternately, *via* divisibility and congruence conditions on d and n.

At the other extreme from the behaviour $\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$ is the condition $\mathbb{Z}[\alpha]^+ = \mathbb{Z}[\alpha]$, *i.e.*, the condition that $\mathbb{Z}[\alpha]$ be seminormal. For the quadratic case, $\alpha = n\omega_d$, Corollary 4.5 characterizes this new condition in the above spirit of elementary number theory. This result generalizes some work of Ooishi [**16**]. Moreover the main result of this paper, Theorem 4.4, identifies $\mathbb{Z}[n\omega_d]^+$ for d, n as above; that is, finds the m so that $(1 \leq m \mid n$ and$)$ $\mathbb{Z}[n\omega_d]^+ = \mathbb{Z}[m\omega_d]$. As $\mathbb{Z}[\omega_d]$ is trivially (semi)normal, this amounts to finding the seminormalization of an arbitrary order in a quadratic algebraic number field.

If one omits universality from the considerations of the first paragraph, one arrives at the condition "Spec $(\mathbb{Z}_\alpha) \to$ Spec $(\mathbb{Z}[a])$ is a homeomorphism and I_α is a radical ideal of $\mathbb{Z}_\alpha$". It is easy to see (*cf.* Proposition 2.1) that this is equivalent to "$\mathbb{Z}[\alpha]$ is a GPVD". (A GPVD, or globalized pseudo-valuation domain, is a particularly tractable type of seminormal integral domain introduced *via* pullbacks in [**4, 5**].) Theorem 2.5, which is the principal result in section 2, identifies all the quadratic orders which are GPVD's. Since all rings of integers are GPVD's, this amounts to characterizing when $\mathbb{Z}[n\omega_d]$ is a GPVD; and this is done, in the above spirit of elementary number theory, *via* conditions on divisibility, congruence, and Legendre symbols. These conditions permit effective calculations, yielding for instance (in Remark 3.5(c)) the first known examples in which $\mathbb{Z}[n\omega_d]$ is seminormal but not a GPVD. In these examples, $n = 2$. It should be noted that the case $n = 2$ of Theorem 2.5 is a completion of the work begun in [**5**, Example 4], where it had been shown that $\mathbb{Z}[\sqrt{d}]$ is a GPVD, if, and only if, either $d \equiv 2, 3 \pmod 4$ or $d \equiv 5 \pmod 8$. As to this paper's organization, we begin by generalizing this result from [**5**], and the techniques lead naturally to seminormalization of quadratic orders.

The notations $\mathbb{Z}_\alpha$, I_α and ω_d will always be used in the senses defined above. For background, the reader is referred to the cited articles on seminormalization, weak normalization, universal properties and GPVD's; and to standard texts such as [**9, 14, 19**]. The interested reader is invited to replace $\mathbb{Z}$ wherever possible with a suitable one-dimensional Noetherian (possibly Dedekind) integral domain.

§2. *GPVD orders.* The main result of this section, Theorem 2.5, determines precisely which of the quadratic orders $\mathbb{Z}[n\omega_d]$ is a GPVD. In preparation for this, we give two propositions, each of some independent interest.

First, for the sake of completeness, we shall recall some background definitions and facts from [**4**]. An integral domain R is said to be a pseudo-domain (PVD) in the case when R has a (necessarily unique) valuation overring V such that Spec $(R) =$ Spec (V) as sets. PVD's may be characterized as the pullbacks $V x_k K$, where (V, M) is a valuation domain and K is a subfield of $k = V/M$. An integral domain R is said to be a locally pseudo-valuation domain (LPVD) if R_M is a PVD for each maximal ideal M of R. Each PVD is an LPVD, as is each Prüfer domain as well. Finally, an integral domain R is said to be a globalized pseudo-valuation domain (GPVD) if R has a (canonically associated) Prüfer overring T such that (a) the canonical contraction map Spec $(T) \to$ Spec (R) is a bijection; and (b) there exists a nonzero radical ideal A common to T and R such that each prime ideal of T (resp., R) which contains A is a maximal ideal of T (resp., R). Each Prüfer domain is a GPVD, and each GPVD is an LPVD. However, an LPVD need not be a GPVD, even in the one-dimensional or Noetherian cases. For quasi-local domains, the notions of PVD, LPVD, and GPVD coincide.

Proposition 2.1. *Let α be an algebraic integer and f*: Spec $(\mathbb{Z}_\alpha) \to$ Spec $(\mathbb{Z}[\alpha])$ *the canonical contraction map. Then the following conditions are equivalent.*

(1) $\mathbb{Z}[\alpha]$ *is seminormal and f restricts to a bijection* $\mathrm{Ass}_{\mathbb{Z}_\alpha}(I_\alpha) \to \mathrm{Ass}_{\mathbb{Z}[\alpha]}(I_\alpha)$.

(2) I_α *is a radical ideal of* $\mathbb{Z}_\alpha$ *and f is a homeomorphism (with respect to Zariski topology).*

(3) I_α *is a radical ideal of* $\mathbb{Z}_\alpha$ *and f is a bijection.*

(4) I_α *is a radical ideal of* $\mathbb{Z}_\alpha$ *and f induces a bijection*

$$\mathrm{Spec}(\mathbb{Z}_\alpha/I_\alpha) \to \mathrm{Spec}(\mathbb{Z}[\alpha]/I_\alpha).$$

(5) $\mathbb{Z}[\alpha]$ *is a* GPVD.

Proof. (1)⇒(4). Assume (1). Then $\mathbb{Z}[\alpha]$ is "seminormal in" $\mathbb{Z}_\alpha$ in the sense of [**17**], and so [**17**, Lemma 1.3] yields that I_α is a radical ideal of $\mathbb{Z}_\alpha$. (*Cf.* also [**10**, Theorem 1.1].) Of course, $I_\alpha \neq 0$ since $\mathbb{Z}_\alpha$ is a module-finite overring of $\mathbb{Z}[\alpha]$. We may assume without loss of generality that I_α is a proper ideal of both $\mathbb{Z}[\alpha]$ and $\mathbb{Z}_\alpha$. Next, we claim that the induced map $\mathrm{Spec}(\mathbb{Z}_\alpha/I_\alpha) \to \mathrm{Spec}(\mathbb{Z}[\alpha]/I_\alpha)$ is a bijection. For, since $\mathbb{Z}[\alpha]$ and $\mathbb{Z}_\alpha$ are each one-dimensional Noetherian domains, an associated prime of I_α in $\mathbb{Z}[\alpha]$ (resp., $\mathbb{Z}_\alpha$) is just a prime ideal of $\mathbb{Z}[\alpha]$ (resp., $\mathbb{Z}_\alpha$) containing I_α. As the set of these primes is in canonical one-to-one correspondence with $\mathrm{Spec}(\mathbb{Z}[\alpha]/I_\alpha)$ (resp., $\mathrm{Spec}(\mathbb{Z}_\alpha/I_\alpha)$), (4) follows.

(4)⇒(3). By the lying-over property for integral extensions (*cf.* [**14**, Theorem 44]), f is surjective in general. Given (4), one need only show that if $P \in \mathrm{Spec}(\mathbb{Z}[\alpha])$ does not contain I_α, then at most one $Q \in \mathrm{Spec}(\mathbb{Z}_\alpha)$ can contract to P. This in turn follows from [**14**, Exercise 41(b), page 46], for any such Q satisfies $(\mathbb{Z}_\alpha)_Q = (\mathbb{Z}[\alpha])_P$, whence $Q = P(\mathbb{Z}[\alpha])_P \cap \mathbb{Z}_\alpha$.

(3)⇒(5). Since $\mathbb{Z}_\alpha$ is a (Dedekind, hence) Prüfer domain, it is a GPVD (*cf.* [**4**, p. 156]). Hence we may assume that $\mathbb{Z}[\alpha] \neq \mathbb{Z}_\alpha$ and, in particular, that I_α is a nonzero proper ideal. Since $\mathbb{Z}[\alpha]$ and $\mathbb{Z}_\alpha$ are each one-dimensional, (3) easily leads to condition (1) in [**4**, Theorem 3.1] (with I_α playing the role of the common radical ideal A), yielding (5).

(5)⇒((1) and (2)). Assume (5). Then, since $\mathbb{Z}[\alpha]$ is Noetherian, [**4**, Proposition 3.6] yields that $\mathbb{Z}_\alpha$ is the Prüfer domain associated to (the GPVD) $\mathbb{Z}[\alpha]$. Then (*cf.* [**4**, p. 156]) f is a homeomorphism. As we have noted *via* one-dimensionality that the associated primes of I_α are just the prime ideals containing I_α, the second assertion in (1) follows easily from the bijectivity of f. Moreover, by [**4**, p. 156 and Remarks 2.4(a)], (the GPVD) $\mathbb{Z}[\alpha]$ is an LPVD and, hence, seminormal. Then, as in the above proof that (1)⇒(4), we see that I_α is a radical ideal of $\mathbb{Z}_\alpha$.

Finally, since (2)⇒(3) trivially, the proof is complete.

As a first step in specializing to the quadratic case, $\alpha = n\omega_d$, we shall characterize the condition "f is a bijection" that appeared above.

PROPOSITION 2.2. *Let d be a square-free integer and let* $n \geq 2$ *be an integer. Let* $f: \mathrm{Spec}(\mathbb{Z}[\omega_d]) \to \mathrm{Spec}(\mathbb{Z}[n\omega_d])$ *be the canonical contraction map. Then the following conditions are equivalent.*

(1) *f is a bijection.*

(2) $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[\omega_d]$ *is an i-extension*; *i.e., f is an injection.*

(3) *f is a homeomorphism.*

(4) *If p is a rational prime such that $p \mid n$, then p is not split in* $\mathbb{Z}[\omega_d]$.

(5) (a) *If p is an odd rational prime such that $p \mid n$ and $p \nmid d$, then* $(d/p) = -1$; *and*

(b) *if $d \equiv 1 \pmod 8$, then n is odd.*

Before proving Proposition 2.2, we collect some useful tools.

LEMMA 2.3. *Let d and n be as in Proposition 2.2.*

(i) *If p is a rational prime such that $p \nmid n$, then* $p\mathbb{Z}[\omega_d] \cap \mathbb{Z}[n\omega_d] = p\mathbb{Z}[n\omega_d]$.

(ii) $n\mathbb{Z}[\omega_d] \cap \mathbb{Z}[n\omega_d] = n\mathbb{Z}[\omega_d] = (\mathbb{Z}[n\omega_d] : \mathbb{Z}[\omega_d]) \supsetneqq n\mathbb{Z}[n\omega_d]$.

(iii) *Let $P \in \operatorname{Spec}(\mathbb{Z}[\omega_d])$ be nonzero, with $P \cap \mathbb{Z} = p\mathbb{Z}$. Then $P \supset n\mathbb{Z}[\omega_d]$, if, and only if, $p \mid n$.*

(iv) *For a rational prime p, the following three conditions are equivalent (and make no mention of n).*

(1) $\mathbb{Z}[p\omega_d] \subset \mathbb{Z}[\omega_d]$ *is an i-extension.*

(2) *p is not split in* $\mathbb{Z}[\omega_d]$.

(3) (a) *If p is odd and $p \nmid d$, then* $(d/p) = -1$; *and*

(b) *if $d \equiv 1 \pmod 8$, then p is odd.*

Proof of Lemma 2.3. (i) One inclusion is clear. Conversely, let $e \in p\mathbb{Z}[\omega_d] \cap \mathbb{Z}[n\omega_d]$. Then $e = p(a + b\omega_d) = a_1 + b_1 n\omega_d$ for suitable a, b, a_1, $b_1 \in \mathbb{Z}$. Then $pa = a_1$ and $pb = b_1 n$. Since $p \nmid n$, the preceding equation yields $p \mid b_1$, say $b_1 = pb_2$, with $b_2 \in \mathbb{Z}$. Then $e = p(a + b_2 n\omega_d) \in p\mathbb{Z}[n\omega_d]$, as desired.

(ii) The second equation follows easily from the fact that $(\omega_d)^2 \in \mathbb{Z} + \mathbb{Z}\omega_d$. The other assertions are evident.

(iii) Note that $n\mathbb{Z}[\omega_d] \cap \mathbb{Z} = n\mathbb{Z}$. Hence if $P \supset n\mathbb{Z}[\omega_d]$ then intersecting with $\mathbb{Z}$ yields $p\mathbb{Z} \supset n\mathbb{Z}$, whence $p \mid n$. Conversely, if $p \mid n$, then $n\mathbb{Z}[\omega_d] \subset (p\mathbb{Z})\mathbb{Z}[\omega_d] = p\mathbb{Z}[\omega_d] \subset P$, as desired.

(iv) Consider the function $g : \mathbb{Z}[p\omega_d] \to \mathbb{Z}/p\mathbb{Z}$ which sends $a + bp\omega_d$ to $a + p\mathbb{Z}$ for all $a, b \in \mathbb{Z}$. Since $(\omega_d)^2 \in \mathbb{Z} + \mathbb{Z}\omega_d$, one readily shows that g is a surjective ring-homomorphism, with $\ker(g) = p\mathbb{Z}[\omega_d]$. In particular, $p\mathbb{Z}[\omega_d]$ is a prime of $\mathbb{Z}[p\omega_d]$. As (ii) established that $p\mathbb{Z}[\omega_d]$ is the conductor of $\mathbb{Z}[\omega_d]$ in $\mathbb{Z}[p\omega_d]$, one may infer from [**14**, Exercise 41(b), page 46], the one-dimensionality of these rings and the lying-over property that (1) holds, if, and only if, (at most) one prime ideal of $\mathbb{Z}[\omega_d]$ meets $\mathbb{Z}[p\omega_d]$ in $p\mathbb{Z}[\omega_d]$. By the incomparability property, $p\mathbb{Z}[\omega_d]$ is the only prime of $\mathbb{Z}[p\omega_d]$ that lies over $p\mathbb{Z}$. Thus, (1) holds, if, and only if, (at most) one prime of $\mathbb{Z}[\omega_d]$ lies over $p\mathbb{Z}$. In other words, (1)$\Leftrightarrow$(2). Finally, (2)$\Leftrightarrow$(3) by classical facts about $\mathbb{Z}[\omega_d]$ (*cf.* [**19**, Chapter 6]). This completes the proof of the lemma.

Proof of Proposition 2.2. By integrality, f is surjective; hence, (1)$\Leftrightarrow$(2). Also by integrality, f is a closed (and continuous) map; hence, (1)$\Leftrightarrow$(3). Moreover, since n factors as a nontrivial product of primes in $\mathbb{Z}$, Lemma 2.3(iv) [(2)$\Leftrightarrow$(3)] easily leads to (4)$\Leftrightarrow$(5).

Notice also, *via* Lemma 2.3(ii) and [**14**, Exercise 41(b), page 46], that f induces a bijection between $\operatorname{Spec}(\mathbb{Z}[\omega_d]) \setminus V(n\mathbb{Z}[\omega_d])$ and $\operatorname{Spec}(\mathbb{Z}[n\omega_d]) \setminus V(n\mathbb{Z}[\omega_d])$. (As usual, if I is an ideal of a ring A, then $V(I)$ denotes the set of prime ideals of A which contain I.)

(2)$\Rightarrow$(4). Assume (2). Then, given $p \mid n$, $\mathbb{Z}[p\omega_d]$ is contained between $\mathbb{Z}[n\omega_d]$ and $\mathbb{Z}[\omega_d]$; then, using (2), we see that $\mathbb{Z}[p\omega_d] \subset \mathbb{Z}[\omega_d]$ is an i-extension. Hence, Lemma 2.3(iv) [(1)$\Leftrightarrow$(2)] yields (4), as desired.

(4)$\Rightarrow$(2). Let $n = \prod p_j^{e_j}$ be the prime-power factorization of n in $\mathbb{Z}$. Assume (4). Then, by Lemma 2.3(iv), $\mathbb{Z}[p_j\omega_d] \subset \mathbb{Z}[\omega_d]$ is an i-extension for each j. Suppose that (2) fails. Then there exist distinct $P_1, P_2 \in \operatorname{Spec}(\mathbb{Z}[\omega_d])$ such that $P_1 \cap \mathbb{Z}[n\omega_d] = P_2 \cap \mathbb{Z}[n\omega_d] =$, say, P; and, by the second paragraph of this proof, $n\mathbb{Z}[\omega_d] \subset P$. Let p be the rational prime such that $P \cap \mathbb{Z} = p\mathbb{Z}$. By Lemma 2.3(iii), $p = p_j$ for some j. Hence $P_1 \cap \mathbb{Z}[p\omega_d]$ and $P_2 \cap \mathbb{Z}[p\omega_d]$ are distinct primes of the one-dimensional ring $\mathbb{Z}[p\omega_d]$, each containing $p\mathbb{Z}[\omega_d]$. Since the proof of Lemma 2.3(iv) showed that $p\mathbb{Z}[\omega_d]$ is a (nonzero) prime of $\mathbb{Z}[p\omega_d]$, we have the desired contradiction, and the proof of Proposition 2.2 is complete.

Remark 2.4. (a) It is useful to record what was just proved. Given d and $n = \prod p_j^{e_j}$ as in Proposition 2.2, then $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[\omega_d]$ is an i-extension, if, and only if, $\mathbb{Z}[p_j\omega_d] \subset \mathbb{Z}[\omega_d]$ is an i-extension for each j.

(b) Let p be a rational prime and d a square-free integer, and consider positive integers $h < k$. Given the above work, it seems interesting to note that $\mathbb{Z}[p^k\omega_d] \subset \mathbb{Z}[p^h\omega_d]$ is an i-extension. We shall now prove this.

To do so, first observe that $(\mathbb{Z}[p^k\omega_d] : \mathbb{Z}[p^h\omega_d]) = p^{k-h}\mathbb{Z}[p^h\omega_d]$. (*Cf.* the proof of Lemma 2.3(ii).) Call this ideal I. Then, by reasoning as in the proof of Proposition 2.2, it will suffice to show that $V(I)$ is a singleton subset of $\operatorname{Spec}(\mathbb{Z}[p^h\omega_d])$. As we saw in the proof of Lemma 2.3(iv), $p\mathbb{Z}[\omega_d]$ is a prime of $\mathbb{Z}[p\omega_d]$; hence, $P = p\mathbb{Z}[\omega_d] \cap \mathbb{Z}[p^h\omega_d]$ is a prime of $\mathbb{Z}[p^h\omega_d]$. As this ring is one-dimensional, it now suffices to show that P is the radical of I (for $\{P\}$ is then the required singleton set). One inclusion is easy since each element of $p\mathbb{Z}[\omega_d]$ has a suitable power in I. Conversely, let $\xi \in \operatorname{rad}(I)$. Then $\xi = a + bp^h\omega_d$ for suitable integers a and b, and there exists an integer $N \geq 1$ such that

$$(a + bp^h\omega_d)^N = \xi^N \in I = p^{k-h}\mathbb{Z} + p^k\mathbb{Z}\omega_d.$$

Since $\omega_d^2 \in \mathbb{Z} + \mathbb{Z}\omega_d$, the left-hand side is in $a^N + p\mathbb{Z} + p\mathbb{Z}\omega_d$. Thus $a^N \in p^{k-h}\mathbb{Z} + p\mathbb{Z}$, whence $p \mid a$ and $\xi \in P$. This completes the proof of the remark.

We are now able to interpret the conditions in Proposition 2.1 for an arbitrary quadratic order.

THEOREM 2.5. *Let d be a square-free integer and let $n \geq 2$ be an integer. Then the following four conditions are equivalent.*

(1) *$\mathbb{Z}[n\omega_d]$ is a GPVD.*

(2) *$\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[\omega_d]$ is an i-extension and $n\mathbb{Z}[\omega_d]$ is a radical ideal of $\mathbb{Z}[\omega_d]$.*

(3) *n is square-free and, for each rational prime p such that $p \mid n$, p is inert in $\mathbb{Z}[\omega_d]$.*

(4) *n is square-free. In addition, if n is even, then $d \equiv 5 \pmod 8$. If p is an odd rational prime such that $p \mid n$, then ($p \nmid d$ and)*

$$\left(\frac{d}{p}\right) = -1.$$

Moreover, if the above conditions hold, then $\mathbb{Z}[\omega_d]$ is the Prüfer domain canonically associated to the GPVD, $\mathbb{Z}[n\omega_d]$.

Proof. (1)$\Rightarrow$(2). In view of Lemma 2.3(ii) and integrality, this equivalence is the special case $\alpha = n\omega_d$ of Proposition 2.1 [(5)$\Leftrightarrow$(3)].

(2)$\Rightarrow$(3). Assume (2). If n were not square-free, $q^2 \mid n$ for some rational prime q, and it would follow that $q \in \text{rad}(n\mathbb{Z}[\omega_d]) = n\mathbb{Z}\omega_d$; then $q \in q^2\mathbb{Z}$, a contradiction. Hence, n is square-free. Next, let p be a rational prime dividing n. By Lemma 2.3(iv) and Remark 2.4(a), p is not split in $\mathbb{Z}[\omega_d]$. If p is not inert, then p ramifies: $p\mathbb{Z}[\omega_d] = P^2$, with $P \in \text{Spec}(\mathbb{Z}[\omega_d])$. Then $n\mathbb{Z}[\omega_d] = (np^{-1})P^2$, which has a unique factorization as a product of prime ideals in the Dedekind domain $\mathbb{Z}[\omega_d]$, cannot be a radical ideal, a contradiction. Hence, p is inert in $\mathbb{Z}[\omega_d]$.

(3)$\Rightarrow$(2). Assume (3). The ideas that were used to prove (2)$\Rightarrow$(3) also work here. To see why $J = n\mathbb{Z}[\omega_d]$ is radical, note that $n = \prod p_i$ for pairwise distinct rational primes p_i; each $p_i\mathbb{Z}[\omega_d] =$, say, $P_i \in \text{Spec}(\mathbb{Z}[\omega_d])$; and $J = \prod P_i = \bigcap P_i$ is evidently radical.

(3)$\Leftrightarrow$(4). This follows from classical quadratic theory, as in [**19**, Chapter 6].

The final assertion follows from the Noetherianness of $\mathbb{Z}[n\omega_d]$ (*cf.* [**4**, Proposition 3.6]). The proof is complete.

Remark 2.6. Let d be a square-free integer, $d \equiv 1 \pmod 4$. By Theorem 2.5, $\mathbb{Z}[\sqrt{d}](= \mathbb{Z}[2\omega_d])$ is a GPVD, if, and only if, $d \equiv 5 \pmod 8$. Thus Theorem 2.5 recovers the motivating result, [**5**, Example 4].

§3. *When the seminormalization is the ring of integers.* A sufficient condition for the property studied in Proposition 2.1 is that $\mathbb{Z}[\alpha] = \mathbb{Z}_\alpha$. This condition has been of recurring interest (*cf.* [**15**, **18**, **1**, **13**, **12**]). We next give a fresh characterization of this condition in terms of seminormalization and other concepts which figured in Section 2.

PROPOSITION 3.1. *For an algebraic integer α, the following conditions are equivalent.*

(1) *$\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$ and I_α is a radical ideal of $\mathbb{Z}_\alpha$.*

(2) *$\mathbb{Z}[\alpha] = \mathbb{Z}_\alpha$.*

Proof. (2)$\Rightarrow$(1) trivially. (1)$\Rightarrow$(2) directly via a result of Traverso [**17**, Corollary 1.8] but, for the sake of completeness, we offer the following alternate proof. It will be convenient to write $R = \mathbb{Z}[\alpha]$ and $T = \mathbb{Z}_\alpha$. Assume (1). Without loss of generality, $R \neq T$, and so I_α is a (nonzero) proper ideal. For each prime ideal P of R containing I_α, the seminormalization hypothesis gives a unique $Q \in \operatorname{Spec}(T)$ contracting to P, and the canonical map $R/P \to T/Q$ is an isomorphism. One then has a commutative diagram

$$\begin{array}{ccc} R/\bigcap P & \longrightarrow & \Pi R/P \\ \downarrow & & \downarrow \\ T/\bigcap Q & \longrightarrow & \Pi T/Q \end{array}$$

in which the horizontal maps are isomorphisms given by the Chinese Remainder Theorem; and the right-vertical map is (an isomorphism) given coordinatewise by the above isomorphisms $R/P \to T/Q$. Then the left-vertical map is also an isomorphism. Hence $T = R + \bigcap Q = R + \operatorname{rad}_T(I_\alpha) = R + I_\alpha = R$, completing the proof.

Remark 3.2. (a) As noted by Ooishi [**16**, Example 1], $\mathbb{Z}[\sqrt{-4}]^+ = \mathbb{Z}[i]$. (Another proof of this would follow from Theorem 3.4 below.) Thus, by taking $\alpha = \sqrt{-4} = 2\sqrt{-1}$, we see that the "radical" hypothesis on I_α cannot be deleted from condition (1) of Proposition 3.1.

(b) Of course, the "$\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$" condition also cannot be deleted. To illustrate this, consider $\alpha = \sqrt{5}$. Then (*cf.* [**5**, Example 4] and Proposition 2.1) I_α is radical, although $\mathbb{Z}[\alpha]^+ = \mathbb{Z}[\alpha] \neq \mathbb{Z}_\alpha$. The (not necessarily GPVD) seminormal quadratic orders are characterized in Theorem 4.1 below.

(c) If $\mathbb{Z}[\alpha] = \mathbb{Z}_\alpha$, then $\mathbb{Z}[\alpha]$ is a (Dedekind domain, hence a) GPVD. Another way to see this is to relate condition (1) of Proposition 3.1 and condition (2) of Proposition 2.1: in this case, the homeomorphic nature of f is explained naturally *via* seminormalization.

In view of Proposition 3.1, we focus next on characterizing the condition, "$\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$". In fact, we shall relate this condition to many of the universal properties in [**11**, p. 240], [**2**, **8**, **6**, and **7**]. (See these references as needed for background information. In particular, $(-)^*$ denotes weak normalization in the sense of [**2**].)

PROPOSITION 3.3. *Let α be an algebraic integer, $i: \mathbb{Z}[\alpha] \to \mathbb{Z}_\alpha$ the inclusion (map), and $f: \operatorname{Spec}(\mathbb{Z}_\alpha) \to \operatorname{Spec}(\mathbb{Z}[\alpha])$ the induced contraction map. Then the following conditions are equivalent.*

(1) *i is universally going-down.*

(2) *i is UGD.*

(3) *i is radiciel.*

(4) *i is universally mated.*

(5) *f is universally open.*

(6) *f is a universal homeomorphism.*

(7) $\mathbb{Z}[\alpha]^* = \mathbb{Z}_\alpha$.

(8) $\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$.

Proof. (8)$\Leftrightarrow$(3). (3) holds, if, and only if, (the surjective map) f is an injection and the extensions of residue class fields induced by f are all (algebraic) purely inseparable. As the residue class fields of R are all perfect,

(3) is thus equivalent to requiring that f is a bijection and each of the extensions of residue class fields induced by f is an isomorphism. In view of the characterization of seminormalization in [**17**], the assertion follows.

(3)$\Leftrightarrow$(4). By [6, Theorems 2.1 and 2.5], (3) holds, if, and only if, $f_1: \mathrm{Spec}\,(\mathbb{Z}_\alpha[X]) \to \mathrm{Spec}\,(\mathbb{Z}[\alpha][X])$ is an injection; and (4) holds, if, and only if, $\mathbb{Z}[\alpha][X] \to \mathbb{Z}_\alpha[X]$ is mated. However, each of these conditions is equivalent to f_1 being a bijection (and so the assertion follows). The point is that f_1 is surjective since i is (universally) integral.

(4)$\Rightarrow$(2). One-dimensionality assures that i satisfies going-down. Accordingly, [**6**, Proposition 3.14] yields the assertion.

(2)$\Rightarrow$(7). [**6**, Remark 3.6] yields this assertion.

(7)$\Rightarrow$(6). By [**2**, Teorema 1], $\mathrm{Spec}\,(\mathbb{Z}[\alpha]^*) \to \mathrm{Spec}\,(\mathbb{Z}[\alpha])$ is a universal homeomorphism. The assertion follows.

(6)$\Rightarrow$(5). Trivial.

(5)$\Rightarrow$(1). It suffices to recall that "open" implies "going-down" [**11**]. (*Cf.* also [**8**].)

(1)$\Rightarrow$(2). Apply [**6**, Corollary 3.20].

(2)$\Rightarrow$(4). Apply [**6**, Corollary 3.12(b)].

The proof is complete.

We next use some of the material in Section 2 to interpret the condition "$\mathbb{Z}[\alpha]^+ = \mathbb{Z}_\alpha$" for a typical quadratic order, with $\alpha = n\omega_d$.

THEOREM 3.4. *Let d be a square-free integer and let $n \geq 2$ be an integer. Then the following conditions are equivalent.*

(1) $\mathbb{Z}[n\omega_d]^+ = \mathbb{Z}[\omega_d]$.

(2) *Each rational prime p such that $p \mid n$ is ramified in $\mathbb{Z}[\omega_d]$.*

(3) *If $d \equiv 1 \pmod 4$, then n is odd. In addition, d is divisible by each odd prime divisor of n.*

Proof. By [**17**], (1) holds, if, and only if, both the following conditions hold: $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[\omega_d]$ is an i-extension, and $k(P \cap \mathbb{Z}[n\omega_d]) \xrightarrow{\cong} k(P)$ for each $P \in \mathrm{Spec}\,(\mathbb{Z}[\omega_d])$. As in the proof of Proposition 2.2, these conditions need only be checked in case P contains the conductor, $n\mathbb{Z}[\omega_d]$; that is, by Lemma 2.3(iii), in case $P \cap \mathbb{Z} = p\mathbb{Z}$ with $p \mid n$. Set $P_1 = P \cap \mathbb{Z}[p\omega_d]$ and $P_2 = P \cap \mathbb{Z}[n\omega_d]$. By incomparability, $P_1 = p\mathbb{Z}[\omega_d]$ and $P_2 = p\mathbb{Z}[\omega_d] \cap \mathbb{Z}[n\omega_d]$. Consider the field extensions

$$\mathbb{Z}/p\mathbb{Z} \hookrightarrow k(P_2) \hookrightarrow k(P_1) \hookrightarrow k(P).$$

As noted in the proof of Lemma 2.3(iv), $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} k(P_1)$. Thus $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} k(P_2)$. Hence, using quadratic ramification theory (concerning $\sum e_i f_i$), we have the following assertions. If p is ramified in $\mathbb{Z}[\omega_d]$, then $k(P_2) \xrightarrow{\cong} k(P)$; if p is inert in $\mathbb{Z}[\omega_d]$, then $k(P_2) \hookrightarrow k(P)$ is two-dimensional. By combining Proposition 2.2 [(2)$\Leftrightarrow$(4)] with the above consequence of [17], we can now conclude that (1)$\Leftrightarrow$(2). Finally, (2)$\Leftrightarrow$(3) by classical quadratic theory [**19**, Chapter 6]. The proof is complete.

Remark 3.5. (a) The conditions in Theorems 2.5 and 3.4 cannot hold simultaneously: ramified primes are not inert! This is to be expected since each GPVD is (equal to its own) seminormal (ization), while $n \geq 2$ forces $\mathbb{Z}[n\omega_d] \neq \mathbb{Z}[\omega_d]$.

(b) Let d be a square-free integer. Then either $d \equiv 2, 3 \pmod 4$ or $d \equiv 1 \pmod 4$. We could use condition (3) in Theorem 3.4 to develop examples showing that this congruence information alone does not determine whether $\mathbb{Z}[3\omega_d]^+$ and $\mathbb{Z}[\omega_d]$ coincide. However, it is enough to observe *via* this condition that $\mathbb{Z}[3\omega_d]^+ = \mathbb{Z}[\omega_d]$, if, and only if, $3 \mid d$.

(c) The case $n=2$ deserves separate treatment. Let d be a square-free integer. If $d \equiv 2, 3 \pmod 4$, then $\mathbb{Z}[2\omega_d] = \mathbb{Z}[2\sqrt{d}]$ is not seminormal, since $\mathbb{Z}[2\omega_d]^+ = \mathbb{Z}[\omega_d] = \mathbb{Z}[\sqrt{d}]$ in this case. However, if $d \equiv 1 \pmod 4$, then we see as above that $\mathbb{Z}[2\omega_d]^+ \neq \mathbb{Z}[\omega_d]$; since $\mathbb{Z}[2\omega_d] \subset \mathbb{Z}[2\omega_d]^+ \subset \mathbb{Z}[\omega_d]$, we have $\mathbb{Z}[2\omega_d]^+ = \mathbb{Z}[2\omega_d]$ in this case. In other words, if $d \equiv 1 \pmod 4$, then $\mathbb{Z}[2\omega_d] = \mathbb{Z}[\sqrt{d}]$ is seminormal. Thus, by Theorem 2.5, $\mathbb{Z}[2\omega_{17}] = \mathbb{Z}[\sqrt{17}]$ is an example of a seminormal quadratic order which is not a GPVD; for a "complex" example, consider $\mathbb{Z}[\sqrt{-7}]$.

By extending this sort of reasoning, we shall determine all seminormal quadratic orders in section 4. More generally, we shall identify the seminormalization of each quadratic order.

§4. *The seminormal quadratic orders.* The rings mentioned in this section's title will be characterized in Corollary 4.5, as a consequence of this section's main result, Theorem 4.4. The latter result computes the seminormalization of an arbitrary quadratic order $\mathbb{Z}[n\omega_d]$; that is, computes the positive integer m such that $\mathbb{Z}[n\omega_d]^+ = \mathbb{Z}[m\omega_d]$. To prepare the way, we next generalize some of the earlier sections' material concerning the inclusion $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[\omega_d]$ to the context $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[m\omega_d]$.

LEMMA 4.1. *Let d be a square-free integer and let m, n be positive integers.*

(i) *If $m \mid n$, then*

$$(n/m)\mathbb{Z}[m\omega_d] \cap \mathbb{Z}[n\omega_d] = (n/m)\mathbb{Z}[m\omega_d] = (\mathbb{Z}[n\omega_d] : \mathbb{Z}[m\omega_d]).$$

(ii) *If p is a rational prime, then $p\mathbb{Z}[m\omega_d]$ is a prime ideal of $\mathbb{Z}[pm\omega_d]$.*

Proof. (i) As in the proof of Lemma 2.3(ii), this follows easily from the fact that $\{1, k\omega_d\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[k\omega_d]$, for each integer $k \geq 1$.

(ii) Consider the function $g: \mathbb{Z}[pm\omega_d] \to \mathbb{Z}/p\mathbb{Z}$ which sends $a + bpm\omega_d$ to $a + p\mathbb{Z}$ for all $a, b \in \mathbb{Z}$. As in the proof of Lemma 2.3(iv), g is a surjective ring-homomorphism, with $\ker(g) = p\mathbb{Z}[m\omega_d]$, and the assertion is immediate.

LEMMA 4.2. *Let d, m, n be as in Lemma 4.1, with $m \mid n$. Then the following three conditions are equivalent.*

(1) *$\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[m\omega_d]$ is an i-extension.*

(2) *If p is a rational prime such that $p \mid (n/m)$ and $p \nmid m$, then p is not split in $\mathbb{Z}[\omega_d]$.*

(3) *Let $n = p_1^{e_1} \ldots p_r^{e_r}$ and $m = p_1^{g_1} \ldots p_s^{g_s}$ be the prime-power decompositions of n and m (with $s \leq r$ and $1 \leq g_i \leq e_i$ for $i \leq s$). Let $p = p_j$ for some $s+1 \leq j \leq r$.*

If $d \equiv 1 \pmod 8$, *then* p *is odd. Moreover, if* p *is odd and* $p \nmid d$, *then*

$$\left(\frac{d}{p}\right) = -1.$$

Proof. By Proposition 2.2, we may assume $m \neq 1$.

(1)$\Rightarrow$(2). Deny this. Then there exists a rational prime p such that $p \mid (n/m)$, $p \nmid m$, and $p\mathbb{Z}[\omega_d] = P_1 P_2$, where P_1 and P_2 are distinct prime ideals of $\mathbb{Z}[\omega_d]$. Put $Q_i = P_i \cap \mathbb{Z}[m\omega_d]$. As $P_i \cap \mathbb{Z} = p\mathbb{Z}$, so also $Q_i \cap \mathbb{Z} = p\mathbb{Z}$. If Q_i contains $(\mathbb{Z}[m\omega_d] : \mathbb{Z}[\omega_d]) = m\mathbb{Z}[\omega_d]$, then intersecting with $\mathbb{Z}$ leads to $p\mathbb{Z} \supset m\mathbb{Z}[\omega_d] \cap \mathbb{Z}$, contradicting $p \nmid m$. Thus Q_i does not contain the conductor, and so (*cf.* [**14**, Exercise 41(b), p. 46]) the fact that $P_1 \neq P_2$ assures that $Q_1 \neq Q_2$. By (1) and the condition $p \mid (n/m)$, $Q_1 \cap \mathbb{Z}[pm\omega_d] \neq Q_2 \cap \mathbb{Z}[pm\omega_d]$. However, we also have equality (and thus the desired contradiction): $p\mathbb{Z}[m\omega_d] \subset Q_i \cap \mathbb{Z}[pm\omega_d]$ are primes of $\mathbb{Z}[pm\omega_d]$ each lying over $p\mathbb{Z}$, whence equality follows via incomparability.

(2)$\Rightarrow$(1). Deny this. Then there exist distinct Q_1, Q_2 in Spec $(\mathbb{Z}[m\omega_d])$ such that $Q_1 \cap \mathbb{Z}[n\omega_d] = Q_2 \cap \mathbb{Z}[n\omega_d] =$, say, P. Let p be the rational prime such that $P \cap \mathbb{Z} = p\mathbb{Z}$. As P must contain $(\mathbb{Z}[n\omega_d] : \mathbb{Z}[m\omega_d])$, Lemma 4.1(i) readily yields $p \mid (n/m)$. Next, choose (distinct) $P_1, P_2 \in$ Spec $(\mathbb{Z}[\omega_d])$ such that $P_i \cap \mathbb{Z}[m\omega_d] = Q_i$. One can now see that $p \nmid m$. Indeed, if $p \mid m$, then incomparability would force the inclusion $p\mathbb{Z}[\omega_d] \subset P_i \cap \mathbb{Z}[p\omega_d]$ to be an equality, so that intersecting with $\mathbb{Z}[m\omega_d]$ would lead to $Q_1 = Q_2$, an absurdity. By (2), only one prime ideal of $\mathbb{Z}[\omega_d]$ lies over $p\mathbb{Z}$. Hence $P_1 = P_2$, the desired contradiction.

(2)$\Rightarrow$(3). Notice that $\{p_i : s+1 \leq i \leq r\}$ is the set of all rational primes p such that $p \mid (n/m)$ and $p \nmid m$. Accordingly, (2)$\Leftrightarrow$(3) follows from classical facts about $\mathbb{Z}[\omega_d]$ (*cf.* [**19**, Ch. 6]). The proof of Lemma 4.2 is complete.

It should be noted that, aside from some trivial set-theoretic and topological observations, setting $m = 1$ in Lemma 4.2 recovers Proposition 2.2.

LEMMA 4.3. *Let* d, m, n *be as in Lemma* 4.1, *with* $m \mid n$, *and suppose that* $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[m\omega_d]$ *is an i-extension.*

(i) *Let* $P \in$ Spec $(\mathbb{Z}[n\omega_d])$ *be nonzero, let* p *be the rational prime such that* $P \cap \mathbb{Z} = p\mathbb{Z}$, *and let* $Q \in$ Spec $(\mathbb{Z}[\omega_d])$ *such that* $Q \cap \mathbb{Z}[m\omega_d] = P$. *Suppose that* $p \mid (n/m)$. *If* $p \mid m$, *then* $k(P \cap \mathbb{Z}[n\omega_d]) \xrightarrow{\cong} k(P) \cong \mathbb{Z}/p\mathbb{Z}$. *If* $p \nmid m$, *then the following four conditions are equivalent.*

(1) $k(P \cap \mathbb{Z}[n\omega_d]) \xrightarrow{\cong} k(P)$.

(2) $[k(Q) : \mathbb{Z}/p\mathbb{Z}] = 1$.

(3) p *is ramified in* $\mathbb{Z}[\omega_d]$.

(4) *If* $d \equiv 1 \pmod 4$, *then* p *is odd. In addition, if* p *is odd, then* $p \mid d$.

(ii) *Let* $n = p_1^{e_1} \ldots p_r^{e_r}$ *and* $m = p_1^{g_1} \ldots p_s^{g_s}$ *be the prime-power decompositions of* n *and* m (*with* $s \leq r$ *and* $1 \leq g_i \leq e_i$ *for* $i \leq s$). *Then the following four conditions are equivalent* (*with* (4) *vacuously satisfied if* $s = r$).

(1) $\mathbb{Z}[m\omega_d]$ *is the seminormalization of* $\mathbb{Z}[n\omega_d]$ *in* $\mathbb{Z}[m\omega_d]$.

(2) $k(P \cap \mathbb{Z}[n\omega_d]) \xrightarrow{\cong} k(P)$ *for each* $P \in$ Spec $(\mathbb{Z}[m\omega_d])$.

(3) *If* p *is a rational prime such that* $p \mid (n/m)$ *and* $p \nmid m$, *then* p *is ramified in* $\mathbb{Z}[\omega_d]$.

(4) *Let* $p = p_i$ *for some* $s+1 \leq i \leq r$. *If* $d \equiv 1 \pmod 4$, *then* p *is odd. In addition, if* p *is odd, then* $p \mid d$.

Proof. (i) It is well known that (3)⇔(4) (*cf.* [**19**, Ch. 6]). Next, note that $p\mathbb{Z}[\omega_d] \subset Q \cap \mathbb{Z}[p\omega_d]$ are prime ideals of $\mathbb{Z}[p\omega_d]$ (*cf.* Lemma 4.1(ii)), each of which lies over $p\mathbb{Z}$. Thus, by incomparability, $p\mathbb{Z}[\omega_d] = Q \cap \mathbb{Z}[p\omega_d]$; moreover, $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} k(p\mathbb{Z}[\omega_d])$ by the proof of Lemma 2.3(iv).

Suppose $p \mid m$. Then we have the tower of fields

$$\mathbb{Z}/p\mathbb{Z} \hookrightarrow k(P \cap \mathbb{Z}[n\omega_d]) \hookrightarrow k(P) \hookrightarrow k(p\mathbb{Z}[\omega_d]).$$

By the above remarks, $\mathbb{Z}/p\mathbb{Z} \xrightarrow{\cong} k(P)$, and so the first assertion holds.

Suppose $p \nmid m$. Then, by Lemma 4.2 [(1)⇒(2)], only one prime of $\mathbb{Z}[\omega_d]$ (namely, Q) lies over $p\mathbb{Z}$. By the fundamental equation of ramification theory, (2)⇔(3). Next, notice that $k(P) \xrightarrow{\cong} k(Q)$ since the condition $p \nmid m$ assures that P does not contain $m\mathbb{Z}[\omega_d] = (\mathbb{Z}[m\omega_d] : \mathbb{Z}[\omega_d])$. In view of the tower of fields

$$\begin{aligned}\mathbb{Z}/p\mathbb{Z} &\hookrightarrow k(P \cap \mathbb{Z}[n\omega_d]) \hookrightarrow k(Q \cap \mathbb{Z}[(n/m)\omega_d]) \hookrightarrow k(Q \cap \mathbb{Z}[p\omega_d]) \\ &= k(p\mathbb{Z}[\omega_d]) \hookrightarrow k(Q)\end{aligned}$$

and the above remarks, (2)⇔(1).

(ii) Since $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[m\omega_d]$ is assumed to be an *i*-extension, the characterisation of seminormalization in [**17**] immediately yields (1)⇔(2). It is easy to see that the remaining equivalences follows from (i). (For some of these, one is given p, and then uses lying-over to arrange a P to which (i) applies.) The proof is complete.

We next present the main result of this section.

THEOREM 4.4. *Let* d *be a square-free integer and let* $m \leq n$ *be positive integers. Then the following four conditions are equivalent.*

(1) $\mathbb{Z}[n\omega_d]^+ = \mathbb{Z}[m\omega_d]$.

(2) *m is minimal among positive integers with respect to these three properties*: $m \mid n$, $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[m\omega_d]$ *is an i-extension, and* $k(P \cap \mathbb{Z}[n\omega_d]) \xrightarrow{\cong} k(P)$ *for each* $P \in \operatorname{Spec}(\mathbb{Z}[m\omega_d])$.

(3) $m = \prod \{p$: *p is a rational prime,* $p \mid n$, *and p is not ramified in* $\mathbb{Z}[\omega_d]\}$.

(4) $m = \prod \{p$ *is a rational prime,* $p \mid n$, $p \nmid d$ *if p is odd, and* $d \equiv 1 \pmod 4$ *if* $p = 2\}$.

Proof. (1)⇔(2). Apply the characterization of semi-normalization in [**17**].

(2)⇔(3). If $m \mid n$, p is a rational prime such that $p \nmid (n/m)$, and $P \in \operatorname{Spec}(\mathbb{Z}[m\omega_d])$ satisfies $P \cap \mathbb{Z} = p\mathbb{Z}$, then $k(P \cap \mathbb{Z}[n\omega_d]) \xrightarrow{\cong} k(P)$. (The point is that P does not contain $(\mathbb{Z}[n\omega_d] : \mathbb{Z}[m\omega_d]) = (n/m)\mathbb{Z}[m\omega_d]$.) Hence, by combining Lemma 4.2 with Lemma 4.3(i) (or by applying Lemma 4.3(ii)), we see that (2) is equivalent to m being minimal among divisors of n such that: whenever p is a rational prime such that $p \mid (n/m)$ and $p \nmid m$, then p is ramified in $\mathbb{Z}[\omega_d]$. Using a prime-power decomposition of n, one readily shows that this latter condition is equivalent to (3).

(3)$\Leftrightarrow$(4). Apply the classical facts about ramification of rational primes in quadratic number fields (*cf.* [**19**, Chapter 6]). The proof is complete.

COROLLARY 4.5. *Let d be a square-free integer and $n \geq 2$ an integer. Then the following three conditions are equivalent.*

(1) *$\mathbb{Z}[n\omega_d]$ is seminormal, that is, $\mathbb{Z}[n\omega_d]^+ = \mathbb{Z}[n\omega_d]$.*

(2) *n is square-free and no rational prime factor of n is ramified in $\mathbb{Z}[\omega_d]$.*

(3) *n is square-free. If n is even, then $d \equiv 1 \pmod 4$. The greatest common divisor of d and n is (either 1 or) an integral power of 2.*

Proof. Interpret $m = n$ for conditions (3) and (4) in the statement of Theorem 4.4.

Remark 4.6 (a) It is conventional to interpret a product of integers indexed by the empty set to be 1. Thus, taking $m = 1$, we see that Theorem 4.4 subsumes Theorem 3.4. Note that the above convention also renders Theorem 4.4 valid in case $n = 1$ $(= m)$, since $\mathbb{Z}[\omega_d]$ is (semi)normal.

(b) The examples in Remark 3.5(c) are illuminated by the following result. Let d be a square-free integer and $n \geq 2$ an integer. Then $\mathbb{Z}[n\omega_d]$ is a GPVD, if, and only if, $\mathbb{Z}[n\omega_d]$ is seminormal such that $\mathbb{Z}[n\omega_d] \subset \mathbb{Z}[\omega_d]$ is an i-extension. The easiest proof of this result is via the elementary number-theoretic criteria in Theorem 2.5(4), Corollary 4.5(3), and Proposition 2.2(5). Since each GPVD is seminormal, another proof follows from Proposition 2.1 [(5)$\Leftrightarrow$(3)] and [**17**, Lemma 1.3].

(c) Corollary 4.5 recovers all the facts implicit in Remark 3.5(c); for instance, $\mathbb{Z}[\sqrt{d}]$ is seminormal, for each square-free integer d. A far-reaching generalization of this, originally discovered by Ooishi, is given in Corollary 4.7 below.

(d) We have seen that if d is a square-free integer, then $A = \mathbb{Z}[\sqrt{d}]$ is seminormal. This is due to a variety of deep reasons. If $d \equiv 2, 3 \pmod 4$, A is actually normal. If $d \equiv 5 \pmod 8$, then A is a GPVD [**5**, Example 4], note, however, in this case that A is not root-closed [**3**, Proposition]. Finally, if $d \equiv 1 \pmod 8$, the above references show that A is root-closed, but not a GPVD. Thus, A is both root-closed and a GPVD, if, and only if, $d \equiv 2, 3 \pmod 4$; that is, if, and only if, A is the maximal order of $\mathbb{Q}(\sqrt{d})$.

More generally, if n is any positive integer and d is as above, then $B = \mathbb{Z}[n\omega_d]$ is both root-closed and a GPVD, if, and only if, $n = 1$. This is a consequence of the following result. Let R be an LPVD (for instance, a GPVD) whose quotient field K is an algebraic number field. Suppose that R is contained in the ring of integers D of K. Then R is integrally closed (that is, $R = D$) if (and only if) R is root-closed.

For a proof, note first, by the Krull-Akizuki theorem, that R is a one-dimensional Noetherian integral domain. Accordingly, by localizing and applying a result of Angermüller [**3**, Theorem 1], we have the desired assertion, *unless* D has some residue field $D/N \cong \mathbb{Z}/2\mathbb{Z}$. In this remaining case, one need only show that $M = N \cap R$ is such that (the *PVD*) R_M is integrally closed. This, in turn, follows from the $(L)PVD$ condition, as the isomorphism $R/M \to D/N (\cong \mathbb{Z}/2\mathbb{Z})$ leads to $R_M \cong D_N x_{D/N} R/M \cong D_N$: an integrally closed (DVR) integral domain.

The "Noetherian" hypothesis is essential above, for one may construct a one-dimensional root-closed PVD which is not integrally closed.

(e) Theorem 4.4(4) permits easy calculation of seminormalizations of quadratic orders, significantly extending Remark 3.5(b). We next list several additional applications.

$$\mathbb{Z}[3\sqrt{2}]^+ = \mathbb{Z}[3\omega_2]^+ = \mathbb{Z}[3\omega_2] = \mathbb{Z}[3\sqrt{2}],$$
$$\mathbb{Z}[3\sqrt{3}]^+ = \mathbb{Z}[3\omega_3]^+ = \mathbb{Z}[1\omega_3] = \mathbb{Z}[\sqrt{3}],$$
$$\mathbb{Z}[4\sqrt{2}]^+ = \mathbb{Z}[2\sqrt{2}]^+ = \mathbb{Z}[\sqrt{2}],$$
$$\mathbb{Z}[4\sqrt{3}]^+ = \mathbb{Z}[2\sqrt{3}]^+ = \mathbb{Z}[\sqrt{3}],$$
$$\mathbb{Z}[4\omega_5]^+ = \mathbb{Z}[2\omega_5] = \mathbb{Z}[\sqrt{5}] \neq \mathbb{Z}[\omega_5],$$
$$\mathbb{Z}[4\sqrt{6}]^+ = \mathbb{Z}[2\sqrt{6}]^+ = \mathbb{Z}[\sqrt{6}],$$
$$\mathbb{Z}[18i]^+ = \mathbb{Z}[6i]^+ = \mathbb{Z}[3\omega_{-1}] = \mathbb{Z}[3i].$$

In [**16**, Example 1, page 7], Ooishi characterized the non-square integers N such that $\mathbb{Z}[\sqrt{N}]$ is seminormal. We next recover this, as our final result.

COROLLARY 4.7. *Let N be a nonsquare integer. Write $N = n^2 d$, with prime-power decompositions $n = 2^{f_0} p_1^{f_1} \dots p_s^{f_s}$ and $d = \pm 2^{g_0} p_{s+1} \dots p_r$. (Here, $\{p_1, \dots, p_s\}$ and $\{p_{s+1}, \dots, p_r\}$ are each sets of pairwise distinct odd primes, $f_0 \geq 0, f_i \geq 1$ for all $1 \leq i \leq s$, and $g_0 \leq 1$. If $s = r$, $d = \pm 2^{g_0}$.) Then $\mathbb{Z}[\sqrt{N}]$ is seminormal, if, and only if, $N = \pm 2^{h_0} p_1^2 \dots p_s^2 p_{s+1} \dots p_r$ with $h_0 \leq 1$ and $p_i \neq p_j$ whenever $1 \leq i \leq s$ and $s+1 \leq j \leq r$.*

Proof. As N is not a perfect square, $d \neq 1$. Thus, d is square-free. By Remark 4.6(c) and the convention regarding products indexed by the empty set, the assertion is verified if $n = 1$. Assume henceforth that $n \neq 1$. There are two cases.

Suppose $d \equiv 2, 3 \pmod 4$. Then $\mathbb{Z}[\sqrt{N}] = \mathbb{Z}[n\sqrt{d}] = \mathbb{Z}[n\omega_d]$. By Corollary 4.5 [(1)$\Leftrightarrow$(3)], $\mathbb{Z}[\sqrt{N}]$ is seminormal, if, and only if, $n = p_1 \dots p_s$ ($\neq 1$) and $(d, n) = 1$. The assertion is evidently verified in this case.

Finally, suppose $d \equiv 1 \pmod 4$. Then $\mathbb{Z}[\sqrt{N}] = \mathbb{Z}[2n\omega_d]$. By another appeal to Corollary 4.5, $\mathbb{Z}[\sqrt{N}]$ is seminormal, if, and only if, $2n$ is square-free and $(d, 2n)$ is (either 1 or) an integral power of 2; that is, if, and only if, $f_0 = 0$, $f_1 = \dots = f_s = 1$, and $\{p_1, \dots, p_s\} \cap \{p_{s+1}, \dots, p_r\} = \emptyset$. The assertion clearly holds in this case, completing the proof.

Acknowledgments. The authors gratefully acknowledge partial support from each other's institution and from a NATO Collaborative Research grant. The second author worked under the auspices of the GNSAGA of the Consiglio Nazionale delle Ricerche.

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12A25: *ALGEBRAIC NUMBER THEORY; Algebraic number theory; global fields; Quadratic fields*

Received on the 11th of November, 1986.